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NORMED CONVEX PROCESSES[1]

BY

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ABSTRACT. We show that several well-known results about continuous linear operators on Banach spaces can be generalized to the wider class of convex processes, as defined by Rockafellar. In particular, the open mapping theorem and the standard bound for the norm of the inverse of a perturbed linear operator can be extended to convex processes. In the last part of the paper, these theorems are exploited to prove results about the stability of solution sets of certain operator inequalities and equations in Banach spaces. These results yield quantitative bounds for the displacement of the solution sets under perturbations in the operators and/or in the right-hand sides. They generalize the standard results on stability of unique solutions of linear operator equations.

1. **Introduction.** The idea of a convex process was introduced by Rockafellar ([9], [10]) in connection with general studies in convexity. If X and Y are real linear spaces, a *convex process* from X into Y is a mapping of points in X into subsets of Y, whose graph is a convex cone in $X \times Y$ containing the origin. If the graph is also closed, then we refer to a *closed convex process*. Here we are using the definition of graph as given in [10]: for a mapping T,

$$\text{graph } T := \{(x, y) \mid y \in Tx\}.$$

An equivalent way of stating the above definition is to say that a mapping T is a convex process if it satisfies the following three requirements:

(a) $T(x+z) \supset Tx + Tz$ for all $x, z \in X$.

(b) $T(\lambda x) = \lambda Tx$ for every $\lambda > 0$ and every $x \in X$.

(c) $0 \in T0$.

It is clear that any linear transformation (considered as a point-to-set mapping) is a convex process, but not vice versa. Just as with linear transformations, we can define the concepts of domain, range and inverse: for a convex process T, dom T is the set of points x for which $Tx \neq \emptyset$, range T is $\bigcup \{Tx \mid x \in \text{dom } T\}$, and T^{-1} is a mapping from range T onto dom T with $T^{-1}y := \{x \mid y \in Tx\}$. Note that dom T and range T are both convex cones containing 0, since they are the projections of graph T into X and Y respectively. Finally, if X and Y

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are normed, we can define the norm of T by

$$\|T\| := \sup\{\inf\{\|y\| \mid y \in Tx\} \mid \|x\| \le 1,\ x \in \operatorname{dom} T\}.$$

The above definitions are taken from [9] and [10], except that the definition of $\|T\|$ is changed slightly from that given in [9]; the change affects only the class of convex processes with domain $\{0\}$.

Note that there are some changes from the theory of linear operators: for one thing, every convex process has an inverse, and it is easy to see that the inverse is itself a convex process. On the other hand, any linear operator between finite-dimensional normed linear spaces has a finite norm, but this is no longer true for convex processes; an example of a closed convex process from $\mathbf{R}^2$ into $\mathbf{R}$ with infinite norm is given in §2.

We shall call a convex process *normed* if its norm is finite. In view of the example just cited, the question naturally arises: When is a convex process normed? Also, if a convex process is normed, when can we be sure that its inverse is also normed? Finally, if T and T^{-1} are normed, and if we perturb T slightly by adding to it another convex process of small norm, can anything be said about the norm of the inverse of the perturbed process; specifically, can that norm be bounded? These are questions that often arise in applications, and in the case of linear operators on Banach spaces they can be answered in a very satisfactory manner.

In this paper we show how these and other questions can be answered for convex processes; in fact, several of the well-known results from the theory of linear operators can be extended to convex processes in very nearly the same form.

We conclude this section by explaining some notational conventions that we shall use in what follows. All linear spaces from this point on will be assumed to be over the real field. If two convex processes, say S and T, are defined from a linear space X into another linear space Y, then their sum, $S + T$, is the mapping defined by $(S + T)(x) := Sx + Tx$. If λ is a real number, then the mapping λT is defined by $(\lambda T)(x) := \lambda(Tx)$. Both of these mappings are convex processes, and if S and T are normed, we have $\|S + T\| \le \|S\| + \|T\|$ and $\|\lambda T\| = |\lambda|\,\|T\|$. One can also define the composition of two convex processes in the obvious way, and show that $\|UT\| \le \|U\|\,\|T\|$. The proofs of these results are omitted; they follow from the important fact that if T is any convex process and, if $x \in \operatorname{dom} T$, then for any $\epsilon > 0$ there is (by the definition of $\|T\|$) some $y \in Tx$ with $\|y\| < \|T\|\,\|x\| + \epsilon$. Here and in what follows we are using the convention $(+\infty) \cdot 0 = 0 = 0 \cdot (+\infty)$.

2. **Characterization of normed convex processes.** In this section we first give an example, mentioned in the introduction, of a closed convex process with infinite norm; we then show that the class of convex processes having finite norms can be characterized in terms of two other equivalent topological properties.

The example is as follows: let T be the convex process from $\mathbf{R}_+$ (the non-negative real numbers) into $\mathbf{R}^2$ given by

$$Tx = \begin{cases} \{(y,z) \mid y^2 \le zx \text{ and } 0 \le z\} & \text{for } x \ge 0, \\ \varnothing \quad \text{for } x < 0. \end{cases}$$

For each $x > 0$, the image Tx is the area in the yz-plane on or above the parabola $y^2 = zx$; $T0$ is the nonnegative z-axis. It is readily verified that this is a closed convex process with norm 0. However, the inverse process is given by

$$T^{-1}(y,z) = \begin{cases} \{x \mid x \ge y^2/z\} & \text{for } z > 0 \text{ and any } y, \\ \mathbf{R}_+ & \text{for } z = 0,\ y = 0, \\ \varnothing & \text{otherwise}, \end{cases}$$

and since for the pair $(1, 1/n)$ we have $T^{-1}(1, 1/n) = \{x \mid x \ge n\}$, it is clear that the norm of T^{-1} must be $+\infty$.

