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ACTIONS OF GROUPS OF ORDER pq

BY

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ABSTRACT. We study the bordism group of stably complex G-manifolds in the case where G is a metacyclic group of order pq and p and q are distinct primes. This bordism group is a module over the complex bordism ring and we compute the projective dimension of this module. We develop some techniques necessary for the study of this module in case G is a more general metacyclic group.

The purpose of this paper is to study the bordism theory of actions of a metacyclic group on stably complex manifolds in a special case. More general and complete results will appear in [8]. However, many of the techniques used can be isolated and simplified in this special case. We will be concerned with actions of a group G which is the semidirect product of two cyclic groups of prime order.

By a family F of subgroups of G we will mean a subset of the subgroups of G with the property that if K is an element of F every subgroup and every conjugate of K is in F. $\Omega_*(G; F)$ is the bordism group of actions of G on stably complex manifolds such that every isotropy group lies in the family F. Ω_* will denote the complex bordism ring.

Theorem A. *Let F be the family consisting of all subgroups of G. Then $\Omega_*(G; F)$ is a free Ω_* module on even dimensional generators.*

Theorem B. *Let F be any family of subgroups of G. Then $\Omega_+(G; F) = \bigoplus \Omega_{2i}(G; F)$ is a free Ω_* module. And $\Omega_-(G; F) = \bigoplus \Omega_{2i+1}(G; F)$ has projective dimension one over Ω_*.*

Basic material on the groups $\Omega_*(G; F)$ can be found in [3], [10]. Our analysis largely follows the lines of [2] but was influenced by [10] and [6].

In §1 we discuss general facts about equivariant bordism groups. In §2 we discuss the cohomology of a cyclic group acting on a polynomial ring. In §3 we discuss $\Omega_*(G; F, F')$ for adjacent families F, F' in G and in §4 we present the proofs of the main theorems.

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1. **Basic facts.** Let p and q be distinct primes. Let f be a homomorphism of Z_p into the automorphism group of Z_q. We denote the resulting semidirect product by $Z_p \times_f Z_q = G$. G is generated by two elements a and b with relations $b^p = a^q = 1$, $bab^{-1} = a^r$ where r is a nonzero element of Z_q. f, of course, is the homomorphism which takes b to the automorphism $a \to a^r$.

Lemma (1.1). *$H_{2k-1}(BG) = Z_p$ if k is not a multiple of p and $H_{2k-1}(BG) = Z_p \oplus Z_q$ if k is a multiple of p.*

Proof. Consider the Hochschild-Serre spectral sequence

$$E^2_{i,j} = H_i(Z_p;\, H_j(Z_q)).$$

$E^2_{i,j} = 0$ unless i or j is zero. $E^2_{i,0} = H_i(Z_p;\, Z) = 0$ or Z_p depending upon whether i is even or odd. To compute $E^2_{0,j} = H_0(Z_p;\, H_j(Z_q))$ we use the resolution

$$Z[Z_p] \xrightarrow{N} Z[Z_p] \xrightarrow{D} Z[Z_p] \tag{1.2}$$

where $D = 1 - b$, $N = 1 + b + \cdots + b^{p-1}$ of [9, p. 121]. Tensoring on the right with $H_j(Z_q)$, we find that $E^2_{0,j} = \operatorname{cokernel}(D \otimes 1)$. A simple computation shows the generator b of Z_p acts on $H_{2j-1}(Z_q) = Z_q$ by multiplication by r^j and $1 - r^j$ is a unit in Z_q unless $j \equiv 0 \bmod p$. It is clear that the inclusion $Z_p \subset G$ induces an injection $H_*(BZ_p) \to H_*(BG)$ onto the p-torsion, and the inclusion $Z_q \subset G$ induces a map $H_*(BZ_q) \to H_*(BG)$ which is onto the q-torsion.

Lemma (1.3). *$\Omega_*(BG)$ has projective dimension one over Ω_*. The left G-manifolds $G \times_{Z_p} S^{2k-1}$ generate the p-torsion and the left G-manifolds $G \times_{Z_q} S^{2kp-1}$ generate the q-torsion where Z_p acts on S^{2k-1} by multiplication of each coordinate in C^k by $e^{2\pi i/p}$ and Z_q acts on S^{2kp-1} by multiplication by $e^{2\pi i/q}$. Let V be the representation of G induced from the one-dimensional representation $e^{2\pi i/q}$ of Z_q (see [5, p. 333]). $V = Z[G] \otimes_{Z[Z_q]} C^*$. Then the left G-manifolds $S(kV)$ also serve as generators for the q-torsion.*

Proof. The statement about dimension follows directly from [7] since $H_*(BG)$ is concentrated in odd dimensions. The manifolds $[Z_p, S^{2k-1}]$ and $[Z_q, S^{2kp-1}]$ give rise, via the Thom map, to generators of $H_{2k-1}(BZ_p)$ and $H_{2kp-1}(BZ_q)$ and so give rise to generators of $H_*(BG)$. Thus the G extensions of these manifolds yield generators of $\Omega_*(BG)$ by standard arguments [1, p. 49]. It is well known that if l is an integer, $\xi_1, \cdots, \xi_l$ are primitive qth roots of unity, and if Z_q acts on C^l by $a(x_1, \cdots, x_l) = (\xi_1 x_1, \cdots, \xi_l x_l)$, then the fundamental class of S^{2l-1}/Z_q is a generator of $H_{2l-1}(BZ_q)$. If $C^{kp} = C^k \times \cdots \times C^k$ and Z_q acts on the jth factor by ξ^{r-j} then S^{2kp-1} gives rise to a generator of $H_{2kp-1}(BZ_q)$. V has $1 \otimes 1, b \otimes 1, \cdots, b^{p-1} \otimes 1$ as basis and

$a(b^j \otimes 1) = \xi^{r-j}(b^j \otimes 1)$. Now we claim that $G \times_{Z_q} S^{2kp-1}$ and $Z_p \times S(kV)$ with the diagonal action of G are G diffeomorphic. It is enough to take the case $k = 1$. Let e_j be a basis for the jth factor in $C \times \cdots \times C$. Send $[b^i, e_j]$ in $G \times_{Z_q} C^p$ to $(b^i, b^{i+j} \otimes 1)$ in $Z_p \times V$. Thus the manifolds $Z_p \times S(kV)$ generate the q-torsion in $\Omega_*(BG)$. Let $\mu: \Omega_*(BG) \to H_*(BG)$ be the Thom map. Then $\{\mu(Z_p \times_G S(kV))\}$ are generators for the q-torsion in $H_*(BG)$ and $\mu(Z_p \times_G S(kV)) = p\mu(S(kV)/G)$. Since p and q are relatively prime, $\{\mu(S(kV)/G)\}$ are generators for the q-torsion in $H_*(BG)$ and so $S(kV)/G$ give generators for the q-torsion in $\Omega_*(BG)$.

