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ABELIAN SUBGROUPS OF FINITE p-GROUPS

BY

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ABSTRACT. Information is obtained about the order of maximal abelian subgroups of central powers and crown products of finite p-groups. This is used to construct groups with "small" maximal abelian subgroups.

1. **Introduction.** G. A. Miller [3] has proved that a group of order p^n has a normal abelian subgroup of order p^m, such that $m(m+1)/2 \geq n$ (for another proof see [2, p. 302]).

In the other direction, J. L. Alperin [1] has shown that, for every odd prime p and positive integer n, there is a group of order p^{3n+2} all of whose abelian subgroups have order at most p^{n+2}. Moreover, there exists a group of order 2^{25} all of whose abelian subgroups have order at most 2^{12}.

In §2 of this paper, maxima are obtained for the orders of abelian subgroups of certain central powers and crown products in terms of those of the factors.

It is possible to improve Alperin's results by applying these constructions. Specifically, the following result will be proved.

Theorem. *For each positive integer n, there exists a group of order 2^{5n+2} all of whose abelian subgroups have order at most 2^{2n+2}. For each odd prime p, there is a group of order $p^{\alpha(n)}$, where $\alpha(n) = 3p^n + 2 + (p^n - 1)/(p-1)$, all of whose abelian subgroups have order at most $p^{\beta(n)}$, where $\beta(n) = p^n + 2$.*

The improvement for $p = 2$ is clear and for p odd

$$0 < \frac{\beta(n)}{\alpha(n)} - \frac{1}{3 + 1/(p-1)} < \frac{1}{p^n}.$$

Except where otherwise stated, the notation used will follow that of B. Huppert [2].

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2. **Central powers and crown products.** In this section, two results are proved about the order of abelian subgroups of groups obtained by taking central and

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crown products. For a description of crown products, see B. H. Neumann [4, pp. 58–60].

Lemma 1. *Let G be a p-group satisfying*

(A) *every subgroup X of G such that $X \geq C_G(X)$ and $X' \leq Z(G)$ contains an abelian subgroup Y such that $|Y|^2 \geq |X|\,|Z(X)|$.*

If $|Z(G)| = p^m$ and G has no abelian subgroup of order greater than p^{l+m}, then a central power of n copies of G, amalgamating $Z(G)$, has no abelian subgroup of order greater than p^{nl+m}.

Proof (Induction on n). If $n > 1$, let P be a central power of n copies of G. Then $P \cong (H \times G)/D$ where H is a central power of $(n-1)$ copies of G and $D = \langle (a\phi)a^{-1}\colon a \in Z(G)\rangle$ for some isomorphism $\phi\colon Z(G) \to Z(H)$.

Let C/D be an abelian subgroup of P of maximal order. Then $Z(G) \leq G \cap C \leq G \cap CH$. If $g, g' \in G \cap C$, then $[g, g'] \in D$. But $G \cap D = E$ and hence $[g, g'] = e$. Thus $G \cap C$ is abelian. If $g = c \in G \cap C$ and $g' = c'h \in G \cap CH$, then $[g, g'] = [g, h][c, c']^h$. Since $[g, h] = e$ and $G \cap D = E$, $[g, g'] = e$ and $g \in Z(G \cap CH)$. Thus $G \cap C \leq Z(G \cap CH)$. Let $a \in C_G(G \cap CH)$ and $c = hg \in C$, where $h \in H$, $g \in G$, then $[a, hg] = [a, g][a, h]^g = e$, since $a \in G$, $h \in H$ and $g = ch^{-1} \in G \cap CH$. Hence $(C\langle a\rangle)/D$ is abelian. Therefore $a \in C$. Thus $C_G(G \cap CH) \leq G \cap C$ and, since $Z(G \cap CH) \leq C_G(G \cap CH)$, $G \cap C = Z(G \cap CH) = C_G(G \cap CH)$.

$$G \cap CH \cong C/(H \cap C) \tag{1}$$

since $G \cap CH \cong H(G \cap CH)/H = CH/H \cong C/(H \cap C)$.

By (A), there exists $Y \leq G \cap CH$ such that $Y' = E$ and $|Y|^2 \geq |G \cap CH|\,|G \cap C| = p^{r+s}$, where $|G \cap C| = p^s$, $|G \cap CH| = p^r$. Hence

$$|Y| \geq p^{(r+s)/2}. \tag{2}$$

If $K = C \cap HY \leq C$ and $F = H \cap KY$, then $H \cap C \leq F \leq H \cap CG$. Let $f, f' \in F$, then $f = ky$, $f' = k'y'$, where $k, k' \in K$ and $y, y' \in Y$. Since $C' \leq D$, $[k, k'] = [fy^{-1}, f'(y')^{-1}] = [f, f']$, because $f, f' \in H$, $y, y' \in G$. But $H \cap D = E$ and hence $[f, f'] = e$. Thus F is abelian.