Before stating the characterization theorem, we mention some topological preliminaries. If X and Y are topological vector spaces with $X_0 \subset X$, and if T is a mapping from X into Y, we say that T is *lower semicontinuous at* $x_0 \in X_0$ *as a mapping from* X_0 *to* Y, if for each open set $Q \subset Y$ with $Q \cap Tx_0 \neq \varnothing$, there is an open neighborhood U of x_0 in X_0 such that, for every $x \in U$, $Q \cap Tx \neq \varnothing$ (see, e.g., [2]). We say that T is *open at* 0 if the image under T of any open neighborhood of 0 in X contains an open neighborhood of 0 in range T. When we speak of a neighborhood in a set we are, as usual, referring to the relative topology on that set.

Theorem 1. *Let X and Y be normed linear spaces, and let T be a convex process from X into Y. Then the following three properties are equivalent:*

(a) *T has a finite norm.*

(b) *T is lower semicontinuous at 0 as a mapping from* dom T *into Y.*

(c) *T^{-1} is open at 0.*

Proof. (a) $\Rightarrow$ (c). Denote the open ball of radius $\epsilon > 0$ about $x \in X$ by $B(x, \epsilon)$, and let $C(y, \epsilon)$ be a similar ball about $y \in Y$. Let $D(x, \epsilon) := B(x, \epsilon) \cap$ dom T and $R(y, \epsilon) := C(y, \epsilon) \cap$ range T. Let $V(0)$ be any open neighborhood of 0 in Y; then $T^{-1}[V(0)] = T^{-1}[V(0) \cap \text{range } T]$. Pick some $\eta > 0$ such that $C(0, \eta) \subset V(0)$; then $R(0, \eta) \subset V(0) \cap \text{range}\, T$. Let $\epsilon > 0$ be so small that $\|T\|\epsilon < \eta/2$. Pick any $x \in D(0, \epsilon)$; then there is some $y \in Tx$ with $\|y\| < \|T\|\epsilon + \eta/2 < \eta$, so $y \in R(0, \eta)$. Since x was arbitrary, it follows that $D(0, \epsilon) \subset T^{-1}[R(0, \eta)] \subset T^{-1}[V(0)]$, so T^{-1} is open at 0.

(c) $\Rightarrow$ (b). Suppose T^{-1} is open at 0; let $Q \subset Y$ be open with $Q \cap T0 \neq \emptyset$. We have to find an $\epsilon > 0$ such that, for each $x \in D(0, \epsilon)$, we have $Q \cap Tx \neq \emptyset$. Let $q \in Q \cap T0$ and suppose $C(q, \delta) \subset Q$ with $\delta > 0$. By the assumption, there is an $\epsilon > 0$ such that $T^{-1}[R(0, \delta)] \supset D(0, \epsilon)$. But then for any $x \in D(0, \epsilon)$ there is a $y \in Tx \cap R(0, \delta)$; then $Tx = T(0 + x) \supset T0 + Tx \ni q + y$, and $q + y \in C(q, \delta) \subset Q$. Hence $q + y \in Q \cap Tx \neq \emptyset$, and so T is lower semicontinuous at 0 as a mapping from dom T into Y.

(b) $\Rightarrow$ (a). Suppose T does not have a finite norm. Then we can find some sequence $\{x_n\} \subset$ dom T with $\|x_n\| \leq 1$ and $\|y\| \geq n$ for all $y \in Tx_n$ and for $n = 1, 2, \cdots$. None of the x_n can be zero (since $0 \in T0$), so we can define a new sequence $\{z_n\} \subset$ dom T by setting $z_n := x_n/(n\|x_n\|)$ for each n. It is clear that $\|z_n\| = 1/n$ and $\|y\| \geq 1$ for all $y \in Tz_n$ and all n. Since $0 \in T0$, we have $C(0, 1) \cap T0 \neq \emptyset$; however, $C(0, 1) \cap Tz_n = \emptyset$ for each n. Since $\{z_n\}$ converges to zero, it follows that T is not lower semicontinuous at 0 as a mapping from dom T into Y. This completes the proof.

3. **Sufficient conditions for a finite norm.** In §2 we found necessary and sufficient conditions for a convex process to have a finite norm. However, these conditions were stated in terms of topological properties of T which will frequently be just as hard to verify as will be the existence of a norm. In this section we develop some sufficient conditions of a simpler kind, which involve various properties of dom T.

Theorem 2 (**generalized open mapping principle**). *Let X and Y be Banach spaces, and let T be a closed convex process from X onto Y. Then the image under T of any open set in X is an open set in Y.*

Proof. Let the neighborhoods B, C, and D be defined as in the proof of Theorem 1. We shall first show that T is open at 0; the conclusion of the theorem then follows easily.