In [10] and [2] the groups $\Omega_*(G; F, F')$, where G is a finite group and F and F' are families of subgroups of G, are studied. If K is a subgroup of G we let $F(K)$ denote the family consisting of all conjugates of subgroups of K and we let $F_0(K)$ denote the family consisting of all conjugates of proper subgroups of K. We call F and F' strictly adjacent families if there is a subgroup K of G such that $F - F'$ consists of the conjugates of K.

Let F and F' be strictly adjacent families of a finite group G which differ by the conjugates of K. We have well-known isomorphisms

$$\Omega_*(G; F, F') \cong \Omega_*(G; F(K), F_0(K)) \cong \Omega_*(N(K); F(K), F_0(K)) \tag{1.4}$$

where $N(K)$ is the normalizer of K in G. We enunciate these isomorphisms. Let W be a stably complex left G-manifold with boundary. Let M be the points in W with isotropy group conjugate to K and N a closed G-invariant tubular neighborhood of K. Then the class of W in $\Omega_*(G; F, F')$ is sent to the class of N in $\Omega_*(G; F(K), F_0(K))$. If we let ν be the normal bundle of M in W, M_0 the points in M with isotropy group equal to K, ν_0 the restriction of ν to M_0, then the disk bundle $D(\nu_0)$ is an $N(K)$ manifold with boundary which represents an element in $\Omega_*(N(K); F(K), F_0(K))$ as we assign the class of $D(\nu_0)$ to the class of N.

The analysis of these groups is carried out in [11] for unoriented bordism and in more detail in a special case in [2]. We present a treatment of $\Omega_*(G; F(K), F_0(K))$ which we will later explicitly need. Let I_x denote inner automorphism by x. Let ρ and ρ' be n-dimensional complex representations of K. We will say that ρ and ρ' are G-equivalent if there are elements g in G and A in $U(n)$ such that $\rho' = I_A \rho I_g$.

Theorem (1.5). *Let K be normal in G. Then*

$$\Omega_m(G; F(K), F_0(K)) \cong \sum \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho)))$$

where n runs from 0 to $[m/2]$, ρ runs over a set of representitives of G-equivalent n-dimensional representations of K, $\Gamma(\rho)$ is the graph of ρ in $G \times U(n)$, and $N(\rho)$ is the normalizer of $\Gamma(\rho)$ in $G \times U(n)$.

Proof. Let W be a manifold with boundary representing an element in $\Omega_m(G; F(K), F_0(K))$, M the set of points with isotropy group equal to K, ν the normal bundle, M_n the union of the $(m-2n)$-dimensional components of M, ν_n the restriction of ν to M_n. Then W and $\Sigma D(\nu_n)$ are bordant and the bordism relation respects this decomposition. Thus $\Omega_m(G; F(K), F_0(K)) \cong \Sigma \Psi_n$, where Ψ_n is the bordism group of complex n-dimensional left G vector bundles E over compact $(m-2n)$-dimensional left G-manifolds M such that every point on the zero section of E has isotropy group equal to K and every point off the zero section has isotropy group properly contained in K. Let P be the principal $U(n)$ bundle corresponding to E. As in [2], P can be regarded as a right $G \times U(n)$ space such that the isotropy group of a point e in P is $\Gamma(\rho_e)$ for some n-dimensional complex representation ρ_e of K. $H = G \times U(n)$ acts differentiably on P so there is a neighborhood V of e in P such that for e' in V there is a (g, A) in H such that $H_{e'} \subset (g^{-1}, A^{-1})H_e(g, A)$. From this it follows that $\rho_{e'} = I_{A^{-1}}\rho_e I_x$. Further, if $e' = eA$ for some A in $U(n)$, then $\rho_{e'} = I_{A^{-1}}\rho_e$. Thus for each x in M there is a neighborhood U of x such that, for any two points e and e' in $P \mid U$, ρ_e and $\rho_{e'}$ are G-equivalent and so H_e and $H_{e'}$ are conjugate in H. Thus $\Psi_n \cong \Sigma \Psi_{n,\rho}$ where $\Psi_{n,\rho}$ is the bordism group of compact right $G \times U(n)$ spaces P such that $U(n)$ acts freely on P, $P/U(n) = M$ is a stably complex G-manifold, every point in P has isotropy group conjugate, in H, to $\Gamma(\rho)$, and ρ runs over a set of representatives of n-dimensional representations of K under G-equivalence. So $M/G/K$ is a stably complex manifold. Now let $P \to M$ represent an element in $\Psi_{n,\rho}$ and let $Q \subset P$ be the points in P with isotropy group equal to $\Gamma(\rho)$. Q is a principal right $N(\rho)/\Gamma(\rho)$ bundle, $Q/N(\rho)/\Gamma(\rho) = M/G/K$, and Q and P determine each other since $Q \times_{N(\rho)} H = P$. Thus

$$\Psi_{n,\rho} \cong \Omega_{m-2n}(B(N(\rho)/\Gamma(\rho))). \tag{1.6}$$

(1.7) We should make a few remarks about extensions of actions. If H is a subgroup of G and F a family of subgroups of H, then we let F also denote that family of subgroups of G consisting of all conjugates of elements of F. We have a map $E: \Omega_*(H; F) \to \Omega_*(G; F)$ which takes the class of a stably complex H-manifold M to the class of $G \times_H M$. Note $G \times_H M$ has a well-defined G-invariant stable complex structure. In a similar manner we have $E: \Omega_*(H; F, F') \to \Omega_*(G; F, F')$. Note that if H_1 and H_2 are conjugate in G, then the image of $\Omega_*(H_1; F_0(H_1))$ is equal, in $\Omega_*(G; F)$, to the image of $\Omega_*(H_2; F_0(H_2))$ where F is any family of G containing $F_0(H_1) = F_0(H_2)$.

2. **Actions of cyclic groups on polynomial rings.** We will develop some algebra which will be relevant to the analysis of $\Omega_*(G; F, F')$. Let $P_1, \cdots, P_t$

be polynomial rings over the integers, $P_j^{(p)}$ the p-fold tensor product. Z_p acts on $P_j^{(p)}$ by letting the generator b of Z_p act by $b(x_0 \otimes \cdots \otimes x_{p-1}) = x_{p-1} \otimes x_0 \otimes \cdots \otimes x_{p-2}$. Let $P = P_1^{(p)} \otimes \cdots \otimes P_t^{(p)}$. P is a polynomial ring and Z_p acts on P via the tensor product of the actions on $P_j^{(p)}$. $\Omega_* \otimes P$ is a polynomial ring over Ω_* on which Z_p acts. It follows directly from the definition of homology via resolutions that $H_*(Z_p; \Omega_* \otimes P) \cong \Omega_* \otimes H_*(Z_p; P)$.

