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QUASI-COMPLEMENTED ALGEBRAS

BY

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ABSTRACT. In this paper we introduce a class of algebras which we call quasi-complemented algebras. A structure and representation theory is developed. We also study the uniformly continuous quasi-complementors on B^*-algebras.

1. **Introduction.** Complemented Banach algebras were introduced in [11] and have been studied by various authors. The present work is an attempt to generalize these algebras.

The concept of quasi-complemented algebra is introduced in §2. Let A be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed. We show that the socle of A is dense in A. This enables us to establish a structure theorem for A if A has the property $x \in \mathrm{cl}(xA)$ for all $x \in A$. We also give a representation theorem for a primitive Banach algebra in which every maximal closed right ideal is modular and $x \in \mathrm{cl}(xA)$ for all $x \in A$. In §5, we study quasi-complementors induced by given quasi-complementors.

We introduce the concept of continuous quasi-complementors in §6. Then we show that if A is a B^*-algebra which has no minimal left ideals of dimension less than three, then every uniformly continuous quasi-complementor on A is a complementor.

As we observed above, many fundamental properties of a complemented algebra hold for a quasi-complemented algebra. However a quasi-complemented algebra, in general, is not complemented as shown by the examples in §2.

2. **Notation and preliminaries.** For any subset S in an algebra A, let $l(S)$ and $r(S)$ denote the left and right annihilators of S in A, respectively. Let A be a topological algebra. Then A is called an annihilator algebra if, for every closed left ideal J and for every closed right ideal R, we have $r(J) = (0)$ if and only if $J = A$ and $l(R) = (0)$ if and only if $R = A$. If $l(r(J)) = J$ and $r(l(R)) = R$, then A is called a dual algebra.

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Let A be a topological algebra and let L_r be the set of all closed right ideals in A. Then A is called a right quasi-complemented algebra if there exists a mapping q: $R \to R^q$ of L_r into itself having the following properties:

(2.1) $R \cap R^q = (0)$ $(R \in L_r)$;

(2.2) $(R^q)^q = R$ $(R \in L_r)$;

(2.3) if $R_1 \supset R_2$, then $R_2^q \supset R_1^q$ $(R_1, R_2 \in L_r)$.

We call the mapping q a right quasi-complementor on A and R^q the right quasi-complement of R in A. It is clear that the concept of quasi-complementation extends that of orthogonal complementation when A is a Hilbert algebra.

A right quasi-complemented algebra A is called a right complemented algebra if it satisfies:

(2.4) $R + R^q = A$ $(R \in L_r)$.

In this case, the mapping q is called a right complementor on A (see [11, p. 615, Definition 1]). A right quasi-complemented algebra may not be right complemented as shown by the following examples:

Example 2.1. Let B and p be given in [1, p. 396, Example 1]. Then p is a right quasi-complementor on B. But p is not a right complementor. However B is a right complemented algebra under the right complementor $R \to l(R)^*$ (see [3, p. 463, Theorem 3.6]).

Example 2.2. Let G be the compact group of real numbers mod 1 and $A = L_p(G)$, where $1 < p < \infty$ and $p \neq 2$. It is well known that A is a commutative dual A^*-algebra which is not an ideal in the completion of its auxiliary norm (see [9, p. 35]). By Theorem 6.5, the mapping q: $R \to l(R)$ is the only right quasi-complementor on A. It follows from [4, p. 233, Theorem 3.8] and [9, p. 35, Theorem 23] that p is not a right complementor on A. Since A has a unique right quasi-complementor, A is not a right complemented algebra.

Analogously we define left quasi-complemented algebras. In this paper, we limit our attention to right quasi-complemented algebras with the remark that similar properties hold for left quasi-complemented algebras. From now on a quasi-complemented (resp. complemented) algebra will always mean a right quasi-complemented (resp. right complemented) algebra.

Let X be a topological space and S a subset in X. Then $\mathrm{cl}(S)$ will denote the closure of S in X.

In this paper, all algebras and linear spaces under consideration are over the complex field C. Definitions not explicitly given are taken from Rickart's book [10].

We shall need the following result.

Lemma 2.1. *Let A be a semisimple dual algebra in which every maximal modular right ideal is closed. Then for each nonzero closed right ideal R of A,* *we have* $R = \mathrm{cl}(\Sigma_\alpha e_\alpha A)$, *where* $\{e_\alpha\}$ *is the family of all minimal idempotents of* A *contained in* R.

Proof. By [5, p. 569, Theorem 4.2], $\{e_\alpha\}$ is not an empty set. Let $J = \mathrm{cl}(\Sigma_\alpha e_\alpha A)$. By a similar argument in the proof of [5, p. 570, Theorem 5.1], we have $l(J)R = (0)$ and so $R \subset r(l(J)) = J$. Therefore $R = J$. This completes the proof.

3. **A structure theorem.**

Lemma 3.1. *Let* A *be a quasi-complemented algebra with a quasi-complementor* q. *Then*

(i) *For any family of closed right ideals* $\{R_\lambda\}$ *in* A, *we have* $\mathrm{cl}(\Sigma_\lambda R_\lambda) = (\bigcap_\lambda R^q)^q$.

