

INTEGRAL REPRESENTATION OF FUNCTIONS AND DISTRIBUTIONS POSITIVE DEFINITE RELATIVE TO THE ORTHOGONAL GROUP

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ABSTRACT. A continuous function f on an open ball B in R^N is called positive definite relative to the orthogonal group $O(N)$ if f is radial and $\iint f(x-y)\phi(x)\overline{\phi(y)} dx dy \geq 0$ for all radial $\phi \in C_0^\infty(B/2)$. It is shown that f is positive definite in B relative to $O(N)$ if and only if f has an integral representation $f(x) = \int e^{ix \cdot t} d\mu_1(t) + \int e^{x \cdot t} d\mu_2(t)$, where μ_1 and μ_2 are bounded, positive, rotation invariant Radon measures on R^N and μ_2 may be taken to be zero if, in addition to f being positive definite relative to $O(N)$, $\iint f(x-y)(-\Delta\phi)(x)\overline{\phi(y)} dx dy \geq 0$ for all radial $\phi \in C_0^\infty(B/2)$. Both conditions are satisfied if f is a radial positive definite function in B . Thus the theorem yields as a special case Rudin's theorem on the extension of radial positive definite functions. The result is extended further to distributions.

1. Introduction. Let N be a fixed positive integer. We denote by R^N the N -dimensional Euclidean space. $|x|$ denotes the usual norm: If $x = (x_1, x_2, \dots, x_N) \in R^N$ and $y = (y_1, y_2, \dots, y_N) \in R^N$, $x \cdot y = \sum x_i y_i$ and $|x| = (x \cdot x)^{1/2}$.

Suppose now that G is an open symmetric neighborhood of the origin in R^n (i.e. $x \in G$ implies that $-x \in G$) and f a complex valued function defined on G . f is said to be *positive definite* in G if

$$(1) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \overline{\alpha_j} f(x_i - x_j) \geq 0$$

for all choices of complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and points x_1, x_2, \dots, x_n in $G/2 = \{x \in R^N \mid 2x \in G\}$.

For a continuous complex valued function f defined on G , (1) is clearly equivalent to

$$(2) \quad \int_{G/2} \int_{G/2} f(x-y)\phi(x)\overline{\phi(y)} dx dy \geq 0 \text{ for all } \phi \in C_0^\infty(G/2).$$

$C_0^\infty(G/2)$ denotes the vector space of all complex valued infinitely differentiable functions in $G/2$ with compact support.

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Does every continuous positive definite function f in G have an extension to R^N which is positive definite in R^N ?

If $N = 1$ and G is an interval the answer is yes. The result is due to Krein [11].

If $N \geq 2$ and G is an N -dimensional cube the answer is in general no. This negative result was first proved by Calderon and Pepinsky [3] for the group of lattice points in R^N and then extended by Rudin [17] to R^N . For further results on this extension problem of positive definite functions we refer the reader to the papers by Devinatz [4] and Eskin [5] and the monograph by Berezanskiĭ [1].

Recently W. Rudin [18] has proved that if G is a ball in R^N and f is a continuous radial positive definite function in G then the answer to the extension problem is affirmative; there exists a radial positive definite function F in R^N such that $F(x) = f(x)$ for all $x \in G$. (Note. If a positive definite function is continuous at the origin it is continuous in its domain. Thus F is necessarily continuous.) It follows from this by Bochner's theorem that there exists a finite, positive, rotation invariant Radon measure μ on R^N such that

$$f(x) = \int_{R^N} e^{ix \cdot t} d\mu(t) \quad \text{for all } x \in G.$$

One main purpose of this paper is to prove that if f is a continuous radial function on a ball $G = S_N(a) = \{x \in R^N \mid |x| < a\}$, $0 < a \leq \infty$, such that

$$(3) \quad \int_{G/2} \int_{G/2} f(x-y) \phi(x) \overline{\phi(y)} dx dy \geq 0$$

for all radial functions $\phi \in C_0^\infty(G/2)$, then there exist finite, positive, rotation invariant Radon measures μ_1 and μ_2 on R^N such that

$$(4) \quad f(x) = \int_{R^N} e^{ix \cdot t} d\mu_1(t) + \int_{R^N} e^{x \cdot t} d\mu_2(t) \quad \text{for all } x \in G.$$

The measure μ_2 may be taken to be the zero measure if in addition to (3) the inequality

$$(5) \quad \int_{G/2} \int_{G/2} f(x-y) (-\Delta\phi)(x) \overline{\phi(y)} dx dy \geq 0$$

holds for all radial functions $\phi \in C_0^\infty(G/2)$.

Here Δ is the Laplace operator in R^N : $\Delta\phi = \sum \partial^2\phi/\partial x_i^2$.

Condition (5) can be rewritten in the form

$$(6) \quad \sum_{i=1}^N \int_{G/2} \int_{G/2} f(x-y) \phi_{x_i}(x) \overline{\phi_{x_i}(y)} dx dy \geq 0$$

for all radial functions $\phi \in C_0^\infty(G/2)$: Indeed

$$\begin{aligned} \int_{G/2} \int_{G/2} f(x-y)(-\Delta\phi)(x)\overline{\phi(y)} dx dy &= \int_G f(x)((-\Delta\phi) * \phi^*)(x) dx \\ &= \int_G f(x) \sum_{i=1}^N (\phi_{x_i} * \phi_{x_i}^*)(x) dx = \sum_{i=1}^N \int_{G/2} \int_{G/2} f(x-y)\phi_{x_i}(x)\overline{\phi_{x_i}(y)} dx dy \end{aligned}$$

for $\phi \in C_0^\infty(G/2)$. Here $\phi^*(x) = \overline{\phi(-x)}$ and $*$ denotes convolution.

Since for a continuous radial function f condition (2) clearly implies (3) and (6), our theorem not only yields the theorem of Rudin but actually strongly improves it.

If $N = 1$, the first part of our theorem, viz. (3) \Rightarrow (4), is precisely the theorem of Krein [12] on the integral representation of evenly positive definite functions (cf. also [8, p. 196] and [1, Theorem 318, p. 689]).

§§ 2-4 are devoted to preliminaries. The main result on functions positive definite relative to the orthogonal group are proved in §5. In §6 the results of §5 are extended to distributions. In particular Rudin's theorem on the extension of radial positive definite functions is extended to rotation invariant positive definite distributions.

In a sequel to this paper the main results of §5 will be further extended.

2. **Expansion into generalized eigenvectors.** To prove our result we shall use the theory of expansions into generalized eigenvectors first developed in the fundamental paper by Gel'fand and Kostjučenko [7] and expanded later by others, notably Ju. M. Berezanskiĭ. For a detailed account, reference to the literature and a more motivated discussion we refer the reader to the monographs by Berezanskiĭ [1] and Gel'fand and Vilenkin [8].

The theorem we need has been proved in one form or another by various authors. However, we have been unable to find a precise reference for the theorem we need. The theorems in the literature are either different from ours or imprecise or faulty in detail. We shall therefore give a proof of the pertinent theorem.

When referring to *nuclear spaces* we shall always mean "nuclear in the sense of Grothendieck" [9]. See also Trèves [20, p. 510] or Pietch [16].

If E is a locally convex space we denote by E' its topological dual, that is to say the vector space of all continuous linear functionals on E . If $x' \in E'$ we shall denote by $\langle x', x \rangle$ its value at the point x of E . If $A: E \rightarrow F$ is a continuous linear mapping of a locally convex space E into a locally convex space F we denote by tA the transpose of A . That is, tA is the linear mapping of F' into E' defined by

$$\langle {}^tAy', x \rangle = \langle y', Ax \rangle \text{ for all } x \in E \text{ and } y' \in F'.$$

An *involution* in E is a mapping $x \rightarrow x^*$ of E into E such that $(x^*)^* = x$, $(x + y)^* = x^* + y^*$ and $(\lambda x)^* = \bar{\lambda}x^*$ for all x and y in E and all complex numbers λ .

By a *pseudo-inner product* we shall mean a positive Hermitian sesquilinear form, i.e. a Hermitian sesquilinear form f on E such that $f(x, x) \geq 0$ for all $x \in E$. We shall write (x, y) for $f(x, y)$.

We assume the reader is familiar with J. von Neumann's theory of a direct integral of Hilbert spaces [14, pp. 404–438]. A short but precise account is given by L. Garding in [6, pp. 1–2].

Theorem 1 (Expansion into generalized eigenvectors). *Suppose*

1. E is a separable nuclear space,
2. (x, y) is a pseudo-inner product on E which is continuous (in both variables jointly in the topology of $E \times E$), and
3. A is a continuous linear mapping of E into E such that $(Ax, y) = (x, Ay)$ for all x and y in E .

If in addition

4. there exists an involution $x \rightarrow x^*$ in E such that $(x^*, y^*) = (y, x)$ and $Ax^* = (Ax)^*$ for all x and y in E , or
- 4'. $(Ax, x) \geq 0$ for all $x \in E$,

then there exists a finite positive Radon measure μ on \mathbb{R} and a μ -measurable function d with values in $\{1, 2, 3, \dots, \infty\}$, and, for every real number λ , a sequence $e'_k(\lambda)$, $k = 1, 2, \dots$, of elements in E' such that $e'_k(\lambda) = 0$ for $k > d(\lambda)$, and

5. $\lambda \rightarrow \langle e'_k(\lambda), x \rangle$ is μ -measurable for every $x \in E$ and every $k = 1, 2, \dots$,
6. there exists a μ -null set N_0 such that for $\lambda \notin N_0$, the sequence $e'_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, is linearly independent,
7. $\lambda e'_k(\lambda) = \lambda e'_k(\lambda)$ for all $k = 1, 2, \dots$ and $\lambda \in \mathbb{R}$,
8. the series $\sum_{k=1}^{d(\lambda)} \langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle}$ converges absolutely for every $\lambda \in \mathbb{R}$ and x and y in E ,
9. the function $\lambda \rightarrow \sum_{k=1}^{d(\lambda)} |\langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle}|$ belongs to $L^1(\mu)$, and

$$(x, y) = \int_{-\infty}^{\infty} \sum_{k=1}^{d(\lambda)} \langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle} d\mu(\lambda)$$

for all x and y in E .