We have, since T is onto Y,

$$Y = T(\text{dom } T) = T\left[\bigcup_{n=1}^{\infty} D(0, n)\right] = \bigcup_{n=1}^{\infty} T[D(0, n)],$$

so by the Baire category theorem [5] there is some N such that $\overline{T[D(0, N)]}$ contains an open ball in Y, say $C(p, \eta)$. By assumption there is some $x \in$ dom T with $-p \in Tx$. For any $y \in C(0, \eta)$ and any $\epsilon > 0$, we can find some $x' \in D(0, N)$ and $z \in Tx'$ such that $\|(p + y) - z\| < \epsilon$; then $z - p \in Tx' + Tx \subset T(x' + x) \subset T[D(0, N + \|x\|)]$ and $\|y - (z - p)\| = \|(p + y) - z\| < \epsilon$. Thus

$C(0, \eta) \subset \overline{T[D(0, N + \|x\|)]}$, and if we define $\delta := \eta/(N + \|x\|)$ it follows from the homogeneity of T that $C(0, \delta) \subset \overline{T[D(0, 1)]}$, and in fact that $C(0, 2^{-k}\delta) \subset \overline{T[D(0, 2^{-k})]}$ for $k = 0, 1, \cdots$. Choose an arbitrary $\bar{y} \in C(0, \delta/2)$; then by the last observation we can find some $x_1 \in D(0, \frac{1}{2})$ and $y_1 \in Tx_1$ such that $\|\bar{y} - y_1\| < \delta/4$. Suppose that for some $k \geq 1$ we have $x_1, \cdots, x_k$ and $y_1, \cdots, y_k$ with $x_j \in D(0, 2^{-j})$ and $y_j \in Tx_j$ for each j, and with $\|\bar{y} - \sum_{j=1}^{k} y_j\| < 2^{-(k+1)}\delta$. Then we can find an $x_{k+1} \in D(0, 2^{-(k+1)})$ and $y_{k+1} \in Tx_{k+1}$ with

$$\left\| \left(\bar{y} - \sum_{j=1}^{k} y_j \right) - y_{k+1} \right\| = \left\| \bar{y} - \sum_{j=1}^{k+1} y_j \right\| < 2^{-(k+2)}\delta.$$

Hence, by induction we can construct sequences $\{x_j\}$ and $\{y_j\}$ having the stated properties for each j. Let $w_k := \sum_{j=1}^{k} x_j$ and $z_k := \sum_{j=1}^{k} y_j$ for $k = 1, 2, \cdots$. It is easily seen that $\{w_k\}$ is a Cauchy sequence and therefore must converge to some $\bar{x}$ (since X is complete). Also, by construction, $\{z_k\}$ converges to $\bar{y}$. We have, for each k,

$$Tw_k = T\left(\sum_{j=1}^{k} x_j \right) \supset \sum_{j=1}^{k} Tx_j \ni \sum_{j=1}^{k} y_j = z_k,$$

so the pair (w_k, z_k) belongs to the graph of T. Since T was assumed to be a closed mapping, it follows that $(\bar{x}, \bar{y})$ also belongs to graph T, or in other words, that $\bar{y} \in T\bar{x}$. Thus $\bar{x} \in \operatorname{dom} T$, and since for each j, $\|x_j\| < 2^{-j}$, we must have $\|\bar{x}\| < \sum_{j=1}^{\infty} 2^{-j} = 1$. Hence $\bar{x} \in D(0, 1)$, and since $\bar{y}$ was an arbitrary element of $C(0, \delta/2)$ we have shown that $T[D(0, 1)] \supset C(0, \delta/2)$; therefore T is open at 0.

Now let Q be any open set in X. Let y be any point of $T(Q)$, and let $x \in Q$ be such that $y \in Tx$. Choose $\epsilon > 0$ so that $B(x, \epsilon) \subset Q$. Then

$$T(Q) \supset T[B(x, \epsilon)] = T[x + B(0, \epsilon)] \supset Tx + T[B(0, \epsilon)]$$

$$= Tx + T[D(0, \epsilon)] \supset y + C(0, \delta\epsilon/2) = C(y, \delta\epsilon/2),$$

so $T(Q)$ must be open. This completes the proof.

Corollary (generalized closed graph theorem). *Lex X and Y be Banach spaces, and let T be a closed convex process from X into Y. If* dom $T = X$, *then T has a finite norm.*

Proof. The convex process T^{-1} takes Y onto X, and is closed since its graph is a reorientation of that of T. Applying Theorem 2 to T^{-1}, we conclude that T^{-1} is open at 0; it follows from Theorem 1 that T must then have a finite norm.

It is *not* true that if dom $T = X$ and T has a finite norm, then graph T is closed. For example, the convex process from $\mathbf{R}$ into $\mathbf{R}^2$ given by $Tx := \{(y, z) \mid y > 0, z > 0\} \cup \{(0, 0)\}$ for each x has domain $\mathbf{R}$ and norm equal to zero, but its graph is not closed.

The following theorem is often useful in dealing with systems of linear equations and inequalities in finite-dimensional spaces.

Theorem 3. *Let X and Y be normed linear spaces, and let T be a convex process from X into Y. If* dom T *is the sum of a finite number of half-lines, then T has a finite norm.*