(ii) *For every closed right ideal* R *of* A, $R + R^q$ *is dense in* A.

Proof. (i) follows from the proof of [3, p. 461, Lemma 2.1].

(ii) Since $A^q = A^q \cap A = (0)$, we have $(0)^q = A$. Therefore it follows from (i) that

$$\mathrm{cl}(R + R^q) = (R^q \cap R)^q = (0)^q = A.$$

Therefore $R + R^q$ is dense in A.

Corollary 3.2. *A finite dimensional quasi-complemented normed algebra is a complemented algebra.*

Proof. This follows easily from Lemma 3.1(ii).

Lemma 3.3. *Let* A *be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed. Then the socle of* A *is dense in* A.

Proof. Let $\{R_\lambda : \lambda \in \Lambda\}$ be the family of all maximal modular right ideals of A. By the semisimplicity of A, $\bigcap_\lambda R_\lambda = (0)$ and therefore by Lemma 3.1, $A = \mathrm{cl}(\Sigma_\lambda R_\lambda^q)$. Clearly $R_\lambda^q \neq (0)$; for otherwise $R_\lambda = (R_\lambda^q)^q = (0)^q = A$, a contradiction. Since $R_\lambda + R_\lambda^q$ is a right ideal which contains R_λ properly, it follows that $R_\lambda + R_\lambda^q = A$. Therefore by Lemma 3.1 in [7], R_λ^q is a minimal right ideal. Hence R_λ^q is contained in the socle S of A and therefore S is dense in A. This completes the proof.

Lemma 3.4. *Let* A *be a semisimple quasi-complemented algebra such that* $x \in \mathrm{cl}(xA)$ *for all* $x \in A$. *Then each closed two-sided ideal* J *in* A *is a quasi-complemented algebra.*

Proof. Let R be a closed right ideal in J. Since $l(J) = r(J)$ (see [14, p. 37]) and $J^q J \subset J \cap J^q = (0)$, it follows that $J^q \subset l(J) = r(J) \neq (0)$. Therefore by the proof of [10, p. 99, Lemma (2.8.11)], R is a closed right ideal in A. Let q be a given quasi-complementor on A and let $R^{q_J} = R^q \cap J$. We show that q_J is a quasi-complementor on J. By Lemma 3.1, we have

$$(R^{q_J})^{q_J} = (R^q \cap J)^q \cap J = \mathrm{cl}(R + J^q) \cap J.$$

Let $x \in (R^{q_J})^{q_J}$ and write $x = \lim_\alpha (a_\alpha + b_\alpha)$ with $a_\alpha \in R$ and $b_\alpha \in J^q$. Since $x \in J$, it follows from Lemma 3.1 that

$$xA = x\,\mathrm{cl}(J + J^q) \subset \mathrm{cl}(x(J + J^q)) = \mathrm{cl}(xJ).$$

Since $x \in \mathrm{cl}(xA)$, we have $x \in \mathrm{cl}(xJ)$. Therefore we can write $x = \lim_\beta xy_\beta$ with $y_\beta \in J$. Since

$$xy_\beta = \lim_\alpha (a_\alpha y_\beta + b_\alpha y_\beta) = \lim_\alpha a_\alpha y_\beta,$$

it follows that $xy_\beta \in R$ and consequently $x \in R$. Therefore $(R^{q_J})^{q_J} \subset R$. Since $R^q \cap J \subset R^q$, we have $R \subset (R^{q_J})^{q_J}$ and hence $(R^{q_J})^{q_J} = R$. It is easy to see that the mapping q_J satisfies the conditions (2.1) and (2.3). Therefore it is a quasi-complementor on J and this completes the proof.

We shall need the following result in §7.

Corollary 3.5. *Let A, J and q_J be as in Lemma 3.4. If M is a closed right ideal in A, then $M^q \cap J = (M \cap J)^{q_J}$.*

Proof. By Lemma 3.1, we have

$$(M^q \cap J)^{q_J} = (M^q \cap J)^q \cap J = \mathrm{cl}(M + J^q) \cap J.$$

Hence by the proof of Lemma 3.4, we have $(M^q \cap J)^{q_J} \subset M \cap J$ and so $M^q \cap J \supset (M \cap J)^{q_J}$. Since $M \cap J \subset M$, it follows that $(M \cap J)^q \supset M^q$. Therefore $(M \cap J)^{q_J} \supset M^q \cap J$. Hence $M^q \cap J = (M \cap J)^{q_J}$.

Now we have the following structure theorem.

Theorem 3.6. *Let A be a semisimple quasi-complemented algebra in which every maximal modular right ideal is closed and x belongs to $\mathrm{cl}(xA)$ for all $x \in A$. Then A is the direct topological sum of its minimal closed two-sided ideals, each of which is a simple quasi-complemented algebra.*

Proof. By Lemma 3.3, the socle of A is dense in A. Therefore by [14, p. 31, Lemma 3.11], A is the topological direct sum of its minimal closed two-sided ideals. By Lemma 3.4, each minimal closed two-sided ideal of A is a quasi-complemented algebra and this completes the proof.