Since $H \lhd H \times G$,

$$H(C \cap HY)/H \cong (C \cap HY)/(H \cap C) = K/(H \cap C).$$

Also, $H(C \cap HY)/H = HY/H \cong Y/(H \cap Y) \cong Y$, since $H \cap Y = E$. Thus

$$K/(H \cap C) \cong Y. \tag{3}$$

Similarly, $(C \cap FG)/(G \cap C) \cong G(C \cap FG)/G = FG/G \cong F/(G \cap F) \cong F$, since $G \cap F = E$. Thus

$$(C \cap FG)/(G \cap C) \cong F. \tag{4}$$

Clearly $K = C \cap HY \geq C \cap FY$. If $k \in K$, then $k = hy$ and $h = ky^{-1} \in H \cap KY = F$. Hence $k \in C \cap FY$ and $C \cap HY = C \cap FY$. Also, $K = C \cap FY \leq C \cap FG$. If $c \in C \cap FG$, then $c = fg = kyg$, where $k \in K$, $y \in Y$, $ky \in H$ and $g \in G$. Thus $k^{-1}c = yg \in G \cap C \leq Y$ and so $g \in Y$. Hence $c \in C \cap FY$ and thus $K = C \cap FG$. Hence, from (3) and (4) it follows that $F/(H \cap C) \cong Y/(G \cap C)$, and, by (2), $|F/(H \cap C)| \geq p^{(r-s)/2}$. Since F is abelian, it follows from the inductive hypothesis that $|F| \leq p^{(n-1)l+m}$. Hence

$$|H \cap C| \leq p^{(n-1)l+m-(r-s)/2}. \tag{5}$$

Since Y is abelian, $|Y| \leq p^{l+m}$. Hence

$$|G \cap C| \leq p^{l+m-(r-s)/2}. \tag{6}$$

Now

$$\begin{aligned} |C| &= |C/(H \cap C)|\,|H \cap C| = |G \cap HC|\,|H \cap C|, && \text{by (1)},\\ &= |(G \cap HC)/(G \cap C)|\,|G \cap C|\,|H \cap C| \\ &\leq p^{r-s} \cdot p^{(n-1)l+m-(r-s)/2} \cdot p^{l+m-(r-s)/2}, && \text{by (5) and (6)},\\ &= p^{nl+2m}. \end{aligned}$$

Hence $|C/D| \leq p^{nl+m}$, which completes the proof.

Lemma 2. *Let p be an odd prime and G a p-group satisfying*

(B) *every subgroup X of G such that $X \geq Z(G)$ and $X' \leq Z(G)$ contains an abelian subgroup Y such that $|Y|^2 \geq |X|\,|Z(X)|$. If $|Z(G)| = p^m$ and G has no abelian subgroup of order greater than p^{l+m}, then the crown product, H, of G with a cyclic group of order p, amalgamating $Z(G)$, has no abelian subgroup of order greater than p^{pl+m}.*

Proof. $H = BC_p$ where B is the base group of the crown product, that is, a central product of p copies of G, amalgamating $Z(G)$.

Since G satisfies the weaker condition (A), it follows from Lemma 1 that an abelian subgroup of H contained in B has order at most p^{pl+m}.

If A is an abelian subgroup of H of maximal order such that $A \nleq B$, then $Z(B) = Z(H) \leq A$ and $A/Z(H)$ is isomorphic to a subgroup of $(G/Z(G))$ wr C_p. Since $A/Z(H) \nleq B/Z(H)$ and $A/Z(H)$ is abelian, it follows from Alperin's result [1, Theorem 2] that $|A| = |Z(H)|\,|A/Z(H)| \leq |Z(H)|\,dp$, where d is the order of the largest abelian subgroup of $G/Z(G)$. Since G has no abelian subgroup of order greater than p^{l+m}, condition (B) implies that if C is a subgroup of G such that $C \geq Z(G)$ and $C' \leq Z(G)$, then $|C| \leq p^{2l+m}$. Hence $|A| \leq p^{2l+m+1} \leq p^{pl+m}$, since $p \geq 3$.

3. **Proof of the theorem.** The theorem will be proved by constructing suitable groups using central powers and crown products. It is often tedious to show that a group satisfies the conditions of Lemmas 1 and 2, in particular when taking iterated crown products, hence the following brief lemma is most useful.

Lemma 3. *A p-group G, whose centre is elementary abelian of order p^2, satisfies condition* (B).