Proof. Let dom T be the sum of n half-lines. If $n = 0$, then dom $T = \{0\}$ and it is easily seen that $\|T\| = 0$. Suppose then that $n > 0$, that there exist vectors $x_1, \cdots, x_n$ with $\|x_j\| = 1$ for each j, and that any $x \in \text{dom } T$ is representable in the form $x = \Sigma_{j=1}^n \lambda_j x_j$ with each λ_j nonnegative. Then dom T lies in the subspace V generated by $x_1, \cdots, x_n$, and this subspace has dimension no higher than n. The restriction of the norm $\|\ \|$ of X to V is a norm on V. Define a function $f(x)$ on dom T by $f(x) := \inf\{\|y\| \mid y \in Tx\}$. It is clear that $f(x)$ has a finite value at each point of dom T; in fact, as we shall see in the following argument, it is even convex there. Let x_1 and x_2 be any two points of dom T; let $\epsilon > 0$ be arbitrary, and pick $y_1 \in Tx_1$ and $y_2 \in Tx_2$ with $\|y_1\| < f(x_1) + \epsilon$ and $y_2 < f(x_2) + \epsilon$. For any $\lambda \in [0, 1]$ we have $\lambda y_1 + (1 - \lambda)y_2 \in \lambda Tx_1 + (1 - \lambda)Tx_2 \subset T[\lambda x_1 + (1 - \lambda)x_2]$, so $f[\lambda x_1 + (1 - \lambda)x_2] \leq \|\lambda y_1 + (1 - \lambda)y_2\| \leq \lambda\|y_1\| + (1 - \lambda)\|y_2\| < \lambda[f(x_1) + \epsilon] + (1 - \lambda)[f(x_2) + \epsilon] = \lambda f(x_1) + (1 - \lambda)f(x_2) + \epsilon$. Since ϵ was arbitrary, it follows that $f(x)$ is convex on dom T.

Let $\|\ \|_p$ be any polyhedral norm on V (that is, any norm whose closed unit ball $\overline{B}_p$ is a polyhedron). Since dom T, being the sum of a finite number of half-lines, is a polyhedral convex cone [3], the intersection $\overline{B}_p \cap \text{dom } T$ will be a polyhedron (nonempty, since $0 \in \overline{B}_p \cap \text{dom } T$). Therefore $f(x)$, being convex, must attain its maximum at one of the extreme points of $\overline{B}_p \cap \text{dom } T$, so the quantity

$$\sup\{\inf\{\|y\| \mid y \in Tx\} \mid \|x\|_p \leq 1,\ x \in \text{dom } T\}$$
$$= \sup\{f(x) \mid x \in \overline{B}_p \cap \text{dom } T\}$$

is finite. However, since V is of finite dimension the norms $\|\ \|$ and $\|\ \|_p$ are equivalent on V [2], and thus

$$\|T\| := \sup\{\inf\{\|y\| \mid y \in Tx\} \mid \|x\| \leq 1,\ x \in \text{dom } T\}$$

is also finite. This completes the proof.

The next theorem is a partial converse to Theorem 3. In the theorem, we shall speak of an *extreme half-line* in a cone; this term is to be understood to mean a half-line (from 0) in the cone which is not the sum of any two distinct half-lines in the cone.

Theorem 4. *Let X be a normed linear space and let K be a convex cone containing the origin in X. If K contains an infinite number of extreme half-lines, then there is a convex process having domain K whose norm is $+\infty$.*

Proof. Let $\{L_n\}$ be a sequence of extreme half-lines in K. Define a function $g(x)$ on K as follows: if for some n, $x \in L_n$, then $g(x) := n\|x\|$; otherwise $g(x) := 0$. Let p_1 and p_2 be any two distinct points in K. If p_1, p_2 and 0 are not collinear, then the "open" line segment (p_1, p_2) cannot contain any point lying on an extreme half-line, so $g(x)$ is zero on the entire segment and hence convex on the closed segment $[p_1, p_2]$. If p_1, p_2 and 0 are collinear, then by enumeration of cases $g(x)$ is easily seen to be convex on $[p_1, p_2]$. Thus $g(x)$ is a convex function on the cone K.

Now for $x \in K$ define $Tx := \{\lambda \in \mathbf{R} \mid \lambda \geq g(x)\}$, or in the notation of [9], $Tx := g(x)\check{}$. Since g is positively homogeneous and convex, with $g(0) = 0$, the epigraph [10] of g is a convex cone containing the origin; but this is also the graph of T, so T is a convex process with dom $T = K$. Consider the points x_n defined by $x_n \in L_n$ and $\|x_n\| = 1$. For each n we have $\inf\{\|y\| \mid y \in Tx_n\} = n$, so it follows that $\|T\| = +\infty$, as was to have been shown.

4. **Perturbation of a convex process.** In this section we shall obtain bounds for the norm of the inverse of a perturbed convex process in terms of the norm of the inverse of the unperturbed process and the norm of the perturbing process, the latter being assumed to be small. These bounds will be applied in the following section to develop a stability theory for certain operator inequalities and equations. The results we shall obtain here generalize the well-known norm bounds for perturbations of a nonsingular linear operator mapping a Banach space into itself.

If T is a convex process and K is a convex cone containing the origin, then we shall denote by T_K the *restriction of T to K*; that is, the convex process defined by

$$T_K x := \begin{cases} Tx, & x \in K, \\ \emptyset, & x \notin K. \end{cases}$$

Theorem 5. *Let X be a Banach space and Y be a normed linear space. Let T and Δ be convex processes from X into Y; denote* dom T *by K and* range T *by R. Assume that T, T^{-1}, and Δ are normed, and that $\|T^{-1}\|\,\|\Delta\| < 1$. Suppose further that $K \subset$ dom Δ, $\Delta(K) \subset R$, K is closed, and $(T - \Delta)(x)$ is closed for each $x \in K$.*

Then the convex process $T - \Delta$ has the following properties:

(a) range $T \subset$ range $(T - \Delta)$.

(b) $(T - \Delta)_R^{-1}$ *is a normed convex process, and* $\|(T - \Delta)_R^{-1}\| \leq \|T^{-1}\|/(1 - \|T^{-1}\|\,\|\Delta\|)$.

