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IRREDUCIBLE REPRESENTATIONS OF THE C^*-ALGEBRA GENERATED BY AN n-NORMAL OPERATOR

BY

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ABSTRACT. For A an n-normal operator on Hilbert space, we determine the irreducible representations of $C^*(A)$, the C^*-algebra generated by A and the identity. For A a binormal operator, we determine an explicit description of the topology on the space of unitary equivalence classes of irreducible representations of $C^*(A)$.

1. **Introduction and preliminaries.** For A a bounded linear operator on a Hilbert space H, let $C^*(A)$ denote the C^*-algebra generated by A and I. The set of unitary equivalence classes of irreducible representations of a C^*-algebra equipped with the hull-kernel topology is called the spectrum of the C^*-algebra [2, paragraph 3]. If A is a normal operator, then the spectrum of $C^*(A)$ is merely the spectrum of the operator A. For noncommutative C^*-algebras very few examples of the spectrum of the algebra have been calculated other than the paper of Fell [3]. In this paper we determine the irreducible representations of the C^*-algebra generated by an n-normal operator and explicitly calculate the spectrum of the C^*-algebra generated by a binormal operator. To calculate the topology on the spectrum we use the methods that Fell used to calculate the topology on the duals of the complex unimodular groups [3].

A W^*-algebra $\mathfrak{R}$ is said to be n-normal [5] if it satisfies the identity

$$\sum \operatorname{sgn} \sigma \, A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(2n)} = 0$$

where $A_1, A_2, \cdots, A_{2n}$ are arbitrary elements of $\mathfrak{R}$ and the summation is taken over all permutations σ of $(1, 2, 3, \cdots, 2n)$. A bounded linear operator A on a Hilbert space H is called n-normal if the W^*-algebra generated by A is n-normal. A 2-normal operator is also called binormal [1]. If A is an n-normal operator and π is an irreducible representation of $C^*(A)$ on a Hilbert space H_0, then the standard identity

$$\sum \operatorname{sgn} \sigma \, X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(2n)} = 0$$

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is satisfied on $C^*(\pi(A))$, and since π is irreducible the standard identity is also satisfied on $B(H_0)$, the set of all bounded linear operators on H_0. Hence the dimension of H_0 is less than or equal to n. We note that a representation π of $C^*(A)$ is completely determined by the value of π at A.

For $1 \leq k \leq n$ let $\mathcal{C}_k$ be a commutative W^*-algebra on a Hilbert space H_k, and let $M_k(\mathcal{C}_k)$ denote the W^*-algebra on $H_k \oplus \cdots \oplus H_k$ (k-times) consisting of $k \times k$ matrices with elements from $\mathcal{C}_k$. Then $M_k(\mathcal{C}_k)$ is n-normal, and $M_1(\mathcal{C}_1) \oplus M_2(\mathcal{C}_2) \oplus \cdots \oplus M_n(\mathcal{C}_n)$ is also n-normal. In fact, any n-normal W^*-algebra is unitarily equivalent to one of this form [5]. Thus any n-normal operator A is unitarily equivalent to an operator of the form $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where A_k is a $k \times k$ matrix whose entries generate a commutative W^*-algebra; that is, whose entries are commuting normal operators. Such an A_k is called a k-homogeneous n-normal operator.

Recall [4,3.1.13] that $(\beta_1, \beta_2, \cdots, \beta_l) \in \sigma(B_1, B_2, \cdots, B_l) \equiv$ the joint spectrum of commuting normal operators $B_1, B_2, \cdots, B_l$ if and only if there is a character ω (i.e. multiplicative linear functional) on the C^*-algebra, $C^*(\{B_i\}_{i=1}^l)$, generated by $B_1, B_2, \cdots, B_l$ and I, such that $\omega(I) = 1$, $\omega(B_i) = \beta_i$ for $i = 1, 2, \cdots, l$. We will show that the spectrum of the C^*-algebra generated by a homogenous n-normal operator is closely related to the joint spectrum of its matrix elements.