If 4' holds μ may be chosen such that the support of μ is contained in $[0, \infty)$.

Proof. We first assume that the pseudo-inner product (x, y) is nondegenerate, i.e., is an inner product. Let H be the Hilbert space which is the completion of E with respect to the norm defined by the inner product (x, y) . We denote the inner product in H again by (x, y) . The natural injection $j: x \rightarrow x$ of E into H is continuous since the inner product (x, y) is continuous. It follows that j is

a nuclear mapping [9, Chapter II, p. 35]. See also [20, Theorem 50.1, p. 511]. Thus there exists a sequence $(\lambda_n)_{n \geq 1}$ of complex numbers such that $\sum_{k=1}^{\infty} |\lambda_n| < \infty$, a bounded sequence $(y_n)_{n \geq 1}$ of elements in H (i.e. $\|y_n\| = (y_n, y_n)^{1/2} \leq M$ for all n) and an equicontinuous sequence $(x'_n)_{n \geq 1}$ of elements in E' (i.e., $|\langle x'_n, x \rangle| \leq p(x)$ for all n and $x \in E$, where p is a continuous seminorm on E) such that

$$(7) \quad x = j(x) = \sum_{n=1}^{\infty} \lambda_n \langle x'_n, x \rangle y_n \quad \text{for all } x \in E,$$

where the series converges in the metric of H .

If the condition 4 holds we may extend the involution $x \rightarrow x^*$ in E by continuity to an involution in H which satisfies 4 for all x and y in H . It follows that A —considered as a symmetric operator in H with domain E —has a selfadjoint extension [13, p. 41]. Let T be any such selfadjoint extension.

If condition 4' holds, A —considered as a symmetric operator in H with domain E —has a positive selfadjoint extension by Friedrich's Theorem [13, p. 35]. In this case we denote by T any positive selfadjoint extension of A .

In either case let $\hat{H} = \int_{-\infty}^{\infty} \oplus \hat{H}(\lambda) d\mu(\lambda)$ be a direct integral Hilbert space which diagonalizes T . (Note that H is a separable Hilbert space because the range of a nuclear mapping is clearly always separable and therefore $j(E)$ is a separable dense subspace of H . We have not used the fact that E is separable (in its initial topology) this fact of course also implies that H is separable.) Thus there exists a unitary mapping U of H onto \hat{H} such that $\hat{T} = UTU^{-1}$ is the multiplication operator $[\hat{x}(\lambda)] \rightarrow [\lambda \hat{x}(\lambda)]$. Here $[\hat{x}(\lambda)]$ denotes the equivalence class \hat{x} in \hat{H} which contains the measurable vector field $\lambda \rightarrow \hat{x}(\lambda)$. We also write $\int \oplus \hat{x}(\lambda) d\mu(\lambda)$ for $[\hat{x}(\lambda)]$. The domain $D(\hat{T})$ consists of all $\hat{x} = [\hat{x}(\lambda)] \in \hat{H}$ such that $\int_{-\infty}^{\infty} \lambda^2 \|\hat{x}(\lambda)\|^2 d\mu(\lambda) < \infty$. μ is a finite positive Radon measure on R whose support is equal to the spectrum of T . Thus, in case condition 4' is satisfied, the spectrum of μ is contained in $[0, \infty)$.

If $x \in H$ we denote by \hat{x} the element $Ux \in \hat{H}$. If $x \in E$, then

$$Ux = \sum_{n=1}^{\infty} \lambda_n \langle x'_n, x \rangle U y_n \quad \text{by (7).}$$

Thus

$$\hat{x} = \sum_{n=1}^{\infty} \lambda_n \langle x'_n, x \rangle \hat{y}_n \quad \text{for all } x \in E,$$

where the series converges in the metric of \hat{H} .

Suppose $\hat{y}_n = \int \oplus \hat{y}_n(\lambda) d\mu(\lambda)$ for $n = 1, 2, \dots$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} |\lambda_n| \|\hat{y}_n(\lambda)\|^2 d\mu(\lambda) &= \sum_{n=1}^{\infty} |\lambda_n| \int_{-\infty}^{\infty} \|y_n(\lambda)\|^2 d\mu(\lambda) = \sum_{n=1}^{\infty} |\lambda_n| \|\hat{y}_n\|^2 \\ &= \sum_{n=1}^{\infty} |\lambda_n| \|y_n\|^2 \leq M^2 \sum_{n=1}^{\infty} |\lambda_n| < \infty. \end{aligned}$$

Therefore,

$$C^2(\lambda) = \sum_{n=1}^{\infty} |\lambda_n| \|\hat{y}_n(\lambda)\|^2 < \infty \quad \mu\text{-a.e.}$$

Hence, if we modify the vectors $\hat{y}_1(\lambda), \hat{y}_2(\lambda), \dots$ at a μ -null set (for example we may set $\hat{y}_n(\lambda) = 0$ for every $n = 1, 2, \dots$ and for λ such that $C^2(\lambda) = \infty$), we may assume without loss of generality that $C^2(\lambda) < \infty$ for all $\lambda \in R$, $C(\lambda) \geq 0$.

For every real number λ and $x \in E$, set

$$u_\lambda(x) = \sum_{n=1}^{\infty} \lambda_n \langle x'_n, x \rangle \hat{y}_n(\lambda).$$

$u_\lambda(x) \in \hat{H}(\lambda)$ and its $\hat{H}(\lambda)$ -norm

$$\begin{aligned} \|u_\lambda(x)\| &\leq \sum_{n=1}^{\infty} |\lambda_n| |\langle x'_n, x \rangle| \|\hat{y}_n(\lambda)\| \leq p(x) \sum_{n=1}^{\infty} |\lambda_n| \|\hat{y}_n(\lambda)\| \\ &\leq p(x) \left(\sum_{n=1}^{\infty} |\lambda_n| \right)^{1/2} \left(\sum_{n=1}^{\infty} |\lambda_n| \|\hat{y}_n(\lambda)\|^2 \right)^{1/2} \\ &\leq C(\lambda) \left(\sum_{n=1}^{\infty} |\lambda_n| \right)^{1/2} p(x) = M(\lambda) p(x), \end{aligned}$$

where $M(\lambda) = C(\lambda) (\sum_{n=1}^{\infty} |\lambda_n|)^{1/2} < \infty$ for every $\lambda \in R$.

Thus $u_\lambda: E \rightarrow \hat{H}(\lambda)$ defines a continuous linear mapping of E into $\hat{H}(\lambda)$ for every λ .

We claim that

$$(8) \quad \hat{x} = \int \bigoplus u_\lambda(x) d\mu(\lambda) \quad \text{for every } x \in E.$$

Indeed, for a given $x \in E$ suppose $\hat{x} = \int \bigoplus \hat{x}(\lambda) d\mu(\lambda)$.

$$S_n(x) = \sum_{k=1}^n \lambda_k \langle x'_k, x \rangle \hat{y}_k \rightarrow \hat{x} \quad \text{in the metric of } \hat{H}$$

as $n \rightarrow \infty$. It follows (as in the case of an ordinary L^2 -space) that there exists a subsequence $(S_{n_k}(x))$ of $(S_n(x))$ such that

$$S_{n_k}(x)(\lambda) = \sum_{m=1}^{n_k} \lambda_m \langle x'_m, x \rangle \hat{y}_m(\lambda) \rightarrow \hat{x}(\lambda) \quad \mu\text{-a.e.}$$

But $S_{n_k}(x)(\lambda) \rightarrow u_\lambda(x)$ for every $\lambda \in R$. Therefore $u_\lambda(x) = \hat{x}(\lambda)$ μ -a.e. for a fixed $x \in E$. This proves (8).

For every $\lambda \in R$ let $d(\lambda) = \dim \hat{H}(\lambda)$ (Hilbert space dimension). d is μ -measurable—it is “the dimension function of T ”. We may assume that, for every $\lambda \in R$, $\hat{H}(\lambda)$ is the sequence Hilbert space of dimension $d(\lambda)$ consisting of all square summable sequences $c(\lambda) = (c(\lambda)_k)_{k \geq 1}$ with $\overline{c(\lambda)}_k = 0$ for $k > d(\lambda)$ and the canonical inner product $(b(\lambda), c(\lambda)) = \sum_{k=1}^{\infty} b(\lambda)_k \overline{c(\lambda)}_k$.

The mapping

$$(9) \quad c(\lambda) \rightarrow c(\lambda)_k$$

is clearly a continuous linear functional on $\hat{H}(\lambda)$. Therefore the mapping

$$e'_k(\lambda): x \rightarrow u_\lambda(x)_k,$$

as the composite of the two continuous linear mappings u_λ and (9) is a continuous linear functional on E for every $\lambda \in R$ and every $k = 1, 2, 3, \dots$ ($e'_k(\lambda) = 0$ if $k > d(\lambda)$). Thus $e'_k(\lambda) \in E'$ for all $\lambda \in R$ and $k = 1, 2, 3, \dots$ and $\lambda \rightarrow \langle e'_k(\lambda), x \rangle = u_\lambda(x)_k$ is μ -measurable for every $x \in E$ and every k .