Proof. Let $\tau := \|T^{-1}\|$ and $\delta := \|\Delta\|$. Let $\bar{y} \in \text{range } T$ and $\epsilon > 0$ be chosen arbitrarily, and let θ be any positive real number with $\tau\delta \leq \theta < 1$. We shall construct a Cauchy sequence $\{x_k\}$ converging to a vector $\bar{x} \in K$ with the property that $\bar{y} \in (T - \Delta)\bar{x}$, showing that range $T \subset \text{range}(T - \Delta)$, and with $\|\bar{x}\| \leq \tau(1-\theta)^{-1}\|\bar{y}\| + \epsilon(1-\theta)^{-1}$. It will follow, upon letting $\epsilon \downarrow 0$ and $\theta \downarrow \tau\delta$, that $\|(T-\Delta)_R^{-1}\| \leq \tau/(1-\tau\delta)$. The number θ is introduced in order to deal with the exceptional case in which $\tau\delta = 0$.

To construct the sequence $\{x_k\}$ we proceed inductively, beginning with the choice $x_0 = 0$. Next, using the fact that $\bar{y} \in \text{range } T$ and the definition of τ, we choose $x_1 \in K$ such that $\bar{y} \in Tx_1$ and $\|x_1\| \leq \tau\|\bar{y}\| + \epsilon/2$. Then the following three statements hold:

(1′) $x_1 - x_0 \in K$.

(2′) $\|x_1 - x_0\| \leq \tau\theta^0\|\bar{y}\| + \epsilon(1 - 2^{-1})\theta^0$.

(3′) $(Tx_1 - \bar{y}) \cap \Delta x_0 \neq \varnothing$ (it contains 0).

Now let $k \geq 1$, and suppose that x_{k-1} and x_k are given with x_{k-1} and x_k in K and the following conditions satisfied:

(1) $x_k - x_{k-1} \in K$.

(2) $\|x_k - x_{k-1}\| \leq \tau\theta^{k-1}\|\bar{y}\| + \epsilon(1 - 2^{-k})\theta^{k-1}$.

(3) $(Tx_k - \bar{y}) \cap \Delta x_{k-1} \neq \varnothing$.

Let η_1 and η_2 be positive real numbers with the property that $\tau\eta_1 + \eta_2 = \theta^k \epsilon/2^{k+1}$. Since $x_k - x_{k-1} \in K$, the set $\Delta(x_k - x_{k-1})$ is nonempty; let z be a member of this set with $\|z\| \leq \delta\|x_k - x_{k-1}\| + \eta_1$. Next select $w \in K$ such that $z \in Tw$ and $\|w\| \leq \tau\|z\| + \eta_2$. Let $x_{k+1} := x_k + w$. Clearly $x_{k+1} - x_k \in K$, and we have, using the second induction hypothesis and the definition of η_1 and η_2,

$$\begin{aligned}\|x_{k+1} - x_k\| &\leq \tau\|z\| + \eta_2 \leq \tau\delta\|x_k - x_{k-1}\| + \tau\eta_1 + \eta_2 \\ &\leq \theta\|x_k - x_{k-1}\| + \theta^k\epsilon/2^{k+1} \\ &\leq \tau\theta^k\|\bar{y}\| + \epsilon(1 - 2^{-k})\theta^k + \epsilon 2^{-(k+1)}\theta^k = \tau\theta^k\|\bar{y}\| + \epsilon(1 - 2^{-(k+1)})\theta^k.\end{aligned}$$

Finally, let p be any member of the set $(Tx_k - \bar{y}) \cap \Delta x_{k-1}$, which was assumed to be nonempty. Then $p + z \in \Delta x_{k-1} + \Delta(x_k - x_{k-1}) \subset \Delta x_k$, and, since $w = x_{k+1} - x_k$, $p + z \in (Tx_k - \bar{y}) + T(x_{k+1} - x_k) \subset Tx_{k+1} - \bar{y}$, so that the set $(Tx_{k+1} - \bar{y}) \cap \Delta x_k$ is nonempty. Thus, by induction, the properties (1), (2), and (3) must hold for $k = 0, 1, 2, \cdots$. We therefore have, for $m \geq 1$,

$$\begin{aligned}\|x_{k+m} - x_k\| &\leq \sum_{j=0}^{m-1} \|x_{k+j+1} - x_{k+j}\| \leq \sum_{j=0}^{m-1} [\tau\|\bar{y}\| + \epsilon(1 - 2^{-(k+j+1)})]\theta^{k+j} \\ &\leq (\tau\|\bar{y}\| + \epsilon)\sum_{j=0}^{m-1} \theta^{k+j} = \theta^k(\tau\|\bar{y}\| + \epsilon)(1-\theta^m)/(1-\theta),\end{aligned} \tag{1}$$

and the latter quantity converges to zero as $k \to \infty$, regardless of m. Thus $\{x^k\}$ is a Cauchy sequence. For each k, $x_k = x_0 + \sum_{j=0}^{k-1}(x_{j+1} - x_j)$, a finite sum of terms in K; hence $\{x_k\} \subset K$. Since X is a Banach space and K was assumed to be closed, the sequence $\{x_k\}$ converges to some $\bar{x} \in K$.