2. **Irreducible representations of n-normal operators.** If $\mathcal{C}$ is a commutative W^*-algebra, then there is a natural n-dimensional irreducible representation of $M_n(\mathcal{C})$ defined in the following manner: For ρ a nonzero character on $\mathcal{C}$ define $\hat{\rho}$ on $M_n(\mathcal{C})$ by

$$\hat{\rho}((C_{ij})) = (\rho(C_{ij})).$$

Then $\hat{\rho}$ is obviously an irreducible representation of $M_n(\mathcal{C})$ in the $n \times n$ scalar matrices. The following well-known result states that every irreducible representation is of this form.

Proposition 1 (see [7, p. 114]). *If $\mathcal{C}$ is a commutative W^*-algebra and π is an irreducible representation of $M_n(\mathcal{C})$ on a Hilbert space H, then the dimension of H is n, and there exists a nonzero character ρ on $\mathcal{C}$ such that π is unitarily equivalent to $\hat{\rho}$.*

Proposition 2. *Let $A = (A_{ij})$ be a homogeneous n-normal operator, where $\{A_{ij}\}_{i,j=1}^n$ are commuting normal operators. Suppose π is an irreducible representation of $C^*(A)$ on a Hilbert space H, whose dimension k is necessarily less than or equal to n. Then there exists a nonzero character ρ on $C^*(\{A_{ij}\})$ and a k-dimensional reducing subspace M for $\hat{\rho}(A)$ such that $\pi(A)$ is unitarily equivalent to $\hat{\rho}(A)|_M$. Conversely every such character and reducing subspace* *produces a representation of* $C^*(A)$, *which is however not necessarily irreducible.*

Proof. Let $\mathcal{C} = C^*(\{A_{ij}\}_{i,j=1}^n)$ and suppose π is an irreducible representation of $C^*(A)$ on H. By Proposition 2.10.2 in [2], π can be extended to an irreducible representation $\tilde{\pi}$ of $M_n(\mathcal{C})$ on K, where $K \supseteq H$, H reduces $\tilde{\pi}(C^*(A))$, and the dimension of K is necessarily n. But now Proposition 1 implies that there is a nonzero character ρ on $\mathcal{C}$ such that $\tilde{\pi}$ is unitarily equivalent to $\hat{\rho}$. Since H reduces $\tilde{\pi}(A)$ there is a k-dimensional reducing subspace M for $\hat{\rho}(A)$ such that $\pi(A) = \tilde{\pi}(A)|_H$ is unitarily equivalent to $\hat{\rho}(A)|_M$. The converse is clear.

Proposition 3. *Let* $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *be an* n*-normal operator with* A_k *a* k*-homogeneous* n*-normal operator of the form* $(A_{ij}^{(k)})$ *for* $1 \leq k \leq n$, *where* $\{A_{ij}^{(k)}\}_{i,j=1}^k$ *are commuting normal operators. Suppose* π *is an irreducible representation of* $C^*(A)$ *on* H, *whose dimension is* h. *Then there exists an integer* k, $h \leq k \leq n$, *and a nonzero character* ρ_k *on* $C^*(\{A_{ij}^{(k)}\}_{i,j=1}^k)$ *and a* h*-dimensional reducing subspace* M_k *for* $\hat{\rho}_k(A_k)$ *such that* $\pi(A)$ *is unitarily equivalent to* $(\hat{\rho}_k(A_k))|_{M_k}$. *Conversely, any such integer, nonzero character, and reducing subspace produces a* h*-dimensional representation of* $C^*(A)$, *which is however not necessarily irreducible.*

