TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 169, July 1972

MODULAR REPRESENTATIONS OF METABELIAN GROUPS

BY

B. G. BASMAJI

ABSTRACT. The irreducible modular representations, the blocks, and the defect groups of finite metabelian groups are determined. Also the dimensions of the principal indecomposable modules are computed.

1. **Introduction.** Let G be a finite metabelian group and Ω be an algebraically closed field with characteristic p dividing the order $|G|$ of G. The purpose of this paper is to determine the irreducible representations over Ω, the blocks (p-blocks), the defect groups of G and the dimensions of the indecomposable components of ΩG. This is done by applying a number of fundamental results due to Brauer, found in Curtis-Reiner [6, Chapter XII], to the results of [1]. In §2 we fix the notations. We prove a lemma in §3 and apply it to determine the irreducible inequivalent modular representations of G over Ω. Assuming the knowledge of linear representations of some subgroups, we are able to give in §4 the blocks and their defect groups. This makes it possible to determine all blocks containing a linear representation. We prove that all principal indecomposable modules belonging to a block have the same dimension and we compute this dimension. In §5 we compute the decomposition and Cartan matrices of a block of the metacyclic group.

2. **Notations.** Let G be a finite group and A be a normal abelian subgroup of G and assume G/A is abelian. Let W be a subgroup of A and let $K(W)/W$ be a maximal abelian subgroup of $N(W)/W$ containing A/W where $N(W)$ is the normalizer of W in G. The subgroup $K(W)$ is not unique and can also be defined as a subgroup of G containing A, the derived group $K(W)' \subseteq W$; and $K(W)$ is maximal in the sense that if K_1 is a subgroup of G containing $K(W)$ and $K_1' \subseteq W$ then $K_1 = K(W)$. If W_1 is a subgroup of W, then for each $K(W_1)$ we can choose a $K(W)$ containing $K(W_1)$. If W_1 is conjugate to W then we may let $K(W_1) = K(W)$. Also if $W_0 = \bigcap_{x \in G} x^{-1} W x$ then we may choose $K(W_0) = K(W)$.

Let H be a subgroup of A such that A/H is cyclic and let $z_1 A, \cdots, z_w A$ be a basis of $K(H)/A$ with $z_i A$ of order t_i. Let $\overline{Q}$ be the algebraic closure of the field Q of the rationals and let T be a linear representation

Received by the editors August 27, 1971.

AMS 1970 *subject classifications*. Primary 20C15, 20C20.

Key words and phrases. Modular representations, blocks, defect groups, principal indecomposable modules, Brauer characters, decomposition matrices, Cartan matrices.

(over $\bar{Q}$) of A with kernel H. There exist $|K(H)/A|$ distinct representations T' which are extensions of T to $K(H)$ and $T'(z_i)^{t_i} = T(z_i^{t_i})$. From [1] the induced representation T'^G of G is irreducible of degree $|G/K(H)|$. All the irreducible inequivalent representations of G are given by the set of all T'^G with $T' \in \bigcup R_c(H, K(H))$ where the union is over all nonconjugate H such that A/H is cyclic and $R_c(H, K(H))$ is a complete set of representatives of the conjugate classes of the set $R(H, K(H))$ of all extensions T' of every possible T with kernel H. In this paper conjugacy will mean G-conjugacy unless stated otherwise.

If S is a linear representation of a subgroup K of G, $K \supseteq A$, then S can be extended to a subgroup $K_1 \supseteq K$ provided that $K_1' \subseteq \ker S \cap A$. In this case there are $|K_1/K|$ distinct such extensions of S.

Let p (fixed all through this paper) be a prime dividing the order $|G|$ of G and let ν_p be a fixed valuation of $\bar{Q}$ extending the p-adic valuation of Q, $\nu_p(p) = 1$. Let O be the local ring of ν_p in $\bar{Q}$, $\mathfrak{p}$ the corresponding unique maximal prime ideal, and $\Omega = O/\mathfrak{p}$. The residue class map of O onto Ω is denoted by bar; $v \longrightarrow \bar{v}$. We let $\bar{1} = 1$ and $\bar{0} = 0$. If S is a representation of G over O and $S(g) = (v_{ij})$, $g \in G$, then the representation $\bar{S}$ of G over Ω is defined by $\bar{S}(g) = (\bar{v}_{ij})$. If S is a representation of a subgroup K of G over O then $\bar{S}^G = \overline{S^G}$, where in both cases the same coset representatives of K in G are taken. Note that the representations T, T', and T'^G are over O and thus $\bar{T}$, $\bar{T}'$, and $\bar{T}'^G$ are defined. For convenience we let $\tau = \bar{T}$, $\tau' = \bar{T}'$, and $\tau'^G = \bar{T}'^G$, and the subscripts on T are carried on to τ as in $\tau_i = \bar{T}_i$ and $\tau'_{1x} = \bar{T}'_{1x}$. Note that in all cases by $\bar{S}'$ we mean $\bar{S'}$. Representations over $\bar{Q}$ are called ordinary and will be denoted by capital Latin letters without bars, and those over Ω are called modular and will be denoted by capital Latin letters with bars or by small Greek letters.

