TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 173, November 1972

LOCALIZATIONS OF HNP RINGS

BY

JAMES KUZMANOVICH

ABSTRACT. In this paper it is shown that every hereditary Noetherian prime ring is the intersection of a hereditary Noetherian prime ring having no invertible ideals with a bounded hereditary Noetherian prime ring in which every nonzero two-sided ideal contains an invertible two-sided ideal. Further, it is shown that this intersection corresponds to a decomposition of torsion modules over such a ring; if R is an HNP ring with enough invertible ideals, then this decomposition coincides with that of Eisenbud and Robson.

If M is a maximal invertible ideal of R where R is as above, then an overring of R is constructed which is a localization of R at M in a "classical sense"; that is, it is a ring of quotients with respect to a multiplicatively closed set of regular elements satisfying the Ore conditions. The localizations are shown to have nonzero radical and are also shown to satisfy a globalization theorem. These localizations are generalizations of ones constructed by A. V. Jategaonkar for HNP rings with enough invertible ideals.

The main tool used throughout the paper is that of a ring of quotients with respect to an idempotent kernel functor. Each such kernel functor is determined by a filter of right ideals, or equivalently a class of cyclic modules. The technique used in construction of each of the above rings will be to define a class of cyclic modules by the presence or absence of certain composition factors, form the ring of quotients with respect to the kernel functor thus defined, and then show that the ring of quotients has the desired properties.

$\mathbf{Mod}(R_R)$ ($\mathbf{Mod}({}_R R)$) will denote the category of right (left) R-modules, and all ring theoretic conditions (Noetherian, Artinian, hereditary, etc.) will be two-sided unless otherwise stated. During much of the remainder of this paper R will be a hereditary Noetherian prime ring (we will often shorten this to HNP ring). In this situation R will be a prime Goldie ring and hence will have a simple Artinian classical quotient ring which we will call Q. We will assume throughout that R is not Artinian; that is, $R \neq Q$.

The author would like to thank the Faculty Research Committee of the University of Northern Colorado for a grant in support of this research.

1. **Preliminaries.** A right R-submodule of Q is called a *right R-ideal* (fractional) if I contains a regular element of Q and if there is a regular element b of

Received by the editors June 4, 1971.

AMS (MOS) subject classifications (1970). Primary 16A08, 16A14.

Key words and phrases. Dedekind prime ring, hereditary Noetherian prime ring, ring of quotients, overring, hereditary torsion theory, idempotent kernel functor.

Q such that $bI \subset R$. The reader may want to refer to Jacobson [8] who has a full account of these definitions. If $I \subset R$, then I is called *integral*. If I is both a left and a right R-ideal, then I is called an R-ideal. If I is a right (left) R-ideal, define I^* $({}^*I)$ by $I^* = \{q \in Q: qI \subset R\}$ $({}^*I = \{q \in Q: Iq \subset R\})$. Note that I^* is isomorphic to $\mathrm{Hom}_R(I, R)$, the dual of I. If I is an R-ideal, then it is not in general true that ${}^*I = I^*$. An R-ideal I is called *invertible* if $(I^*)I = I({}^*I) = R$. In this case it will be true that $I^* = {}^*I$, denote it by I^{-1}. An HNP ring in which every R-ideal is invertible is called a *Dedekind prime ring* (Robson [15]).

If m is an element of a right R-module M, then m is called a *torsion element* if $mb = 0$ for a regular element b of R. Levy [12] showed that if R had a classical right quotient ring, then the set of all torsion elements of M, denoted by $t(M)$, was a submodule. Later in the paper we will be concerned with various other torsion submodules relative to other torsion theories, but when the word torsion is used without adjectives, we will mean the torsion just defined.

Webber [17] has shown that if I is an essential right ideal of an HNP ring R, then R/I is Artinian. Since in a prime Goldie ring a right ideal is essential if and only if it contains a regular element, it easily follows that a finitely generated torsion module over an HNP ring is Artinian. Hence a finitely generated torsion module would have a composition series; in particular if I is an essential right ideal, then R/I has a composition series.

Goldman [7] has called a covariant functor σ from $\mathbf{Mod}(R_R)$ to $\mathbf{Mod}(R_R)$ an an *idempotent kernel functor* if σ satisfies the following properties:

(1) $\sigma(M) \subset M$ for all right R-modules M.

(2) If $f \in \mathrm{Hom}_R(M', M)$ for right R-modules M' and M, then $f(\sigma(M')) \subset \sigma(M)$ and $\sigma(f)$ is $f|_{\sigma(M')}$.

(3) If $M' \subset M$, then $\sigma(M') = \sigma(M) \cap M'$.

(4) $\sigma(M/\sigma(M)) = 0$ for every right R-module M.

Note that $\mathcal{G}_\sigma = \{M_R: \sigma(M) = M\}$ is a hereditary torsion class in the sense of Dickson [2]. In fact the concepts of torsion theories and kernel functors are equivalent in the sense that each torsion theory gives rise to a kernel functor whose torsion class is that of the original torsion theory. $\sigma(M)$ is called the *σ-torsion submodule* of M. If $\sigma(M) = M$, then M is called *σ-torsion*. If $\sigma(M) = 0$, then M is called *σ-torsion-free*. See also Gabriel [6], Maranda [13], Fuchs [5], and Walker and Walker [16]. Lambek [11] has recently given a very good account of torsion theories (kernel functors) and rings of quotients. There is a filter, $\mathcal{F}_\sigma$, of right ideals associated with σ defined by $\mathcal{F}_\sigma = \{I_R \subset R: R/I \in \mathcal{G}_\sigma\}$. Then $\mathcal{F}_\sigma$ determines σ in the following sense: if $x \in M_R$, then $x \in \sigma(M)$ if and only if there exists $I \in \mathcal{F}_\sigma$ such that $xI = 0$. See Gabriel [6] and Goldman [7]. Given a right R-module M such that $\sigma(M) = 0$, Goldman defines the *module of quotients* of M with respect to σ, $Q_\sigma(M)$, in the following way: let E be an injective hull of M, then let $Q_\sigma(M) = \{x \in E: [x + M]$ is an element of $\sigma(E/M)\}$. $Q_\sigma(M)$ is unique up to a unique isomorphism over M. If $\sigma(R_R) = 0$, then $Q_\sigma(R)$ has a unique ring structure extending the R-module structure. $Q_\sigma(R)$ (often shortened to Q_σ) is called the *ring of quotients* of R with respect to σ. In our situation where R is a prime Goldie ring with classical quotient ring Q, Q_R is the injective hull of R and the ring of quotients of R with respect to the filter of all essential right ideals of R. If σ is an idempotent kernel functor such that $\sigma(R) = 0$, then by the above construction it is clear that $Q_\sigma = \{q \in Q: qI \subset R$ for some $I \in \mathcal{T}_\sigma\}$, since Q is the injective hull of R. It can be seen that Q_σ is in fact a subring of Q. If every right ideal in $\mathcal{T}_\sigma$ is essential in R, then $q \in Q_\sigma$ if and only if $q \in I^*$ for some $I \in \mathcal{T}_\sigma$. This is summarized in the following proposition.

Proposition 1.1. *Let σ be an idempotent kernel functor such that $\sigma(R) = 0$ and every right ideal in $\mathcal{T}_\sigma$ is essential in R. Then $Q_\sigma = \bigcup I^*$ where I ranges over the elements of $\mathcal{T}_\sigma$.* □

An idempotent kernel functor σ is said to have property (T) if it satisfies any of the equivalent conditions of the following theorem; the theorem is basically Theorem 4.3 of [7].

Theorem 1.2. *Let σ be an idempotent kernel functor. The following statements are equivalent:*

(1) *Every Q_σ-module is σ-torsion free.*

(2) *$IQ_\sigma = Q_\sigma$ for every $I \in \mathcal{T}_\sigma$.*

(3) *$M \otimes_R Q_\sigma \simeq Q_\sigma(M)$ for every right R-module M.* □

In this paper we will be concerned exclusively with HNP rings and with kernel functors σ such that Q_σ is an overring of R. It is not hard to see that over a hereditary Noetherian ring every kernel functor has property (T). In fact in our situation every Q_σ is an HNP ring and is flat as an R-module (both left and right); see [10]. Let I be an integral right R-ideal and suppose that $IQ_\sigma = Q_\sigma$. Take $r \in R$. Then it is easy to see that there is a J in $\mathcal{T}_\sigma$ such that $rJ \subset I$. Hence $\sigma(R/I) = R/I$ and $I \in \mathcal{T}_\sigma$. Together with (2) of Theorem 1.2 we have that $I \in \mathcal{T}_\sigma$ if and only if $IQ_\sigma = Q_\sigma$. If $IQ_\sigma = Q_\sigma$, then we will say that I explodes in Q_σ and in general IQ_σ is called the *expansion* of I in Q_σ. If J is an integral right Q_σ-ideal, then $J^c = J \cap R$ is called the *contraction* of J to R. Proposition 4.6 of [7] shows that if σ has property (T), then every right ideal of Q_σ is the expansion of its contraction. This property will be important in the rest of the paper.