Proof. Let $\mathcal{C}_k = C^*(\{A_{ij}^{(k)}\}_{i,j=1}^k)$ and suppose π is an irreducible representation of $C^*(A)$ on H. Again by Proposition 2.10.2 in [2], π can be extended to an irreducible representation $\tilde{\pi}$ of $M_1(\mathcal{C}_1) \oplus M_2(\mathcal{C}_2) \oplus \cdots \oplus M_n(\mathcal{C}_n)$ on K, where $K \supseteq H$, H reduces $\tilde{\pi}(C^*(A))$. Each $E_i = \tilde{\pi}(0 \oplus \cdots 0 \oplus I_i \oplus 0 \cdots \oplus 0)$, for $0 \leq i \leq n$, is a projection in the commutant of the image of $\tilde{\pi}$, so that E_i is either 0 or I, since $\tilde{\pi}$ is irreducible. Hence there exists an integer k, $h \leq k \leq n$, such that $\tilde{\pi}(C_1 \oplus C_2 \oplus \cdots \oplus C_n) = 0$ whenever $C_k = 0$. Thus $\tilde{\pi}$ can be considered as a k-dimensional irreducible representation of $M_k(\mathcal{C}_k)$ which extends π. Thus π can be considered as a h-dimensional irreducible representation of $C^*(A_k)$. Hence Proposition 2 implies that there exists a nonzero character ρ_k on $\mathcal{C}_k$ and a h-dimensional reducing subspace M_k for $\hat{\rho}_k(A_k)$ such that $\pi(A)$ is unitarily equivalent to $\hat{\rho}_k(A_k)|_{M_k}$. Again the converse is clear.

Remark. Although the previous two propositions characterize all the irreducible representations of the C^*-algebra generated by a homogeneous or general n-normal operator, they do not give us a description of the set of unitary equivalence classes of irreducible representations. For it is quite difficult to determine whether a given $n \times n$ scalar matrix is irreducible, and it is also quite difficult to determine when two $n \times n$ matrices are unitarily equivalent.

3. **The spectrum of the C^*-algebra generated by a binormal operator.** In this section we are able to give a complete description of the spectrum of a C^*-algebra generated by a binormal operator A, using Brown's characterization [1] of

binormal operators. It turns out that the spectrum of $C^*(A)$ given its hull-kernel topology is homeomorphic to the quotient of a set in $\mathbf{C}^3$, related to the joint spectrum of the matrix elements of A, modulo an equivalence relation. We remark that the spectrum need not be Hausdorff.

We begin by recalling a result from [1]. Let $\mathrm{Tri}(X, Y, Z)$ denote the triangular 2×2 matrix (A_{ij}) where $A_{11} = X$, $A_{22} = Z$, $A_{12} = Y$, and $A_{21} = 0$. Then Brown proves that every binormal operator is unitarily equivalent to an operator of the form $B \oplus \mathrm{Tri}(X, Y, Z)$ where B is normal, X, Y, Z are commuting normal operators, and Y is positive and one-to-one.

Proposition 4. *Let $A = B \oplus \mathrm{Tri}(X, Y, Z)$ be a binormal operator, with $Y \geq 0$. Then*

(i) *If π is a two-dimensional irreducible representation of $C^*(A)$ on H then $\pi(A)$ is unitarily equivalent to* $\mathrm{Tri}(\alpha, \beta, \gamma)$ *where $(\alpha, \beta, \gamma) \in \sigma(X, Y, Z)$, and $\beta > 0$. Conversely, every such triple gives rise to a two-dimensional irreducible representation in this manner.*

(ii) *If π is a nonzero character on $C^*(A)$ then $\pi(A) = \lambda$ where $\lambda \in \sigma(B)$ or there exists a $\mu \in \mathbf{C}$ such that either $(\lambda, 0, \mu) \in \sigma(X, Y, Z)$ or $(\mu, 0, \lambda) \in \sigma(X, Y, Z)$. Conversely, every such λ gives rise to a character in this manner.*

Of course, every irreducible representation of $C^*(A)$ for A binormal has dimension less than or equal to 2.