The notations $S \sim Z$ and $\sigma \sim \zeta$ mean equivalence of these representations over $\bar{Q}$ and over Ω respectively. If K is a subgroup of G then S_K and σ_K mean the restrictions of S and σ respectively to K.

We make a note about Brauer characters that will be used several times in this paper. Let S be a linear representation of a subgroup K of G, $K \supseteq A$. Then there exists a linear representation Z of K such that $Z(k) = 1$ if k is a p-element and $Z(k) = S(k)$ if k is p-regular. Let χ and ψ be the characters of S^G and Z^G respectively. Then $\chi(g) = \psi(g) = 0$ if $g \notin K$, and $\chi(k) = \psi(k) = \sum_{x \in G/K} S(k^x)$ if k is a p-regular element of K, where $x \in G/K$ means x runs over a coset representative of K in G and $k^x = x^{-1}kx$. Thus the Brauer characters of $\bar{S}^G$ and $\bar{Z}^G$ are equal and by [6, (82.7)], $\bar{S}^G$ and $\bar{Z}^G$ have the same composition factors.

3. **Irreducible representations.** Assume the notations of §2. We have

Lemma. *Let* $p \nmid |A/H|$.

(a) τ'^G *is an irreducible representation of* G.

(b) *Let* T_1' *be a linear representation of* $K(H)$ *with* $\ker T_1' \cap A = H_1$ *conjugate to* H. *Then* $\tau'^G \sim \tau_1'^G$ *if and only if* τ' *and* τ_1' *are conjugate.*

(c) *Let* T_1' *be a linear representation of* $K(H_1)$, $\ker T_1' \cap A = H_1$, $p \nmid |A/H_1|$, *and* H_1 *not conjugate to* H. *Then* τ'^G *and* $\tau_1'^G$ *are inequivalent.*

Proof. (a) Let $K = K(H)$. For $x \in G$ assume $\tau'(k^x) = \tau'(k)$ for all $k \in K$. Then $\tau'(k^{-1}k^x) = 1$ or $k^{-1}k^x = k^{-1}x^{-1}kx$ is in $\ker \tau' \cap A = \ker T' \cap A = H$. Thus $k^{-1}x^{-1}kx \in H$ and $x \in K$. Hence for each $x \notin K$ there exists at least one $k \in K$ such that $\tau'(k^x) \neq \tau'(k)$. Thus K is the inertia group of τ' and τ' has $|G/K|$ distinct conjugates which we denote by τ_x', $\tau_x'(k) = \tau'(k^x)$. Note that $(\tau'^G)_K \sim \sum_{x \in G/K} \tau_x'$ where the summation is direct sum of representations. Assume τ'^G is reducible and let σ be an (irreducible) composition factor of τ'^G such that τ' is a composition factor of σ_K. By Clifford's theorem ([5, Theorem 1], [6, (49.7)]), σ_K is equivalent to e copies of $\sum_{x \in G/K} \tau_x'$. Thus $\sigma \sim \tau'^G$ and τ'^G is irreducible. An elementary but considerably longer proof of (a) could also be given.

(b) Assume $\tau'^G \sim \tau_1'^G$. Since $(\tau'^G)_K \sim \sum_{x \in G/K} \tau_x'$ and $(\tau_1'^G)_K \sim \sum_{x \in G/K} \tau_{1x}'$, it follows that τ_1' is conjugate τ'. The converse is easy.

(c) Let $K = K(H)$, $K_1 = K(H_1)$, and assume $|K| = |K_1|$, for otherwise there is nothing to prove. Let $m = |G/K|$ and let $G = x_1 K \cup \cdots \cup x_m K = y_1 K_1 \cup \cdots \cup y_m K_1$. For any x_j and y_i there exists $d \in A$ such that $d^{x_j} \in H$ and $d^{y_i} \notin H_1$. Assume $S = (v_{ij})$ is an $m \times m$ matrix over Ω such that $S\tau'^G(g) = \tau_1'^G(g)S$ for all $g \in G$. Then, in particular, this is true for $g = d$ or $v_{ij}\tau'(d^{x_j}) = \tau_1'(d^{y_i})v_{ij}$. Since $1 = \tau'(d^{x_j}) \neq \tau_1'(d^{y_i})$ we have $v_{ij} = 0$ and thus $S = 0$ which proves the result.

Using the notations of §2 let $p \nmid |A/H|$ and $M(H, K(H))$ be the set of all τ' where $T' \in R(H, K(H))$. Let $M_c(H, K(H))$ be a complete set of representatives of the conjugate classes of $M(H, K(H))$.

Theorem 1. *All the irreducible inequivalent representations of* G *over* Ω *are given by the set of all* τ'^G *where* $\tau' \in \bigcup M_c(H, K(H))$ *and the union is over all nonconjugate* H, A/H *cyclic, and* $p \nmid |A/H|$.

Proof. The lemma implies that all the representations τ'^G in the theorem are irreducible and inequivalent. Thus the p-rank of the decomposition matrix is greater than or equal to $|\bigcup M_c(H, K(H))|$.