Now, $\langle e'_k(\lambda), Ax \rangle = u_\lambda(Ax)_k$ and

$$(Ax) = \int \bigoplus \lambda u_\lambda(x) d\mu(\lambda) \quad \text{for every } x \in E.$$

Hence $u_\lambda(Ax)_k = \lambda u_\lambda(x)_k (= \lambda \langle e'_k(\lambda), x \rangle)$ μ -a.e.; i.e. for all $\lambda \notin N_{x,k}$, where $N_{x,k}$ is a μ -null set. Thus, $\langle e'_k(\lambda), Ax \rangle = \lambda \langle e'_k(\lambda), x \rangle$ if $\lambda \notin N_{x,k}$, $x \in E$, $k = 1, 2, \dots$.

Since E is separable there exists a countable dense set $\{x_1, x_2, x_3, \dots\}$ in E . Let

$$N' = \bigcup_{n,k=1}^{\infty} N_{x_n,k}.$$

N' is a μ -null set and

$$\langle e'_k(\lambda), Ax_n \rangle = \lambda \langle e'_k(\lambda), x_n \rangle$$

for all $\lambda \notin N'$ and every $n = 1, 2, 3, \dots$ and $k = 1, 2, 3, \dots$.

Therefore, since the x_n 's are dense in E , by continuity $\langle e'_k(\lambda), Ax \rangle = \lambda \langle e'_k(\lambda), x \rangle$ for all $\lambda \notin N'$, every $k = 1, 2, 3, \dots$, and all $x \in E$.

We now modify $e'_k(\lambda)$ for $\lambda \in N'$ by setting $e'_k(\lambda) = 0$ for $\lambda \in N'$ and every $k = 1, 2, \dots$. Then

$$\langle e'_k(\lambda), Ax \rangle = \lambda \langle e'_k(\lambda), x \rangle$$

for all $\lambda \in R$, all $x \in E$, and every $k = 1, 2, 3, \dots$,

and

$$\sum_{k=1}^{d(\lambda)} |\langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle}| \leq \sum_{k=1}^{d(\lambda)} |u_\lambda(x)_k \overline{u_\lambda(y)_k}| \leq \|u_\lambda(x)\| \|u_\lambda(y)\|$$

for all $\lambda \in R$ and all x and y in E . We have thus proved 5, 7 and 8.

$$\int_{-\infty}^{\infty} \|u_\lambda(x)\| \|u_\lambda(y)\| d\mu(\lambda) \leq \left(\int_{-\infty}^{\infty} \|u_\lambda(x)\|^2 d\mu(\lambda) \right)^{1/2} \left(\int_{-\infty}^{\infty} \|u_\lambda(y)\|^2 d\mu(\lambda) \right)^{1/2} = \|\hat{x}\| \|\hat{y}\|.$$

Therefore $\lambda \rightarrow \sum_{k=1}^{d(\lambda)} |\langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle}|$ belongs to $L^1(\mu)$ for every x and y in E and since $\langle e'_k(\lambda), x \rangle = u_\lambda(x)_k$ for all $x \in E$, $k = 1, 2, 3, \dots$, and $\lambda \notin N'$, $\mu(N') = 0$,

$$(x, y) = \int_{-\infty}^{\infty} (u_\lambda(x), u_\lambda(y)) d\mu(\lambda) = \int_{-\infty}^{\infty} \sum_{k=1}^{d(\lambda)} \langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle} d\mu(\lambda)$$

for all x and y in E . This establishes the theorem for the case the pseudo-inner product is nondegenerate except for 6.

To prove 6 let $\Lambda_k = \{\lambda \in R \mid d(\lambda) \geq k\}$. Λ_k is a μ -measurable set. Let χ_k be the characteristic function of Λ_k (relative to R) and let $v_k(\lambda)$ be the vector in $\hat{H}(\lambda)$ whose k th component is $\chi_k(\lambda)$ and all other components are zero. The sequence $v_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, is an orthonormal basis in $\hat{H}(\lambda)$ for every $\lambda \in R$.

Let $v_k = \int \bigoplus v_k(\lambda) d\mu(\lambda)$ for $k = 1, 2, 3, \dots$. The set $\{x_1, x_2, x_3, \dots\}$ is clearly dense in H ; hence for a fixed n there exists a subsequence (x_{n_k}) of (x_n) which converges to $U^{-1}v_n$ in the metric of H . Therefore $\hat{x}_{n_k} = Ux_{n_k} \rightarrow v_n$ in the metric of \hat{H} as $k \rightarrow \infty$. Since $\hat{x}_{n_k} = \int \bigoplus u_\lambda(x_{n_k}) d\mu(\lambda)$ there exists (as noted above) a subsequence $(u_\lambda(x_{m_k}))$ of $(u_\lambda(x_{n_k}))$ such that $u_\lambda(x_{m_k}) \rightarrow v_n(\lambda)$ in the metric of $\hat{H}(\lambda)$ as $k \rightarrow \infty$ for every $\lambda \notin N_n$, where N_n is a μ -null set. Let $N'' = \bigcup_{n=1}^{\infty} N_n$, then it follows from what precedes that $v_n(\lambda) \in \overline{u_\lambda(E)}$ for all n if $\lambda \notin N''$ ($\overline{u_\lambda(E)}$ denotes the closure of $u_\lambda(E)$ in $H(\lambda)$). Thus $u_\lambda(E) \supset \hat{H}(\lambda)$ and therefore $u_\lambda(E) = \hat{H}(\lambda)$ for $\lambda \notin N''$.

Let $N_0 = N' \cup N''$. N_0 is a μ -null set and the elements $e'_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, are linearly independent if $\lambda \notin N_0$. Otherwise there exists for some $\lambda \notin N_0$ a sequence $(c_k)_{k \geq 1}$ of complex numbers which are not all zero and such that $c_k = 0$ for $k > n$, where n is an integer $\leq d(\lambda)$ and $\sum_{k \geq 1} c_k e'_k(\lambda) = 0$. Therefore

$$\sum_{k \geq 1} c_k \langle e'_k(\lambda), x \rangle = \sum_{k \geq 1} c_k u_\lambda(x) = 0 \text{ for all } x \in E.$$

Thus $c = (\overline{c_k})_{k \geq 1} \in \hat{H}(\lambda)$, $c \neq 0$ and c is orthogonal to $u_\lambda(E)$. This contradicts $\overline{u_\lambda(E)} = \hat{H}(\lambda)$.

Assume now that the pseudo-inner product is degenerate. Let $N = \{x \in E \mid (x, x) = 0\}$. N is a closed linear subspace of E and therefore the quotient space $\dot{E} = E/N$ is nuclear [9, Chapter II, p. 47] (see also [20, Proposition 50.4, p. 514] or [16, Satz 5.1.3, p. 77]). \dot{E} is separable because E is separable. $AN \subset N$ for if $x \in N$ then $(Ax, Ax) = (A^2x, x) \leq (A^2x, A^2x)^{1/2}(x, x)^{1/2} = 0$.

Let $\dot{A}: \dot{E} \rightarrow \dot{E}$ be the quotient mapping of $A \bmod N: \dot{A}\dot{x} = \dot{A}\dot{x}$ where $\dot{x} = x + N$ is the coset in \dot{E} which contains x . \dot{A} is a continuous linear mapping of \dot{E} into \dot{E} .

On \dot{E} define an inner product by setting $(\dot{x}, \dot{y}) = (x, y)$. Clearly (\dot{x}, \dot{y}) is continuous in \dot{x} and \dot{y} , and if condition 4' holds for A it also holds for \dot{A} because $(\dot{A}\dot{x}, \dot{x}) = (Ax, x)$ for all $\dot{x} \in \dot{E}$. If condition 4 holds define an involution in \dot{E} by setting $\dot{x}^* = \widehat{x^*}$. \dot{x}^* is well defined for $x \in N \Rightarrow x^* \in N$ since $(x^*, x^*) = (x, x)$. Moreover,

$$(\dot{x}^*, \dot{y}^*) = (x^*, y^*) = (y, x) = (\dot{y}, \dot{x})$$

and

$$\dot{A}\dot{x}^* = \dot{A}\widehat{x^*} = \widehat{Ax^*} = \widehat{(Ax)^*} = (\widehat{Ax})^* = (\dot{A}\dot{x})^*.$$

Thus condition 4 holds for \dot{A} .

It follows from what precedes that the conclusions of the theorem are valid for E and A , etc., replaced by \dot{E} and \dot{A} , etc., respectively. Thus there exists a finite, positive Radon measure μ on R whose support is contained in $[0, \infty)$ if 4' holds (for A), a μ -measurable function d with values in $\{1, 2, 3, \dots, \infty\}$, and for every $\lambda \in R$, a sequence $\dot{e}'_k(\lambda)$, $k = 1, 2, 3, \dots$, in \dot{E}' such that $\dot{e}'_k(\lambda) = 0$ for $k > d(\lambda)$ and conditions 5-9 hold.

Let $e'_k(\lambda) \in E'$ be the linear functional on E defined by

$$\langle e'_k(\lambda), x \rangle = \langle \dot{e}'_k(\lambda), \dot{x} \rangle \quad \text{for every } x \in E;$$

then $e'_k(\lambda) = 0$ for $k > d(\lambda)$.

5. $\lambda \rightarrow \langle \dot{e}'_k(\lambda), \dot{x} \rangle = \langle e'_k(\lambda), x \rangle$ is μ -measurable for every $k = 1, 2, \dots$.

6. $e'_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, are linearly independent for $\lambda \notin N_0$ (because $\dot{e}'_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, are linearly dependent if and only if $\dot{e}'_k(\lambda)$, $1 \leq k < d(\lambda) + 1$, are linearly dependent).

7. ${}^t A e'_k(\lambda) = \lambda e'_k(\lambda)$ for all $\lambda \in R$ and all k , because $\langle \dot{e}'_k(\lambda), Ax \rangle = \langle \dot{e}'_k(\lambda), \dot{A}\dot{x} \rangle = \lambda \langle \dot{e}'_k(\lambda), \dot{x} \rangle = \lambda \langle e'_k(\lambda), x \rangle$ for all $x \in E$, $\lambda \in R$ and $k = 1, 2, \dots$.