2. **R as an intersection, decompositions of torsion modules.** In this section it is shown that R is the intersection of two overrings, one of which has no invertible ideals while the other is bounded with enough invertible ideals. These rings are constructed as rings of quotients with respect to two torsion theories. It is also shown that each torsion module over R is the direct sum of the two torsion submodules corresponding to the above overrings. This decomposition generalizes the

fact that over a Dedekind prime ring a finitely generated torsion module is a direct sum of a completely faithful module and a bounded module, as a result of Eisenbud and Robson [4].

If M is a right R-module, define $b(M)$, the *invertible torsion submodule*, by $b(M) = \{m \in M: mB = 0$ for some invertible two-sided ideal of R, $B\}$. If $f: M' \longrightarrow M$ is an R-homomorphism of right R-modules, let $b(f)$ be the restriction of f to $b(M')$.

Proposition 2.1. *b is an idempotent kernel functor.*

Proof. Let M be a right R-module. It will first be shown that $b(M)$ is a submodule. Let x and y be elements of $b(M)$. Then there are invertible two-sided ideals of R, A and B, such that $xA = 0$ and $xB = 0$. AB is an invertible ideal contained in $A \cap B$ so that $(x + y)AB \subset (x + y)(A \cap B) = 0$ and $x + y \in b(M)$. If $r \in R$, then $(xr)A = x(rA) \subset xA = 0$ and $xr \in b(M)$. Hence $b(M)$ is a submodule of M. Let $f: M' \longrightarrow M$ be an R-homomorphism of right R-modules. Take $x \in b(M')$; then there is an invertible two-sided ideal B of R such that $xB = 0$. $f(x)B = f(xB) = 0$ so that $f(x) \in b(M)$. Hence $f(b(M')) \subset b(M)$. It is clear that if $M' \subset M$, then $b(M') = M' \cap b(M)$. Thus b is a kernel functor. Let M be a right R-module and consider $M/b(M)$. Suppose that $[x + b(M)] \in M/b(M)$ is b-torsion. Then there is a two-sided invertible ideal B of R such that $xB \subset b(M)$. R is Noetherian so that $B = r_1R + \cdots + r_nR$ and $xB = xr_1R + \cdots + xr_nR$. Let A_i $(i = 1, \cdots, n)$ be a two-sided invertible ideal of R such that $xr_iA_i = 0$. Then $x(BA_1 \cdots A_n) = (xr_1R + \cdots + xr_nR)(A_1 \cdots A_n) = xr_1A_1 \cdots A_n + \cdots + xr_nA_1 \cdots A_n \subset xr_1A_1 + \cdots + xr_nA_n = 0$. Therefore, since $BA_1 \cdots A_n$ is invertible, $x \in b(M)$ and $[x + b(M)] = [0 + b(M)]$ and $M/b(M)$ is b-torsion free. That is, b is idempotent. □

Lemma 2.2. *$\mathcal{T}_b = \{I: I$ is a large right ideal containing an invertible two-sided ideal of $R\}$.*

Proof. $\mathcal{T}_b = \{I_R \subset R: b(R/I) = R/I\}$. Hence if $I \in \mathcal{T}_b$, then there is an invertible two-sided ideal of B such that $1 \cdot B \subset I$. I is large since all two-sided ideals are large. □

Form the quotient ring with respect to b, Q_b. By Proposition 1.1 Q_b is just the union of all I^* where I is an element of $\mathcal{T}_b$. In fact, more can be said as is seen in the following proposition.

Proposition 2.3. *Q_b is the union of all B^{-1} where B is an invertible two-sided ideal of R. Furthermore, the quotient ring resulting from considering the invertible torsion theory in the category of left R-modules is precisely Q_b.*

Proof. $Q_b = \bigcup \{I^*: I \in \mathcal{T}_b\}$. If $I \in \mathcal{T}_b$, then by Lemma 2.2 $I \supset B$, where B is an invertible two-sided ideal of R. But $B \subset I$ implies that $I^* \subset B^*$ and thus it follows that $Q_b = \bigcup \{B^*: B$ a two-sided invertible ideal of $R\}$.

The furthermore follows since the quotient ring with respect to the analogous left torsion theory will just be the union of all $^*B$ where B is an invertible ideal of

R. But if B is invertible, then $^*B = B^{-1} = B^*$ so that taking unions yields equality. □

Call a right (left) R-module M *completely b-torsion free* (*cbf*) if every submodule of every factor module of M is b-torsion free. A right (left) integral R-ideal I is called *cbf* if R/I is *cbf*. Note that I will be *cbf* if and only if R/I has no b-torsion composition factors.

By the results of the first section we know that every right (left) integral Q_b-ideal is the expansion of its contraction and that a right (left) R-ideal explodes in Q_b if and only if it contains an invertible two-sided ideal of R. (The parenthetical version follows from Proposition 2.3.) The following proposition expands on the relationships between the ideals of R and those of Q_b.

Proposition 2.4.

(i) *If I is a cbf right (left) integral R-ideal, then $R/I \simeq Q_b/IQ_b$ and $I = (IQ_b)^c$.*

(ii) *Let I be an integral right (left) Q_b-ideal. Then I^c is cbf and $R/I^c \simeq Q_b/I$.*

(iii) *If I is a cbf integral right (left) R-ideal, then $I = (IQ_b)^c$.*

(iv) *If M is a cbf integral R-ideal, then $Q_bM = MQ_b = Q_bMQ_b$ is a proper Q_b-ideal and $R/M \simeq Q_b/MQ_b$ (as rings).*

Proof. (i) Let I be an integral right R-ideal such that R/I is *cbf*. Then R/I has no b-torsion composition factors and hence $I + B = R$ for every invertible two-sided ideal B. Thus $IB^{-1} + BB^{-1} = RB^{-1}$ or $IB^{-1} + R = B^{-1}$. Since Q_b is the union of all such B^{-1}, it follows that $IQ_b + R = Q_b$. The second isomorphism theorem yields that Q_b/IQ_b is isomorphic to $R/(IQ_b)^c$. Let $x \in (IQ_b)^c$; then $x \in IB^{-1}$ for some invertible two-sided integral R-ideal B and hence $xB \subset I$. That is, $(IQ_b)^c/I$ is b-torsion; this implies that $I = (IQ_b)^c$ since R/I had no b-torsion composition factors. Combining results yields that $R/I \simeq Q_b/IQ_b$.

(ii) The proof of (ii) will be an induction argument on the composition length of R/I^c where I is an integral right Q_b-ideal. If the composition length of R/I^c is one, then R/I^c is simple. R/I^c must be b-torsion free (and hence *cbf*) for otherwise I^c would explode in Q_b. Hence by (i) $R/I^c \simeq Q_b/I^cQ_b = Q_b/I$. Now suppose that the result is true whenever the composition length of the contraction is $k - 1$ and assume that I is such that the composition length of R/I^c is k. Then there is a composition series for R/I^c of the form $I^c = J_0 \subset \cdots \subset J_k = R$. By the second isomorphism theorem $(I + R)/I \simeq R/I^c$ and $(I + R)/I$ is an R-submodule of the Q_b-module Q_b/I. Q_b-modules are b-torsion free so that R/I^c is b-torsion free; hence J_1/I^c is also b-torsion free. $(J_1 + I)/I \simeq J_1/I^c$ is thus a b-torsion free R-submodule of Q_b/I and hence is a Q_b-submodule of Q_b/I. $I + J_1$ is then a right Q_b-ideal whose contraction (J_1) has composition length $k - 1$. By the induction hypothesis it follows that R/J_1 is *cbf*. Hence R/I^c has no b-torsion composition factors. By (i) $R/I^c \simeq Q_b/IQ_b = Q/I$ and (ii) is proven.

(iii) Let I be a *cbf* integral right R-ideal. Certainly $I \subset (IQ_b)^c$. By (i) the composition length of R/I is equal to the composition length of Q_b/IQ_b as a right R-module; but by (ii) this is the same as the composition length of $R/(IQ_b)^c$. Hence $I = (IQ_b)^c$.

Note that the parenthetical versions of (i), (ii), and (iii) follow by Proposition 2.3.