Any irreducible ordinary representation of G is given by $T_1'^G$ where T_1' is a linear representation of $K(H_1)$ with $\ker T_1' \cap A = H_1$. Now H_1 determines a unique subgroup H of A such that $H_1 \subseteq H$, $p \nmid |A/H|$, and H/H_1 is a (cyclic) p-group. Pick $K(H) \supseteq K(H_1)$ and let S be a linear representation of $K(H_1)$ such that $S(k) = 1$ if k is a p-element and $S(k) = T_1'(k)$ if k is p-regular. Now $\ker S \cap A = H$ and thus there exist $|K(H)/K(H_1)|$ extensions S' of S to $K(H)$. Since $S'_{K(H_1)} = S$, it follows that $S^{K(H)} \sim \sum S'$ and thus $S^G \sim \sum S'^G$ where the summation (direct sum of representations) is over the $|K(H)/K(H_1)|$ extensions S' of S to $K(H)$. Each $\overline{S}'$ is conjugate to some $\tau' \in M_c(H, K(H))$ and thus each $\overline{S}'^G$ is equivalent to some τ'^G. From §2, the Brauer character of $\overline{T}_1'^G$ is a sum of Brauer characters of some τ'^G with $\tau' \in M_c(H, K(H))$. Thus the decomposition matrix has p-rank equal to $|\bigcup M_c(H, K(H))|$, and by [6, (83.5)] the result follows.

Remark. Let A/H be cyclic (p may divide $|A/H|$) and let K_1/H and K_2/H be two maximal abelian subgroups of $N(H)/H$, $K_i/H \supseteq A/H$, $i = 1, 2$. Set $K = K_1 \cap K_2$ and let S be a linear representation of K with $\ker S \cap A = H$. Let S_1 be an extension of S to K_1. Then, as done in the proof of Theorem 1, $S^G \sim \sum S_1^G$ where the summation is over the distinct $|K_1/K|$ extensions S_1 of S to K_1. Similarly $S^G \sim \sum S_2^G$ where S_2 runs over the $|K_2/K|$ extensions of S to K_2. Since S_1^G and S_2^G are irreducible, it follows that each S_1^G is equivalent to some S_2^G and conversely each S_2^G is equivalent to some S_1^G. In particular $|K_1| = |K_2|$. Let χ be the character of S_1^G. Then χ vanishes outside K and thus it depends only on S. Thus $S_1^G \sim S_1'^G$ for any two extensions S_1 and S_1' of S to K_1. Therefore all S_1^G and S_2^G are equivalent. This implies that the Brauer characters of $\overline{S}_1^G$ and $\overline{S}_2^G$ are equal, and by [6, (82.7)] they have the same composition factors. In particular if $p \nmid |A/H|$, then $\overline{S}_1^G$ and $\overline{S}_2^G$ are irreducible and thus equivalent.

4. **Blocks.** Let $p \nmid |A/H|$, A/H cyclic, and let $\mathrm{Sub}(H)$ be the set of all subgroups L of H such that A/L is cyclic and H/L is a p-group. Let $L_1, \cdots, L_t$ be the nonconjugate minimal subgroups in $\mathrm{Sub}(H)$ and set $\Lambda = \bigcap_{i=1}^t L_i$ and $K = K(\Lambda)$. If $\Lambda_1 = \bigcap' L$ and $\Lambda_2 = \bigcap'(\bigcap_{x \in G} x^{-1}Lx)$ where the intersection $\bigcap'$ is of all elements L of $\mathrm{Sub}(H)$, then we may pick $K(\Lambda_1) = K(\Lambda_2) = K(\Lambda) = K$. For each $L \in \mathrm{Sub}(H)$ choose $K(L) \supseteq K$. Let $C(H, K(\Lambda))$ be a complete set of representatives of the conjugate classes of the set of all linear representations σ of K with $\ker\sigma \cap A = H$. Let $B(\sigma, H)$ be the set of all representations T'^G where T' is a linear representation of $K(L)$, $L \in \mathrm{Sub}(H)$, with $\ker T' \cap A = L$ and $\overline{T}'_K$ conjugate to σ. Include in $B(\sigma, H)$ the irreducible composition factors of $\overline{T}'^G$, $T'^G \in B(\sigma, H)$ and identify equivalent representations. Note that two irreducible representations S and Z are in the same block if and only if they are linked. We say S and Z are linked if there exist irreducible representations $S = S_0, S_1, \cdots, S_n = Z$ such that $\overline{S}_i$ and $\overline{S}_{i+1}$ have a composition factor in common for $i = 0, 1, \cdots, n-1$. We prove

Theorem 2. *All the distinct blocks of G are given by the collection of the sets $B(\sigma, H)$ where H runs over all nonconjugate subgroups of A, A/H cyclic, $p \nmid |A/H|$, and σ runs over the elements of $C(H, K(\Lambda))$.*

Proof. If L is any subgroup of A such that A/L is cyclic, then L is conjugate to some element in $\mathrm{Sub}(H)$ for some H. From §2 we may take $L \in \mathrm{Sub}(H)$. Thus any irreducible ordinary representation of G is given by S^G where S is a linear representation of $K(L)$, $\ker S \cap A = L$, with $L \in \mathrm{Sub}(H)$ for some H. From the proof of Theorem 1 we have the composition factors of $\overline{S}^G$ equivalent to representations τ'^G where $\tau' \in M_c(H, K(H))$. Hence the set of all irreducible representations S^G, S any linear representation of $K(L)$ with $\ker S \cap A = L$ and L any element of $\mathrm{Sub}(H)$, forms a collection of blocks. This means we only need to study this set of irreducible representations.