Now choose an arbitrary $\delta > 0$, and let k be so large that we have

$$\max [(\|T\| + \|\Delta\|)\|\bar{x} - x_k\|, \|\Delta\| \|x_k - x_{k-1}\|] < \delta/4.$$

Since $\bar{x} - x_k$ is the limit of the sequence $\{x_{k+j} - x_k\}$ as $j \to \infty$, and since for each j, $x_{k+j} - x_k = \sum_{i=0}^{j-1}(x_{k+i+1} - x_{k+i}) \in K$, the point $\bar{x} - x_k$ lies in K. Therefore

$$\begin{aligned}(T - \Delta)(\bar{x}) - \bar{y} &\supset (T - \Delta)(\bar{x} - x_k) + (T - \Delta)(x_k) - \bar{y}\\ &\supset (T - \Delta)(\bar{x} - x_k) + \{[Tx_k - \bar{y}] - \Delta x_{k-1}\} - \Delta(x_k - x_{k-1})\\ &\supset (T - \Delta)(\bar{x} - x_k) - \Delta(x_k - x_{k-1}),\end{aligned}$$

the last inclusion following since $0 \in (Tx_k - \bar{y}) - \Delta x_{k-1}$ by property (3) of the induction. As noted in §1, we have $\|T - \Delta\| \le \|T\| + \|\Delta\|$, so we can select $z_1 \in (T - \Delta)(\bar{x} - x_k)$ with $\|z_1\| < (\|T\| + \|\Delta\|)\|\bar{x} - x_k\| + \delta/4 < \delta/2$, and $z_2 \in -\Delta(x_k - x_{k-1})$ with $\|z_2\| < \|\Delta\| \|x_k - x_{k-1}\| + \delta/4 < \delta/2$. Therefore $z_1 + z_2 \in (T - \Delta)(\bar{x}) - \bar{y}$ with $\|z_1 + z_2\| < \delta$, but since δ was arbitrary and $(T - \Delta)\bar{x}$ was assumed to be closed, we must have $\bar{y} \in (T - \Delta)\bar{x}$. Since $\bar{y}$ was an arbitrary element of range T, we have range $T \subset$ range $(T - \Delta)$.

Taking $k = 0$ and letting $m \to \infty$ in (1), we obtain $\|\bar{x}\| = \|\bar{x} - x_0\| \le (\tau\|\bar{y}\| + \epsilon)/(1 - \theta)$. Since ϵ was arbitrary, we have $\inf\{\|x\| \mid \bar{y} \in (T - \Delta)x\} \le [\tau/(1 - \theta)]\|\bar{y}\|$, and since $\bar{y}$ was any element of range T, we see that $\|(T - \Delta)_R^{-1}\| \le \tau/(1 - \theta)$. Letting $\theta \downarrow t\delta$, we obtain $\|(T - \Delta)_R^{-1}\| \le \tau/(1 - \tau\delta)$, as was to have been shown.

It is not difficult to see that the conclusion of this theorem fails if the process $(T - \Delta)^{-1}$ is not restricted to range T. For example, if $X = Y = \mathbf{R}^2$ with the l_∞ norm, and if we set

$$Tx := \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}x\right\} \quad \text{and} \quad \Delta x := \left\{\begin{bmatrix}.5 & 0\\ 0 & -.1\end{bmatrix}x\right\}$$

for every $x \in \mathbf{R}^2$, then $\|T\| = \|T^{-1}\| = 1$, $\|\Delta\| = ½$, but $\|(T - \Delta)^{-1}\| = 10$. However, $\|(T - \Delta)_R^{-1}\| = 2$, as stated in the theorem.

For the case in which T and Δ are continuous linear operators from a Banach space into itself with T and $T - \Delta$ invertible, we can obtain also a lower bound for $\|(T - \Delta)^{-1}\|$, namely

$$\|(T-\Delta)^{-1}\| \geq \|T^{-1}\|/(1+\|T^{-1}\|\,\|\Delta\|). \tag{2}$$

However, this inequality is generally false for convex processes. For example, let $X = Y = \mathbf{R}$, and let T be the identity mapping on $\mathbf{R}$. Let Δ be defined by $\Delta x := \mathbf{R}$ for each $x \in \mathbf{R}$. Then $\|T\| = \|T^{-1}\| = 1$, but $\|(T-\Delta)^{-1}\| = 0$, so the inequality (2) does not hold. In order to be able to prove that (2) is valid, we require additional conditions. We have followed Rockafellar [10] in denoting by 0^+S the *recession cone* of a convex set S; that is, the cone made up of all points x with the property that $x + S \subset S$. Intuitively, this is the "set of directions in which S is unbounded."

To establish (2), we first note that if $\|(T-\Delta)^{-1})\| = +\infty$ there is nothing to prove, and that if $\|(T-\Delta)^{-1}\|\,\|\Delta\| \geq 1$ the inequality follows from

$$\|(T-\Delta)^{-1}\| \geq \|T^{-1}\|(1-\|\Delta\|\,\|(T-\Delta)^{-1}\|), \tag{3}$$

which is equivalent to (2). We need therefore be concerned only with the case in which $\|(T-\Delta)^{-1}\|\,\|\Delta\| < 1$. If we now assume that for each $x \in \operatorname{dom} T \cap \operatorname{dom} \Delta$, we have

$$(\Delta - \Delta)x \subset 0^+ Tx, \tag{4}$$

it follows that, for each such x, $(T-\Delta)x - (-\Delta)x \subset Tx$; but since $0 \in (\Delta - \Delta)x$, the reverse inclusion is trivial. Therefore, we have $(T-\Delta) - (-\Delta) = T$, and now by making the assumptions necessary to apply Theorem 5 to $T - \Delta$ and $-\Delta$, and by assuming that $T - \Delta$ is onto, we can establish (3), from which (2) follows.