Proof. Follows immediately from Proposition 3.

Now let

$$S_0 = \{(\alpha, \beta, \gamma): \text{either } (\alpha, \beta, \gamma) \in \sigma(X, Y, Z) \text{ or } (\gamma, \beta, \alpha) \in \sigma(X, Y, Z)\}$$

and let

$$S = \sigma(B) \cup S_0.$$

Then define an equivalence relation $\sim$ on S by saying $s_1 \sim s_2$ if and only if one of the following four conditions is satisfied: (1) $s_1 = s_2$, (2) $s_1 = (\alpha, \beta, \gamma)$ and $s_2 = (\gamma, \beta, \alpha)$, $\beta > 0$, (3) $s_1 = (\alpha, 0, \gamma)$ and $s_2 = (\alpha, 0, \delta)$, or (4) $s_1 = \alpha \in \sigma(B)$ and $s_2 = (\alpha, 0, \gamma)$ or vice versa. We use the set S_0 instead of the set $\sigma(X, Y, Z)$ for two reasons. The first reason is that an element of the form $(\alpha, 0, \gamma) \in \sigma(X, Y, Z)$ yields two characters on $C^*(A)$ and the second is because the final topologies would not agree otherwise.

Let X denote the set of unitary equivalence classes of irreducible representations of $C^*(A)$. We define a map $\theta: S \to X$ as follows: If $s = (\alpha, \beta, \gamma)$ with $\beta > 0$ and $(\alpha, \beta, \gamma) \in \sigma(X, Y, Z)$, let $\theta(s)$ be the two-dimensional irreducible representation given in Proposition 4(i) by $(\theta(s))(A) = \mathrm{Tri}(\alpha, \beta, \gamma)$. If $s = (\alpha, \beta, \gamma)$ with $\beta > 0$ and $(\gamma, \beta, \alpha) \in \sigma(X, Y, Z)$, let $\theta(s)$ be the two-dimensional irreducible representation given by $(\theta(s))(A) = \mathrm{Tri}(\gamma, \beta, \alpha)$. Notice that if $\beta > 0$ and both (α, β, γ) and (γ, β, α) are in $\sigma(X, Y, Z)$ then this does give a single valued definition of $\theta(s)$ since $\mathrm{Tri}(\alpha, \beta, \gamma)$ and $\mathrm{Tri}(\gamma, \beta, \alpha)$ are unitarily equivalent. If $\alpha \in \sigma(B)$ let $\theta(\alpha)$ be the character on $C^*(A)$ given in Proposition 4(ii) by $(\theta(\alpha))(A) = \alpha$. If $s = (\alpha, 0, \beta) \in S_0$ let $\theta(s)$ be the character on $C^*(A)$ given in Proposition 4(ii) by $(\theta(s))(A) = \alpha$. Since $\mathrm{Tri}(\alpha, \beta, \gamma)$ and $\mathrm{Tri}(\alpha', \beta, \gamma')$ are unitarily equivalent if and only if $\{\alpha, \gamma\} = \{\alpha', \gamma'\}$ we have that $s_1 \sim s_2$ if and only if $\theta(s_1) = \theta(s_2)$. Thus θ induces a one-to-one mapping θ_0 from $S/\sim$ to X, which is onto by Proposition 4.

Give S_0 the topology it inherits from $\mathbf{C}^3$ and $\sigma(B)$ its natural topology. Let S have the disjoint union topology, and let $S/\sim$ have its quotient topology. Finally let X have its hull-kernel topology. Our goal is to show that θ_0 is a homeomorphism of $S/\sim$ onto X. Recall [2, 3.3.3] that if $\{D_i\}$ is a dense subset of $C^*(A)$, then a base for the hull-kernel topology on X is given by the sets $U_i = \{\pi \in X: \|\pi(D_i)\| > 1\}$.