Remark. Let A be an algebra. The condition that $x \in \mathrm{cl}(xA)$ for all $x \in A$ is automatically satisfied if A has an approximate identity or A is a semisimple complemented algebra. Also, if A is a semisimple dual algebra, it has this property.

4. **A representation theorem.** The following lemma is implicit in [2, p. 40, Proposition 1].

Lemma 4.1. *Let A be a semisimple Banach algebra and I a minimal left ideal in A. Then*

(i) *For each closed right ideal R in A, $RI = R \cap I$.*

(ii) *For each closed subspace E in I, $E = \mathrm{cl}(EA) \cap I$.*

Proof. (i). We can write $I = Ae$, where e is a minimal idempotent of A (see [14, p. 37]). Let R be a closed right ideal in A and let $x \in R \cap I$. Since $x = xe \in RI$, we have $R \cap I \subset RI$. But $RI \subset R \cap I$ and so $RI = R \cap I$. This proves (i).

(ii) Let E be a closed subspace in I and let $R = \mathrm{cl}(EA)$. Since $Ee = E$, we have $E \subset R \cap I$. It follows from (i) that

$$R \cap I = \mathrm{cl}(EA)I \subset E(eAe) = Ee = E.$$

Therefore $E = \mathrm{cl}(EA) \cap I$ and this completes the proof.

Let A be a primitive quasi-complemented Banach algebra and I a minimal left ideal of A. Then $I = Ae$ for some minimal idempotent e of A. By [10, p. 68, Corollary (2.4.16)], the left regular representation $a \longrightarrow T_a$ of A is a faithful, continuous, strictly dense representation on I. Let $A' = \{T_a : a \in A\}$. Then by [10, p. 67, Theorem (2.4.12)], the image of the socle of A is the set of all operators of finite rank in A'. Since by Lemma 3.3, the socle of A is dense in A, it follows that A is a simple algebra (see [10, p. 65]).

Lemma 4.2. *Let A be a primitive Banach algebra with a quasi-complementor q such that $x \in \mathrm{cl}(xA)$ for all $x \in A$. For each closed subspace E in I, let $E' = [\mathrm{cl}(EA)]^q \cap I$. Then an inner product (x, y) can be introduced in I having the following properties*:

(i) *I becomes a Hilbert space under (x, y).*

(ii) *The norm $|x| = (x, x)^{1/2}$ is equivalent to the given norm $\|x\|$ in I.*

(iii) *If A is infinite-dimensional, then E' is the orthogonal complement of E in I.*

Proof. Let $R = [\mathrm{cl}(EA)]^q$. Since A is a simple algebra and since IA is a two-sided ideal in A, IA is dense in A. Let $x \in R$. Then

$$xA = x\,\mathrm{cl}(IA) \subset \mathrm{cl}(xIA) \subset \mathrm{cl}(RIA).$$

Since $x \in \mathrm{cl}(xA)$, it follows that $x \in \mathrm{cl}(RIA)$ and so $R \subset \mathrm{cl}(RIA)$. Clearly $R \supset \mathrm{cl}(RIA)$. Hence $R = \mathrm{cl}(RIA)$. Therefore by Lemma 4.1(i), $R = \mathrm{cl}((R \cap I)A) = \mathrm{cl}(E'A)$. Hence it follows from Lemma 4.1(ii) that

$$E'' = [\mathrm{cl}(E'A)]^q \cap I = R^q \cap I = \mathrm{cl}(EA) \cap I = E.$$

If $x \in E \cap E'$, then by Lemma 4.1(ii) $x \in \mathrm{cl}(EA) \cap [\mathrm{cl}(EA)]^q$ and so $x = 0$.

Therefore $E \cap E' = (0)$. If E_1 and E_2 are closed subspaces of I such that $E_1 \subset E_2$, then clearly $E_1' \supset E_2'$. Therefore by [8, p. 731, Theorem 2], an inner product (x, y) can be introduced in I having properties (i) and (ii). If A is infinite dimensional, then so is I. Hence (iii) follows from [8, p. 729, Theorem 1].

We have the following representation theorem.

Theorem 4.3. *Let A be a primitive quasi-complemented Banach algebra in which every maximal closed right ideal is modular and $x \in \mathrm{cl}(xA)$ for all $x \in A$. Then there exists a continuous isomorphism of A onto an algebra A' of completely continuous operators on a Hilbert space. Also A is a dual algebra.*