(iv) Let M be a *cbf* integral R-ideal and suppose that $Q_bMQ_b = Q_b$. Then $\Sigma x_im_iy_i = 1$ for x_i's and y_i's in Q_b and m_i's in M. By an argument similar to that of Proposition 2.1 there is an invertible integral R-ideal B such that all the x_i's and y_i's are in B^{-1}. $B^2 = B \cdot 1 \cdot B = B(\Sigma x_im_iy_i)B = \Sigma Bx_im_iy_iB$. Bx_i and y_iB are contained in R for all i so that $B^2 = \Sigma Bx_im_iy_iB$ is contained in M. This contradicts the fact that R/M is *cbf* since B^2 is invertible. Hence $Q_bMQ_b \neq Q_b$. Let $N = (Q_bMQ_b)^c$. Every integral Q_b-ideal is the expansion of its contraction so that $NQ_b = Q_bMQ_b = Q_bN$. N is finitely generated as a right R-module so that $NQ_b = (\Sigma x_iR)Q_b = \Sigma x_iQ_b$ where the x_i's are in N. $Q_b(MQ_b) = NQ_b$ so that for each i there is an invertible two-sided ideal B_i of R such that $x_i \in B_i^{-1}(MQ_b)$. Again it is easy to see that there is a single invertible ideal B such that $x_i \in B^{-1}(MQ_b)$ for all i. As a result $MQ_b \supset B(\Sigma x_iQ_b) = BNQ_b = BQ_bN = Q_bN = NQ_b$. Thus $MQ_b = NQ_b$. Similarly $Q_bM = Q_bN = Q_bMQ_b = MQ_b$. By (iii) it follows that $M = N$. The map $f: R/M \longrightarrow Q_b/MQ_b$ defined by $f([r + M]) = [r + MQ_b]$ is a well-defined and a one-to-one ring homomorphism since $(MQ_b)^c = M$. It is onto since $R + MQ_b = Q_b$ as in (i). □

Corollary 2.5. *Every cyclic Q_b-module is cbf as an R-module.*

Theorem 2.6. *Q_b is a hereditary Noetherian prime ring with no invertible ideals. The one-sided (two-sided) Q_b-ideal lattices are isomorphic to the one-sided (two-sided) lattices (lattice) of cbf R-ideals, and the corresponding factor modules (rings) are isomorphic.*

Proof. Every overring of R is hereditary and Noetherian (Proposition 1.6 of [10]); hence so is Q_b. The last statement is a restatement of parts of Proposition 2.4. It remains to be shown that Q_b has no invertible ideals.

Let I be an invertible (two-sided) ideal of Q_b. Then $I^{-1} = \{q \in Q: qI \subset Q_b\}$. We will first show that $I^{-1} = Q_b(I^c)^*$. $I = I^cQ_b$ so that $Q_b((I^c)^*)I^cQ_b$ is contained in $Q_bRQ_b = Q_b$ and $Q_b(I^c)^* \subset I^{-1}$. Let q be in I^{-1}. Then $qI \subset Q_b$ and in particular $qI^c \subset Q_b$. qI^c is a finitely generated R-submodule of Q_b so that by a simple argument similar to that in Proposition 2.1 there is an invertible integral R-ideal B such that $qI^c \subset B^{-1}$. Hence $qI^c(I^c)^* \subset B^{-1}(I^c)^*$. $1 \in (I^c)(I^c)^*$ so that $q \in B^{-1}(I^c)^*$; that is, $q \in Q_b(I^c)^*$. As a result both containments hold and we have that $I^{-1} = Q_b(I^c)^*$.

I^c is *cbf* so that it cannot be contained in any invertible ideal of R. Then there is an integer k such that $(I^c)^k$ is idempotent by Proposition 4.5 of Eisenbud and Robson [3]. By (iv) of Proposition 2.4 $I = I^c Q_b$ so from (iii) of Proposition 2.4 it follows that $(I^k)^c = (I^c)^k$ since $I^k = (I^c)^k Q_b$. The proof above shows that $Q_b((I^c)^k)^* = (I^k)^{-1}$. $(I^c)^k$ is idempotent so that $(((I^c)^k)^*)(I^c)^k$ by Lemma 1.5 of [3]. Hence $Q_b = (I^k)^{-1}(I^k) = Q_b(((I^c)^k)^*)(I^c)^k Q_b = Q_b(I^c)^k Q_b = I^k$. Thus I was not a proper ideal and Q_b has no invertible ideals. □

Lemma 2.7. *Let M be a right Q_b-torsion module. Then the structure of M as an R-module is identical to the structure of M as a Q_b-module; that is, every R-submodule of M is a Q_b-submodule of M and vice versa.*

Proof. The fact that the result is true for cyclic M is basically the content of (ii) of Proposition 2.4. In fact $xQ_b = xR$ for all x in M by the same reason. Since any R-submodule of M is a sum of cyclic R-submodules of M, it follows that every R-submodule of M is a sum of cyclic Q_b-submodules of M, and thus is a Q_b-submodule of M. Clearly, every Q_b-submodule is an R-submodule. □

Proposition 2.8. *Let M be a simple right cbf R-module. Then the R-injective hull of M is cbf.*

Proof. M is simple so that M is isomorphic to R/I for a maximal right ideal I. We have that $M = R/I \simeq Q_b/IQ_b$. The Q_b-injective hull H of Q_b/IQ_b is Q_b-torsion (in the sense of Levy) and is thus Q_b-divisible. Since R is contained in Q_b, H is R-divisible. By a theorem of Levy [12] H is thus injective as an R-module. Thus E, the R-injective hull of M, is an R-submodule of E; by Lemma 2.7 E is in fact a Q_b-submodule of H. Let N be an R-submodule of E, by Lemma 2.7 N is a Q_b-submodule of E and hence E/N is a Q_b-module. It follows that every R-submodule of E/N is b-torsionfree and hence, since N was arbitrary, that E is *cbf*. □

The following corollary is one of the main results of this section and will be used again in the paper.

Corollary 2.9. *Let B and C be simple right R-modules such that B is b-torsion and C is cbf. Then* $\mathrm{Ext}^1_R(B, C) = 0$.

Proof. $\mathrm{Ext}^1(B, C) = 0$ if and only if every short exact sequence of R-modules

$$(*) \qquad 0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0$$

splits. Assume that there is a sequence $(*)$ that does not split. C is then an essential submodule of A so that $E(C) = E(A)$ (injective hulls). By Proposition 2.8 $E(A)$ is a *cbf* R-module. But this leads to a contradiction since B is a b-torsion submodule of $E(A)/C$. □

Much of the remainder of this section is concerned with the construction of the ring of quotients which is complementary to the ring Q_b, the ring of quotients with respect to the torsion theory determined by the *cbf* integral right R-ideals.

Let $\mathcal{T}_c = \{I_R \subset R: R/I \text{ is } cbf\}$. In review, if M is a right R-module, then $c(M) = \{x \in M: xI = 0 \text{ for some } I \in \mathcal{T}_c\}$.

Lemma 2.10. *$\mathcal{T}_c$ determines a kernel functor.*

Proof. Let $I \subset J \subset R$ be right ideals of R with $I \in \mathcal{T}_c$. Then R/I is *cbf* and has no b-torsion composition factors; hence neither does R/J since it is a homomorphic image of R/I; hence $J \in \mathcal{T}_c$. Now let I and J be elements of $\mathcal{T}_c$. $I/(I \cap J) \simeq (I + J)/J$ which is contained in R/J which has no b-torsion composition factors. R/I has no b-torsion composition factors either; consequently neither does $R/(I \cap J)$ and $I \cap J \in \mathcal{T}_c$. Take $I \in \mathcal{T}_c$ and $x \in R$. Let $(I:x) = \{r \in R: xr \in I\}$. It is easy to see that $R/(I:x) \simeq (I + xR)/I$ (take $[r + (I:x)]$ to $[xr + I]$). R/I has no b-torsion composition factors so that neither can $R/(I:x)$. Thus $(I:x) \in \mathcal{T}_c$ and there is an element of $\mathcal{T}_c$ which annihilates $[x + I]$ in R/I. We have shown that the three properties of Proposition 2.1 of [7] hold. It follows that $\mathcal{T}_c$ defines a kernel functor. □

Lemma 2.11. *Let M be an Artinian right c-torsion module, then M is cbf.*

Proof. Let $x \in M$; then x is c-torsion; that is, there exists $I \in \mathcal{T}_c$ such that $xI = 0$. Hence xR which is a homomorphic image of R/I is *cbf*. Let $x_1, \cdots, x_n$ be a set of generators for M; then $M = x_1R + \cdots + x_nR$. $M' = x_1R \oplus \cdots \oplus x_nR$ is *cbf* and hence M which is a homomorphic image of M' is *cbf*. □

The next proposition shows that $\mathcal{T}_c$ defines an idempotent kernel functor.

Proposition 2.12. *c is an idempotent kernel functor.*

Proof. c is a kernel functor by Lemma 2.10. By Theorem 2.5 of [7] c will be idempotent if it can be shown that if $I \subset J \subset R$ are right ideals of R such that $J \in \mathcal{T}_c$ and J/I is c-torsion, then $I \in \mathcal{T}_c$. Let I and J be such. J/I is Artinian and a *cbf* module by Lemma 2.11 so that J/I has no b-torsion composition factors. $J \in \mathcal{T}_c$ so that R/J has no b-torsion composition factors. Hence R/I has no b-torsion composition factors and $I \in \mathcal{T}_c$. □

c is an idempotent kernel functor so that we can construct the corresponding ring of quotients Q_c. By Proposition 1.1 Q_c is just the union of all I^* where $I \in \mathcal{T}_c$. Before giving any properties of Q_c it will be necessary to introduce a duality for finitely generated torsion modules which is due to G. M. Bergman [1] and P. M. Cohn. The following theorem is a restatement of Proposition 51 of Bergman [1] for the case of a hereditary Noetherian ring. The proof may be found there.