Proposition 5. *The map $\theta: S \rightarrow X$ is continuous.*

Proof. Consider the dense set in $C^*(A)$ consisting of operators of the form $D = p(A, A^*)$ where p is a polynomial in two noncommuting variables. Suppose that $s_n \in S$ converges to $s \in S$. We need to show that $\theta(s_n)$ converges to $\theta(s)$ in X. There are three cases to be considered.

First assume that $s \in \sigma(B)$. So we may assume $s_n \in \sigma(B)$ for all n. Then $\rho_n = \theta(s_n)$ and $\rho = \theta(s)$ are characters on $C^*(A)$ such that $\rho_n(A) = s_n$ and $\rho(A) = s$. Thus $|\rho_n(D)| = |p(\rho_n(A), \rho_n(A^*))| = |p(s_n, \overline{s}_n)|$ converges to $|p(s, \overline{s})| = |\rho(D)|$ so that ρ_n converges to ρ in X.

Second assume that $s = (\alpha, \beta, \gamma) \in S_0$ with $\beta > 0$. Then we may assume $s_n = (\alpha_n, \beta_n, \gamma_n) \in S_0$ with $\beta_n > 0$ for all n. By Proposition 4(i) there exist unitary operators U_n and V such that $V(\theta(S)(A))V^* = \mathrm{Tri}(\alpha, \beta, \gamma)$ and $U_n(\theta(s_n)(A))U_n^* = \mathrm{Tri}(\alpha_n, \beta_n, \gamma_n)$. Hence $U_n(\theta(s_n)(D))U_n^* = U_n p(\theta(s_n)(A), \theta(s_n)(A^*))U_n^* = p(\mathrm{Tri}(\alpha_n, \beta_n, \gamma_n), \mathrm{Tri}(\alpha_n, \beta_n, \gamma_n)^*)$ which converges to $p(\mathrm{Tri}(\alpha, \beta, \gamma), \mathrm{Tri}(\alpha, \beta, \gamma)^*) = V(\theta(s)(D))V^*$. Thus $\|\theta(s_n)(D)\|$ converges to $\|\theta(s)(D)\|$, so that $\theta(s_n)$ converges to $\theta(s)$ in X.

Lastly assume that $s = (\alpha, 0, \gamma) \in S_0$. Then we may assume $s_n = (\alpha_n, \beta_n, \gamma_n) \in S_0$ for all n. Let N_1 be the set of integers n such that $\beta_n \neq 0$, and let N_2 be the set of integers n such that $\beta_n = 0$. For $n \in N_2$, $\theta(s_n)(D) = p(\alpha_n, \overline{\alpha}_n)$. So that if N_2 is infinite, $|\theta(s_n)(D)| = |p(\alpha_n, \overline{\alpha}_n)|$ converges to $|p(\alpha, \overline{\alpha})| = |\theta(s)(D)|$. If $n \in N_1$ then $\theta(s_n)(A)$ is unitarily equivalent to $\mathrm{Tri}(\alpha_n, \beta_n, \gamma_n)$. So that if N_1 is infinite, $\|\theta(s_n)(D)\| = \|p(\mathrm{Tri}(\alpha_n, \beta_n, \gamma_n), \mathrm{Tri}(\alpha_n, \beta_n, \gamma_n)^*)\|$ converges to $\|p(\mathrm{Tri}(\alpha, 0, \gamma), \mathrm{Tri}(\alpha, 0, \gamma)^*)\|$, which is greater than or equal to $|p(\alpha, \overline{\alpha})| = \|\theta(s)(D)\|$. So that, if $\|\theta(s)(D)\| > 1$, there exists an integer N such that, for all $n \in N_1 \cup N_2$, $n \geq N$, $\|\theta(s_n)(D)\| > 1$.