Lemma (2.1). *$H_i(Z_p; P) = 0$ for i even.*

Proof. We use the resolution of (1.2) for Z over $Z[Z_p]$. We must calculate kernel(N)/image(D), where N and D act on P. Let X be a monomial in P. Let $Z\{X\}$ be the subgroup of P which has basis consisting of the distinct images of X under Z_p. $P = \bigoplus Z\{X\}$. It is enough to show $H_i(Z_p; Z\{X\}) = 0$. Either $bX = X$ or $X, bX, \cdots, b^{p-1}X$ are distinct. In the first case, the kernel of N is clearly zero. In the second case

$$N\left(\sum_{j=0}^{p-1} m_j(b^jX)\right) = \sum_{j=0}^{p-1} m_j N(X) = 0 \quad \text{implies} \quad \sum_{j=0}^{p-1} m_j = 0.$$

Now for any integer k, $X - b^kX = \sum_{j=0}^{k-1} (b^jX - b^{j+1}X)$. So $X \equiv b^kX$ mod image(D). Thus $\sum_{j=0}^{p-1} m_j(b^jX) \equiv \sum_{j=0}^{p-1} m_jX$ mod image(D).

Lemma (2.2). *$H_0(Z_p; P)$ is a free Z module.*

Proof. $H_0(Z_p; P) = P/\text{image}(D)$. As in (2.1) it is enough to compute $Z\{X\}/\text{image}(D)$. If $bX = X$, image$(D) = 0$ and $Z\{X\}$ is a free module with basis X. In the other case, $Z\{X\}$ has, as basis $X, (1-b)X, \cdots, (1-b)b^{p-2}X$. From the formula $(1-b)b^{p-1}X = -(1-b)X - (1-b)bX - \cdots - (1-b)b^{p-2}X$ it follows that $(1-b)X, \cdots, (1-b)b^{p-2}X$ is a basis for image(D) and so $Z\{X\}/\text{image}(D)$ is a free Z module with basis $X + \text{image}(D)$.

Now let us consider the map $\phi_j: P_j \to P_j^{(p)}$

(2.3) $$\phi_j(X) = \sum X_0 \otimes \cdots \otimes X_{p-1}$$

where the sum is taken over all $X_0 \otimes \cdots \otimes X_{p-1}$ such that $X_0X_1 \cdots X_{p-1} = X$ in P_j. We then let $\overline{P} = P_1 \otimes \cdots \otimes P_t$ and let

(2.4) $$\Phi: \overline{P} \to P$$

be the tensor product of the $\phi_j: P_j \to P_j^{(p)}$. $\overline{P}$ is a trivial Z_p module. Φ is a map of Z_p modules.

Theorem (2.5). *$\Phi_*: H_{2i-1}(Z_p; \overline{P}) = \overline{P} \to H_{2i-1}(Z_p; P)$ is surjective. Let $X = X_1 \otimes \cdots \otimes X_t$ be a monomial in $\overline{P}$ and let $\overline{P}_1$ be the subgroup with basis consisting of those X such that each monomial X_j is a pth power. Let $\overline{P}_2$ be* *the submodule with basis consisting of the remaining monomials. Then* $P_2 = \text{kernel}(\Phi_*)$ *and* $\Phi_*: \bar{P}_1 \xrightarrow{\cong} H_{2i-1}(Z_p; P)$.

Proof. Let $H_-(Z_p; P)$ denote $H_{2i-1}(Z_p; P)$. Consider $\Phi(X) = \phi(X_1) \otimes \cdots \otimes \phi(X_t)$ where $\phi_j(X_j) = \Sigma X_{j0} \otimes \cdots \otimes X_{jp-1}$. X_j is a pth power in P_j if and only if for one term in this sum $X_{j0} = \cdots = X_{jp-1}$. Call this common value Y_j. Thus $\Phi(X)$ has a term

$$\underbrace{Y_1 \otimes \cdots \otimes Y_1}_{p} \otimes \cdots \otimes \underbrace{Y_t \otimes \cdots \otimes Y_t}_{p} = Y$$

if and only if each X_j is a pth power. Furthermore, a monomial in P is invariant under Z_p if and only if it is such a Y. Thus $\Phi(X) = Y + \Sigma T$ where $bY = Y$ and $bT \neq T$ if each X_j is a pth power, and $\Phi(X) = \Sigma T$ where $bT \neq T$ if some X_j is not a pth power.

Now if $bY = Y$, then $D(Y) = 0$ and $N(Y) = pY$ and so $H_-(Z_p; Z\{Y\}) = Y \bmod pY$. If $bT \neq T$, then the kernel of D restricted to $Z\{T\}$ is the subgroup generated by $T + bT + \cdots + b^{p-1}T = N(T)$ and so $H_-(Z_p; Z\{T\}) = 0$. Thus $H_-(Z_p; P) = \bigoplus H_-(Z_p; Z\{Y\}) = \bigoplus Y \bmod pY$ for those Y such that $bY = Y$. If $Y = Y_1 \otimes \cdots \otimes Y_1 \otimes \cdots \otimes Y_t \otimes \cdots \otimes Y_t$, let $X_j = Y_j^p$ and $X = X_1 \otimes \cdots \otimes X_t$. Then $\Phi_*(X) = \Phi(X) \bmod \text{image } N = Y \bmod \text{image } N$. And if $\Phi(X) = \Sigma T$, then $\Phi(X) = 0 \bmod \text{image } N$.

3. $\Omega_*(G; F, F')$. Throughout this section $G = Z_p \times_f Z_q$, $\eta = e^{2\pi i/p}$, $\xi = e^{2\pi i/q}$. Let F_α be the family of all subgroups of G, F_p be $F_0(Z_p)$, F_q be $F_0(Z_q)$, F_0 be the family consisting of all proper subgroups, and F_1 the family consisting of the identity subgroup

$$\Omega_*(G; F_0, F_q) \cong \Omega_*(G; F_p, F_1) \cong \Omega_*(Z_p; F(Z_p), F_0(Z_p)),$$

$$\Omega_*(G; F_0, F_p) \cong \Omega_*(G; F(Z_q), F_0(Z_q)) \quad \text{by (1.4)}.$$

First we study $\Omega_*(Z_p; F(Z_p), F_0(Z_p))$. The first lemma is well known.