Let $1 \le g \le t$, $\Lambda_g = \bigcap_{i=1}^{g} L_i$ and let μ_g be a linear representation of $K(\Lambda_g) \supseteq K$ such that $\mu_{gK} = \sigma$. Let $\mathrm{Rep}(\mu_g)$ be the set of all irreducible representations S^G where S is any linear representation of $K(L) \supseteq K(\Lambda_g)$, $\ker S \cap A = L$, the restriction of $\overline{S}$ to $K(\Lambda_g)$ is conjugate to μ_g, and L is any element in $\mathrm{Sub}(H)$ with $L \supseteq L_i$ for some i, $1 \le i \le g$. We use induction on g to prove that any two elements of $\mathrm{Rep}(\mu_g)$ are linked, $1 \le g \le t$. Note that $\mu_t = \sigma$ and $\mathrm{Rep}(\sigma)$ forms the set of ordinary representations of $B(\sigma, H)$.

Assume $g = 1$. Let $\mu = \mu_1$ and $J = K(L_1)$ and pick $K(L) \supseteq J$ whenever $L \in \mathrm{Sub}(H)$ and $L \supseteq L_1$. Let $\theta_1, \cdots, \theta_m$ be all the nonconjugate linear representations of $K(H) \supseteq J$ such that $\theta_{iJ} = \mu$, $1 \le i \le m$. Then θ_i^G, $1 \le i \le m$, are all irreducible and inequivalent. Let Z be a linear representation of J, with $\ker Z \cap A = L_1$ and $\overline{Z}$ conjugate to μ. Then Z^G is irreducible. Let M be a linear representation of J, $M(k) = 1$ if k is a p-element and $M(k) = Z(k)$ if k is p-regular. Since $\ker M \cap A = H$ we have $|K(H)/K(L_1)|$ extensions M' of M to $K(H)$. Every $\overline{M}'$ is conjugate to some θ_i and conversely every θ_i, $1 \le i \le m$, is conjugate to some $\overline{M}'$. Also $M^G \sim \sum M'^G$ where the summation is over all the extensions M' of M to $K(H)$. From the lemma every $\overline{M}'^G$ is equivalent to some θ_i^G and conversely every θ_i^G, $1 \le i \le m$, is equivalent to some $\overline{M}'^G$. Thus every θ_i^G, $1 \le i \le m$, and no others, appears as a composition factor of $\overline{Z}^G$.

Now let S be any linear representation of $K(L) \supseteq J$, $\ker S \cap A = L$, $\overline{S}_J$ conjugate to μ, $L \supseteq L_1$, and $L \in \mathrm{Sub}(H)$. Take $K(H) \supseteq K(L)$. Define a linear representation N of $K(L)$ such that $N(k) = 1$ if k is a p-element and $N(k) = S(k)$ if k is p-regular. Using the same method as above, we have some θ_i^G, and no others, appear as composition factors of $\overline{S}^G$. Thus $\overline{S}^G$ and $\overline{Z}^G$ have a composition factor in common. Since S was arbitrary it follows that any two elements of $\mathrm{Rep}(\mu)$ are linked.

If $t = 1$ then there is nothing more to prove.

Let $t > 1$, $1 < g \le t$, $\Delta = \Lambda_g = \bigcap_{i=1}^{g} L_i$, and $\Gamma = \Lambda_{g-1} = \bigcap_{i=1}^{g-1} L_i$. Let $J = K(\Delta) \supseteq K$, $E = K(\Gamma) \supseteq J$ and $F = K(L_g) \supseteq J$. Let μ be a linear representation of J such that $\mu_K = \sigma$, i.e. $\mu = \mu_g$, and let η and ζ be any linear representations of

E and F respectively such that $\eta_J = \zeta_J = \mu$. (Note $\eta = \mu_{g-1}$.) Define the set $\mathrm{Rep}(\zeta)$ as done above for μ_1, i.e. by replacing ζ for μ_1 and L_g for L_1. Let the sets $\mathrm{Rep}(\mu)$ and $\mathrm{Rep}(\eta)$, $\mu = \mu_g$ and $\eta = \mu_{g-1}$, be as in the beginning of this proof. From the above any two elements of $\mathrm{Rep}(\zeta)$ are linked. Assume any two elements of $\mathrm{Rep}(\eta)$ are linked. We shall construct two equivalent irreducible representations of G one belonging to $\mathrm{Rep}(\eta)$ and the other to $\mathrm{Rep}(\zeta)$. Since η and ζ were arbitrarily chosen provided that $\eta_J = \zeta_J = \mu$, this will prove that any two elements of $\mathrm{Rep}(\mu)$ are linked, which completes the induction process defined above.