Proof. Let I be a minimal left ideal in A. By Lemma 4.2, I is a Hilbert space. Let $a \longrightarrow T_a$ be the left regular representation of A on I and $A' = \{T_a: a \in A\}$. Then $a \longrightarrow T_a$ is a continuous isomorphism of A onto A'. Letting A' have the given norm of A, we can identify A with A'. Let q be a given quasi-complementor on A and R a proper closed right ideal of A. Since the socle of A is dense in A, by [14, p. 37, Lemma 3.1], R^q contains a minimal right ideal M. It is easy to see that M^q is a maximal closed right ideal and so modular by the assumption. Therefore by [14, p. 38, Lemma 3.3], $l(M^q) \neq (0)$. Since $R \subset M^q$, it follows that $l(R) \neq (0)$. Therefore by the proof of [10, p. 101, Lemma (2.8.20)], A contains all operators of finite rank on I. Hence A is an algebra of completely continuous operators on I (see the proof of [11, p. 657, Theorem 7]). By [10, p. 104, Theorem (2.8.23)] A is an annihilator algebra. Since I is reflexive and since $x \in \mathrm{cl}(xA)$, it follows from the proof of [10, p. 105, Theorem (2.8.27)] that A is a dual algebra.

Corollary 4.4. *Let A be a semisimple quasi-complemented Banach algebra in which $x \in \mathrm{cl}(xA)$ for all $x \in A$. Then A is an annihilator algebra if and only if every maximal closed right ideal of A is modular.*

Proof. Suppose every maximal closed right ideal of A is modular. Let I be a minimal closed two-sided ideal of A and M a maximal closed right ideal of I. By the proof of Lemma 3.4, M^{qI} is a minimal right ideal of I and A. Therefore $N = (M^{qI})^q$ is a maximal modular right ideal of A. Since $M^{qI} \oplus N = A$, by [3, p. 462, Lemma 3.1] $N = (1 - e)A$ and $M^{qI} = eA$, where e is a minimal idempotent. Since $e \in I$, $M = (M^{qI})^{qI} = N \cap I = (1 - e)I$. Therefore M is modular. By the proof of Lemma 3.4, we have $x \in \mathrm{cl}(xI)$ for all $x \in I$. Hence by Theorem 4.3, I is an annihilator algebra and so is A by [10, p. 106, Theorem (2.8.29)]. The converse of the corollary follows from [10, p. 98, Corollary (2.8.7)].

Theorem 4.5 (we use the notation in Theorem 4.3.). *If A' is a two-sided ideal of $B(I)$, the set of all continuous linear operators on I, then every quasi-complementor q on A is a complementor.*

Proof. By Corollary 3.2, we can assume that A is infinite dimensional. In this proof, we identify A with A'. Let R be a closed right ideal in A. To complete the proof, it suffices to show that $R + R^q$ is closed by Lemma 3.1. Let $E = R \cap I$ and let $E' = [\mathrm{cl}(EA)]^q \cap I$. By Lemma 4.2(iii), E' is the orthogonal complement of E in I. Denote the orthogonal projection on E by P. Let $a \in \mathrm{cl}(R + R^q)$ and write $a = \lim_n (b_n + c_n)$ with $b_n \in R$ and $c_n \in R^q$. Since $b_n I \subset RI = R \cap I = E$, we have $(Pb_n)(h) = b_n(h)$ for all $h \in I$. Hence $Pb_n = b_n$. Since

$$c_n I \subset R^q \cap I = [\mathrm{cl}(RIA)]^q \cap I = \mathrm{cl}(EA)^q \cap I = E',$$

we have $Pc_n = 0$. By the proof of [2, p. 41, Theorem 3], we have $\|Pa - b_n\| \leq k\,\|a - b_n - c_n\|$, where k is a constant. Hence we have $Pa \in R$ and so $a - Pa \in R^q$. Therefore $a = Pa + (a - Pa) \in R + R^q$. Hence $R + R^q$ is closed and this completes the proof.

5. **Induced quasi-complementors.** In this section, unless otherwise stated, A will be a semisimple Banach algebra with norm $\|\cdot\|$ which is a dense subalgebra of a semisimple Banach algebra B with norm $|\cdot|$. Further A and B have the following properties:

(5.1) There exists a constant k such that $k\|x\| \geq |x|$ for all $x \in A$, i.e., $\|\cdot\|$ majorizes $|\cdot|$.

(5.2) Every proper closed left (right) ideal in B is the intersection of maximal modular (right) ideals in B.

Notation. For any subset E of A, $\mathrm{cl}_A(E)$ (resp. $\mathrm{cl}(E)$) will denote the closure of E in A (resp. B) and $l_A(E)$ and $r_A(E)$ (resp. $l(E)$ and $r(E)$) the left and right annihilators of E in A (resp. B).

Lemma 5.1. *Let A be an annihilator algebra. Then*

(i) *For each closed right ideal R of A, we have* $\mathrm{cl}(R) \cap A = r_A(l_A(R))$.

(ii) *If M is a closed right ideal of B, then* $M = \mathrm{cl}(M \cap A)$.

Proof. First we note that B is a dual algebra [13, p. 81] and A and B have the same socle S (Lemma 4.1 in [7]).