Conditions 8 and 9 follow from the fact that

$$\sum_{k=1}^{d(\lambda)} |\langle e'_k(\lambda), x \rangle \overline{\langle e'_k(\lambda), y \rangle}| = \sum_{k=1}^{d(\lambda)} |\langle \dot{e}'_k(\lambda), \dot{x} \rangle \overline{\langle \dot{e}'_k(\lambda), \dot{y} \rangle}|.$$

This completes the proof.

3. **Rotation invariant distributions.** Let $O(N)$ be the orthogonal group operating in R^N , f a function defined on the ball $S_N(a)$. For every $\rho \in O(N)$ we define the function ρf by $\rho f(x) = f(\rho^{-1}x)$ for $x \in S_N(a)$. f is said to be *rotation invariant* or *radial* if $\rho f = f$ for all $\rho \in O(N)$. In that case $f(x) = F(|x|)$ for all $x \in S_N(a)$, where $F(r)$ is a function on $0 \leq r < a$. In fact, $F(r) = f(r, 0, \dots, 0)$.

Let $d\rho$ be the Haar measure on $O(N)$ normalized such that $\int_{O(N)} d\rho = 1$; we recall that $O(N)$ is compact. If f is a continuous function on $S_N(a)$ we set for every $x \in S_N(a)$

$$f^\#(x) = \int_{O(N)} \rho f(x) d\rho.$$

Clearly $f^\#$ is a continuous rotation invariant function on $S_N(a)$ which is a C^∞ -function if f is a C^∞ -function.

Let $\mathcal{D}^\#(S_N(a))$ be the space of rotation invariant complex valued C^∞ -functions on $S_N(a)$ with compact support. $\mathcal{D}^\#(S_N(a))$ is a vector subspace of $\mathcal{D}(S_N(a))$ (the usual space of test functions on $S_N(a)$ of distribution theory with the Schwartz topology). We endow $\mathcal{D}^\#(S_N(a))$ with the topology induced by $\mathcal{D}(S_N(a))$.

If a sequence $(\phi_n)_{n \geq 1}$ in $\mathcal{D}(S_N(a))$ converges to 0 in $\mathcal{D}(S_N(a))$, i.e. $\phi_n \rightarrow 0$ uniformly, as well as all its derivatives and keeps its support in a fixed compact set, the same is true for the sequence $(\phi_n^\#)_{n \geq 1}$. That is, the mapping $\#: \phi \rightarrow \phi^\#$ is a linear sequentially continuous mapping of $\mathcal{D}(S_N(a))$ onto $\mathcal{D}^\#(S_N(a))$, whence—since $\mathcal{D}(S_N(a))$ is bornological— $\#$ is a continuous projection of $\mathcal{D}(S_N(a))$ onto $\mathcal{D}^\#(S_N(a))$. Clearly $(\rho\phi)^\# = \phi^\#$ for all $\phi \in \mathcal{D}(S_N(a))$ and $\rho \in O(N)$ since the Haar measure $d\rho$ is unimodular.

If T is a distribution in $S_N(a)$, i.e. $T \in \mathcal{D}'(S_N(a))$, and $\rho \in O(N)$ we define a new distribution ρT on $S_N(a)$ by $\phi \rightarrow \langle T, \rho^{-1}\phi \rangle$ for all $\phi \in \mathcal{D}(S_N(a))$. We note that if T corresponds to a locally integrable function f on $S_N(a)$, ρT corresponds to ρf . T is said to be *rotation invariant* if $\rho T = T$ for all $\rho \in O(N)$. The rotation invariant distributions on $S_N(a)$ form a closed vector subspace $\mathcal{D}'^\#(S_N(a))$ of $\mathcal{D}'(S_N(a))$ ($\mathcal{D}'(S_N(a))$ is endowed with the strong dual topology).

Let $\mathcal{D}^\#(S_N(a))'$ be the strong dual of $\mathcal{D}^\#(S_N(a))$ and ${}^t\#: \mathcal{D}^\#(S_N(a))' \rightarrow \mathcal{D}'(S_N(a))$ the transpose of $\#$.

Proposition 1. ${}^t\#$ is a topological isomorphism of $\mathcal{D}^\#(S_N(a))'$ onto $\mathcal{D}'^\#(S_N(a))$; its inverse is the mapping which associates with every rotation invariant distribution T in $S_N(a)$ its restriction to $\mathcal{D}^\#(S_N(a))$. Thus every element in $\mathcal{D}^\#(S_N(a))'$ is the restriction to $\mathcal{D}^\#(S_N(a))$ of a unique element in $\mathcal{D}'^\#(S_N(a))$.

Proof. $\#: \mathcal{D}(S_N(a)) \rightarrow \mathcal{D}^\#(S_N(a))$ is continuous and its range is $\mathcal{D}^\#(S_N(a))$. Therefore ${}^t\#$ is continuous and 1-1.

Suppose $T = {}^t\#(S)$, $S \in \mathcal{D}^\#(S_N(a))'$. Then for every $\phi \in \mathcal{D}(S_N(a))$ and $\rho \in O(N)$,

$$\langle \rho T, \phi \rangle = \langle T, \rho^{-1}\phi \rangle = \langle S, (\rho^{-1}\phi)^\# \rangle = \langle S, \phi^\# \rangle = \langle T, \phi \rangle,$$

whence $T \in \mathcal{D}'^\#(S_N(a))$. Thus the range of ${}^t\#$ is contained in $\mathcal{D}'^\#(S_N(a))$.

Suppose $T \in \mathcal{D}'^\#(S_N(a))$. Let S be its restriction to $\mathcal{D}^\#(S_N(a))$. Then $\langle T, \phi \rangle = \langle T, \rho\phi \rangle$ for every $\phi \in \mathcal{D}(S_N(a))$ and $\rho \in O(N)$. Therefore

$$\langle T, \phi \rangle = \int_{O(N)} \langle T, \phi \rangle d\rho = \int_{O(N)} \langle T, \rho\phi \rangle d\rho = \left\langle T, \int_{O(N)} \rho\phi d\rho \right\rangle (= \langle T, \phi^\# \rangle)$$

for every $\phi \in \mathcal{D}(S_N(a))$. The last equality holds if the distribution T is a function by Fubini's theorem. The general case is obtained by regularization: Choose a sequence $(u_k)_{k \geq 1}$ in $\mathcal{D}^\#(S_N(a))$ such that $u_k \rightarrow T$ in $\mathcal{D}'(S_N(a))$; then

$$\langle T, \phi \rangle = \lim_{k \rightarrow \infty} \langle u_k, \phi \rangle = \lim_{k \rightarrow \infty} \langle u_k, \phi^\# \rangle = \langle T, \phi^\# \rangle$$

for all $\phi \in \mathcal{D}(S_N(a))$. Thus $\langle T, \phi \rangle = \langle T, \phi^\# \rangle = \langle S, \phi^\# \rangle = \langle {}^t\#(S), \phi \rangle$ for all $\phi \in \mathcal{D}(S_N(a))$, whence $T = {}^t\#(S)$.

Finally, if $j: \mathcal{D}^\#(S_N(a)) \rightarrow \mathcal{D}(S_N(a))$ is the natural injection of $\mathcal{D}^\#(S_N(a))$ into $\mathcal{D}(S_N(a))$, the transpose ${}^tj: \mathcal{D}'(S_N(a)) \rightarrow \mathcal{D}'^\#(S_N(a))'$ is continuous. If $T \in \mathcal{D}'^\#(S_N(a))$, then $\langle {}^tj(T), \phi \rangle = \langle T, \phi \rangle$ for all $\phi \in \mathcal{D}^\#(S_N(a))$. Thus the restriction of tj to $\mathcal{D}'^\#(S_N(a))$ is precisely the inverse of ${}^t\#$ (restricted to $\mathcal{D}'^\#(S_N(a))$). This completes the proof.

4. Further preliminaries. If a function $f \in L^1(\mathbb{R}^N)$ is radial, $f(x) = F(|x|)$, then its Fourier transform $\hat{f}(t) = \int_{\mathbb{R}^N} e^{-it \cdot x} f(x) dx$ is equal to [2, p. 187]

$$\hat{f}(t) = (2\pi)^{N/2} \int_0^\infty \Omega_N(|t|r) F(r) d\mu_N(r) = \int_{\mathbb{R}^N} \Omega_N(|t||x|) f(x) dx.$$

Here $d\mu_N(r) = [2^{(N-2)/2} \Gamma(N/2)]^{-1} r^{N-1} dr$ and

$$\begin{aligned} \Omega_N(z) &= \Gamma\left(\frac{N}{2}\right) \left(\frac{z}{2}\right)^{(N-2)/2} J_{(N-2)/2}(z) \quad \text{if } z \neq 0 \text{ and} \\ &= 1 \quad \text{if } z = 0, \\ &= 1 - \frac{z^2}{2N} + \frac{z^4}{2 \cdot 4N(N+2)} - \frac{z^6}{2 \cdot 4 \cdot 6N(N+2)(N+4)} + \dots \end{aligned}$$

$J_\nu(z)$ is the Bessel function of index ν .

(The function $\tilde{F}(\lambda) = \int_0^\infty \Omega_N(\lambda r) F(r) d\mu_N(r)$, $0 \leq \lambda < \infty$, is called the *Hankel transform* of F of order $(N-2)/2$. If F has compact support $\tilde{F}(\lambda)$ exists for all complex numbers λ ; in this case \tilde{F} is an even entire analytic function.)