Lemma (3.1). *Let* ρ *be an n-dimensional representation of* Z_p. *Let* $N(\rho)$ *be the normalizer of* $\Gamma(\rho)$ *in* $Z_p \times U(n)$. *Then*

$$\Omega_*(B(N(\rho)/\Gamma(\rho))) \cong \Omega_*(BU(k_1) \times \cdots \times BU(k_{p-1})).$$

The isomorphism $\Omega_*(Z_p; F(Z_p), F_0(Z_p)) \cong \Omega_*(G; F_0, F_q)$ *of* (1.4) *is induced by extension.*

Proof. Let $\rho(b) = k_1\eta \oplus \cdots \oplus k_{p-1}\eta^{p-1}$. Then $N(\rho) = Z_p \times C(\rho)$, where

$C(\rho)$ is the centralizer of ρ in $U(n)$. By Schur's lemma, $C(\rho) = U(k_1) \times \cdots \times U(k_{p-1})$. $N(\rho)/\Gamma(\rho) \cong U(k_1) \times \cdots \times U(k_{p-1})$, an explicit isomorphism is obtained by sending $(b^j, A_1, \cdots, A_{p-1})$ to $(\eta^{-1}A_1, \cdots, \eta^{-(p-1)}A_{p-1})$. Now let W represent an element of $\Omega_*(Z_p; F(Z_p), F_0(Z_p))$. $G \times_{Z_p} W$ is the extension. Let M be the fixed point set of Z_p in W and D an invariant tubular neighborhood. Then $G \times_{Z_p} M = G/Z_p \times M$ is the points with isotropy group conjugate to Z_p, $G \times_{Z_p} D = D'$ is an invariant tubular neighborhood of $G \times_{Z_p} M$. The points with isotropy group equal to Z_p is $Z_p \times_{Z_p} M = M$ and D' restricted to M is D. So, the last statement follows.

Now we study $\Omega_*(G; F(Z_q), F_0(Z_q))$. Let σ be an n-dimensional representation of Z_q. Suppose $\sigma(a) = k_1\xi \oplus \cdots \oplus k_{q-1}\xi^{q-1}$. Then σI_{b^j}, $j = 0, 1, \cdots, p-1$, are all G-equivalent to σ. Let $C(\sigma)$ be the centralizer of σ in $U(n)$. Then $[\Gamma(\sigma), 1 \times C(\sigma)] \subset N(\sigma)$. (Here [ ] denotes the group generated by .)

Lemma (3.2). *(b, B) is in $N(\sigma)$ for some B in $U(n)$ if and only if $\sigma I_b = I_B\sigma$, and this holds if and only if $k_j = k_{jr^s}$ for all j and all $s = 0, 1, \cdots, p-1$.*

Proof. $(b, B)\Gamma(\sigma)(b^{-1}, B^{-1}) = \Gamma(\sigma)$ means that $B\sigma(a)B^{-1} = \sigma(a^r) = \sigma I_b(a)$. So $\sigma I_b = I_B\sigma$. Let $V = C^n$ be a representation space for σ, and V_j the ξ^j eigenspace for $\sigma(a)$. Suppose $\sigma(a)B^{-1} = B^{-1}\sigma(a^r)$. Then x is in V_j if and only if $\sigma(a)B^{-1}(x) = B^{-1}\sigma(a)x = \xi^{jr}B^{-1}(x)$. Thus $B^{-1}: V_j \cong V_{jr}$, and this can hold only if $k_j = k_{jr^s}$. On the other hand, if $k_j = k_{jr}$ for all j, choose a basis for V by choosing a basis for each V_j. Let B^{-1} be the permutation matrix relative to this basis which takes the basis in V_j to the basis in V_{jr}. Then (b, B) is in $N(\sigma)$.

Lemma (3.3). *If $\sigma I_b = I_B\sigma$ for some B, then*

$$N(\sigma) = [(b, B), \Gamma(\sigma), 1 \times C(\sigma)].$$

If not, $N(\sigma) = [\Gamma(\sigma), 1 \times C(\sigma)]$. $C(\sigma) \cong U(k_1) \times \cdots \times U(k_{q-1})$. If $A = (A_1, \cdots, A_{q-1})$ is the matrix of an element in $C(\sigma)$ relative to the basis indicated in (3.2), *we can choose B to be the permutation matrix of* (3.2) *and then*

$$(b, B)(1, A_1, \cdots, A_{q-1})(b^{-1}, B^{-1}) = (1, A'_1, \cdots, A'_{q-1})$$

where $A'_j = A_{jr}$.

Proof. It is clear that the B of (3.2) gives us an element (b, B) in $N(\sigma)$. If (b, B') is in $N(\sigma)$, then $B^{-1}B'$ is in $C(\sigma)$ and so $(b, B') = (b, B)(1, B^{-1}B')$.

Let Z_q^* be the multiplicative group of nonzero elements of Z_q, and $[r]$ the subgroup generated by r. Let $\{s_j\}$, $j = 1, \cdots, (q-1)/p$, be representatives for $Z_q^*/[r]$. Then

$$\prod_{j=1}^{q-1} U(k_j) \quad \prod_{j=1}^{q-1/p} \prod_{t=0}^{p-1} U(k_{s_j r} t).$$

Inner automorphism by B in (3.3) sends $\Pi_{t=0}^{p-1} U(k_{s_j r} t)$ to itself and takes $(A_0, \cdots, A_{p-1})$ to $(A_1, \cdots, A_{p-1}, A_0)$. Thus

Lemma (3.4). *If $\sigma I_b = I_B \sigma$, then*

$$\frac{N(\sigma)}{\Gamma(\sigma)} \cong Z_p \times_f \prod_{j=1}^{q-1/p} \prod_0^{p-1} U(l_j)$$

where l_j is the common value of $k_{s_j r} t$, and inner automorphism by the generator b of Z_p preserves $\Pi_0^{p-1} U(l_j)$ and takes $(A_0, \cdots, A_{p-1})$ to $(A_1, \cdots, A_{p-1}, A_0)$.

Now we will analyze $\Omega_*(B(N(\sigma)/\Gamma(\sigma)))$ for such σ. Let

$$U = \prod_{j=1}^{q-1/p} \prod_0^{p-1} U(l_j). \tag{3.5}$$

$\Omega^*(BU)$ is a power series ring in Chern classes.

$$\Omega_*(BU) = \mathrm{Hom}_{\Omega^*}(\Omega^*(BU); \Omega^*).$$

For each monomial C_α in the Chern classes, let X_α be the element in $\Omega_*(BU)$ such that $\langle X_\alpha, C_\beta \rangle = \delta_\alpha^\beta$. Then $\Omega_*(BU)$ is a free Ω_* module with basis $\{X_\alpha\}$. We can make $\Omega_*(BU)$ into a polynomial ring by defining $X_\alpha X_\beta = X_\gamma$ where X_γ is dual to $C_\alpha C_\beta$. And $\Omega_*(BU) \cong \Omega_* \otimes P$ where P is the polynomial ring over the integers with monomials X_α. Consider the diagonal map $U(l_j) \to \Pi_0^{p-1} U(l_j)$. Let

$$\overline{U} = \prod_{j=1}^{q-1/p} U(l_j) \tag{3.6}$$

and let

$$\Psi : \overline{U} \to U \tag{3.7}$$

be the product of the diagonal maps. Let Ψ also denote the induced map $B\overline{U} \to BU$. Let $t = (q-1)/p$.