Using the notations of the preceding paragraph, let K_1/H and K_2/H be maximal abelian subgroups of $N(H)/H$ with $K_1 \supseteq E$ and $K_2 \supseteq F$. Moreover, let $R = K_1 \cap K_2$, $C = R \cap E$, $D = R \cap F$, $V = DC$, and $W = RE$. Let M be a linear representation of J with $\ker M \cap A = H$ and $\overline{M} = \mu$. From the definitions of E and F we have $E \cap F = C \cap D \doteq J$. Thus there exist bases $x_1J, \cdots, x_nJ$ of C/J and $y_1J, \cdots, y_mJ$ of D/J such that $V/J = \langle x_iJ, y_jJ \mid 1 \le i \le n,\ 1 \le j \le m\rangle$ and $\langle x_iJ\rangle \cap \langle y_jJ\rangle = J/J$ for $1 \le i \le n$, $1 \le j \le m$. This implies that there exists an extension M' of M to V such that $\overline{M}'_C = \eta_C$ and $\overline{M}'_D = \zeta_D$. Let N be an extension of M' to R. Since $W/R \cong E/C$, there exists a basis $z_1R, \cdots, z_sR$ of W/R such that $z_1C, \cdots, z_sC$ is a basis of E/C and the orders of z_iR in W/R and z_iC in E/C are equal. Thus there exists an extension Z of N to W such that $\overline{Z}_E = \eta$. Let Z' be any extension of Z to K_1. Then Z'^G and $\overline{Z}'^G$ are irreducible and $Z'^G \in \mathrm{Rep}(\eta)$. Similarly we can find an extension Y' of N to K_2 such that $\overline{Y}'_F = \zeta$. Again Y'^G and $\overline{Y}'^G$ are irreducible and $Y'^G \in \mathrm{Rep}(\zeta)$. From the remark in §3 it follows that Y'^G and Z'^G are equivalent, and thus any two elements of $\mathrm{Rep}(\mu)$ are linked. Now the inductive hypothesis implies that any two ordinary representations in $B(\sigma, H)$ are linked.

Let $\mu_1, \cdots, \mu_m$ be the set of all nonconjugate linear representations of $K(H) \supseteq K(\Lambda) = K$ such that $\mu_{iK} = \sigma$. Let $T'^G \in B(\sigma, H)$, T' as in first part of this section. Let S be a linear representation of $K(L)$ such that $S(k) = 1$ if k is a p-element and $S(k) = T'(k)$ if k is p-regular. Then $\ker S \cap A = H$ and $S^G \sim \sum S'^G$ where the summation is over all extensions S' of S to $K(H) \supseteq K(L) \supseteq K$. Since $\overline{S}'_K$ is conjugate to σ it follows that only representations μ_i^G appear as composition factors of $\overline{T}'^G$. Since for each μ_i there exists a linear representation S of $K(H)$ such that $\overline{S} = \mu_i$, it follows that μ_i^G, $i = 1, \cdots, m$, are the only modular representations in $B(\sigma, H)$. Now consider some $B(\sigma_1, H_1)$ and let ζ_j^G, $j = 1, \cdots, n$, be its modular representations where $\zeta_1, \cdots, \zeta_n$ are all the nonconjugate linear representations of $K(H_1) \supseteq K(\Lambda_1) = K_1$ such that $\zeta_{jK_1} = \sigma_1$. Here Λ_1 is defined for H_1 as Λ was done for H. Now if H_1 is not conjugate to H or σ_1 is not conjugate to σ then μ_i^G and ζ_j^G are inequivalent. If Z is an irreducible representation of G and $Z \notin B(\sigma, H)$, then $Z \in B(\sigma_1, H_1)$ with either H_1 not conjugate to H or σ_1 not conjugate to σ. Thus only representations ζ_j^G appear as composition factors of Z or Z is not linked to any element of $B(\sigma, H)$. Hence $B(\sigma, H)$ is a block which completes the proof of the theorem.

Note that the blocks $B(\sigma, H)$ where $H \supseteq G'$, G' the derived group of G, are the only blocks of G containing linear representations. If we set $A = G'$ then these blocks are simply given by $B(\sigma, G')$. By Brauer [4, Proposition (4E)] these blocks are flat (as defined in [4]).

Now consider $K(H) \supseteq K(\Lambda) = K$. Since $K(H)$ is metabelian and A is a normal subgroup of $K(H)$, the blocks of $K(H)$ are given by $b = b(\mu, \overline{H})$ where $\overline{H}$ runs over all nonconjugate (in $K(H)$) subgroups of A such that $A/\overline{H}$ is cyclic and $p \nmid |A/\overline{H}|$. Let ω_b be the corresponding linear character of the center $Z(\Omega K(H))$ of $\Omega K(H)$. Consider the block $B = B(\sigma, H)$ of G and let ω_B be the corresponding linear character of $Z(\Omega G)$.