5. Functions positive definite relative to the orthogonal group.

Definition 1. A continuous complex valued function f (see the note following

the proof of Theorem 2) on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N is said to be positive definite relative to the orthogonal group $(O(N))$ if it is radial and if

$$(10) \quad \int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)\phi(x)\overline{\phi(y)} dx dy \geq 0$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

Theorem 2. A continuous function f on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N is positive definite relative to the orthogonal group if and only if there exists a positive bounded Radon measure γ on R such that

$$(11) \quad f(x) = \int_{-\infty}^\infty \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } x \in S_N(a).$$

The measure γ may be chosen such that its support is contained in $[0, \infty)$ if in addition to (10)

$$(12) \quad \int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)(-\Delta\phi)(x)\overline{\phi(y)} dx dy \geq 0$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

Proof. Let $E = \mathcal{D}^\#(G)$ where $G = S_N(a/2)$. E is a subspace of $\mathcal{D}(G)$ which is nuclear [9, Chapter II, p. 55] (see also [20, p. 530]), hence E is nuclear. Clearly E is separable. On E we define a pseudo-inner product by setting

$$(\phi, \psi) = \int_G \int_G f(x-y)\phi(x)\overline{\psi(y)} dx dy = \int_{S_N(a)} f(x)\phi * \psi^*(x) dx$$

for ϕ and ψ in $\mathcal{D}^\#(G)$. Here $\psi^*(x) = \overline{\psi(-x)} = \overline{\psi(x)}$.

Let A be the linear operator $-\Delta$ in E . Here Δ denotes the Laplacian in R^N . A is continuous in E and $(A\phi, \psi) = (\phi, A\psi)$ for all ϕ and ψ in E because $\phi_{x_i} * \psi = -\phi * \psi_{x_i}$, $\phi_{x_i}^* = -(\phi_{x_i})^*$, $i = 1, \dots, N$, and therefore $\Delta\phi * \psi^* = \phi * \Delta\psi^* = \phi * (\Delta\psi)^*$ for all ϕ and ψ in $C_0^\infty(R^N)$. If we introduce in E the involution $\phi \rightarrow \phi^*$ then all the conditions 1-4 of Theorem 1 are satisfied. Let $\mu, d, e'_k(\lambda)$ and N_0 be as in Theorem 1. The measure μ has its support in $[0, \infty)$ if (12) holds. $e'_k(\lambda) \in E' = \mathcal{D}^\#(G)'$ for all λ and all k . By Proposition 1 there exists for every λ and k a unique rotation invariant distribution $e_k(\lambda)$ in G whose restriction to $\mathcal{D}^\#(G)$ is $e'_k(\lambda)$. A simple calculation shows that the Laplacian commutes with every rotation: $\Delta(\rho\phi) = \rho(\Delta\phi)$ for all $\phi \in C_0^\infty(R^N)$ and $\rho \in O(N)$ and hence $(\Delta\phi)^\# = \Delta\phi^\#$ for every $\phi \in \mathcal{D}(G)$. Hence for every $\phi \in \mathcal{D}(G)$,

$$\langle e_k(\lambda), -\Delta\phi \rangle = \langle e'_k(\lambda), (-\Delta\phi)^\# \rangle = \langle e'_k(\lambda), A\phi^\# \rangle$$

by Proposition 1. Similarly $\langle e_k(\lambda), \phi \rangle = \langle e'_k(\lambda), \phi^\# \rangle$ for $\phi \in \mathcal{D}(G)$. But $\langle e'_k(\lambda), A\phi^\# \rangle = \lambda \langle e_k(\lambda), \phi^\# \rangle$ for all $\phi^\# \in \mathcal{D}^\#(G) = E$ by 7 of Theorem 1. Hence, $\langle e_k(\lambda), -\Delta\phi \rangle = \lambda \langle e_k(\lambda), \phi \rangle$ for all $\phi \in \mathcal{D}(G)$ and all $\lambda \in R$ and $k = 1, 2, \dots$. That is—since the transpose of the differential operator Δ is Δ —

$$(13) \quad (\Delta + \lambda)e_k(\lambda) = 0 \quad \text{for all } \lambda \in R, \text{ and } k = 1, 2, \dots,$$

in the sense of distributions. It follows since $\Delta + \lambda$ is an elliptic differential operator that $e_k(\lambda)$ is a C^∞ -function $e_k(\lambda, x)$ in G which satisfies the differential equation (13) in the classical sense. In fact $e_k(\lambda, x)$ is an analytic function of x since the coefficients of $\Delta + \lambda$ are analytic [10, Theorem 7.5.1, p. 178].

Since $e_k(\lambda, x) = e_k(\lambda; x_1, x_2, \dots, x_N)$ is radial,

$$e_k(\lambda, x) = u_k(\lambda, r), \quad r = |x| < a/2,$$

where $u_k(\lambda, r) = e_k(\lambda; r, 0, \dots, 0)$ is a C^∞ -function on $0 \leq r < a/2$ which satisfies the equation

$$(14) \quad (\Delta + \lambda)e_k(\lambda, x) = \frac{d^2 u_k(\lambda, r)}{dr^2} + \frac{N-1}{r} \frac{du_k(\lambda, r)}{dr} + \lambda u_k(\lambda, r) = 0$$

for $0 < r < a/2$.

If $\lambda = 0$, the general solution of (14) is

$$e_k(0, x) = u_k(0, r) = A_0 + B/r^{N-2} \quad \text{if } N \neq 2, \text{ and}$$

$$e_k(0, x) = u_k(0, r) = A_0 + B \log r \quad \text{if } N = 2,$$

for $0 < r = |x| < a/2$. (A_0 and B are constants.) It follows since $e_k(\lambda, x)$ is a C^∞ -function in G that $e_k(0, x) = A_0$ for every $N = 1, 2, 3, \dots$.

If $\lambda \neq 0$ the general solution of (14) is $e_k(\lambda, x) = u_k(\lambda, r) = r^{-(N-2)/2} w(\lambda^{1/2} r)$, where $w(r)$ is the general solution of the Bessel equation

$$r^2 w''(r) + r w'(r) + (r^2 - ((N-1)/2)^2) w(r) = 0$$

(the simple computation is left to the reader). It follows that if $\lambda \neq 0$,

$$(15) \quad e_k(\lambda, x) = u_k(\lambda, r) = C_1 r^{-(N-2)/2} J_{(N-2)/2}(\lambda^{1/2} r) + C_2 r^{-(N-2)/2} N_{(N-2)/2}(\lambda^{1/2} r)$$

for $0 < r = |x| < a/2$. Here J_ν is Bessel's function of index ν and N_ν is Neumann's function (or Bessel's function of the second kind) of index ν . C_1 and C_2 are constants.

If $m = (N-2)/2$ is not an integer,

$$N_m(r) = \frac{\cos m\pi J_m(r) - J_{-m}(r)}{\sin m\pi}$$

and therefore (15) becomes

$$\begin{aligned} e_k(\lambda, x) &= u_k(\lambda, r) = C_1^1 r^{-m} J_m(\lambda^{1/2} r) + C_2^1 r^{-m} J_{-m}(\lambda^{1/2} r) \\ &= A_1 \Omega_N(\lambda^{1/2} r) + C_2^1 r^{-m} J_{-m}(\lambda^{1/2} r) \end{aligned}$$

for $0 < r = |x| < a/2$. Here C_1^1 , C_2^1 and A_1 are constants. Now $e_k(\lambda, x)$ is a C^∞ -function in $S_N(a)$ and so is $\Omega_N(\lambda^{1/2}|x|)$ (for every N). Therefore $C_2^1 r^{-m} J_{-m}(\lambda^{1/2}r)$ may be extended to $r = 0$ so as to be infinitely differentiable at $x = 0$. But $|r^{-m} J_{-m}(\lambda^{1/2}r)| \rightarrow \infty$ as $r \rightarrow 0$ if $m > 0$ (i.e. $N \geq 3$); and if $N = 1$, $r^{-m} J_{-m}(\lambda^{1/2}r) = (2/\pi(\lambda)^{1/2})^{1/2} \sin(\lambda^{1/2}|x|)$ which is not differentiable at $x = 0$. Therefore $C_2^1 = 0$ in all cases. Thus

$$e_k(\lambda, x) = A_1 \Omega_N(\lambda^{1/2}|x|) \text{ for } |x| < a/2.$$

If $m = (N - 2)/2$ is an integer then $|r^{-m} J_m(\lambda^{1/2}r)| \rightarrow \infty$ as $r \rightarrow 0$. See [21]. It follows for the same reason as above that C_2 in (15) must be zero. Thus

$$e_k(\lambda, x) = C_1 r^{-m} J_m(\lambda^{1/2}r) = A_2 \Omega_N(\lambda^{1/2}r),$$

A_2 constant, for $r = |x| < a/2$. Since $\Omega_N(0|x|) = 1$ we have

$$e_k(\lambda, x) = A(\lambda) \Omega_N(\lambda^{1/2}|x|) \text{ for } |x| < a/2 \text{ and all } \lambda \in R,$$

where $A(\lambda)$ is a constant depending upon λ . Therefore $d(\lambda) = 1$ μ -a.e. and

$$\langle e_1(\lambda), \phi \rangle = A(\lambda) \int_{S_N(a/2)} \Omega_N(\lambda^{1/2}|x|) \phi(x) dx.$$

By 9 of Theorem 1,

$$\langle e_1(\lambda), \phi \rangle \overline{\langle e_1(\lambda), \psi \rangle} = |A(\lambda)|^2 \int_G \int_G \Omega_N(|x|\lambda^{1/2}) \Omega_N(|y|\lambda^{1/2}) \phi(x) \overline{\psi(y)} dx dy$$

belongs to $L^1(\mu)$ for all ϕ and ψ in $\mathcal{D}^\#(G)$. Hence $\lambda \rightarrow |A(\lambda)|^2$ is locally μ -integrable. Let γ be the positive Radon measure $d\gamma(\lambda) = |A(\lambda)|^2 d\mu(\lambda)$ on R ; then by 9 of Theorem 1,

$$\begin{aligned} \langle \phi, \psi \rangle &= \int_G \int_G f(x - y) \phi(x) \overline{\psi(y)} dx dy \\ (16) \quad &= \int_{-\infty}^{\infty} \int_G \int_G \Omega_N(|x|\lambda^{1/2}) \Omega_N(|y|\lambda^{1/2}) \phi(x) \overline{\psi(y)} dx dy d\gamma(\lambda) \end{aligned}$$

for all ϕ and ψ in $\mathcal{D}^\#(G)$.