Lemma (3.8). *The map $\Omega_*(BU(l_j)) \to \bigotimes_1^{p-1} \Omega_*(BU(l_j))$ induced by the diagonal takes a monomial X in $\Omega_*(BU(l_j))$ to $\Sigma X_0 \otimes \cdots \otimes X_{p-1}$ where the sum is taken over all $X_1 \otimes \cdots \otimes X_t$ such that $X_0 \cdots X_{p-1} = X$ in $\Omega_*(BU(l_j))$. The map $\Omega_*(B\overline{U}) \to \Omega_*(BU)$ is the tensor product of these maps.*

Proof. Consider the diagonal map $\Delta: BU(l) \to \Pi_0^{p-1} BU(l)$. Let X be a monomial in $\Omega_*(BU(l))$. Then $\Delta_*(x) = \Sigma\, M(X_0 \otimes \cdots \otimes X_{p-1})$ where X_i is in

$\Omega_*(BU(l))$ and M is in Ω_*. Consider $C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}}$ in $\Omega^*(\Pi_0^{p-1} BU(l))$.

$$\langle \Delta_*(X), C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}} \rangle = \langle X, \Delta^*(C_{\alpha_0}) \cdots \Delta^*(C_{\alpha_{p-1}}) \rangle$$

$$= \langle X, C_{\alpha_0} \cdots C_{\alpha_{p-1}} \rangle = 1$$

if X is dual to $C_{\alpha_0} \cdots C_{\alpha_{p-1}}$ and zero otherwise. A term $M\langle X_0 \otimes \cdots \otimes X_{p-1}, C_{\alpha_0} \otimes \cdots \otimes C_{\alpha_{p-1}} \rangle$ is one only if $M = 1$ and X_j is dual to C_{α_j}. Then by definition, $X_0 \cdots X_{p-1}$ is dual to $C_{\alpha_0} \cdots C_{\alpha_{p-1}}$ and so equals X. Now the lemma follows.

Remark. Let $P_j = \Omega_*(BU(l_j))$, $\Omega_* \otimes P_j^{(p)} = \Omega_*(\Pi_0^{p-1} BU(l_j))$, and $\Omega_* \otimes P_1^{(p)} \otimes \cdots \otimes P_t^{(p)} = \Omega_* \otimes P = \Omega_*(BU)$.

$$\Omega_* \otimes P_1 \otimes \cdots \otimes P_t = \Omega_* \otimes \overline{P} = \Omega_*(B\overline{U}).$$

And the map $\Omega_*(B\overline{U}) \to \Omega_*(BU)$ is precisely the map $1 \otimes \Phi$ where Φ is the map of (2.4).

The map (3.7) induces a group homomorphism $Z_p \times \overline{U} \to Z_p \times_f U$ and so a map

$$BZ_p \times B\overline{U} \to B(Z_p \times_f U). \tag{3.9}$$

Lemma (3.10). *The maps* $\Omega_-(BZ_p \times B\overline{U}) \to \Omega_-(B(Z_p \times_f U))$ *and* $H_-(BZ_p \times B\overline{U}) \to H_-(B(Z_p \times_f U))$ *are surjective.*

Proof. Let us consider the two Atiyah spectral sequences

$$\overline{E}^2_{i,*} = H_i(Z_p; \Omega_*(B\overline{U})) \Rightarrow \Omega_*(B(Z_p \times \overline{U})),$$

$$E^2_{i,*} = H_i(Z_p; \Omega_*(BU)) \Rightarrow \Omega_*(B(Z_p \times_f U)).$$

The map (3.9) induces a map of these spectral sequences $\overline{E}_{*,*} \to E_{*,*}$. In the remark following (3.8) we noted that $\Omega_*(B\overline{U}) = \Omega_* \otimes \overline{P}$, $\Omega_*(BU) = \Omega_* \otimes P$ and the map $H_i(Z_p; \Omega_* \otimes \overline{P}) = \overline{E}^2_{i,*} \to E^2_{i,*} = H_i(Z_p; \Omega_* \otimes P)$ is the map $1 \otimes \Phi_*$. $\overline{E}_{i,j} = E_{i,j} = 0$ if i is even, $i \neq 0$ by (2.1). $\overline{E}_{i,j} = E_{i,j} = 0$ for i odd, j odd, since $\Omega_*(B\overline{U})$, $\Omega_*(BU)$ is concentrated in even dimensions. $\overline{E}_{i,j}$, $E_{i,j}$ are p-torsion and $\overline{E}_{i,j} \to E_{i,j}$ is surjective for i odd, j even, by (2.5). $\overline{E}_{0,j} = \Omega_j(BU)$ is free abelian for j even, and $E_{0,j} = H_0(Z_p; \Omega_* \otimes P)$ is free abelian by (2.2). For j odd $\overline{E}_{0,j} = E_{0,j} = 0$. Thus no differential could possibly be nonzero and so both spectral sequences collapse. Consider the filtrations $\overline{F}_{*,*}$ on $\Omega_*(BZ_p \times B\overline{U})$ and $F_{*,*}$ on $\Omega_*(B(Z_p \times_f U))$.

$$\overline{F}_{i,j}/\overline{F}_{i-1,j+1} = F_{i,j}/F_{i-1,j+1} = 0$$

for $i + j$ even and $i \neq 0$. For m odd,

$$\Omega_m(BZ_p \times B\overline{U}) = \overline{F}_{m,0} \supset \cdots \supset \overline{F}_{1,m-1} \supset 0,$$

$$\Omega_m(B(Z_p \times_f U)) = F_{m,0} \supset F_{m-2,2} \supset \cdots \supset F_{1,m-1} \supset 0.$$

And

$$\overline{F}_{i,j}/\overline{F}_{i-2,j+2} = H_i(Z_p;\, \Omega_j(B\overline{U})), \qquad F_{i,j}/F_{i-2,j+2} = H_i(Z_p;\, \Omega_j(BU))$$

for $i+j$ odd. We have maps

$$\begin{array}{ccccccccc} 0 & \to & \overline{F}_{i-2,j+2} & \to & \overline{F}_{i,j} & \to & H_i(Z_p;\, \Omega_j(B\overline{U})) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \to & F_{i-2,j+2} & \to & F_{i,j} & \to & H_i(Z_p;\, \Omega_j(BU)) & \to & 0 \end{array}$$

For $i=1$, $j=m-1$, $\overline{F}_{1,m-1} = H_1(Z_p; \Omega_{m-1}(B\overline{U})) \to H_1(Z_p; \Omega_{m-1}(BU)) = F_{1,m-1}$ is onto. By induction α' is surjective, and α'' is surjective by computation. Hence α is surjective. Thus by induction $\overline{F}_{m,0} \to F_{m,0}$ is surjective. The same argument applied to $H_*(BZ_p \times B\overline{U})$ and $H_*(B(Z_p \times_f U))$ gives the corresponding conclusion for homology.

Lemma (3.11). $\Omega_+(BU) \to \Omega_+(B(Z_p \times_f U))$ *and* $H_+(BU) \to H_+(B(Z_p \times_f U))$ *are surjective.*

Proof. In the spectral sequence $E_{*,*}$ of (3.10), $E^2_{0,*} = H_0(Z_p; \Omega_*(BU))$ and $E^2_{i,j} = 0$ for $i+j$ even, $i \neq 0$. Thus $\Omega_+(B(Z_p \times_f U)) = H_0(Z_p; \Omega_*(BU))$ and $\Omega_*(BU) \to H_0(Z_p; \Omega_*(BU))$ is surjective by definition of H_0. Similarly for homology.