Theorem 2.13. *Let R be a hereditary Noetherian ring. There exists an anti-isomorphism, α, between the categories of finitely generated right torsion R-modules and finitely generated left torsion R-modules, such that if M is the cokernel of*

$P \xrightarrow{i} Q$ where P and Q are finitely generated projectives, then the anti-isomorphic image of M, $\alpha(M)$, will be isomorphic to the cokernel of the map $Q^* \xrightarrow{i^*} P^*$. □

We will be concerned with the situation when P and Q are fractional R-ideals. Namely, if I is an integral right R-ideal, then $\alpha(R/I)$ is I^*/R since $R^* = R$. In the following we will implicitly use that $I \rightsquigarrow I^*/R$ is an anti-isomorphism. We are of course also using the fact that I^* is isomorphic to $I^* = \mathrm{Hom}_R(I, R)$. Note also that $I^{**} = I$.

Lemma 2.14. *Let I be an integral left R-ideal such that R/I is cbf. Then ${}^*I/R$ is cbf.*

Proof. The proof is by induction on the composition length of R/I. Suppose R/I is simple and *cbf* but ${}^*I/R$ is not *cbf*. ${}^*I/R$ is simple so that this means that ${}^*I/R$ is b-torsion. Then there is an invertible two-sided ideal B of R such that $({}^*I/R)B = 0$; that is ${}^*IB \subset R$ or ${}^*IBB^{-1} = {}^*IR = {}^*I \subset RB^{-1} = B^{-1} = {}^*B$. Taking duals yields that $I = {}^{**}I \supset {}^{**}B = B$ contradicting the fact that R/I was *cbf*. Hence ${}^*I/R$ is *cbf*. Inductively suppose that the result holds for all *cbf* left ideals J such that the composition length of R/J is less than or equal to k and let I be such that R/I is *cbf* and has composition length $k + 1$. Then there is a left ideal J such that $I \subset J \subset R$ where R/J is *cbf* and has composition length k. $R \subset {}^*J \subset {}^*I$. By the induction hypothesis it follows that ${}^*J/R$ is *cbf*. Also note that ${}^*I/{}^*J$ is just $\alpha(J/I)$ which is simple. Again by the induction hypothesis ${}^*I/{}^*J$ is *cbf*. As a result ${}^*I/R$ has no b-torsion composition factors and is *cbf*. The result follows by induction. □

Proposition 2.15. *Q_c is the ring of quotients of R with respect to both the left and right c-torsion theories.*

Proof. To prevent confusion let ${}_c\mathfrak{I}$ be the family of left ideals that generates the c-torsion theory in $\mathrm{Mod}({}_RR)$ and let ${}_cQ$ be the corresponding ring of quotients. Let $I \in {}_c\mathfrak{I}$; by Lemma 2.14 ${}^*I/R$ is *cbf* and hence is c-torsion. Consequently if $x \in I^*$, then there is a $J \in \mathfrak{I}_c$ such that $xJ \subset R$. But this just says that $x \in J^*$. Q_c is just the union of all such J^* so that $x \in Q_c$; however, ${}_cQ$ is just the union of all such ${}^*I$. Hence ${}_cQ \subset Q_c$; similarly it follows that $Q_c \subset {}_cQ$ and $Q_c = {}_cQ$. □

Lemma 2.16. *Let $M \subset N$ be R-submodules of a Q_c-module K. If $MQ_c = NQ_c$, then N/M is c-torsion.*

Proof. Take $x \in N$, x is then in $NQ_c = MQ_c$ so that $x = \sum m_i y_i$ where the m_i's are in M and the y_i's are in Q_c. Q_c is the union of all elements of $\mathfrak{I}_c$; therefore, for each i there is an $I_i \in \mathfrak{I}_c$ such that $y_i \in I_i^*$. Hence $y_i \in I_1^* + \cdots + I_n^* = (I_1 \cap \cdots \cap I_n)^*$ for each i. $I = I_1 \cap \cdots \cap I_n \in \mathfrak{I}_c$. $xI = (\sum m_i y_i)I = \sum m_i y_i I \subset \sum m_i R \subset M$; therefore, $[x + M]$ was c-torsion in N/M. x was arbitrary in N so that N/M is c-torsion. □

Proposition 2.17. (i) *Let $I \in \mathfrak{J}_b$, that is, let R/I be b-torsion. Then $R/I \simeq Q_c/IQ_c$ and $I = (IQ_c)^c$.*

(ii) *Let B be a two-sided of $\mathfrak{J}_b$. Then $Q_c MQ_c \neq Q_c$ and $Q_c M = Q_c MQ_c = MQ_c$. Furthermore, R/M, and Q_c/MQ_c are isomorphic as rings.*

Proof. (i) Let $I \in \mathfrak{J}_b$; I will then contain an invertible two-sided ideal B of R. If $J \in {}_c\mathfrak{J}$ (J is a *cbf* integral left R-ideal), then $R = J + B \subset J + I$. It follows that $^*J \subset R^*J = J^*J + I^*J \subset R + IQ_c$. Q_c is the union of all such $^*J$ so that $R + IQ_c = Q_c$. The second isomorphism theorem yields that $Q_c/IQ_c \simeq R/(IQ_c)^c$. The proof of (i) will be completed once it is shown that $I = (IQ_c)^c$. $IQ_c = (IQ_c)^c Q_c$ so that by Lemma 2.16 $(IQ_c)^c/I$ is c-torsion. But $(IQ_c)^c/I$ is b-torsion since it is a submodule of R/I; therefore $(IQ_c)^c/I$ is both c-torsion and b-torsion and hence the zero module since the c-torsion cyclic modules are precisely the completely b-torsion free cyclic modules.

(ii) Suppose that B is a two-sided ideal of $\mathfrak{J}_b$ and that $Q_c BQ_c = Q_c$. Lemma 2.16 yields that Q_c/BQ_c is c-torsion since $Q_c(BQ_c) = Q_c = Q_c Q_c$. In the proof (i) we have that $(BQ_c)^c = B$ and that $Q_c/BQ_c \simeq R/B$. The map giving the isomorphism is that of the second isomorphism theorem and in this case will be an R-R bimodule map. In particular Q_c/BQ_c is isomorphic to R/B as a left R-module; Q_c/BQ_c is thus b-torsion. Q_c/BQ_c is both b-torsion and c-torsion and is therefore the zero module so that $Q_c = BQ_c$. But this contradicts the fact that $Q_c/BQ_c \simeq R/B$; whence $Q_c BQ_c = BQ_c \neq Q_c$ and similarly $Q_c BQ_c = Q_c B$. The fact that Q_c/BQ_c and R/B are isomorphic as rings follows as in Proposition 2.4. □

We are now able to prove a proposition complementary to Corollary 2.9.

Proposition 2.18. *Let B and C be simple right R-modules such that B is b-torsion and C is cbf (hence c-torsion). Then* $\operatorname{Ext}^1_R(C, B) = 0$.

Proof. $\operatorname{Ext}^1(C, B) = 0$ if and only if every short exact sequence

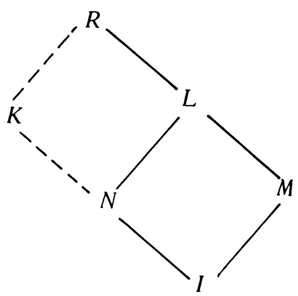
$$(*) \qquad 0 \longrightarrow B \longrightarrow A \longrightarrow C \longrightarrow 0$$

splits. Assume that $(*)$ is such a sequence that does not split. $B \simeq R/I$ where $I \in \mathfrak{J}_b$. By Proposition 2.17 $B \simeq R/I \simeq Q_c/IQ_c$, a simple right Q_c-module; identify B with Q_c/IQ_c. $(*)$ does not split so that B is an essential submodule of A; hence $E(A) = E(B)$. Let F be the Q_c-injective hull of B. F is Q_c-divisible, thus R-divisible, and by a theorem of Levy [12] is then R-injective. Consequently $E(A) \subset F$. B is a Q_c-submodule of F so that F/B is a Q_c-module and hence is c-torsion free; this is a contradiction of the fact that F/B is c-torsion. This implies that B is not large in A and that $(*)$ splits. □

Corollary 2.9 and Proposition 2.18 are brought together in a stronger form in the following theorem.