(i) Let $\{e_\alpha\}$ be the family of all minimal idempotents of B contained in $l(R)$. Since B is a dual algebra, it follows from Lemma 2.1 that $\mathrm{cl}(\Sigma_\alpha Be_\alpha) = l(R)$. Since $e_\alpha \in l(R) \cap S \subset l(R) \cap A = l_A(R)$, we have $\mathrm{cl}(l_A(R)) \supset l(R)$. Clearly $l(R) \supset \mathrm{cl}(l_A(R))$ and therefore $\mathrm{cl}(l_A(R)) = l(R) = l(\mathrm{cl}(R))$. Hence by the duality of B, we have

$$r_A(l_A(R)) = r(l_A(R)) \cap A = r(\mathrm{cl}(l_A(R))) \cap A = r(l(\mathrm{cl}(R))) \cap A = \mathrm{cl}(R) \cap A.$$

This proves (i).

(ii) Let $\{e_\beta\}$ be the family of all minimal idempotents of B contained in M.

By Lemma 2.1, $M = \mathrm{cl}(\Sigma_\beta e_\beta B)$. Since each $e_\beta B \subset M \cap S \subset M \cap A$, we have $\Sigma_\beta e_\beta B \subset M \cap A$. It is now easy to see that $M = \mathrm{cl}(M \cap A)$. This completes the proof.

Lemma 5.2. *Let A be an annihilator algebra. Then the following statements are equivalent*:

(i) *A is a dual algebra.*

(ii) *For each element $x \in A$, we have $x \in \mathrm{cl}_A(xA) \cap \mathrm{cl}_A(Ax)$.*

(iii) *For each closed right (left) ideal R of A, we have $R = \mathrm{cl}(R) \cap A$.*

Proof. (i) $\Rightarrow$ (ii). This follows immediately from [10, p. 97, Corollary (2.8.2)].

(ii) $\Rightarrow$ (iii). Suppose (ii) holds. Let S be the socle of A. By Lemma 4.1 in [7], S is also the socle of B. Let R be a closed right ideal of A. We show that $\mathrm{cl}(R)S \subset R$. In fact, let $x \in \mathrm{cl}(R)$, $y \in A$ and e a minimal idempotent in A. Let $\{x_n\}$ be a sequence in R such that $x_n \to x$ in $|\cdot|$. By the proof of [13, p. 82, Lemma 3.2], the norms $\|\cdot\|$ and $|\cdot|$ are equivalent on $Ae = Be$. Hence it follows easily that $x_n ye \to xye$ in $\|\cdot\|$. Therefore $xye \in R$ and so $\mathrm{cl}(R)S \subset R$. Let $a \in \mathrm{cl}(R) \cap A$. Then we have

$$a \in \mathrm{cl}_A(aA) = \mathrm{cl}_A(aS) \subset \mathrm{cl}_A(\mathrm{cl}(R)S) \subset R.$$

Hence $\mathrm{cl}(R) \cap A \subset R$. Clearly $R \subset \mathrm{cl}(R) \cap A$ and so $R = \mathrm{cl}(R) \cap A$. This proves (iii).

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Let R be a closed right ideal of A. By Lemma 5.1, we have $R = \mathrm{cl}(R) \cap A = r_A(l_A(R))$. Similarly we can show that $J = l_A(r_A(J))$ for all closed left ideals J of A. Therefore A is a dual algebra and the proof is complete.

Theorem 5.3. *Let A be a dual algebra. Then for every quasi-complementor p on B, the mapping $q: R \to [\mathrm{cl}(R)]^p \cap A$ on the closed right ideals R of A is a quasi-complementor on A.*

Proof. Let R be a closed right ideal of A. Since A is a dual algebra, by Lemma 5.2, $R = \mathrm{cl}(R) \cap A$. Therefore

$$R \cap R^q = \mathrm{cl}(R) \cap [\mathrm{cl}(R)]^p \cap A = (0).$$

By Lemma 5.1, we have $[\mathrm{cl}(R)]^p = \mathrm{cl}([\mathrm{cl}(R)]^p \cap A)$. Therefore it follows that

$$(R^q)^q = [\mathrm{cl}([\mathrm{cl}(R)]^p \cap A)]^p \cap A = [\mathrm{cl}(R)]^{pp} \cap A = \mathrm{cl}(R) \cap A = R.$$

If R_1 and R_2 are closed right ideals of A such that $R_1 \supset R_2$, then clearly $R_1^q \subset R_2^q$. Therefore q is a quasi-complementor on A.

We now establish the converse of Theorem 5.3.

Theorem 5.4. *Let A be a dual algebra. Then for every quasi-complementor q*

on A, the mapping p: $M \to \mathrm{cl}([M \cap A]^q)$ on the closed right ideals M of B is a quasi-complementor on B.

Proof. Let M be a closed right ideal of B. Then it follows from Lemma 5.2 that $M \cap M^p \cap A = [M \cap A] \cap [M \cap A]^q = (0)$. Hence it follows from Lemma 5.1 that $M \cap M^p = \mathrm{cl}(M \cap M^p \cap A) = (0)$. We also have

$$(M^p)^p = \mathrm{cl}([\mathrm{cl}([M \cap A]^q) \cap A]^q) = \mathrm{cl}([M \cap A]^{qq}) = M.$$

If M_1 and M_2 are closed right ideals of B such that $M_1 \supset M_2$, then clearly $M_1^p \subset M_2^p$. Therefore p is a quasi-complementor on B and this completes the proof.