Theorem (3.12). $\Omega_-(B(Z_p \times_f U))$ *has projective dimension one over* Ω_*.

Proof. First consider

$$\begin{array}{ccc} \Omega_+(BU) & \to & \Omega_+(B(Z_p \times_f U)) \\ \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\ H_+(BU) & \to & H_+(B(Z_p \times_f U)) \end{array}$$

where $\overline{\mu}$ and μ are the Thom maps. $\overline{\mu}$ is surjective since $\Omega_*(BU)$ is a free Ω_* module, and the two horizontal maps are surjective. Hence μ is surjective. Now consider

$$\begin{array}{ccc} \Omega_-(BZ_p \times BU) & \to & \Omega_-(B(Z_p \times_f U)) \\ \downarrow{\scriptstyle \overline{\mu}} & & \downarrow{\scriptstyle \mu} \\ H_-(BZ_p \times BU) & \to & H_-(B(Z_p \times_f U)). \end{array}$$

Now $\Omega_-(BZ_p \times B\bar{U}) \cong \bigoplus \Omega_-(BZ_p)$ and so has projective dimension one. Then from [4], [7], $\bar{\mu}$ is onto, and since the horizontal maps are onto, μ is onto. Thus $\mu\colon \Omega_*(B(Z_p \times_f U)) \to H_*(B(Z_p \times_f U))$ is onto and so again by [4], [7], $\Omega_*(B(Z_p \times_f U))$ and $\Omega_-(B(Z_p \times_f U))$ have projective dimension one over Ω_*.

Corollary (3.12). *$\Omega_-(G;\, F(Z_q),\, F_0(Z_q))$ has projective dimension one over Ω_*.*

Proof. From (1.5), $\Omega_-(G;\, F(Z_q),\, F_0(Z_q)) \cong \bigoplus \Omega_-(B(N(\sigma)/\Gamma(\sigma)))$. From (3.3), $\Omega_-(B(N(\sigma)/\Gamma(\sigma))) \cong \Omega_-(BU) = 0$ or $\Omega_-(B(Z_p \times_f U))$ which has dimension one over Ω_*.

As an indication of a technique necessary in the case of the more general metacyclic group we give more information on the module $\Omega_-(B(Z_p \times_f U))$ of (3.12). As in the remark following (3.8), $\Omega_*(B\bar{U}) = \Omega_* \otimes \bar{P}$, and as in (2.5) let $\bar{P}_1$ be the free Z-module with basis consisting of $X_1 \otimes \cdots \otimes X_t$ where each X_j is a pth power in $\Omega_*(BU(l_j))$.

Theorem (3.13). $\Omega_-(BZ_p) \otimes \Omega_* \otimes \bar{P}_1 \cong \Omega_-(B(Z_p \times_f U))$.

Proof. Let $\Lambda = \Omega_* \otimes \bar{P}_1$. Let $h'_*(X)$ be the homology theory $\Omega_*(X) \otimes \Lambda$, $h'''_*(X)$ the homology theory $\Omega_*(X) \otimes \Omega_*(B\bar{U})$, and $h''_*(X) = \Omega_*(X \times B\bar{U})$. We have a natural transformation $h'_*(X) \to h'''_*(X)$ induced by $\Lambda \subset \Omega_*(B\bar{U})$, and $h'''_*(X) \to h''_*(X)$ induced by $[M, f] \otimes [W, g] \to [M \times W, f \times g]$. So we have induced maps on the skeleton filtrations

$$(Fh')_{*,*}(X) \to (Fh''')_{*,*}(X) \to (Fh'')_{*,*}(X).$$

The second natural transformation is a natural isomorphism. Now consider the fibration $B\bar{U} \to X \times B\bar{U} \to X$. The spectral sequence of this fibration is $\bar{E}^2(X) = H_*(X; \Omega_*(B\bar{U})) \Rightarrow \Omega_*(X \times B\bar{U})$ and the associated filtration on $\Omega_*(X \times B\bar{U})$ is precisely $(Fh'')_{*,*}(X)$. Now let $X = BZ_p$ and consider the map of fibrations of (3.9).

$$\begin{array}{ccc}
B\bar{U} & \longrightarrow & BU \\
\downarrow & & \downarrow \\
BZ_p \times B\bar{U} & \longrightarrow & B(Z_p \times_f U) \\
\downarrow & & \downarrow \\
BZ_p & \longrightarrow & BZ_p
\end{array}$$

gives rise to a map $\bar{E}^2(BZ_p) = \bar{E}^2 \to E^2 = H_*(Z_p;\, \Omega_*(BU))$ and we know from (3.10) that both spectral sequences collapse. Let

$$E'^2 = H_*(BZ_p;\, h'_*), \qquad E'''^2 = H_*(BZ_p;\, h'''_*), \qquad E''^2 = H_*(BZ_p;\, h''_*).$$

All of these spectral sequences collapse. We have a map of spectral sequences $E'^2 \to E'''^2 \to E''^2 = \bar{E}^2 \to E^2$ where the first two maps are induced by the natural transformations. Thus we have a map $E'^2_{i,j} \to E^2_{i,j}$. Now $E'^2_{i,j} = E^2_{i,j} = 0$ if i is even and $i \neq 0$, or if j is odd. If i is odd and j is even, this map is an isomor- by (2.5). Thus $E'^2_{i,j} = E^2_{i,j}$ unless $i = 0$ and j is even. Let $F'_{*,*}$ be the filtration on $h'_*(BZ_p)$ and $F_{*,*}$ on $\Omega_*(B(Z_p \times_f U))$. Then for m odd,

$$\begin{array}{ccccccc} 0 \to F'_{m-(2k+2),*} & \to & F'_{m-2k,*} & \to & H_{m-2k}(BZ_p;\, h'_*) = E'^2_{m-2k,*} & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to F_{m-(2k+2),*} & \to & F_{m-2k,*} & \to & H_{m-2k}(Z_p;\, \Omega_*(BU)) = E^2_{m-2k,*} & \to & 0 \end{array}$$

The left most map is an isomorphism by induction, and the right most by computation. Thus, the middle map is an isomorphism and so

$$h'_-(BZ_p) \cong \Omega_-(B(Z_p \times_f U)).$$

We need a description of the image of $\Omega_-(B(Z_p \times_f U))$ in $\Omega_-(G;\, F(Z_q),\, F_0(Z_q))$. In view of (3.10) we need only describe the image of $\Omega_-(BZ_p \times BU(l_1) \times \cdots \times BU(l_t))$.