Theorem 2.19. *If M is an Artinian finitely generated right (left) R-module, then $M = b(M) \oplus c(M)$.*

Proof. The proof will be by induction on the composition length of M. If the composition length of M is one, then M is simple and the result is trivially true. Suppose now that the result is valid for modules of composition length less than or equal to n, and also suppose that the composition length of M is $n+1$. $b(M) \cap c(M) = 0$ so it is sufficient to show that if $x \in M$, then $x = b + c$ where $b \in b(M)$ and $c \in c(M)$; this implies that it is sufficient to assume that M is cyclic of composition length $n+1$. In this case $M \simeq R/I$ and there exists $I \subset L \subset R$ such that R/L is simple and the composition length of L/I is n. Throughout the rest of the proof we will refer to the following diagram:



By the induction hypothesis there are right ideals of R, N and M, such that $N \cap M = I$, $N + M = L$, N/I is c-torsion, and M/I is b-torsion. $L/M \simeq N/I$ is c-torsion and $L/N \simeq M/I$ is b-torsion. There are two cases: either R/L is c-torsion or R/L b-torsion. We will consider the case in which R/L is c-torsion; the proof in the other case is very similar. Again by the induction hypothesis (or if $N = I$ and the composition length of R/I is 2, then by Proposition 2.18) $N = K \cap L'$ where $K + L' = R$, K/N is c-torsion, and L'/N is b-torsion. $K \cap M \supset N \cap M = I$. Also, $(K \cap M)/I$ is a submodule of N/I and is thus c-torsion; at the same time however $(K \cap M)/I$ is a submodule of M/I and thus is b-torsion. Therefore $(K \cap M)/I$ is the zero module and $K \cap M = I$. $(K + M)/I$ is isomorphic to $(K/I) \oplus (M/I)$ and therefore has composition length equal to that of L/I plus one, that is, $n+1$. R/I also has composition length $n+1$ so that $K + M = R$. We have shown that $R/I = (K/I) \oplus (M/I)$ where K/I is c-torsion and M/I is b-torsion which is the desired result. □

As a corollary to Proposition 2.19 we have the next theorem which is a generalization of Theorem 3.9 of Eisenbud and Robson [4] since in a Dedekind prime ring the concepts of c-torsion and b-torsion reduce to those of completely faithful and bounded.

Theorem 2.20. *Let M be a right (left) torsion module over R. Then $M = b(M) \oplus c(M)$.*

Proof. It need only be shown that if $x \in M$, then $x = b + c$ where $b \in b(M)$ and $c \in c(M)$, for $b(M) \cap c(M) = 0$. Take $x \in M$; xR is Artinian so that by Proposition 2.19 $x = b + c$ where $b \in b(xR)$ and $c \in c(xR)$. $b(xR) \subset b(M)$ and $c(xR) \subset c(M)$;

therefore $b \in b(M)$ and $c \in c(M)$ as desired. □

The previous results are now used to deduce some further properties of the overring Q_c.

Proposition 2.21. *If I is an integral right Q_c-ideal, then $I^c \in \mathfrak{J}_b$ and $Q_c/I \simeq R/I^c$.*

Proof. Let I be an integral right Q_c-ideal and consider R/I^c. By Proposition 2.19 R/I^c is the direct sum of its b-torsion submodule and its c-torsion submodule so that there are right ideals J and K of R such that $J \cap K = I^c$, $J + K = R$, R/J is b-torsion and R/K is c-torsion. $R/K \simeq J/I^c$ by the second isomorphism theorem which means that J/I^c is c-torsion. If $x \in J$, then there exists $L \in \mathfrak{J}_c$ such that $xL \subset I^c$. $x = x1 \in xO_l(L) = x(LL^*) = xLL^* \subset I^cL^* \subset I^cQ_c$ and since x was arbitrary in J, $J \subset I^cQ_c$; but $J \subset R$, therefore $J \subset I^cQ_c \cap R = I^c$. As a result $R/I^c = R/J$ is b-torsion. The fact that $Q_c/I \simeq R/I^c$ now follows from Proposition 2.17. □

A hereditary Noetherian prime ring is said to have *enough invertible ideals* if every nonzero two-sided ideal of R contains an invertible two-sided ideal. The following theorem completes the structure of Q_c.

Theorem 2.22. *Q_c is a bounded hereditary Noetherian prime ring with enough invertible ideals. The one-sided (two-sided) Q_c-ideal lattices are isomorphic to the one-sided (two-sided) ideal lattices (lattice) of R-ideals in $\mathfrak{J}_b$, and the corresponding factor modules (rings) are isomorphic.*

Proof. Clearly Q_c is hereditary and Noetherian, and the last statement is a statement of Propositions 2.17 and 2.21. It still must be shown that Q_c is bounded with enough invertible ideals.

If B is a two-sided invertible ideal of R, then $Q_cB = BQ_c$ is a two-sided ideal of Q_c. Consider Q_cB^{-1}; $Q_cB^{-1} = Q_cRB^{-1} = RQB^{-1} = B^{-1}BQ_cB^{-1} = B^{-1}Q_cBB^{-1} = B^{-1}Q_cR = B^{-1}Q_c$; that is, $Q_cB^{-1} = B^{-1}Q_c$. $(Q_cB^{-1})(BQ_c) = Q_cRQ_c = Q_c = (Q_cB)(B^{-1}Q_c) = (BQ_c)(Q_cB^{-1})$; whence BQ_c is an invertible two-sided ideal of Q_c.

Let I be an integral right (left) Q_c-ideal. By Proposition 2.21 $I^c \in \mathfrak{J}_b({}_b\mathfrak{J})$ and hence contains an invertible two-sided ideal B of R. I then contains $B^{-1}Q_c$ an invertible two-sided Q_c-ideal. As a result Q_c is bounded with enough invertible ideals. □

Proposition 2.23. *Let B be a simple b-torsion right R-module. Then $E(B)$ is b-torsion.*

Proof. The proof is much the same as those of Lemma 2.7 and Proposition 2.8. □

The main results of this section are combined in the following theorem.

Theorem 2.24. *Let R be an hereditary Noetherian prime ring. Then R is the intersection of a bounded hereditary Noetherian prime ring with enough invertible* *ideals and an hereditary Noetherian prime ring with no invertible ideals.*

Proof. $R \subset (Q_b \cap Q_c)$ and the proof will be complete once it is shown that $Q_b \cap Q_c \subset R$ and thus that equality holds. Take $x \in (Q_b \cap Q_c)\backslash R$. $[x + R]$ will then be an element of Q/R which is both c-torsion and b-torsion. Hence $[x + R] = [0 + R]$ and $x \in R$. □

3. **Localizations at maximal invertible ideals.** In this section it is shown that if M is a maximal invertible ideal of an HNP ring R, then R can be localized at M. That is, an overring of R is constructed which is a bounded HNP with only finitely many maximal ideals. Also, the Jacobson radical of this overring is the expansion of M and is invertible. Finally, it is shown that these localizations are classical in nature and that a globalization theorem holds.

Jategaonkar [9] has shown that localizations can be constructed at maximal invertible ideals in an HNP ring with enough invertible ideals, thus generalizing previous results of the author [10]. The results of this section generalize both those of Jategaonkar and the author.

Many of the proofs in this section are similar to those in §2. When this occurs, it will be stated in order to keep repetitions at a minimum.

Let M be a maximal invertible ideal of R. Let $\mathfrak{I}_m = \{I\colon I$ is an integral right R-ideal and each composition factor of R/I is annihilated by $M\}$. Clearly $I \in \mathfrak{I}_m$ if and only if I contains a power of M. If A is a right R-module, define $m(A)$ by $m(A) = \{a \in A\colon aI = 0$ for some $I \in \mathfrak{I}_m\}$. The left analogs ${}_m\mathfrak{I}$ and $m(B)$ for B a right R-module are defined in a similar manner. A module A is called *completely m-torsion free* if all submodules of all factor modules of A are m-torsion free. Let $\mathfrak{I}_f = \{I\colon I$ is an integral right R-ideal and R/I is completely m-torsion free$\}$. $I \in \mathfrak{I}_f$ if and only if R/I has no m-torsion composition factors. If A is a right R-module, $f(A)$ is defined by $f(A) = \{a \in A\colon aI = 0$ for some $I \in \mathfrak{I}_f\}$. Again ${}_f\mathfrak{I}$ is defined in analogous fashion. Note that if $I \in \mathfrak{I}_f$, then $I + M = R$. From this it follows that if A is a right R-module, then $m(A) \cap f(A) = 0$.

Proposition 3.1. *f and m are idempotent kernel functors.*

Proof. The proof is roughly the same as that of Lemma 2.10 and Proposition 2.12. □

The corresponding rings of quotients Q_m and Q_f are now formed. Q_m is the union of the duals of all the right ideals in $\mathfrak{I}_m$ while Q_f is the union of the duals of all the right ideals in $\mathfrak{I}_f$. The next proposition shows that these quotient rings are also the left-handed versions.