6. Continuous quasi-complementors on A^*-algebras.

Theorem 6.1. *Let A be a dual A^*-algebra. Then A is a quasi-complemented algebra under the quasi-complementor q: $R \to l(R)^*$.*

Proof. Let R be a closed right ideal of A. Since $l(R)^* = r(R^*)$, by the duality of A, we have $(R^q)^q = R$. It is easy to see that q has properties (2.1) and (2.3). Therefore q is a quasi-complementor on A and this completes the proof.

It is known that a B^*-algebra is complemented if and only if it is dual (see [3, p. 463, Theorem 3.6]). A similar result is true for quasi-complemented algebras. In fact we have the following:

Corollary 6.2. *Let A be an A^*-algebra which is a dense two-sided ideal of a B^*-algebra B. Then A is a dual algebra if and only if A is quasi-complemented and $x \in \mathrm{cl}(xA)$ for all $x \in A$.*

Proof. Suppose A is quasi-complemented and $x \in \mathrm{cl}(xA)$ for all $x \in A$. Let e be a minimal idempotent of A. Clearly $Ae = Be$. Therefore by Lemma 3.3 and Theorem 4.3 in [7], A is an annihilator algebra. Hence by Lemma 5.2, A is a dual algebra. The converse of the corollary follows from Theorem 6.1 and Lemma 5.2.

Corollary 6.3. *Let A be a B^*-algebra. Then A is a dual if and only if A is quasi-complemented.*

Proof. Since a B^*-algebra has an approximate identity, it follows that $x \in \mathrm{cl}(xA)$. Therefore Corollary 6.3 follows immediately from Corollary 6.2.

Lemma 6.4. *Let A be an annihilator semisimple Banach algebra with a quasi-complementor q. Then for every maximal closed right ideal R of A, there exists a unique minimal idempotent f such that $R^q = fA$ and $R = (1 - f)A$.*

Proof. By [10, p. 97, Theorem (2.8.5)], R is a maximal modular right ideal of A. Since $R + R^q = A$, by [3, p. 462, Lemma 3.1] we have the desired result.

Definition. Let A be a quasi-complemented Banach algebra. A minimal idempotent f in A is called a q-projection if $(fA)^q = (1 - f)A$.

We now introduce the concept of continuous quasi-complementor on annihilator A^*-algebras. This is similar to the concept of continuous complementor on B^*-algebras (see [3, p. 463, Definition 3.7]).

Definition. Let A be an annihilator A^*-algebra with a quasi-complementor q. Let E denote the set of all hermitian minimal idempotents and E_q the set of all q-projections in A. For each $e \in E$, let $Q(e)$ be the unique element of E_q such that $Q(e)A = eA$ (Lemma 6.4). The mapping Q: $e \to Q(e)$ is called the q-derived mapping of E into E_q. The quasi-complementor q is said to be continuous if Q is continuous in the relative topologies of E and E_q induced by the given norm on A.

Remark 1. Since by [10, p. 261, Lemma (4.10.1)] every minimal right ideal of A is of the form eA with a unique $e \in E$, it follows that Q maps E onto E_q.

Remark 2. Let A and q be as in Theorem 6.1. Then $E = E_q$ and so the q-derived mapping Q of q is the identity mapping. Hence q is uniformly continuous.

For commutative dual A^*-algebras, the study of quasi-complementor becomes very trivial.

Theorem 6.5. *Let A be a commutative dual A^*-algebra. Then there is only one quasi-complementor q on A; q is uniformly continuous.*

Proof. Let B be the completion of A in an auxiliary norm. We use the notation introduced in §5. The existence of a quasi-complementor on A is given by Theorem 6.1. Let q be any given quasi-complementor on A. By Theorem 5.6, q induces a quasi-complementor p on B. Let M be a closed ideal in B. Since $M \cap M^q = (0)$, it follows from [10, p. 259, Corollary (4.9.22)] that $M + M^q$ is a closed ideal in B. Therefore, by Lemma 3.1, $M + M^p = B$. Since $MM^p \subset M \cap M^p = (0)$, $M^p \subset l(M) = r(M)$. Since $M + l(M) = B$, it follows that $M^p = l(M)$. Let R be an ideal in A. Then we see that $R = R^*$ and $R^q = [\mathrm{cl}(R)]^p \cap A = l_A(R)$. Therefore q is uniquely determined. By Remark 2, q is uniformly continuous and this completes the proof.

Corollary 6.6. *Let A be a commutative dual A^*-algebra which is a dense two-sided ideal of a B^*-algebra. Then there is a unique complementor q on A; q is uniformly continuous.*

Proof. This follows easily from Theorem 6.5, [4, p. 233, Theorem 3.8] and [9, p. 30, Theorem 16].

7. **Quasi-complementors on B^*-algebras.** In this section, unless otherwise stated, A will be a B^*-algebra with a quasi-complementor q. By Corollary 6.3, A is a dual algebra.