Proposition 6. *The mapping θ_0 is a homeomorphism of $S/\sim$ onto X.*

Proof. Because of Proposition 5 and the remarks preceding it, we need only show that θ_0 is a closed mapping. Since S is compact, $S/\sim$ is also compact. Let F be a closed, hence compact, subset of $S/\sim$. Since θ_0 is continuous, $\theta_0(F)$ must be compact. Let $\{\pi_n\}_1^\infty$ be a sequence in $\theta_0(F)$ converging to $\pi_0 \in X$. We need to show that $\pi_0 \in \theta_0(F)$. Since X is second countable and $\theta_0(F)$ is compact, there is a subsequence $\{\pi_{n_k}\}$ which converges to $\pi_1 \in \theta_0(F)$. Hence we may assume $\{\pi_n\}$ converges to π_0 and π_1. If $\pi_0 = \pi_1$ we are done. Assume $\pi_0 \neq \pi_1$. By Corollary 1 in [3, p. 388], we have that $\dim \pi_0 + \dim \pi_1 \leq 2$, thus $\dim \pi_0 = 1 = \dim \pi_1$, and π_0, π_1 are the only limit points of $\{\pi_n\}$. Let $J = \bigcap \{\rho^{-1}(0): \rho$ is a character on $C^*(A)\}$. Then $J \neq \emptyset$, $C^*(A)/J$ is commutative and the spectrum of $C^*(A)/J$ is just the space of characters of $C^*(A)$ [2, 3.6.3 and 3.2.1], and hence is Hausdorff. Then since π_n converges to the characters π_0, π_1 we must have that $\dim \pi_n = 2$ for large n. Hence we can assume $\dim \pi_n = 2$ for all n. Now let q be the quotient map of S onto $S/\sim$. Then $\pi_n = \theta(s_n)$ where $s_n \in q^{-1}(F)$ and $s_n = (\alpha_n, \beta_n, \gamma_n) \in \sigma(X, Y, Z)$ with $\beta_n > 0$. Suppose that $\pi_0(A) = \alpha_0$ and $\pi_1(A) = \alpha_1$. Since $\sigma(X, Y, Z)$ is compact [4], there is a subsequence $s_{nk} = (\alpha_{nk}, \beta_{nk}, \gamma_{nk})$ which converges, say to (α, β, γ). Since θ is continuous, π_{nk} converges to $\theta(\alpha, \beta, \gamma)$. But π_0 and π_1 are the only limit points of π_n, thus $\beta = 0$ and $\alpha = \alpha_0$ or α_1. Thus α_{nk} converges to either α_0 or α_1. If α_{nk} converges to α_0, then $(\alpha_0, 0, \gamma) \in \sigma(X, Y, Z)$ and $(\alpha_0, 0, \gamma) \in q^{-1}(F)$ since $q^{-1}(F)$ is closed. Thus $\pi_0 \in \theta_0(F)$ and we are done. On the other hand, if α_{nk} converges to α_1, then $(\gamma_{nk}, \beta_{nk}, \alpha_{nk}) \in S$ and converges to $(\gamma, 0, \alpha_1)$. Since θ_0 is continuous, π_{nk} converges to $\theta(\gamma, 0, \alpha_1)$. Hence $\gamma = \alpha_1$ or α_0. If $\gamma = \alpha_0$, then we would have $\pi_0 \in \theta_0(F)$ since $s_{nk} \in q^{-1}(F)$ and we would be done. Finally, suppose $\gamma = \alpha_1$. Then $(\alpha_{nk}, \beta_{nk}, \gamma_{nk})$ converges to $(\alpha_1, 0, \alpha_1)$. Now by the lower semicontinuity of the map π to $\|\pi(D)\|$ for $D \in C^*(A)$ [2, 3.3.2], since π_n converges to π_0, we have that

$$\begin{aligned}\|\pi_0(A - \alpha_1)\| &\leq \liminf \|\pi_n(A - \alpha_1)\| \\ &\leq \liminf \|\mathrm{Tri}\,(\alpha_{n_k}, \beta_{n_k}, \gamma_{n_k}) - \mathrm{Tri}\,(\alpha_1, 0, \alpha_1)\| = 0,\end{aligned}$$

thus $\pi_0(A) = \alpha_0 = \alpha_1$ and $\pi_0 \in \theta_0(F)$.