Proposition 3.2. *$Q_m = {}_mQ$ and $Q_f = {}_fQ$.*

Proof. $I \in \mathfrak{I}_m$ if and only if I contains some power of M. Thus Q_m is the union of all powers of M^{-1}; but ${}_mQ$ is also the union of all powers of M^{-1}. Hence

$Q_m = {}_mQ$. The proof of the fact that $Q_f = {}_fQ$ is much the same as that of Proposition 2.15 using a lemma similar to Lemma 2.14. □

We can now give some properties of Q_m.

Proposition 3.3. (i) *If $I \in \mathcal{T}_m$, then $Q_f/IQ_f \simeq R/I$ and $I = (IQ_f)^c$.*

(ii) *If B is a two-sided ideal in $\mathcal{T}_m$, then $Q_fB = Q_fBQ_f = BQ_f$, and Q_f/BQ_f and R/B are isomorphic as rings.*

Proof. The proof is again essentially that of §2. □

Proposition 3.4. *Let S and S' be simple right R-modules such that S is m-torsion and S' is f-torsion. Then $\mathrm{Ext}_R^1(S', S) = 0$ and $\mathrm{Ext}_R^1(S, S') = 0$.*

Proof. The proof that $\mathrm{Ext}_R^1(S', S) = 0$ is much the same as that of Proposition 2.18. S' is either b-torsion or c-torsion. If S' is c-torsion, then $\mathrm{Ext}_R^1(S, S') = 0$ by Corollary 2.9. If S' is b-torsion, then S' is annihilated by some maximal invertible ideal M' and $M' \neq M$. In this case S if f'-torsion and S' is m'-torsion where f' and m' are the kernel functors. Hence $\mathrm{Ext}_R^1(S, S') = 0$. □

Proposition 3.5. *Let I be an integral right Q_f-ideal. Then $I^c \in \mathcal{T}_m$ and $R/I^c \simeq Q_f/I$.*

Proof. The proof is essentially that of Proposition 2.21; it requires a lemma which is the analog of Proposition 2.20. We will not give it now since a generalization will be given later in the paper. □

We are now ready to give the structure of the rings Q_m. The structure of Q_m depends on the nature of the maximal invertible ideal M. By Theorem 2.6 of Eisenbud and Robson [3] either M is a maximal ideal or M is the intersection of a finite number of distinct idempotent maximal ideals $M_1, \cdots, M_n$ where $O_r(M_1) = O_l(M_2)$, $\cdots, O_r(M_n) = O_l(M_1)$. Such a set of maximal ideals is called a *nontrivial cycle*, or just a *cycle*.

Theorem 3.6. *Let M be a maximal invertible ideal of an HNP ring R, and let m and f be the associated kernel functors. Then Q_f is a bounded HNP with enough invertible ideals whose Jacobson radical is MQ_f. Furthermore:*

(i) *If M is a maximal ideal, then Q_f is a Dedekind prime (in fact a PIR) with a unique maximal ideal MQ_f. Every ideal of Q_f is a power of MQ.*

(ii) *If M is an intersection of a cycle, say $M = M_1 \cap \cdots \cap M_n$ where $M_1, \cdots, M_n$ is a cycle, then $MQ_f = M_1Q_f \cap \cdots \cap M_nQ_f$ where $M_1Q_f, \cdots, M_nQ_f$ is a cycle. In fact the M_iQ_f's are the only maximal ideals of Q_f and all are idempotent. Also, every invertible ideal of Q_f is a power of MQ_f.*

Proof. Q_f is bounded since by Proposition 3.5 if I is an integral right Q_f-ideal, then $I^c \in \mathcal{T}_m$ so that I^c contains some power, M^k, of M. I then contains the two-sided ideal of M^kQ_f.

(i) Let I be a two-sided ideal of Q_f; I^c is then a two-sided ideal in $\mathcal{T}_m$ and hence contains some power of M; suppose that k is the least such power. We show by induction on k that if J is any R-ideal with k the least positive integer such that $M^k \subset J$, then $M^k = J$. If $k = 1$, then $M \subset J \subset R$ so that $M = J$ since M is a maximal ideal. Inductively suppose that the result holds for all powers less than k and that k is the least positive integer such that $M^k \subset J$. J must be contained in M so that $M^{-1}M^k \subset M^{-1}J \subset M^{-1}M$ and $M^{k-1} \subset M^{-1}J \subset R$. Then the least power of M contained in J is no bigger than $k-1$ and the induction hypothesis implies that J is a power of M. It follows that $I^c = M^k$ for some k. Every ideal is the power of its contraction, and $MQ_f = Q_fM$; therefore $I = M^kQ_f = (MQ_f)^k$ and every ideal is a power of MQ_f. MQ_f is the Jacobson radical of Q_f since it is the unique maximal ideal of Q_f and since Q_f is bounded. As in the case of Q_c, expansions of invertible ideals are invertible and $(M^kQ_f)^{-1} = QM^{-k}Q_f$. This means that Q_f is a Dedekind prime ring. Let I be an integral right Q_f-ideal; then there is a power k of MQ_f which is properly contained in I, in fact small in I since $MQ_f = \mathrm{rad}(Q_f)$. $Q_f/(MQ)^k$ is an Artinian PIR so that $I/(MQ_f)^k$ is principal. $(MQ_f)^k$ is small in I which implies that I is principal and that Q_f is a PIR.

(ii) Every ideal is the expansion of its contraction and every ideal in $\mathcal{T}_m$ has a proper expansion, therefore the maximal (two-sided) ideals of Q_f are precisely $M_1Q_f, \cdots, M_nQ_f$. Since $Q_f/MQ_f \simeq R/M$, $\bigcap M_iQ_f = MQ_f$; Q_f is bounded so that MQ_f is the Jacobson radical of Q_f.

First we note that an ideal I of Q_f is idempotent if and only if its contraction I^c is idempotent in R, for $I = I^2 = (I^cQ_f)(I^cQ_f) = (I^c)^2Q_f$ and $(((I^c)^2)Q_f)^c = (I^c)^2 = I^c$. This also means that if J is an idempotent ideal in $\mathcal{T}_m$, then JQ_f is idempotent. Clearly then an ideal of Q_f is eventually idempotent if and only if its contraction is eventually idempotent. Note also that all powers of MQ_f are invertible since $M^{-k}Q_f = Q_fM^{-k}$ and $(Q_fM^{-k})(M^kQ_f) = Q_f = (Q_fM^k)(M^{-k}Q_f)$. These are in fact the only invertible ideals of Q_f; for assume that I is an invertible ideal of Q_f, then I^c contains some power of M, say M^k. If $k = 1$, then either $I^c = M$ (in which case $I = MQ_f$) or else I^c properly contains M. M is a maximal invertible ideal which implies that I^c is eventually idempotent and hence by the above that so is I. But this contradicts the fact that I is invertible; therefore, $I^c = M$. Inductively assume that if J is any invertible ideal of Q_f such that the least power of M contained in J^c is less than the kth, then J is a power of MQ_f. Suppose that I is an invertible ideal of Q_f and that M^k is the least power of M contained in I^c with k greater than one. $(MQ_f)^k$ is then an invertible ideal contained in I. I being an invertible ideal is contained in a maximal invertible ideal of Q_f, say L. L is a finite intersection of idempotent maximal ideals and hence must contain MQ_f the intersection of all the maximal ideals. In this case $L^c \supset M$ and by the induction hypothesis $L^c = M$ and $L = MQ_f$. Hence $(MQ_f)^k \subset I \subset MQ_f$ and $(MQ_f)^{k-1} \subset (MQ_f)^{-1}I \subset Q_f$.

$(MQ_f)^{-1}I$ is again invertible and the least power of M contained in $((MQ_f)^{-1}I)^c$ is less than k so that $(MQ_f)^{-1}I$ is a power of MQ_f and hence so is I. □

Let $S = \{m: m$ is the kernel functor associated to a maximal invertible ideal $M\}$. The following theorem gives a decomposition theory for torsion modules over a hereditary Noetherian prime ring.