Let H be a Hilbert space with inner product $(\ ,\ )$. If x and y are elements of H, then $x \otimes y$ will denote the operator on H given by the relation $(x \otimes y)(h) = (h, y)x$ for all $h \in H$. $LC(H)$ will denote the algebra of all completely continuous linear operators on H. If A is a simple dual B^*-algebra, then it is well known that $A = LC(H)$ for some Hilbert space H. H can be chosen as a minimal left ideal in A with the inner product given in [10, p. 261, Theorem (4.10.3)].

Lemma 7.1. *Let A be a simple B^*-algebra. Then every quasi-complementor q on A is a complementor.*

Proof. Since A has the form $LC(H)$, it follows from Theorem 4.5 that q is a complementor on A.

Notation. Let $A = LC(H)$. For every closed subspace X of H, let $J(X) = \{a \in A: a(H) \subset X\}$. For every closed right ideal R of A, let $S(R)$ be the smallest closed subspace of H that contains the range $a(H)$ of each operator a in R.

Let $A = LC(H)$. For each closed right ideal R of A, by Lemma 7.1, the projection P_R on R along R^q is continuous. Let P'_R be the projection on $S(R)$ along $S(R^q)$. Since by [3, p. 464, Lemma 4.1], $S(R) \oplus S(R^q) = H$, it follows that P'_R is continuous.

Lemma 7.2. *Let R be a closed right ideal of $A = LC(H)$. Then $\|P_R\| = \|P'_R\|$.*

Proof. Let $k > 0$ be given. Choose $x \in A$ such that $\|x\| \le 1$ and $\|P_R(x)\| \ge \|P_R\| - k/2$. Hence there exists some $h \in H$ such that $\|h\| \le 1$ and $\|(P_R(x))(h)\| > \|P_R\| - k$. Write $x = y + z$ with $y \in R$ and $z \in R^q$. Then $y(h) \in S(R)$ and $z(h) \in S(R^q)$ and so

$$\|P'_R(x(h))\| = \|y(h)\| = \|(P_R(x))(h)\| > \|P_R\| - k.$$

Since $\|x(h)\| \le 1$ and k is arbitrary, it follows that $\|P'_R\| \ge \|P_R\|$. By using [3, p. 464, Lemma 4.1] and a similar argument, we can show that $\|P'_R\| \ge \|P_R\|$. Therefore $\|P_R\| = \|P'_R\|$.

Lemma 7.3. *Suppose $A = LC(H)$ with* $\dim H \ge 3$, *q a continuous quasi-complementor on A and R a closed right ideal of A. If $\|P_R\| > k$ for some constant k, then there exists a q-projection $f \in R$ such that $\|f\| > k$.*

Proof. By Lemma 7.2, $\|P'_R\| > k$. Hence there exists an element $h \in H$ such that $\|h\| = 1$ and $\|P'_R(h)\| > k$. Write $h = u + v$ with $u \in S(R)$ and $v \in S(R^q)$. It is clear that $u \neq 0$. Let Q be a q-representing operator on H (see [3, p. 467, Definition 5.4]) and put $f = (u \otimes Qu)/(u, Qu)$. Then f is a q-projection (see [3, p. 467]). Since $u \in S(R)$, $f \in R$. Let $\langle x, y\rangle = (x, Qy)$ for all $x, y \in H$. Since q is a continuous complementor, by the proof of [3, p. 473, Theorem 6.11], $S(R)$ is the orthogonal complement of $S(R^q)$ in H relative to the inner product $\langle x, y\rangle$. Since

$v \in S(R^q)$ and since $u \in S(R)$, we have $(v, Qu) = \langle v, u\rangle = 0$ and therefore $f(h) = u$. Hence we have $\|f(h)\| = \|u\| = \|P_R'(h)\| > k$. Since $\|h\| = 1$, $\|f\| > k$ and this completes the proof.

Let A be a B^*-algebra with a quasi-complementor q. Let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. Since A is a dual B^*-algebra, $A = (\Sigma_\lambda I_\lambda)_0$, the $B^*(\infty)$-sum of $\{I_\lambda: \lambda \in \Lambda\}$. Since each I_λ is a simple dual B^*-algebra, $I_\lambda = LC(H_\lambda)$ for some Hilbert space H_λ $(\lambda \in \Lambda)$. By Corollary 3.5, q induces a quasi-complementor q_λ on each I_λ. By Lemma 7.1, q_λ is a complementor on I_λ.

Let E (resp. E_λ) be the set of all hermitian minimal idempotents in A (resp. I_λ) and let E_q (resp. E_q^λ) be the set of all q-projections in A (resp. I_λ). Clearly $E_\lambda = E \cap I_\lambda$ and $E_q^\lambda = E_q \cap I_\lambda$ $(\lambda \in \Lambda)$.

Lemma 7.4. *A quasi-complementor q on A is continuous if and only if each q_λ is continuous.*

Proof. By a similar argument in [3, p. 464, Theorem 3.9], we have the desired result.