We remark that (4) is always satisfied when Δ is a single-valued function.

5. **Action of a convex process on sets.** In this section we examine the following question: if two sets P and Q are "close" to each other (in a sense to be made precise), and if these sets are mapped by a convex process T into sets TP and TQ, then how "close" to each other are TP and TQ?

We first consider a measure of the distance between two sets. If Z is a normed linear space, we define for any $z \in Z$ and any nonempty subset $A \subset Z$,

$$d(z, A) := \inf\{\|z - a\| \mid a \in A\}.$$

For any nonempty set $B \subset Z$, define

$$d(B, A) := \sup\{d(b, A) \mid b \in B\}.$$

If we now let

$$\rho(A, B) := \max\{d(A, B), d(B, A)\},$$

then ρ is a generalized pseudometric on the family of nonempty subsets of Z;

that is, a pseudometric [6] which may assume the value $+\infty$. If A and B are required to be closed, then ρ is the Hausdorff metric [2].

Theorem 6. *Let X and Y be normed linear spaces. Let P and Q be nonempty subsets of X, and let T be a normed convex process from X into Y, with TP and TQ nonempty.*

(a) *If $Q - P \subset \operatorname{dom} T$, then $d(TP, TQ) \leq \|T\| d(P, Q)$.*

(b) *If $(Q - P) \cup (P - Q) \subset \operatorname{dom} T$, then $\rho(TP, TQ) \leq \|T\| \rho(P, Q)$.*

Proof. We shall prove (a); (b) then follows by symmetry. Choose $\epsilon > 0$, and let y be any point in TP. Let $p \in P$ be such that $y \in Tp$, and find some $q \in Q$ with $\|T\| \, \|p - q\| < \|T\| d(P, Q) + \epsilon/2$. Since $Q - P \subset \operatorname{dom} T$, there is some $y' \in T(q - p)$ with $\|y'\| < \|T\| \, \|q - p\| + \epsilon/2$. Then

$$Tq - y \supset T(q - p) + Tp - y \ni y' + y - y = y'$$

and

$$\|y'\| < \|T\| \, \|q - p\| + \epsilon/2 < \|T\| \, d(P, Q) + \epsilon.$$

Therefore,

$$d(y, TQ) := \inf\{\|z - y\| \mid z \in TQ\}$$
$$= \inf\{\|w\| \mid w \in TQ - y\} < \|T\| \, d(P, Q) + \epsilon,$$

and, since y was any element of TP, we have $d(TP, TQ) < \|T\| d(P, Q) + \epsilon$. But ϵ was arbitrary, so the result follows. This completes the proof.

If the conditions on the difference of Q and P are not satisfied, then the conclusions of Theorem 6 can fail to hold. This can be seen by considering the following example: define a convex process T from $\mathbf{R}^2$ (with the l_∞ norm) into $\mathbf{R}$ by

$$T(x, y) := \begin{cases} \{z \mid z \geq 0\} & \text{for } y > 0, \\ \{z \mid z \geq |x|\} & \text{for } y = 0, \\ \varnothing & \text{for } y < 0. \end{cases}$$

It is easy to verify that $\|T\| = 1$. However, for any $\epsilon > 0$ we have $d[T(1, \epsilon), T(1, 0)] = 1$, although $d[(1, \epsilon), (1, 0)] = \epsilon$, so that the conclusions of Theorem 6 do not hold. In this case, the difficulty arises from the fact that $(0, -\epsilon)$ is not in $\operatorname{dom} T$ for $\epsilon > 0$.

6. **Stability of solution sets of certain systems.** Suppose that we are given two Banach spaces X and Y, and that Y contains a nonempty closed convex cone K. Let f be a function from a nonempty closed convex cone $\operatorname{dom} f \subset X$ into Y. Suppose f is closed, convex with respect to K, and positively homogeneous of degree 1; that is, the epigraph of f, $\operatorname{epi} f := \bigcup \{(x, f(x) + K) \mid x \in \operatorname{dom} f\}$,

is a closed convex cone containing the origin in $X \times Y$. Suppose further that the *bound* of f, $\|f\| := \sup\{\|f(x)\| \mid \|x\| \le 1,\ x \in \operatorname{dom} f\}$, is finite. Let δf be another function from dom f into Y satisfying the same assumptions as f. Let b and δb be points in Y. We shall consider the following two questions:

(a) Suppose $\bar{x}$ is a solution of

$$f(x) \in b - K. \tag{5}$$

Is there any $\hat{x} \in \operatorname{dom} f$ such that $\hat{x}$ solves

$$f(x) + \delta f(x) \in b + \delta b - K, \tag{6}$$

and, if so, how small can $\|\bar{x} - \hat{x}\|$ be made; i.e., how close to $\bar{x}$ is there a solution of (6)?

(b) Suppose $\hat{x}$ solves (6). If (5) is known to be solvable, how close is $\hat{x}$ to the set of solutions of (5)?

The expressions in (5) and (6) are general enough to include a number of problems of importance in applications. One example would be a mixed system of linear inequalities and equations.

Question (a) arises in such areas as the analysis of convergence for iterative processes (see, e.g., [8]). Question (b) formulates the fundamental problem of determining error bounds for the approximate solution of certain systems (including linear systems). We shall see that both questions can be answered in a convenient way by applying Theorem 6 and some of our previous results.