Let x_0 be an arbitrary but fixed point in G , and let, for every positive integer n , ϕ_n be a function in $\mathcal{D}^\#(G)$ such that $\phi_n \geq 0$, $\text{supp } \phi \subset \{x \mid ||x| - |x_0|| < 1/n\}$ and $\int \phi_n dx = 1$, then by standard arguments

$$\int_G \Omega_N(|x|\lambda^{1/2}) \phi_n(x) dx \rightarrow \Omega_N(|x_0|\lambda^{1/2}) \text{ as } n \rightarrow \infty,$$

and therefore if we substitute in (16) ϕ_n for ϕ and ψ we obtain by Fatou's lemma

$$\begin{aligned} \int_{-\infty}^{\infty} (\Omega_N(|x_0|\lambda^{1/2}))^2 d\gamma(\lambda) &\leq \liminf_{n \rightarrow \infty} \int_G \int_G f(x - y) \phi_n(x) \overline{\phi_n(y)} dx dy \\ &= \liminf_{n \rightarrow \infty} \int_{S_N(a)} f(x) \phi_n * \phi_n(x) dx = \liminf_{n \rightarrow \infty} \int_{|x| \leq 2|x_0| + 2/n} f(x) \phi_n * \phi_n(x) dx \\ &\leq \sup_{|x| \leq 2|x_0|} |f(x)| = M(x_0) < \infty. \end{aligned}$$

Therefore if x and y are arbitrary points in G then

$$(17) \quad \int_{-\infty}^{\infty} |\Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2})| d\gamma(\lambda) \leq \left(\int_{-\infty}^{\infty} (\Omega_N(|x|\lambda^{1/2}))^2 d\gamma(\lambda)\right)^{1/2} \left(\int_{-\infty}^{\infty} (\Omega_N(|y|\lambda^{1/2}))^2 d\gamma(\lambda)\right)^{1/2} \\ \leq M(x)^{1/2} M(y)^{1/2} < \infty.$$

In particular, $\int_{-\infty}^{\infty} d\gamma(\lambda) \leq M(0) = |f(0)|$; i.e. γ is a finite measure.

Now

$$\int_G \int_G |\Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2})\phi(x)\overline{\psi(y)}| dx dy \\ \leq \left(\int_G |\Omega_N(|x|\lambda^{1/2})\phi(x)|^2 dx\right)^{1/2} \left(\int_G |\Omega_N(|y|\lambda^{1/2})\psi(y)|^2 dy\right)^{1/2}$$

and therefore

$$\int_{-\infty}^{\infty} \int_G \int_G |\Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2})\phi(x)\overline{\psi(y)}| dx dy d\gamma(\lambda) \\ \leq \left(\int_{-\infty}^{\infty} \int_G |\Omega_N(|x|\lambda^{1/2})\phi(x)|^2 dx d\gamma(\lambda)\right)^{1/2} \left(\int_{-\infty}^{\infty} \int_G |\Omega_N(|y|\lambda^{1/2})\psi(y)|^2 dy d\gamma(\lambda)\right)^{1/2} \\ = (\phi, \phi)(\psi, \psi) < \infty \text{ for all } \phi \text{ and } \psi \text{ in } \mathcal{D}^{\#}(G).$$

Hence from (16) by Fubini's theorem

$$(18) \quad \int_G \int_G f(x-y)\phi(x)\overline{\psi(y)} dx dy = \int_G \int_G K(x, y)\phi(x)\overline{\psi(y)} dx dy$$

for all ϕ and ψ in $\mathcal{D}^{\#}(G)$. Here $K(x, y) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2}) d\gamma(\lambda)$ for x and y in G .

$K(x, y)$ is continuous on $G \times G$, for $(x, y) \rightarrow \int_0^{\infty} \Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2}) d\gamma(\lambda)$ is continuous on $R^N \times R^N$ by the Lebesgue bounded convergence theorem, since $|\Omega_N(r)| \leq 1$ for all $0 \leq r < \infty$ (cf. (22) below), and

$$(x, y) \rightarrow \int_{-\infty}^0 \Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2}) d\gamma(\lambda)$$

is continuous on $G \times G$ by the Lebesgue dominated convergence theorem, since $1 \leq \Omega_N(ir) \leq \Omega_N(ir_1)$ if $0 \leq r \leq r_1$.

If we choose a sequence (ϕ_n) in $\mathcal{D}^{\#}(G)$ as above for $x_0 = 0$ and substitute ϕ_n for ψ in (18) and let $n \rightarrow \infty$, we obtain

$$(19) \quad \int_G f(x)\phi(x) dx = \int_G K(x, 0)\phi(x) dx$$

for all $\phi \in \mathcal{D}^{\#}(G)$. $K(x, 0) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda)$.

From (19) follows, by Proposition 1 and continuity of $f(x)$ and $K(x, 0)$ on G , that

$$(20) \quad f(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \text{ for all } |x| < a/2.$$

To complete the proof we must show that (20) holds for all $|x| < a$. (If $a = \infty$ there is nothing to prove.)

Let

$$(21) \quad F(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \text{ for } |x| < a.$$

The integral exists for $|x| < a$. This is seen as follows:

Using the formula [21, p. 367],

$$\Omega_N(zu)\Omega_N(zv) = C_N \int_0^\pi \Omega_N(z(u^2 + v^2 - 2uv \cos \theta)^{1/2}) (\sin \theta)^{N-2} d\theta,$$

where $u, v \in R$, z complex and $C_N = \Gamma(N/2)[\Gamma((N-1)/2)\pi^{1/2}]^{-1}$, we have, for $0 < |x| < a/2$,

$$\begin{aligned} K(x, x) &= \int_{-\infty}^{\infty} (\Omega_N(|x|\lambda^{1/2}))^2 d\gamma(\lambda) \\ &= C_N \int_{-\infty}^{\infty} \int_0^\pi \Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2}) (\sin \theta)^{N-2} d\theta d\gamma(\lambda), \end{aligned}$$

from which follows that

$$\int_{-\infty}^0 \int_0^\pi \Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2}) (\sin \theta)^{N-2} d\theta d\gamma(\lambda) < \infty.$$

(Note that the integrand is nonnegative.) Therefore,

$$\int_{-\infty}^{\infty} \int_0^\pi |\Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2})| (\sin \theta)^{N-2} d\theta d\gamma(\lambda) < \infty.$$

It follows by Fubini's theorem that

$$\int_{-\infty}^{\infty} |\Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2})| d\gamma(\lambda) < \infty$$

for almost all $0 \leq \theta \leq \pi$. But

$$\int_0^\infty |\Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2})| d\gamma(\lambda) < \infty$$

for all $0 \leq \theta \leq \pi$, and $\Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2}) \leq \Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta_1)^{1/2})$ for $0 \leq \theta \leq \theta_1 \leq \pi$ if $\lambda < 0$. Therefore

$$\int_{-\infty}^{\infty} |\Omega_N(\lambda^{1/2}|x|(2 - 2 \cos \theta)^{1/2})| d\gamma(\lambda) < \infty$$

for $0 \leq \theta < \pi$. Thus

$$\int_{-\infty}^{\infty} |\Omega_N(\lambda^{1/2}|y|)| d\gamma(\lambda) < \infty \quad \text{for all } |y| < 2|x|,$$

and since $|x| < a/2$ was arbitrary

$$\int_{-\infty}^{\infty} |\Omega_N(|y|\lambda^{1/2})| d\gamma(\lambda) < \infty \quad \text{for all } |y| < a.$$

From the preceding argument also follows, using the Lebesgue dominated convergence theorem, that $F(x)$ is continuous on $S_N(a)$.

Using the well-known formula ([15, (5)], [21, p. 51])

$$(22) \quad \Omega_N(z|t|) = \int_{|\xi|=1} e^{iz(\xi \cdot t)} d\sigma_N(\xi),$$

where z is any complex number, $t \in R^N$ and σ_N the surface measure of the unit sphere $\{\xi \in R^N \mid |\xi| = 1\}$ in R^N normalized so that $\int d\sigma_N = 1$, one verifies routinely that

$$(23) \int_G \int_G \Omega_N(|x-x|\lambda^{1/2})\phi(x)\overline{\psi(y)} dx dy = \int_G \int_G \Omega_N(|x|\lambda^{1/2})\Omega_N(|y|\lambda^{1/2})\phi(x)\overline{\psi(y)} dx dy$$

for all ϕ and ψ in $\mathcal{D}^\#(G)$. From this follows using Fubini's theorem that

$$\int_G \int_G F(x-y)\phi(x)\overline{\psi(y)} dx dy = \int_G \int_G K(x,y)\phi(x)\overline{\psi(y)} dx dy$$

for all ϕ and ψ in $\mathcal{D}^\#(G)$, and hence, by (18),

$$\int_G \int_G f(x-y)\phi(x)\overline{\psi(y)} dx dy = \int_G \int_G F(x-y)\phi(x)\overline{\psi(y)} dx dy$$

for all ϕ and ψ in $\mathcal{D}^\#(G)$. Let $b(x) = f(x) - F(x)$ for $x \in S_N(a)$. Then b is a continuous rotation invariant function and

$$(24) \int_G \int_G b(x-y)\phi(x)\psi(y) dx dy = 0$$

for all ϕ and ψ in $\mathcal{D}^\#(G)$. We assert that this implies that $b = 0$, which will establish the "if" part of the theorem because of (21).