Assume $\overline{H}$ is conjugate (in G) to H and define $\overline{\Lambda}$ for $\overline{H}$ as Λ was done for H. Take $K(\overline{\Lambda}) = K(\Lambda) = K$ and assume μ is conjugate to σ. There exists a linear representation S of $K(H)$ in $b = b(\mu, \overline{H})$ such that $S(k) = 1$ if k is a p-element, $\overline{S}_K = \mu$, and $\ker S \cap A = \overline{H}$. Now S^G is irreducible and $S^G \in B = B(\sigma, H)$. Using the characters χ and χ^G of S and S^G respectively it is easy to show that $\omega_b^G = \omega_B$ and thus b^G is defined and $b^G = B$. (The definitions of ω_b^G and b^G can be found in [2, §2] or [3, §2, 7] and need not be confused with induced representations.) Also, by Brauer [3, (4D)], B covers b. Assume a block b_1 of $K(H)$ is covered by B, then by [3, (4A)] some constituent of $(S^G)_{K(H)}$ belongs to b_1. Thus we have

Corollary. *Let $K(H) \supseteq K(\Lambda)$. The blocks $b = b(\mu, \overline{H})$ of $K(H)$ with μ conjugate to σ and $\overline{H}$ conjugate to H are the only blocks of $K(H)$ such that $b^G = B(\sigma, H)$. Furthermore, these are the only blocks of $K(H)$ covered by $B(\sigma, H)$.*

Now since $b(\sigma, H)$ contains a linear representation S of $K(H)$ with $\overline{S}_K = \sigma$ and $\ker S \cap A = H$, it follows that any p-Sylow subgroup of $K(H)$ can be taken as a defect group of $b(\sigma, H)$. Since S^G is an irreducible representation and $S^G \in B(\sigma, H)$ it follows that the defect d of $B(\sigma, H)$ is given by $p^d \,||\, |K(H)|$. Thus from Brauer [2, (2B)] we have

Corollary. *Let $K(H) \supseteq K(\Lambda)$. The defect group of $B(\sigma, H)$ is the p-Sylow subgroup of $K(H)$.*

A principal indecomposable module belongs to a block B if all its composition factors afford (modular) representations in B.

Theorem 3. *The dimension of any principal indecomposable module belonging to the block $B(\sigma, H)$ is $p^\alpha|G/K(H)|$ where $p^\alpha \,||\, |K(H)|$.*

Proof. Let $L \in \mathrm{Sub}(H)$ and let $\mu^G \in B(\sigma, H)$ where μ is a linear representation of $K(H) \supseteq K(L) \supseteq K(\Lambda) = K$ with $\mu_K = \sigma$. Set $K(L) = J$ and $p^l = |H/L|$.

Let T be a linear representation of A with $\ker T = L$ and $\overline{T} = \mu_A$. Then there exist $\phi(p^l)$ such representations T where ϕ is Euler's function.

Let T' be an extension of T to J such that $\overline{T}' = \mu_J$ and let $p^j \,||\, |J/A|$. For each T there are p^j such extensions T'.

Let $\nu(L)$ be the number of conjugates of L in $\mathrm{Sub}(H)$. Then there are $\nu(L)p^j\phi(p^l)$ representations T' of J such that $\ker T' \cap A = L_1 \in \mathrm{Sub}(H)$, L_1 is conjugate to L, and $\overline{T}' = \mu_J$.

Let $I = I\{\mu_J\}$ be the inertia group of μ_J in G. Then $I \supseteq K(H)$. Thus, since the inertia group of T' is J, there are

$$g_L = \nu(L)p^j\phi(p^l)/|I/J|$$

conjugate classes in the set of all representations T' defined above.

Now fix a representation T' satisfying the above conditions. Let M be a representation of J such that $M(k) = 1$ if k is a p-element and $M(k) = T'(k)$ if k is p-regular. Then $\overline{M}^G$ and $\overline{T}'^G$ have the same composition factors. Also $M^G \sim \sum M'^G$ where the summation is over all the $|K(H)/J|$ extensions M' of M to $K(H)$. Let $p^s \,||\, |K(H)/J|$; then there are p^s extensions M' such that $\overline{M}' = \mu$. Note that for all M', $\overline{M}'_J = \mu_J = \overline{M}$. Since $K(H)$ is the inertia group of μ it follows that the number of linear representations θ of $K(H)$ such that θ is conjugate to μ and $\theta_J = \mu_J$ is $|I/K(H)|$. This implies that the multiplicity in which μ^G appears in the composition factors of $\overline{T}'^G$ is $m_L = p^s|I/K(H)|$.

Combining the above results it follows that the part of the column of the decomposition matrix D corresponding to μ^G and the above representations T'^G is $m_L I(g_L)$ where $I(g_L)$ is the $(g_L \times 1)$ matrix ${}^t(1, \cdots, 1)$. (The part of this column corresponding to the remaining representations that we get from $K(L)$ and L is zero.)

Let U be a principal indecomposable module such that U/U' affords μ^G where U' is the unique maximal submodule of U. From [6, (84.4)] the dimension of U is

$$u = \sum{}' m_L g_L \,|G/J|$$

where the summation $\sum'$ is over all nonconjugate elements L of $\mathrm{Sub}(H)$. A short computation gives

$$u = p^{\alpha - a}\,|G/K(H)| \sum \phi(p^l)$$

where $p^a \,||\, |A|$ and the summation is over all elements of $\mathrm{Sub}(H)$. The proof will be complete if we show $\sum \phi(p^l) = p^a$.

Since $p \nmid |A/H|$ we have $p^a \parallel |H|$. Let $H = Q \times P$ with $|P| = p^a$ and $p \nmid |Q|$. Then H/L is a p-group if and only if $L \supseteq Q$. If H/L is a p-group then H/L is cyclic if and only if A/L is cyclic. Thus without loss of generality we may take $H = P$ and $Q = 1$. Now $\sum \phi(p^b) = p^a$ follows from the formula in [1, §3] which completes the proof.