Lemma (3.14). *Let P_0 be a left principal Z_p bundle, and P_j a right principal $U(l_j)$ bundle. Let $E_j = P_j \times_{U(l_j)} C^{l_j}$. Then $P_0 \times P_1 \times \cdots \times P_t$ in $\Omega_-(BZ_p \times B\bar{U})$ goes under the map*

$$\Omega_-(BZ_p \times B\bar{U}) \to \Omega_-(B(Z_p \times_f U)) \to \Omega_-(G;\, F(Z_q),\, F_0(Z_q))$$

to the G vector bundle $P_0 \times \bigoplus_0^{p-1} E_1 \times \cdots \times \bigoplus_0^{p-1} E_t \to P_0$ where G acts on P_0 by $G \to G/Z_q = Z_p$ acting on P_0, and G acts on $\bigoplus_0^{p-1} E_j$ by $b(x_0, \cdots, x_{p-1}) = (x_1, x_2, \cdots, x_{p-1}, x_0)$, $a(x_0, \cdots, x_{p-1}) = (\xi^{s_j} x_0, \cdots, \xi^{s_j r^{p-1}} x_{p-1})$.

Proof. Let $P = P_0 \times P_1 \times \cdots \times P_t$. P is a right $Z_p \times \bar{U}$ bundle. Let $Q = P \times_{Z_p \times \bar{U}} Z_p \times_f U$. It is easy to see that $Q \cong P_0 \times \Pi(P_j \times_{U(l_j)} U(l_j)^p)$, where Z_p acts diagonally on the right and on $P_j \times_{U(l_j)} U(l_j)^p$, Z_p acts by $[e;\, A_0, \cdots, A_{p-1}]b = [e;\, A_{p-1}, A_0, \cdots, A_{p-2}]$. Now $N(\sigma)/\Gamma(\sigma) \cong Z_p \times_f U$, the isomorphism takes $(b, B)\Gamma(\sigma)$ to b and $(1 \times U)\Gamma(\sigma)$ to $1 \times U$. Thus Q becomes a principal $N(\sigma)/\Gamma(\sigma)$ bundle. Then $Q \times_{N(\sigma)} (G \times U(n)) \times_{U(n)} C^n$ is a left G vector bundle, and from (1.5) is the element in $\Omega_-(G;\, F(Z_q),\, F_0(Z_q))$ coming from P. By a series of obvious isomorphisms,

$$Q \times_{N(\sigma)} (G \times C^n) \cong Q \times_U C^n \cong P_0 \times \prod_{j=1}^{t} P_j \times_{U(l_j)} C^{pl_j}$$

where, following the left action of G along, we find G acting on $P_0 \times \prod P_j \times_{U(l_j)} C^{pl_j}$ by acting diagonally. G acts on P_0 by $G \to G/Z_q = Z_p$ and on $P_j \times_{U(l_j)} C^{pl_j}$ by

$$b[e; x_0, \cdots, x_{p-1}] = [e; x_1, x_2, \cdots, x_{p-1}, x_0],$$

$$a[e; x_0, \cdots, x_{p-1}] = [e; \xi^{s_j} x_0, \cdots, \xi^{s_j r^{p-1}} x_{p-1}].$$

Finally $P_j \times_{U(l_j)} C^{pl_j} = \bigoplus_0^{p-1} E_j$ with the indicated action.

Corollary (3.15). *$\Omega_-(G; F(Z_q), F_0(Z_q))$ is generated by the G vector bundles $S^{2k-1} \times \bigoplus_0^{p-1} E_1 \times \cdots \times \bigoplus_0^{p-1} E_t \to S^{2k-1}$ where G acts on S^{2k-1} by $G \to Z_p$ and Z_p acts by $b(x_1, \cdots, x_k) = (\eta x_1, \cdots, \eta x_k)$ and G acts on $\bigoplus_0^{p-1} E_j$ as in* (3.14).

Proof. This follows from the fact that S^{2k-1} with the indicated action of Z_p for $k = 1, 2, \cdots$ are generators for $\Omega_-(BZ_p)$.

Remark.(3.16). If G is any finite group, F the family of all subgroups, F_0 the family of all proper subgroups, then it is well known that $\Omega_*(G; F, F_0)$ is the bordism group of G vector bundles E over trivial G manifolds. $E \cong \bigoplus_\pi V_\pi \otimes \mathrm{Hom}_G(\pi, E)$ where V_π runs over the irreducible representation of G. So $\Omega_*(G; F, F_0) \cong \bigoplus \Omega_*(BU(k_1) \times \cdots \times BU(k_t))$ is a free Ω_*-module.

4. **Proofs of Theorems A and B.** We start with the families F_p and F_1 which are adjacent:

$$\Omega_+(G; F_1) = \Omega_+, \qquad \Omega_-(G; F_p, F_1) = 0.$$

The long exact sequence

$$(4.1) \qquad \to \Omega_*(G; F') \to \Omega_*(G; F) \to \Omega_*(G; F, F') \to \Omega_*(G; F') \to$$

for any pair of families breaks up as

$$0 \to \Omega_+ \to \Omega_+(G; F_p) \xrightarrow{\gamma} \Omega_+(G; F_p, F_1) \to \Omega_-(G; F_1) \to \Omega_-(G; F_p) \to 0.$$

Now we have, from (1.3) and (3.1),

$$\begin{array}{ccc}
\Omega_+(G; F_p, F_1) & \to & \Omega_-(G; F_1) = \Omega_-(Z_p; F_0) \oplus \mathrm{Image}(\Omega_-(Z_q; F_0)) \\
\uparrow \cong & & \uparrow \\
\Omega_+(Z_p; F(Z_p), F_0(Z_p)) & \to & \Omega_-(Z_p; F_0).
\end{array}$$

The map $\Omega_+(Z_p; F(Z_p), F_0(Z_p)) \to \Omega_-(Z_p; F_0)$ is well known to be onto. Thus we get short exact sequences

$$0 \to \Omega_+ \to \Omega_+(G;\, F_p) \xrightarrow{\gamma} \Omega_+(G;\, F_p,\, F_1) \to \Omega_-(Z_p;\, F_0) = \Omega_-(BZ_p) \to 0,$$

$$\text{Image}\,(\Omega_-(Z_q;\, F_0)) \cong \Omega_-(G;\, F_p).$$

Now Image$(\Omega_-(Z_q;\, F_0))$ being a summand of $\Omega_-(BG)$ has projective dimension one. Hence so does $\Omega_-(G;\, F_p)$. Now $\Omega_-(BZ_p)$ has dimension one and $\Omega_+(G;\, F_p,\, F_1)$ is free by (3.1), hence Image(γ) is projective over Ω_* and hence free [4, 3.2]. Thus $0 \to \Omega_+ \to \Omega_+(G;\, F_p) \to \text{Image}(\gamma) \to 0$ and so $\Omega_+(G;\, F_p)$ is free.