Proof. Let X be a subgroup of G such that $X \geq Z(G)$ and $X' \leq Z(G)$. If $x \in X - Z(X)$, $y \in X$, then $[x^p, y] = [x, y]^p = e$ and hence $x^p \in Z(X)$. Thus $X/Z(X)$ is elementary abelian and, from Alperin's result on two alternating forms [1, Theorem 3], it follows that X has an abelian subgroup Y such that $|Y|^2 \geq |X|\,|Z(X)|$.

For p an odd prime, the required groups are constructed from the same group as Alperin's, a group of order p^5, with elementary abelian centre of order p^2 and no abelian subgroups of order greater than p^3. It is the free group on two generators of the variety of groups of exponent p and nilpotency class three and will be denoted by F.

Let $F_0 = F$ and, for $n \geq 1$, let F_n be the crown product of F_{n-1} with C_p, amalgamating the centre of F_{n-1}. Clearly $|F_1| = p^{3p+1+2}$ and, in general, $|F_n| = p^{\alpha(n)}$, where $\alpha(n) = 3p^n + p^{n-1} + \cdots + p + 1 + 2 = 3p^n + 2 + (p^n - 1)/(p - 1)$. Since $Z(F_k) \cong Z(F)$ for all $k \leq n$, by Lemma 3, F_k satisfies condition (B). Hence, an application of Lemma 2 at each stage of the construction assures that no abelian subgroup of F_n has order greater than $p^{\beta(n)}$, where $\beta(n) = p^n + 2$.

For $p = 2$ the construction uses a group of order 2^7, with elementary abelian centre of order 4 and no abelian subgroups of order greater than 2^4. It can be constructed as follows.

Let $H_1 = \langle x, y \rangle \cong C_4 \times C_4$. The automorphism group of H_1 contains a subgroup $K = \langle \sigma, \tau \rangle \cong C_2 \times C_2$ such that $x\sigma = x^{-1}$, $y\sigma = x^2y^{-1}$, $x\tau = x^{-1}y^2$ and $y\tau = y^{-1}$. Let H_2 be the splitting extension of H_1 by K and let u, v denote the elements corresponding to σ, τ respectively. Then H_2 has an automorphism ρ of order 2, defined by $x\rho = x^{-1}$, $y\rho = y^{-1}$, $u\rho = ux$ and $v\rho = vy$. Let G be the splitting extension of H_2 by $\langle \rho \rangle$ and let w denote the element corresponding to ρ.

Now $Z(G) = \langle x^2, y^2 \rangle$. Hence, by Lemma 3, G satisfies condition (B) and thus also the weaker condition (A).

Let A be an abelian subgroup of G of maximal order. Since $G' = \langle x, y \rangle$, $|A| \leq 2^6$. If $|A| > 2^4$, G has a subgroup B, containing A, of order 2^6 and $B > G'$. Since

$$|AG'| = |A|\,|G'|/|A \cap G'| \leq |B| = 2^6,$$

it follows that $|A \cap G'| \geq 2^3$. Hence A contains an element $g \in G' - Z(G)$ and $A \leq C_G(g)$. It requires only routine calculation to show that if $g = x^a y^b \in G' - Z(G)$, then $C_G(g) = \langle x, y, u^\alpha v^\beta w \rangle$, where $\alpha \equiv a \mod 2$ and $\beta \equiv b \mod 2$, and so $|C_G(g)| = 2^5$. Thus $A = C_G(g)$. But $C_G(g)$ is not abelian for any choice of α, β, since

$$[x, u^\alpha v^\beta w] = x^{2(\alpha+\beta+1)} y^{2\beta} \quad \text{and} \quad [y, u^\alpha v^\beta w] = x^{2\alpha} y^{2(\alpha+\beta+1)}.$$

Thus $|A| \leq 2^4$.

Let G_n be a central power of n copies of G, amalgamating $Z(G)$. Then $|G_n| = 2^{5n+2}$ and from Lemma 1 it follows that G_n has no abelian subgroup of order greater than 2^{2n+2}. This completes the proof of the theorem.

It should perhaps be pointed out that, since the group F satisfies condition (A), Lemma 1 gives an alternative proof to Alperin's result for p odd, since a central power of n copies of F has the required properties.

REFERENCES

1. J. L. Alperin, *Large abelian subgroups of p-groups*, Trans. Amer. Math. Soc. 117 (1965), 10–20. MR 30 #1180.

2. B. Huppert, *Endliche Gruppen*. I, Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin and New York, 1967. MR 37 #302.

3. G. A. Miller, *On the number of abelian subgroups whose order is a power of a prime*, Messenger Math. 36 (1906/07), 79–80.

4. B. H. Neumann, *Ascending derived series*, Compositio Math. 13 (1956), 47–64; Corrigendum, ibid., p. 128. MR 19, 632.

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