Examples and remarks. (1) Proposition 6 can be used to give a variety of examples of non-Hausdorff spaces that are the spectrums of singly generated C^*-algebras. For example let H be $L^2[-1, 1]$ and let X be multiplication by the function $g(t) = t$, $Z = -X$, and Y be multiplication by a nonnegative continuous bounded function f such that $f^{-1}(0)$ is nonempty. Then $A = \mathrm{Tri}\,(X, Y, Z)$ is a homogenous binormal operator, in fact A^2 is normal [6]. Also, $\sigma(X, Y, Z) = \{(t, f(t), -t): -1 \leq t \leq 1\}$. Let $T_0 = \{(t, f(t)): -1 \leq t \leq 1\} \cup \{(-t, f(t)): -1 \leq t \leq 1\}$, and define $\sim$ on T_0 by $(\alpha, \beta) \sim (\alpha_1, \beta_1)$ if and only if $\beta = \beta_1 \neq 0$ and $\alpha^2 = \alpha_1^2$. Then $S_0/\sim$ is easily seen to be homeomorphic to $T_0/\sim$. Thus the spectrum of the algebra $C^*(A)$ is $T_0/\sim$ and is simply obtained by taking a quotient space of a subset of the complex plane. By choosing particular functions f, a number of examples can be obtained.

(2) We recall that if $\mathfrak{A}$ is a C^*-algebra and $A \in \mathfrak{A}$, then the mapping that sends $\pi \in X$ to $\|\pi(A)\|$ is lower semicontinuous. That is, if π_n converges to π then $\|\pi(A)\| \leq \lim\inf \|\pi_n(A)\|$. If A is binormal and $\mathfrak{A} = C^*(A)$, then the proof of Proposition 5 shows that unless π is the image of an element of the form $(\alpha, 0, \gamma) \in S_0$ we actually have $\|\pi(D)\| = \lim \|\pi_n(D)\|$ for all $D \in C^*(A)$. If π is the image of an element of the form $(\alpha, 0, \gamma)$, then since $|\alpha|$ may be strictly less than $|\gamma|$ we may have $\|\pi(D)\| < \lim\inf \|\pi_n(D)\|$.

Added in revision. Carl Pearcy has kindly informed us that this paper is related to a paper of Harry Gonshor [Canad. J. Math. **10** (1958), 97–102]. For A a binormal operator, Gonshor used direct integral theory to determine what Fell [Acta. Math. 26 (1961), 233–280] later called the Hausdorff compactification $\underset{\sim}{Q}$ of the spectrum of $C^*(A)$ and characterized those continuous functions on $\underset{\sim}{Q}$ that come from elements of $C^*(A)$.

REFERENCES

1. A. Brown, *The unitary equivalence of binormal operators*, Amer. J. Math. 76 (1954), 414–434.

2. J. Dixmier, *Les C^*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR 30 #1404.

3. J. M. G. Fell, *The dual spaces of C^*-algebras*, Trans. Amer. Math. Soc. **94** (1960), 365–403. MR 26 #4201.

4. Lars Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, N. J., 1966. MR 34 #2933.

5. C. Pearcy, *A complete set of unitary invariants for operators generating finite W^*-algebras of type* I, Pacific J. Math. 12 (1962), 1405–1416. MR 26 #6816.

6. H. Radjavi and P. Rosenthal, *On roots of normal operators*, J. Math. Anal. Appl. 34 (1971), 653–664. MR 43 #3829.

7. M. Takesaki, *On the cross-norm of the direct product of C^*-algebras*, Tôhoku Math. J. (2) **16** (1964), 111-122. MR 29 #2668.

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