Next consider the families F_q, F_1 and apply (4.1). $\Omega_-(G;\, F_q,\, F_1)$ is all torsion by (3.10). Thus

$$0 \to \Omega_+ \to \Omega_+(G;\, F_q) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \Omega_-(G;\, F_1) \to \Omega_-(G;\, F_q) \to \Omega_-(G;\, F_q,\, F_1) \to 0.$$

Consider

$$\begin{array}{ccc} \Omega_+(G;\, F_q,\, F_1) & \to & \Omega_-(G;\, F_1) = \text{Image}\,(\Omega_-(Z_q;\, F_0)) + \Omega_-(Z_p;\, F_0) \\ \uparrow E & & \uparrow \\ \Omega_+(Z_q;\, F_q,\, F_0) & \to & \Omega_-(Z_q;\, F_0) \to 0. \end{array}$$

From (3.1) and (3.11) it follows that the map E is onto, and we also know, from (3.11), that $\Omega_+(G;\, F_q,\, F_1)$ is free. Thus we get

$$0 \to \Omega_+ \to \Omega_+(G;\, F_q) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \text{Image}\,(\Omega_-(Z_q;\, F_0)) \to 0$$

$$0 \to \Omega_-(Z_p;\, F_0) \to \Omega_-(G;\, F_q) \to \Omega_-(G;\, F_q,\, F_1) \to 0.$$

Now $\Omega_-(Z_p;\, F_0)$ has dimension one and, by (3.12), $\Omega_-(G;\, F_q,\, F_1)$ has dimension one and thus $\Omega_-(G;\, F_q)$ has dimension one. By the same argument as for F_p, F_1, Image(γ) is free and hence $\Omega_+(G;\, F_q)$ is free.

Now consider the families F_0, F_p. By (1.4), $\Omega_*(G;\, F_0,\, F_p) \cong \Omega_*(G;\, F_q,\, F_1)$. Thus we get the sequence of (4.1)

$$\begin{array}{ccccccc} 0 \to \Omega_+(G;\, F_p) \to \Omega_+(G;\, F_0) \xrightarrow{\gamma} & \Omega_+(G;\, F_q,\, F_1) & \to & \Omega_-(G;\, F_p) & \to \Omega_-(G;\, F_0) \to \Omega_-(G;\, F_q,\, F_1) \to 0 \\ & \uparrow \text{onto} & & \uparrow & \\ & \Omega_+(Z_q;\, F_q,\, F_1) & \to & \Omega_-(Z_q;\, F_0) \to 0. & \end{array}$$

Since we have previously shown $\Omega_-(G;\, F_p) \cong \text{Image}\,(\Omega_-(Z_q;\, F_0))$ we get $\Omega_-(G;\, F_0) \cong \Omega_-(G;\, F_q,\, F_1)$ and

$$0 \to \Omega_+(G;\, F_p) \to \Omega_+(G;\, F_0) \xrightarrow{\gamma} \Omega_+(G;\, F_q,\, F_1) \to \text{Image}\,(\Omega_+(BZ_q)) \to 0.$$

From the first isomorphism and (3.12), $\Omega_-(G; F_0)$ has projective dimension one. The isomorphism takes W in $\Omega_-(G; F_0)$ to the G vector bundle $E \to M$ where M is the set of points in W with isotropy group Z_q, and E is the normal bundle of M in W. From the second sequence we conclude, since $\Omega_+(G; F_q, F_1)$ is free and Image$(\Omega_-(BZ_q))$ has dimension one, that Image(γ) is free. Now

$$0 \to \Omega_+(G; F_p) \to \Omega_+(G; F_0) \to \text{Image}(\gamma) \to 0.$$

Since $\Omega_+(G; F_p)$ and Image(γ) are free, it follows that $\Omega_+(G; F_0)$ is free.

Finally consider the families F_α and F_0 of G:

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \to \Omega_-(G; F_\alpha) \to 0.$$

We want to show $\Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0) \cong \Omega_-(G; F_q, F_1)$ is onto. (3.15) tells us generators $S^{2k-1} \times \oplus E_1 \times \cdots \times \oplus E_t \to S^{2k-1}$ for $\Omega_-(G; F_q, F_1)$. Recall $\{s_j\}$ were representatives for $Z_q^*/[r]$, $j = 1, \cdots, t$. Let V_j be the representation of G induced from the one dimensional representation $Z_q \to C^*$ sending a to ξ^{s_j}. So

$$V_j = Z[G] \otimes_{Z[Z_q]} C^* \quad \text{(see [5, p. 333])}.$$

Let G act on C^k by $G \to G/Z_q = Z_p$ and Z_p acts by $b(x_1, \cdots, x_k) = (\eta x_1, \cdots, \eta x_k)$. Then the G vector bundle $C^k \times V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t = E$ represents an element in $\Omega_+(G; F_\alpha, F_0)$. $S(E)$ represents an element in $\Omega_-(G; F_0)$. The set of points with isotropy group Z_q is precisely $S(C^k) = S^{2k-1}$ and the normal bundle to S^{2k-1} is $S^{2k-1} \times V_1 \otimes E_1 \times \cdots \times V_t \otimes E_t$. To check that this is the generator $S^{2k-1} \times \oplus E_1 \times \cdots \times \oplus E_t$ consider $V_j \otimes E_j$. We can take as a basis of V_j, $1 \otimes 1, b^{-1} \otimes 1, \cdots, b^{-(p-1)} \otimes 1$ where $a(b^{-i} \otimes 1) = \xi^{s_j r^i}(b^{-i} \otimes 1)$. The general element of $V_j \otimes E_j = (x_0, \cdots, x_{p-1}) = 1 \otimes 1 \otimes x_0 + \cdots + 1 \otimes b^{-(p-1)} \otimes x_{p-1}$, $b(x_0, \cdots, x_{p-1}) = 1 \otimes 1 \otimes x_1 + b^{-1} \otimes 1 \otimes x_2 + \cdots + 1 \otimes b^{-(p-2)} \otimes x_{p-1} + 1 \otimes b^{-(p-1)} \otimes x_0 = (x_1, x_2, \cdots, x_{p-1}, x_0)$. And $a(x_0, \cdots, x_{p-1}) = \Sigma \xi^{s_j r^i}(1 \otimes b^{-i} \otimes x_i) = (\xi^{s_j} x_0, \cdots, \xi^{s_j r^{p-1}} x_{p-1})$. Thus $\Omega_+(G; F_\alpha, F_0) \to \Omega_-(G; F_0)$ is onto and so $\Omega_-(G; F_\alpha) = 0$. Our exact sequence becomes

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \xrightarrow{\gamma} \Omega_+(G; F_\alpha, F_0) \to \Omega_-(G, F_0) \to 0.$$

We know $\Omega_-(G; F_0)$ has projective dimension one and so Image(γ) is free. We get the exact sequence

$$0 \to \Omega_+(G; F_0) \to \Omega_+(G; F_\alpha) \to \text{Image}(\gamma) \to 0.$$

$\Omega_+(G; F_0)$ and Image(γ) are free, hence $\Omega_+(G; F_\alpha)$ is free. This concludes the proof of Theorems A and B.

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