Theorem 3.7. *Let A be a torsion right R-module. Then $A = c(A) \oplus (\bigoplus_S m(A))$.*

Proof. By Theorem 2.20 $A = c(M) \oplus b(A)$; $m(A) \subset b(A)$ for $m \in S$. Therefore it just remains to be shown that $b(A) = \bigoplus_S m(A)$. Also $m(A) \cap (\bigoplus_{m' \neq m} m'(A)) = 0$ since $\bigoplus_{m' \neq m} m'(A) \subset f(A)$ where f is associated with M. Hence it is enough to show that every element of $b(A)$ is a sum of elements of the desired type, or assume that $b(A)$ is cyclic and hence of finite composition length. From these, one can proceed by induction on the composition length. If the length is one, the result is clear. If not, then a proof similar to that of Proposition 2.19 shows that if m_1 is such that $b(A)$ has at least one m_1-torsion composition factor then $b(A) = m_1(b(A)) \oplus f_1(b(A))$ and $f_1(b(A))$ has composition length strictly less than that of $b(A)$ and the result follows from the induction hypothesis. □

Let M again be a maximal invertible ideal in the HNP ring R. Define the *cancellation set of* M, $C(M)$, by $C(M) = \{b \in R: bx \in M$ implies $x \in M\}$. Hence $b \in C(M)$ if and only if $[b + M]$ is right regular in R/M. R/M is a semisimple Artinian ring so that in R/M being right regular is equivalent to being a unit which is equivalent to being left regular. As a result we have that $C(M) = \{b \in R: xb \in M$ implies $x \in M\}$. Also, $C(M) = \{b \in R: bR + M = R\} = \{b \in R: Rb + M = R\}$.

Let $b \in C(M)$; then $bR + M = R$ which implies that $bM + M^2 = M$. Hence $bR + (bM + M^2) = R$ and $bR + M^2 = R$. Induction yields that $bR + M^n = R$ for all n; that is, $[b + M^n]$ is a unit in R/M^n for all n. Suppose that $bx = 0$; this means then that $x \in M^n$ for all values of n. Since M is invertible, the intersections of its powers is zero; and hence that x is zero. As a result, all the elements of $C(M)$ are regular.

Lemma 3.8. *If $b \in C(M)$, then R/bR is f-torsion; that is, R/bR has no compostion factors which are annihilated by M.*

Proof. Consider R/bR. $R/bR = f(R/bR) \oplus m(R/bR)$ by 3.7. Equivalently, there are right ideals I and J of R such that $I \cap J = bR$, $I + J = R$, R/I is f-torsion and R/J is m-torsion. If R/bR is not f-torsion, then $J \neq R$; hence $bR + M \subseteq J + M \neq R$ which contradicts the fact that $b \in C(M)$. Hence R/bR is f-torsion. □

Lemma 3.9. *Let $I \in \mathfrak{F}_f$. Then I contains an element of $C(M)$.*

Proof. R/I is f-torsion so that R/I has no composition factors annihilated by M. Therefore $I + M = R$, and hence $1 = x + m$ for $x \in I$ and $m \in M$. $xR + M = R$ so that $x \in C(M)$. □

Theorem 3.10. *Q_f is a classical localization of R. That is, every element of Q_f can be written in the form $xb^{-1}(d^{-1}y)$ where $x \in R$ and $b \in C(M)$ ($y \in R$ and $d \in C(M)$); every element of $C(M)$ has its inverse in Q_f. R satisfies the Ore conditions with respect to $C(M)$.*

Proof. If $b \in C(M)$, then $b^{-1} \in Q_f$ by Lemma 3.8. Conversely let $q \in Q_f$, then $q \in I^*$ for $I \in \mathfrak{J}_f$. Then there is $b \in C(M)$ such that $b \in I$. $bR \subset I$ so that $Rb^{-1} \supset I^*$. As a result $q = xb^{-1}$ for $x \in R$. The other representation follows in similar fashion. The Ore conditions are forced by the fact that Q_f is a ring. □

Lemma 3.11. *Let σ be an idempotent kernel functor in* $\mathrm{Mod}(R_R)$ *where R is an HNP ring. Then $\sigma(B) = \ker(B \to B \otimes_R Q_\sigma)$ where Q_σ is the ring of quotients with respect to σ.*

Proof. Q_σ is an overring of R and is the union of the duals of the elements of $\mathfrak{J}_\sigma$ by the results of section one. By [10] ${}_RQ_\sigma$ is flat so that we can just show that a right ideal I is in $\mathfrak{J}_\sigma$ if and only if $(R/I) \otimes Q_\sigma = 0$ ($x \in \sigma(B)$ if and only if $xR \simeq R/I$ for some $I \in \mathfrak{J}_\sigma$). The key to the fact is the following commutative diagram where the horizontal maps are inclusion maps and the vertical maps are defined by $\Sigma\, x_i \otimes y_i \to \Sigma\, x_iy_i$.

$$\begin{array}{ccccc} 0 & \longrightarrow & I \otimes Q_\sigma & \longrightarrow & R \otimes Q_\sigma \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & IQ_\sigma & \longrightarrow & Q_\sigma \end{array}$$

The vertical maps are in fact isomorphisms since $Q \otimes_R Q \simeq Q$ under the same map. $(R/I) \otimes Q_\sigma = 0$ if and only if the map of the top row is onto. But this happens exactly when the bottom map is onto which is when $IQ_\sigma = Q_\sigma$. However, this holds precisely when $I \in \mathfrak{J}_\sigma$. □

We now can prove the globalization theorem.

Theorem 3.12. *Let B be a right R-module such that $B \otimes_R Q_b = 0$ and $B \otimes_R Q_f = 0$ for each maximal invertible ideal M. Then B is the zero module.*

Proof. All of the overrings involved are flat as left R-modules, therefore we may assume that B is a cyclic torsion module. Since $B \otimes Q_b = 0$, B is b-torsion by Lemma 3.12. Since tensor products commute with direct sums we may in view of Theorem 3.7, assume that B is m-torsion where m is associated with the maximal ideal M. Let f be associated with M, then $B \otimes Q_f = 0$ so that by Lemma 3.11 B is f-torsion. Hence B is both f and m-torsion and is thus the zero module. □

4. **The one and one-half generator property.** In this section it is shown as an application that if R (an HNP) has the one and one-half generator property for integral right R-ideals, then R is a Dedekind prime ring. The main tool used will be

the localization technique. Eisenbud and Robson [4] have shown that the one and one-half generator property holds for Dedekind prime rings.

R is said to have the one and one-half generator property for integral right R-ideals if given an integral right R-ideal I and b a regular element of I, then I/bR is principal. Another way of stating this is that I is generated by two elements, one of which may be chosen almost at random. It might also be noted that for HNP's this just means that submodules of cyclic torsion modules are cyclic.

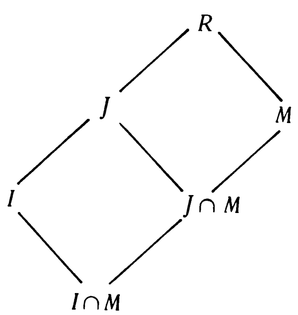
Eisenbud and Robson [3] have shown that an HNP ring will be Dedekind if and only if it has no idempotent maximal ideals. For the rest of the section R will be an HNP ring with the one and one-half generator property and M will be idempotent maximal ideal of R. The proof will be complete once it is shown that the fact that M is idempotent leads to a contradiction.

Let m be the kernel functor determined by M and let f be the associated kernel functor determined by the set of integral right R-ideals I such that R/I is completely m-torsion free.

Lemma 4.1. *$I \in \mathcal{F}_f$ if and only if $I + M = R$.*

Proof. If $\in \mathcal{F}_f$, then R/I has no m-torsion composition factors and hence $I + M = R$.

Conversely, let I be such that $I + M = R$. If $I + M = R$, then $L + M = R$ for any $L \supset I$; therefore, we can assume that $I \subset J \subset R$ such that $J \in \mathcal{F}_f$ and J/I is a simple m-torsion module. Consider the following diagram:



$R/M \simeq I/(I \cap M)$ and J/I is cyclic m-torsion. Hence if the composition length of R/M is k, then the composition length of $J/(I \cap M)$ is $k + 1$. Each composition factor of $J/(I \cap M)$ is m-torsion so that $(J/I \cap M)M^{k+1} = 0 = (J/I \cap M)M$ since M is idempotent. This says that $J/(I \cap M)$ is an R/M-module. By the one and one-half generator property $J/(I \cap M)$ is cyclic as a right R-module; therefore, it is cyclic as an R/M-module. This, however, is a contradiction since cyclic modules over R/M can have composition length at most k. Thus it must be that $J = I$ and $I \in \mathcal{F}_f$. □

Lemma 4.2. *Let $I \in \mathfrak{I}_f$, then there is a regular $a \in I$ such that $aR \in \mathfrak{I}_f$.*

Proof. $I \in \mathfrak{I}_f$ so that $I + M = R$. R has the one and one-half generator property so that there is an element a of I such that $aR + I \cap M = I$. Hence $aR + M = R$ and $aR \in \mathfrak{I}_f$. □

Again we could form the analogous left kernel functors m and f with associated filters of left ideals ${}_m\mathfrak{I}$ and ${}_f\mathfrak{I}$. The next proposition relates the left and right theories; namely, their rings of quotients coincide.