Theorem 7. *Let X and Y be Banach spaces, and let K, f, δf, b and δb be as previously defined. For $x \in \operatorname{dom} f$, define*

$$Tx := f(x) + K \quad \textit{and} \quad (\delta T)x := \delta f(x) + K.$$

Suppose T carries dom f *onto Y. Then T, T^{-1}, and δT are normed convex processes, and we have*

(a) *If $\hat{x}$ solves (6), then*

$$\inf\{\|\hat{x} - x\| \mid x \text{ solves } (5)\} \le \|T^{-1}\|(\|\delta b\| + \|\delta f\|\,\|\hat{x}\|).$$

(b) *If $\bar{x}$ solves (5) and $\|\delta f\|\,\|T^{-1}\| < 1$, then (6) is solvable, $T + \delta T$ is a normed convex process whose domain is all of Y, $\|(T + \delta T)^{-1}\| \le \|T^{-1}\|/(1 - \|\delta f\|\,\|T^{-1}\|)$, and*

$$\inf\{\|\bar{x} - x\| \mid x \text{ solves } (6)\} \le \|T^{-1}\|(\|\delta b\| + \|\delta f\|\,\|\bar{x}\|)/(1 - \|\delta f\|\,\|T^{-1}\|).$$

Proof. Since the graphs of T and δT are the epigraphs of f and δf respectively, it follows immediately from the hypotheses that T, T^{-1} and δT are closed convex processes, and that $(T + \delta T)x$ is closed for each x. We have dom T = dom δT = dom f. Since $\|T\| \le \|f\|$ and $\|\delta T\| \le \|\delta f\|$, and since both of these quantities were assumed to be finite, T and δT are normed. We required T to be onto Y, so by the corollary to Theorem 2, T^{-1} is normed.

For part (a), we rewrite (6) with $x = \hat{x}$ to obtain $b + \delta b - \delta f(\hat{x}) \in f(\hat{x}) + K =: T(\hat{x})$ or $\hat{x} \in T^{-1}[b + \delta b - \delta f(\hat{x})]$. Since the set of points solving (5) is $T^{-1}(b)$, we may apply Theorem 6 to obtain

$$\inf\{\|\hat{x} - x\| \mid x \text{ solves } (5)\} = d(\hat{x}, T^{-1}[b]) \le d(T^{-1}[b + \delta b - \delta f(\hat{x})], T^{-1}[b])$$
$$\le \|T^{-1}\| \|\delta b - \delta f(\hat{x})\| \le \|T^{-1}\|(\|\delta b\| + \|\delta f\| \|\hat{x}\|).$$

which proves (a).

To prove part (b), we shall apply Theorem 5 to the convex process $T + \delta T = T - (-\delta T)$. We have $\|T^{-1}\| \|-\delta T\| \le \|T^{-1}\| \|\delta f\| < 1$ by hypothesis, and $-\delta T(\text{dom } f) \subset Y = \text{range } T$. The other hypotheses of Theorem 5 are satisfied by our previous remarks. We therefore conclude that $(T + \delta T)^{-1}$ is a normed convex process whose domain is all of Y, so that (6) is solvable for any δb. Also, $\|(T + \delta T)^{-1}\| \le \|T^{-1}\|/(1 - \|T^{-1}\| \|\delta f\|)$; no restriction of $(T + \delta T)^{-1}$ is necessary, since T was onto Y.

Rewriting (5) with $x = \bar{x}$, we obtain $\delta f(\bar{x}) + b \in f(\bar{x}) + \delta f(\bar{x}) + K = (T + \delta T)\bar{x}$ or $\bar{x} \in (T + \delta T)^{-1}[b + \delta f(\bar{x})]$. Reasoning as we did in part (a), we find that

$$\inf\{\|\bar{x} - x\| \mid x \text{ solves } (6)\} = d(\bar{x}, (T + \delta T)^{-1}[b + \delta b])$$
$$\le \|(T + \delta T)^{-1}\| \|\delta f(\bar{x}) - \delta b\| \le \|T^{-1}\|(\|\delta b\| + \|\delta f\| \|\bar{x}\|)/(1 - \|T^{-1}\| \|\delta f\|),$$

as was to have been shown.

In applying part (b) of Theorem 7 to the case of a finite system of linear inequalities and equations, one would in general want to include in the function f only those constraints active at $\bar{x}$, and to deal separately with the inactive constraints by finding some ball about $\bar{x}$ in which they remain inactive.

If the space X is reflexive, then the infima in the conclusions of Theorem 7 are actually attained, since the norm is weakly lower semicontinuous. If all spaces involved are finite-dimensional, and if f is taken to be linear, then a result similar to part (a) can be proved with no assumptions at all about the range of T. This type of result was apparently first proved by Hoffman [4]; see [7] for a treatment using the methods of the present paper, with applications to linear programming. An application of Theorem 7 to Newton's method is given in [8]. Ben-Israel has proved analogous perturbation results for generalized inverses in finite-dimensional spaces in [1].

The results of Theorem 7 are expressed in terms of the quantity $\|T^{-1}\|$, defined in this case to be

$$\sup\{\inf\{\|x\| \mid f(x) \in y - K\} \mid \|y\| \le 1\}. \tag{7}$$

Even in the finite-dimensional case, this quantity can be rather difficult to compute. It would be very desirable to have efficient computational methods for evaluating (7), both in the finite-dimensional and infinite-dimensional cases.

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