Suppose $\phi \in C_0^\infty(G)$ and $\psi \in \mathcal{D}^\#(G)$; then for every $\rho \in O(N)$,

$$\begin{aligned} \int_G \int_G b(x-y)\phi(x)\psi(y) dx dy &= \int_G \int_G b(\rho(x-y))\phi(x)\psi(y) dx dy \\ &= \int_G \int_G b(x-y)\phi(\rho^{-1}x)\psi(\rho^{-1}y) dx dy = \int_G \int_G b(x-y)\phi(\rho^{-1}x)\psi(y) dx dy. \end{aligned}$$

Hence

$$\begin{aligned} \int_G \int_G b(x-y)\phi(x)\psi(y) dx dy &= \int_{O(N)} \int_G \int_G b(x-y)\phi(\rho^{-1}x)\psi(y) dx dy d\rho \\ &= \int_G \int_G b(x-y)\phi^\#(x)\psi(y) dx dy = 0 \end{aligned}$$

by Fubini's theorem and (24). Thus $\int_G \int_G b(x-y)\phi(x)\psi(y) dx dy = 0$ for all $\phi \in C_0^\infty(G)$ and $\psi \in \mathcal{D}^\#(G)$. From this follows that

$$(25) \int_G b(x-y)\psi(y) dy = 0 \text{ for every } x \in G \text{ and every } \psi \in \mathcal{D}^\#(G).$$

We shall prove that (25) implies that $b(x) = 0$ for all $|x| < a$. (Note that (25) trivially implies that $b(x) = 0$ for $|x| < a/2$.) For $N = 1$, (25) may be written in the form

$$\int_0^{a/2} [b(x+y) + b(x-y)]\psi(y) dy = 0 \text{ for } |x| < a/2 \text{ and all } \psi \in \mathcal{D}^\#(-a/2, a/2).$$

If $|x_0| < a/2$, substitute in this equation for ψ a function $\psi_n \in \mathcal{D}^\#(-a/2, a/2)$ which has the properties: $\psi_n \geq 0$, $\text{supp } \psi_n \subset \{x \mid |x| - |x_0| < 1/n\}$ and $\int \psi_n dx = 1$.

If we now let $n \rightarrow \infty$ we obtain that

$$(b(x+x_0) + b(x-x_0))/2 = 0 \text{ for } |x| < a/2.$$

Therefore $b(2x_0) + b(0) = 0$ (take $x = x_0$) and $b(0) = 0$ (take $x = x_0 = 0$), whence $b(2x_0) = 0$. Thus $b(x) = 0$ for all $|x| < a$.

If $N \geq 2$, (25) may be rewritten using generalized spherical coordinates in the form⁽²⁾

$$0 = \int_{|y| < a/2} b(x - y)\psi(y) dy$$

$$= \frac{2\pi^{(N-1)/2}}{\Gamma((N-1)/2)} \int_0^{a/2} \int_0^\pi H((|x|^2 - 2|x|r \cos \theta + r^2)^{1/2}) (\sin \theta)^{N-2} \Psi(r)r^{N-1} d\theta dr$$

for $|x| < a/2$ and all $\psi \in \mathcal{D}^{\#}(G)$. Here $\psi(x) = \Psi(|x|)$ and $b(x) = H(|x|)$.

From this we conclude that $\int_0^\pi H((t^2 - 2tr \cos \theta + r^2)^{1/2}) (\sin \theta)^{N-2} d\theta = 0$ for all $0 \leq t < a/2$ and $0 \leq r < a/2$. In particular, if we let $r = t$,

$$\int_0^\pi H(t(2 - 2 \cos \theta)^{1/2}) (\sin \theta)^{N-2} d\theta = 0 \quad \text{for } 0 \leq t < a/2.$$

If we make the substitution $x = (t/2)(2 - 2 \cos \theta)^{1/2}$, $0 < t < a/2$, we obtain (up to the factor $2^{N-1}/t^{2(N-2)}$) the equation

$$(26) \quad \int_0^t H(2x)x^{N-2}((t^2 - x^2)^{1/2})^{N-3} dx = 0 \quad \text{for } 0 < t < a/2,$$

and hence for $0 \leq t < a/2$.

If $N = 3$, equation (26) shows that $\int_0^t H(2x)x^{N-2} dx = 0$ for all $0 \leq t < a/2$ which implies that $H(2x)x^{N-2} = 0$ for $0 \leq x < a/2$ whence $H(2x) = 0$ for $0 < x < a/2$.

If N is an odd integer > 3 then (26) implies that

$$\frac{d}{dt} \int_0^t H(2x)x^{N-2}((t^2 - x^2)^{1/2})^{N-3} dx$$

$$= (N - 3)t \int_0^t H(2x)x^{N-2}((t^2 - x^2)^{1/2})^{N-5} dx = 0$$

for $0 \leq t < a/2$ and therefore

$$\int_0^t H(2x)x^{N-2}((t^2 - x^2)^{1/2})^{N-5} dx = 0 \quad \text{for } 0 \leq t < a/2.$$

Continuing this way until the exponent of $(t^2 - x^2)^{1/2}$ is zero we conclude that

$$\int_0^t H(2x)x^{N-2} dx = 0 \quad \text{for } 0 \leq t < a/2$$

and hence as above $H(2x) = 0$ for $0 \leq x < a/2$.

If N is an even integer we first conclude using the above procedure that

$$(27) \quad \int_0^t H(2x)x^{N-2}(t^2 - x^2)^{-1/2} dx = 0 \quad \text{for all } 0 \leq t < a/2.$$

Multiplying both sides of (27) by t and integrating we get

$$\int_0^t \int_0^s H(2x)x^{N-2} \frac{s}{(s^2 - x^2)^{1/2}} dx ds = \int_0^t \int_x^t H(2x)x^{N-2} \frac{s}{(s^2 - x^2)^{1/2}} ds dx$$

$$= \int_0^t H(2x)x^{N-2}(t^2 - x^2)^{1/2} dx = 0 \quad \text{for } 0 \leq t < a/2.$$

⁽²⁾ Cf. (8) in the following article.

Continuing this way we obtain the equations

$$(28) \quad \int_0^t H(2x)x^{N-2}((t^2-x^2)^{1/2})^{2n-1} dx = 0 \quad \text{for } 0 \leq t < a/2 \text{ and } n = 0, 1, 2, 3, \dots$$

If we now make the substitution $u = (t^2 - x^2)^{1/2}$, equations (28) yield

$$(29) \quad \int_0^t H(2(t^2-u^2)^{1/2})((t^2-u^2)^{1/2})^{N-3}u^{2n} du = 0 \quad \text{for } 0 \leq t < a/2 \text{ and } n = 0, 1, 2, 3, \dots$$

From this follows by the Stone-Weierstrass approximation theorem that, for fixed $0 < t < a/2$,

$$\int_0^t H(2(t^2-u^2)^{1/2})((t^2-u^2)^{1/2})^{N-3}g(u) du = 0$$

for all continuous functions g on $0 \leq u \leq t$. Therefore,

$$H(2(t^2-u^2)^{1/2})((t^2-u^2)^{1/2})^{N-3} = 0 \quad \text{for } 0 \leq u \leq t$$

(note that $H(2(t^2-u^2)^{1/2})((t^2-u^2)^{1/2})^{N-3}$ is continuous on $0 \leq u \leq t$ even if $N = 2$ because $H(x) = 0$ for $0 \leq x < a/2$); in particular, $H(2t) = 0$ for $0 < t < a/2$. We have thus proved that $H(2t) = 0$ for $0 \leq t < a/2$ in all cases.

The "only if" part of the theorem is proved by direct computation. If

$$f(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for } |x| < a$$

and $\phi \in \mathcal{D}^\#(G)$, where $G = S_N(a/2)$, then by (23),

$$(30) \quad \int_G \int_G \Omega_N(|x-y|\lambda^{1/2})\phi(x)\overline{\phi(y)} dx dy = \left| \int_G \Omega_N(|x|\lambda^{1/2})\phi(x) dx \right|^2.$$

Therefore,

$$\begin{aligned} \int_G \int_G f(x-y)\phi(x)\overline{\phi(y)} dx dy &= \int_G \int_G \int_{-\infty}^{\infty} \Omega_N(|x-y|\lambda^{1/2})\phi(x)\overline{\phi(y)} d\gamma(\lambda) dx dy \\ &= \int_{-\infty}^{\infty} \left| \int_G \Omega_N(|x|\lambda^{1/2})\phi(x) dx \right|^2 d\gamma(\lambda) \geq 0 \end{aligned}$$

by Fubini's theorem

$$\begin{aligned} &\left(\int_G \int_G \int_{-\infty}^{\infty} |\Omega_N(|x-y|\lambda^{1/2})\phi(x)\overline{\phi(y)}| d\gamma(\lambda) dx dy \right. \\ &\leq \int_G \int_G \int_{-\infty}^0 \Omega_N(|x-y|\lambda^{1/2})|\phi(x)||\phi(y)| d\gamma(\lambda) dx dy \\ &\quad + \int_G \int_G \int_0^{\infty} |\Omega_N(|x-y|\lambda^{1/2})|\phi(x)||\phi(y)| d\gamma(\lambda) dx dy \\ &\leq \int_G \int_G f(x-y)|\phi(x)||\phi(y)| dx dy + 2 \int_G \int_G \int_0^{\infty} |\Omega_N(|x-y|\lambda^{1/2})|\phi(x)||\phi(y)| d\gamma(\lambda) dx dy \\ &\quad \left. < \infty \text{ because } |\Omega_N(|x-y|\lambda^{1/2})|\phi(x)||\phi(y)| \leq |\phi(x)||\phi(y)| \text{ for } \lambda \geq 0 \right) \end{aligned}$$

and (30). This completes the proof.