Proposition 4.3. *$Q_f = {}_fQ$ and every element of Q_f can be expressed in the form $a^{-1}r$ or sb^{-1} where a, b, r, and s are elements of R and $aR + M = R = Rb + M$.*

Proof. Q_f is the union of all I^* where $I \in \mathfrak{I}_f$; but by Lemma 4.2 $aR \subset I$ where $aR + M = R$. Thus $I^* \subset Ra^{-1}$ and Q_f is the union of all such Ra^{-1}. Similarly ${}_fQ$ is the union of all $b^{-1}R$ where $Rb + M = R$. But $Rb + M = R$ just says that $[b + M]$ has a left inverse in the simple Artinian ring R/M; hence $[b + M]$ has a right inverse and $bR + M = R$. As a result $b^{-1} \in Q_f$ and hence $b^{-1}R \subset Q_f$. Thus ${}_fQ \subset Q_f$; similarly $Q_f \subset {}_fQ$ and $Q_f = {}_fQ$. □

In the next proposition we give a few properties of Q_f.

Proposition 4.4. (i) *$MQ_f = Q_fMQ_f = Q_fM$ and Q_f/MQ_f and R/M are isomorphic as rings.*

(ii) *MQ_f is the unique maximal two-sided ideal of Q_f.*

(iii) *MQ_f is the Jacobson radical of Q_f.*

Proof. Suppose that $Q_fMQ_f = Q_f$. Then $\Sigma\, q_im_ip_i = 1$ where the m_i's are in M, and the q_i's and p_i's are in Q_f. Then there are elements a and b in R such that $Ra + M = R$ and $bR = M = R$ for which $q_i = a^{-1}r_i$ for all i and $p_i = s_ib^{-1}$ for all i where the r_i's and s_i's are in R. Then $1 = \Sigma\, a^{-1}r_im_is_ib^{-1}$ and $ab = \Sigma\, r_im_is_i$ is an element of M. This contradicts, however, the fact that $[a + M]$ and $[b + M]$ are units in R/M. Therefore $Q_fMQ_f \neq Q_f$. $(Q_fMQ_f)^c = M$ since M is maximal and $(Q_fMQ_f)^c$ is a proper two-sided ideal of Q_f that contains M. Every ideal is the expansion of its contraction; therefore, $MQ_f = Q_fMQ_f = Q_fM$. The last statement easily follows.

(ii) Let I be a maximal two-sided ideal of Q_f. Then I^c is a two-sided ideal of R and $I^c + M = M$ and $I = I^cQ_f \subset MQ_f$. Hence $I = MQ_f$ since I is a maximal two-sided ideal of Q_f.

(iii) Let I be an integral right Q_f-ideal and let $J = I^c + M$. $J \neq M$ since if this were the case, $I = I^cQ_f$ would have to be all of Q_f. *Claim.* $JQ_f \neq Q_f$. Suppose that $JQ_f = Q_f$. Then $\Sigma\, x_iq_i = 1$ where the x_i's are in J and the q_i's are in Q_f. Then there is a regular element b of R such that $q_i = r_ib^{-1}$ for r_i's in R. Hence $1 = \Sigma\, x_iq_i = \Sigma\, x_ir_ib^{-1} = (\Sigma\, x_ir_i)b^{-1}$ and $b = \Sigma\, x_ir_i$ an element of J. If this were the case, however, then $bR + M \subset J + M \subset J$ contradicting the fact that $bR + M = R$;

hence $JQ_f \neq Q_f$. $JQ_f \supset I^c Q_f = I$ a maximal right ideal of R; hence $JQ_f = I$. $J \supset M$ so that $I = JQ_f \supset MQ_f$. I was an arbitrary maximal right ideal; therefore MQ_f is the Jacobson radical of Q_f by (ii). □

We are now ready to prove the main result of this section.

Theorem 4.5. *A hereditary Noetherian prime ring with the one and one-half generator property is a Dedekind prime ring.*

Proof. It is enough to show that if R is such a ring, then R has no idempotent maximal ideals. Suppose that M is an idempotent maximal ideal of R. Form the ring Q_f constructed earlier in the section. Q_f is hereditary and in particular Noetherian. MQ_f was also shown to be the Jacobson radical of Q_f; hence by Nakyama's lemma MQ_f is not idempotent. But if M is idempotent, then $(MQ_f)^2 = (MQ_f)(MQ_f) = M^2Q_f = MQ_f$, a contradiction. Therefore R has no idempotent maximal ideals and R is a Dedekind prime ring. □

5. **Remarks.** In this section the results of the previous sections are compared with known results and related examples are given.

A ring is called pre-QF if every proper homomorphic image of it is quasi-Frobenius (QF). For commutative rings Levy has shown that a non-Artinian commutative pre-QF ring is a Dedekind domain. While it is true that every Dedekind prime ring is a pre-QF ring, Robson [14] has recently given an example of a hereditary Noetherian prime ring with precisely one two-sided ideal; hence even in the class of HNP rings a ring can be pre-QF without being Dedekind. The one and one-half generator property of §4 is a strengthening of the hypothesis of being pre-QF. Robson's example shows that the hypothesis of §4 cannot be weakened to pre-QF.

A module is called bounded if every element of it is annihilated by a nonzero two-sided ideal of the ring; a module is called completely faithful if every submodule of every factor module is faithful (see [4]). If R is an HNP with enough invertible ideals, then these concepts are equivalent to being b-torsion and c-torsion respectively. Hence if R is an HNP with enough invertible ideals, then Theorem 2.20 coincides with Theorem 3.9 of [4] which says that every torsion module is the direct sum of a bounded module and a completely faithful module; our proof, however, is different. The above example shows that this theorem cannot be proven for an arbitrary HNP ring; the argument is given below.

Let R be an HNP ring with a unique two-sided ideal M such that every torsion module over R splits into a direct sum of a completely faithful module and a bounded module. Note that the class of completely faithful modules forms a torsion class and that its associated filter $\mathcal{T}_{cf}$, of right ideals is given by $\mathcal{T}_{cf} = \{I \subset R: R/I$ is completely faithful$\} = \{I \subset R: R/I$ has no bounded composition factors$\}$. It easily follows from the assumed splitting property that $\mathcal{T}_{cf} = \{I \subset R: I + M = R\}$. Form the associated ring of quotients Q_{cf}. Then again the assumed splitting property shows that if I is a maximal right ideal of Q_{cf}, then $I^c = I \cap R$ contains M. Hence the intersection of all maximal right ideals of Q_{cf} has nonzero Jacobson radical J. R has only one two-sided ideal, M, and J^c is a two-sided ideal; therefore $J = MQ_{cf} = Q_{cf}M$. Then $J^2 = (MQ_{cf})(Q_{cf}M) = MQ_{cf}M = M^2Q_{cf} = MQ_{cf} = J$; but J is the Jacobson radical of a Noetherian ring and therefore cannot be idempotent by Nakyama's lemma. Hence we have a contradiction, and such a ring cannot enjoy such a splitting property.

REFERENCES

1. G. M. Bergman, *Commuting elements in free algebras and related topics in ring theory*, Thesis, Harvard University, Cambridge, Mass., 1967.

2. S. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. 121 (1966), 223–235. MR 33 #162.

3. D. Eisenbud and J. C. Robson, *Hereditary Noetherian prime rings*, J. Algebra 16 (1970), 86–104.

4. ———, *Modules over Dedekind prime rings*, J. Algebra 16 (1970), 67–85.

5. L. Fuchs, *Torsion preradicals and ascending Loewy series of modules*, J. Reine Angew. Math. 239 (1969), 169–179.

6. P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448. MR 38 #1144.

7. O. Goldman, *Rings and modules of quotients*, J. Algebra 13 (1969), 10–47. MR 39 #6914.

8. N. Jacobson, *The theory of rings*, Math. Surveys, vol. 1, Amer. Math. Soc., Providence, R. I., 1943. MR 5, 31.

9. A. V. Jategaonkar, Unpublished letter.

10. J. Kuzmanovich, *Localizations of Dedekind prime rings*, J. Algebra 21 (1972), 378–393.

11. J. Lambek, *Torsion theories, additive semantics, and rings of quotients*, Lecture Notes in Math., no. 177, Springer-Verlag, Heidelberg, 1971.

12. L. Levy, *Torsion-free and divisible modules over non-integral domains*, Canad. J. Math. 15 (1963), 132–151. MR 26 #155.

13. J. M. Maranda, *Injective structures*, Trans. Amer. Math. Soc. 110 (1964), 98–135. MR 29 #1236.

14. J. C. Robson, *Idealizers and hereditary Noetherian prime rings*, J. Algebra 22 (1972), 45–81.

15. ———, *Non-commutative Dedekind rings*, J. Algebra 9 (168), 249–265. MR 38 #177.

16. C. Walker and E. Walker, *Quotient categories and rings of quotients*, Rocky Mountain J. Math 2 (1972), 513–555.

17. D. B. Webber, *Ideals and modules of simple Noetherian hereditary rings*, J. Algebra 16 (1970), 239–242. MR 42 #305.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTHERN COLORADO, GREELEY, COLORADO 80631

Current address: Department of Mathematics, Wake Forest University, Winston-Salem, North Carolina 27109