Corollary. *Let u be the dimension of a principal indecomposable module. Then $\nu_p(u) = \nu_p(|G|)$.*

Since every modular representation in $B(\sigma, H)$ has the same degree $|G/K(H)|$ we have

Corollary. *The dimension of the unique two-sided indecomposable ideal of ΩG that corresponds to $B(\sigma, H)$ is $hp^\alpha|G/K(H)|^2$ where $p^\alpha \parallel |K(H)|$ and h is the number of the (inequivalent) modular representations in $B(\sigma, H)$.*

By block some authors mean the two-sided ideal in the above corollary. (See [6].)

5. **Metacyclic groups.** Consider the group

$$G = \langle a, b | a^n = b^m = 1, \ a^k = b^t, \ b^{-1}ab = a^r\rangle$$

with $r^t - 1 \equiv kr - k \equiv 0 \pmod{n}$ and $t \mid m$. Set $A = \langle a\rangle$. For $x \mid n$ let t_x be the smallest positive integer such that $r^{t_x} \equiv 1 \pmod{x}$. Let $H_x = \langle a^x\rangle$ and $K_x = K(H_x) = \langle a, b^{t_x}\rangle$. Let $n = p^\alpha n'$, $t = p^\delta t'$ and $t_x = p^{\delta_x} t'_x$, where $(p, n') = (p, t') = (p, t'_x) = 1$. We have $|G/K_x| = t_x = p^{\delta_x} t'_x$ and $|K_x/A| = t/t_x = p^{\delta - \delta_x} t'/t'_x$. If $z \mid x$ then $t_z \mid t_x$, $t'_z \mid t'_x$, and $\delta_z \le \delta_x$.

Let $s \mid n'$ and let τ_s be a linear representation of A such that $\tau_s(a) = \zeta_s^y$, where ζ_s is a primitive sth root of unity in Ω and $(y, s) = 1$. Note that $\ker \tau_s = H_s$. For each τ_s there exist t'/t'_s representations τ'_s of K_s such that $\tau'_{sA} = \tau_s$. Here $\tau'_s(b^{t_s})^{t'/t'_s} = \tau_s(a^k)$. Each $\tau_s'^G$ is irreducible. To get all the irreducible inequivalent representations of G over Ω we let s run over all positive divisors of n', $y \in M_s/R_s$, and τ'_s over all the t'/t'_s representations with $\tau'_{sA} = \tau_s$. Here M_s is the multiplicative group of the reduced residues (mod s) and R_s is the subgroup of M_s generated by r. For each s we have $t'\phi(s)/t_s t'_s$ inequivalent modular representations $\tau_s'^G$ where ϕ is Euler's function.

Let $z = p^\alpha s$ and let $\tau_z = \tau_s$. Let σ_z be a linear representation of $K = K_z$ such that $\sigma_{zA} = \tau_z$. Then $B(\sigma_z, H_s)$ is a block of G. We get different blocks by letting s and y run over the same values as above and σ_z run over the t'/t'_z distinct representations such that $\sigma_{zA} = \tau_z$. Thus for each s we have $t'\phi(s)/t_s t'_z$ distinct blocks. Each block $B(\sigma_z, H_s)$ contains $h^{(s)} = t'_z/t'_s$ irreducible modular representations. These are given by μ_i^G, $1 \le i \le h^{(s)}$, where μ_i is a linear representation of K_s with $\mu_{iK} = \sigma_z$.

Fix s and σ_z and let $\sigma = \sigma_z$, $H = H_s$, and $h = h^{(s)}$. We shall give the decomposition and Cartan matrices of the block $B(\sigma, H)$. Let $\mu_1, \cdots, \mu_h$ be as above. Set $s_i = p^i s$, $t_{s_i} = f_i$, $\delta_{s_i} = \beta_i$, $H_{s_i} = L_i$ and $F_i = K(L_i) = \langle a, b^{f_i}\rangle$ where $i = 0, 1, \cdots, \alpha$. Note that $s_0 = s$, $s_\alpha = z$, $t_s = f_0$, $t_z = f_\alpha$, $\delta_s = \beta_0$, and $\delta_z = \beta_\alpha$. Also $K_s = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_\alpha = K_z = K$ where F_i/F_j is of order $f_j//f_i$, $\alpha \geq j \geq i \geq 0$.

Let $\lambda_i = t_{p^i}$, $\lambda_1 = t_p$, then $r \equiv r_1 r_2 \pmod{p^i}$ where $r_1^{\lambda_1} \equiv 1 \pmod{p^i}$ and $r_2^{p^w} \equiv 1 \pmod{p^i}$, $\lambda_1 \mid p-1$, $0 \leq w < i$. Assuming w is a smallest such positive integer we have $\lambda_i = \lambda_1 p^w$. Thus $\lambda_{i+1}/\lambda_i = p$ or 1 for $i \geq 1$. But for $i \geq 1$, $f_i = \operatorname{lcm}[\lambda_i, f_0]$ which proves that $f_{i+1}//f_i = p$ or 1 for $i \geq 1$. Since $f_1//f_0$ divides λ_1 it follows that $f_1//f_0$ divides $p-1$ and $\beta_1 = \beta_0$. Note that $f_i//f_1$ divides p^{i-1}, $1 \leq i \leq \alpha$. Since $t_z/t_s = f_\alpha//f_0 = (f_\alpha//f_1)(f_1//f_0)$ we have the number of modular representations in $B(\sigma, H)$ equal to $h = t_z'/t_s' = f_1//f_0$ and thus $h \mid p-1$. Since $|F_i/A| = t//f_i$ it follows that the highest power of p dividing $t//f_i$ is $p^{\delta-\beta_i} = p^{\delta-\delta_z} f_\alpha//f_i$, $i \geq 1$.