Lemma 7.5. *Let A be a B^*-algebra which has no minimal left ideal of dimension less than three and q a quasi-complementor on A. If E_q is a closed and bounded subset of A, then q is a complementor on A.*

Proof. For each closed right ideal R_λ of $LC(H_\lambda)$, let P_{R_λ} be the projection on R_λ along $R_\lambda^{q_\lambda}$. Let

$$k_\lambda = \sup\{\|P_{R_\lambda}\|: R_\lambda \subset LC(H_\lambda)\} \qquad (\lambda \in \Lambda),$$

and let

$$k = \sup\{k_\lambda: \lambda \in \Lambda\}.$$

We show that k is finite. Suppose this is not so. Then for each positive integer n, there exists some $k_n \in \{k_\lambda: \lambda \in \Lambda\}$ such that $k_n > n$. Hence there exists a closed right ideal $R_n \subset LC(H_n)$ such that $\|P_{R_n}\| > n$. Since $E_q^n = E_q \cap I_n$, it follows immediately from the assumption that E_q^n is a closed and bounded subset of I_n. Since q_n is a complementor on I_n, by [12, p. 257, Theorem 3], q_n is continuous. Since $\|P_{R_n}\| > n$, it follows from Lemma 7.3 that there exists some $f_n \in E_q^n \subset E_q$ such that $\|f_n\| > n$ $(n = 1, 2, \cdots)$. This contradicts the boundedness of E_q and shows that k is finite.

Let M be a closed right ideal of A and let $M_\lambda = M \cap I_\lambda$ $(\lambda \in \Lambda)$. Since $A = (\Sigma_\lambda I_\lambda)_0$, we see that $M = (\Sigma_\lambda M_\lambda)_0$. Since by Corollary 3.5, $M^q \cap I_\lambda = M_\lambda^{q_\lambda}$, we have

$$M^q = (\Sigma_\lambda M^q \cap I_\lambda)_0 = (\Sigma_\lambda M_\lambda^{q_\lambda})_0.$$

Let $x = (x_\lambda) \in A$ and write $x_\lambda = y_\lambda + z_\lambda$, where $y_\lambda \in M_\lambda$ and $z_\lambda \in M_\lambda^{q_\lambda}$. Then $\|y_\lambda\| = \|P_{M_\lambda} x_\lambda\| \le k\|x_\lambda\|$ $(\lambda \in \Lambda)$. Since k is finite, it follows that $(y_\lambda) \in (\Sigma_\lambda M_\lambda)_0 = M$. Similarly we have $(z_\lambda) \in M^q$. Therefore $A = M + M^q$ and so q is a complementor on A.

We can now prove the main result of this section.

Theorem 7.6. *Let A be a B^*-algebra which has no minimal left ideal of dimension less than three and q a quasi-complementor on A. If q is uniformly continuous, then it is a complementor.*

Proof. By Lemma 7.5, it suffices to show that E_q^λ is a closed and bounded subset of A. By Lemma 7.4 and [12, p. 257, Theorem 3] each E_q^λ is closed and bounded. Hence it follows that E_q is closed. It remains to show that E_q is bounded. Suppose this is not so. Then we can choose a sequence of q-projections f_n such that $f_n \in E_q^n$ and $\|f_n\| > n$ $(n = 1, 2, \cdots)$. Let T_n be a q-representing operator on H_n. Then by [3, p. 470, Theorem 6.4], T_n is a continuous positive linear operator with inverse T_n^{-1}. We may assume that $\|T_n^{-1}\| = 1$ for all n (see [3, p. 472, Corollary 6.10]). We can write

$$(*) \qquad f_n = (u_n \otimes T_n u_n)/(u_n, T_n u_n),$$

where $u_n \in H_n$ and $\|u_n\| = 1$ $(n = 1, 2, \cdots)$ (see [3, p. 467]). Since

$$\inf\{(h_n, T_n h_n): \|h_n\| = 1 \text{ and } h_n \in H_n\} = \|T_n^{-1}\|^{-1} = 1,$$

if follows from $(*)$ that $\|T_n u_n\| > n$ $(n = 1, 2, \cdots)$. Let Q be the q-derived mapping of q. By using the argument in [3, p. 477, Theorem 7.4], we can find minimal idempotents $a_n, b_n \in E$ such that $\|a_n - b_n\| \to 0$ and $\|Q(a_n) - Q(b_n)\| \to \infty$. This contradicts the uniform continuity of Q. Therefore E_q is bounded and this completes the proof.

Remark. Let B and p be given in [1, p. 396, Example 1]. Then p is a continuous quasi-complementor on B. But p is not a complementor. Therefore a continuous quasi-complementor may not be uniformly continuous by Theorem 7.6. However a continuous complementor on a B^*-algebra is uniformly continuous (see [1] and [3]).

Corollary 7.7. *Let A be as in Theorem 7.6. Then a quasi-complementor q on A is uniformly continuous if and only if E_q is a closed and bounded subset of A.*

Proof. The corollary follows immediately from Theorem 7.6 and [12, p. 257, Theorem 3].

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