Fix F_i, $i \geq 1$. There exist $\phi(p^i)$ linear representations T of A with $\ker T = L_i$ and $\overline{T} = \sigma_A$. These partition into $f_0\phi(p^i)//f_i$ conjugate classes. Since $f_\alpha//f_i$ is a power of p, it follows that each T has $p^{\delta-\beta_i} = p^{\delta-\delta_z}(f_\alpha//f_i)$ extensions T' to F_i such that $\overline{T}'_K = \sigma$. Thus there exist exactly

$$h_i = p^{\delta-\delta_z} f_\alpha f_0 \phi(p^i)//f_i^2$$

nonconjugate linear representations T' of F_i such that $\overline{T}'_K = \sigma$. The representations T'^G are irreducible and inequivalent and belong to $B(\sigma, H)$. Using the Brauer character of $\overline{T}'^G$ and the method of this paper, it follows that every μ_j^G, $j = 1, \cdots, h$, appears with a multiplicity $f_i//f_1$ as composition factors of $\overline{T}'^G$.

For $i = 0$ we have $h_0 = p^{\delta-\delta_s}(f_1//f_0) = p^{\delta-\delta_s} h$ nonconjugate representations T' of $K_s = F_0$ such that $\overline{T}'_K = \sigma$. Each $\overline{T}'^G$ is irreducible. Thus for each μ_j^G, $j = 1, \cdots, h$, there are $h_0/h = p^{\delta-\delta_s}$ representations T'^G such that $\overline{T}'^G \sim \mu_j^G$. The number of the ordinary representations in $B(\sigma, H)$ is $\sum_{i=0}^{\alpha} h_i$.

Using the above and arranging the representations T'^G and μ_j^G appropriately, the transpose of the decomposition matrix D of $B(\sigma, H)$ is given by

$${}^tD = ({}^tD_0, {}^tD_1, \cdots, {}^tD_\alpha)$$

where

$$D_0 = \operatorname{diag}(I(h_0/h), \cdots, I(h_0/h))$$

and

$$D_i = f_i//f_1(I(h_i), \cdots, I(h_i)), \qquad i \geq 1,$$

with $I(x)$ the $(x \times 1)$ matrix ${}^t(1, \cdots, 1)$ and D_i, $0 \leq i \leq \alpha$, has h columns. The Cartan matrix is given by

$$C = {}^tD \circ D = \sum_{i=0}^{a} {}^tD_i \circ D_i = p^{\delta - \delta_s} E_h + \sum_{i=1}^{a} (f_i/f_1)^2 h_i(I(h), \cdots, I(h))$$

where E_h is the $h \times h$ identity matrix. A short computation gives $\det C = (p^{\delta - \delta_s})^h p^a$ which is a power of p as expected from [6, (84.17)].

Let U_j be a principal indecomposable module such that U_j/U_j' affords μ_j^G where U_j' is the maximal submodule of U_j. Then U_j is of dimension $p^{\delta - \delta_s + a} t_s$. This could also be computed using the Cartan matrix above. In the composition factors of U_j, $p^{\delta - \delta_s}[1 + (p^a - 1)/h]$ factors appear that afford μ_j^G and for each $q \neq j$, $1 \leq q \leq h$, $p^{\delta - \delta_s}(p^a - 1)/h$ factors appear that afford μ_q^G.

In particular if $p = 2$ then $h = 1$ and each block of G contains one irreducible modular representation, and thus the two-sided ideal of ΩG that corresponds to $B(\sigma, H)$ is the direct sum of t_s isomorphic principal indecomposable modules.

BIBLIOGRAPHY

1. B. G. Basmaji, *Monomial representaions and metabelian groups*, Nagoya Math. J. 35 (1969), 99–107. MR 39 #5709.

2. R. Brauer, *Zur Darstellungstheorie der Gruppen endlicher Ordnung*. II, Math. Z. 72 (1959/60), 25–46. MR 21 #7258.

3. ———, *On blocks and sections in finite groups*. I, Amer. J. Math. 89 (1967), 1115–1136. MR 36 #2716.

4. ———, *Some applications of the theory of blocks of characters of finite groups*. IV, J. Algebra 17 (1971), 489–521.

5. A. H. Clifford, *Representation induced in an invariant subgroup*, Ann. of Math. (2) 38 (1937), 533–550.

6. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. XI, Interscience, New York, 1962. MR 26 #2519.

DEPARTMENT OF MATHEMATICS, CALIFORNIA STATE COLLEGE, LOS ANGELES, CALIFORNIA 90032