Remarks. 1. A continuous function positive definite relative to the orthogonal group is necessarily real valued. This follows for example from the integral representation (11).

2. It is clear from the proof of Theorem 2 that if we replace in the definition of a function f positive definite relative to the orthogonal group continuity by local essential boundedness the conclusions of Theorem 2 hold with

$$f(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } |x| < a$$

replaced by

$$f_1(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{exists for all } |x| < a$$

and

$$f(x) = f_1(x) \quad \text{a.e.}$$

3. The measure γ in (11) may be chosen such that $\text{supp } \gamma \subset (-\infty, 0]$ if in addition to (10)

$$(12') \quad \int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)(\Delta\phi)(x)\overline{\phi(y)} dx dy \geq 0$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

We recall that a continuous function f on a ball $S_N(a)$ in R^N is called (*radial*) *exponentially convex* (if it is radial and) if

$$(31) \quad \sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j f(x_i + x_j) \geq 0$$

for all choices of complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and points x_1, x_2, \dots, x_n in $S_N(a/2)$.

Condition (31) is clearly equivalent to

$$\int_{S_N(a/2)} \int_{S_N(a/2)} f(x+y)\phi(x)\overline{\phi(y)} dx dy \geq 0$$

for all ϕ and ψ in $C_0^\infty(S_N(a/2))$. (It implies that f is real valued.)

Now, a continuous function f on a ball $S_N(a)$ in R^N is radial exponentially convex [15, Proposition 2] if and only if there exists a bounded positive Radon measure γ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty \Omega_N(|x|t) d\gamma(t) \quad \text{for all } x \in S_N(a).$$

If we combine this result with Theorem 2 we obtain the following theorem.

Theorem 3. For a continuous function f on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:

1. f is positive definite relative to the orthogonal group $O(N)$.
2. There exists a bounded positive Radon measure γ on R such that

$$f(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } x \in S_N(a).$$

3. There exist bounded positive rotation invariant Radon measures μ_1 and μ_2 on R^N such that $\int_{R^N} e^{x \cdot t} d\mu_2(t) < \infty$ for all $x \in S_N(a)$ and

$$f(x) = \int_{R^N} e^{ix \cdot t} d\mu_1(t) + \int_{R^N} e^{x \cdot t} d\mu_2(t) \quad \text{for all } x \in S_N(a).$$

4. There exists a continuous radial positive definite function f_1 on R^N and a continuous radial exponentially convex function f_2 on $S_N(a)$ such that

$$f(x) = f_1(x) + f_2(x) \quad \text{for all } x \in S_N(a).$$

Proof. (1) \Rightarrow (2). This is the content of Theorem 2.

(2) \Rightarrow (4). Indeed, the equation

$$f(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } x \in S_N(a)$$

may be rewritten in the form

$$f(x) = \int_0^{\infty} \Omega_N(|x|t) d\gamma_1(t) + \int_0^{\infty} \Omega_N(i|x|t) d\gamma_2(t) \quad \text{for all } x \in S_N(a),$$

where γ_1 and γ_2 are bounded positive Radon measures on $[0, \infty)$ such that

$$\int_0^{\infty} \Omega_N(i|x|t) d\gamma_2(t) < \infty \quad \text{for all } x \in S_N(a).$$

Let

$$f_1(x) = \int_0^{\infty} \Omega_N(|x|t) d\gamma_1(t) \quad \text{for all } x \in R^N$$

and

$$f_2(x) = \int_0^{\infty} \Omega_N(i|x|t) d\gamma_2(t) \quad \text{for all } x \in S_N(a);$$

then f_1 is a radial positive definite function on R^N as one verifies immediately using (22). See [19, Theorem 1, p. 816]. f_2 is radial exponentially convex by the preceding discussion and $f(x) = f_1(x) + f_2(x)$ for all $|x| < a$.

(4) \Rightarrow (3). This follows from the fact that every continuous positive definite function f_1 on R^N and every continuous exponentially convex function f_2 on $S_N(a)$ has an integral representation of the form

$$f_1(x) = \int_{R^N} e^{ix \cdot t} d\mu_1(t) \quad \text{for all } x \in R^N$$

and

$$f_2(x) = \int_{R^N} e^{x \cdot t} d\mu_2(t) \quad \text{for all } x \in S_N(a),$$

respectively, where the Radon measures μ_1 and μ_2 are bounded, positive and unique. See [15]. Clearly μ_1 and μ_2 are rotation invariant if and only if f_1 and f_2 are radial, respectively.

(3) \Rightarrow (1). This is obvious.

Corollary 1. *A continuous function on R^N is radial positive definite if and only if it is positive definite relative to the orthogonal group and bounded.*

Theorem 4. *For a continuous function f on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:*

1. f is radial positive definite in $S_N(a)$.
2. There exists a radial positive definite function f_1 on R^N such that $f(x) = f_1(x)$ for all $x \in S_N(a)$.
3. f is positive definite relative to the orthogonal group $O(N)$ and

$$\int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)(-\Delta\phi)(x)\overline{\phi(y)} dx dy \geq 0$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

Proof. (1) \implies (3). This was shown in the introduction.

(3) \implies (2). By Theorem 2, $f(x) = \int_0^\infty \Omega_N(|x|\lambda^{1/2})d\gamma(\lambda)$ for all $x \in S_N(a)$ where γ is a bounded positive Radon measure on $[0, \infty)$. Let $f_1(x) = \int_0^\infty \Omega_N(|x|\lambda^{1/2})d\gamma(\lambda)$ for $x \in R^N$. Then f_1 is a radial positive definite function in R^N as noted above.

(2) \implies 1. Obvious.

Theorem 4'. *For a continuous function f on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:*

1. f is radial exponentially convex in $S_N(a)$.
2. f is positive definite relative to the orthogonal group and

$$\int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)(\Delta\phi)(x)\overline{\phi(y)} dx dy \geq 0$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

Proof. If 1 holds then

$$\begin{aligned} &\int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)\phi(x)\overline{\phi(y)} dx dy \\ &= \int_{S_N(a/2)} \int_{S_N(a/2)} f(x+y)\phi(x)\overline{\phi(y)} dx dy \geq 0 \end{aligned}$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$, and therefore f is positive definite relative to the orthogonal group $O(N)$. Furthermore,

$$\begin{aligned} &\int_{S_N(a/2)} \int_{S_N(a/2)} f(x-y)(\Delta\phi)(x)\overline{\phi(y)} dx dy = \int_{S_N(a/2)} \int_{S_N(a/2)} f(x+y)(\Delta\phi)(x)\overline{\phi(y)} dx dy \\ &= \int_{S_N(a)} f(x)((\Delta\phi) * \overline{\phi})(x) dx = \int_{S_N(a)} f(x) \sum_{i=1}^N (\phi_{x_i} * \overline{\phi_{x_i}})(x) dx \\ &= \sum_{i=1}^N \int_{S_N(a/2)} \int_{S_N(a/2)} f(x+y)\phi_{x_i}(x)\overline{\phi_{x_i}(y)} dx dy \geq 0 \end{aligned}$$

for all radial $\phi \in C_0^\infty(S_N(a/2))$.

If 2 holds then by Theorem 2 and Remark 3 there exists a bounded positive Radon measure γ on $(-\infty, 0]$ such that

$$f(x) = \int_{-\infty}^0 \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } x \in S_N(a).$$

This equation may be rewritten in the form

$$f(x) = \int_0^\infty \Omega_N(i|x|t) d\alpha(t) \quad \text{for all } x \in S_N(a),$$

where α is a bounded positive Radon measure on $[0, \infty)$. Hence f is radial exponentially convex by [15, Proposition 2]. (This amounts to a simple computation using (22).)

Note. From the integral representation

$$f(x) = \int_0^\infty \Omega_N(|x|t) d\mu(t)$$

of a radial positive definite function on R^N follows that f is $[(N-1)/2]$ -times continuously differentiable [19, Lemma 4]. An exponentially convex function is analytic as seen from its integral representation. Therefore by 4 of Theorem 3 a function f on $S_N(a)$ which is positive definite relative to the orthogonal group $O(N)$ is of class $C^{[(N-1)/2]}$. Therefore if $N \geq 5$ condition 3 of Theorem 4 and condition 2 of Theorem 4' are equivalent to

$-\Delta f$ is positive definite relative to $O(N)$ and Δf is positive definite relative to $O(N)$, respectively.

Theorem 5. A real valued function F defined on $0 \leq r < a$ ($0 < a \leq \infty$) has an integral representation of the form

$$(32) \quad F(r) = \int_{-\infty}^\infty e^{r^2 t} d\mu(t) \quad \text{for all } 0 \leq r < a,$$

where μ is a positive bounded Radon measure on R , if and only if $f_N(x) = F(|x|)$, $x \in S_N(a)$, is a positive definite function relative to the orthogonal group $O(N)$ in $S_N(a)$ for every $N = 1, 2, \dots$.

Proof. If F has an integral representation of the form (32) it may be rewritten in the form

$$F(r) = \int_0^\infty e^{-r^2 t} d\mu_1(t) + \int_0^\infty e^{r^2 t} d\mu_2(t) \quad \text{for all } 0 \leq r < a,$$

where μ_1 and μ_2 are bounded positive Radon measures on $[0, \infty)$ such that $\int_0^\infty e^{r^2 t} d\mu_2(t) < \infty$ for all $0 \leq r < a$.

Let $G(r) = \int_0^\infty e^{-r^2 t} d\mu_1(t)$ for $0 \leq r < \infty$ and $H(r) = \int_0^\infty e^{r^2 t} d\mu_2(t)$ for $0 \leq r < a$. Then $g_N(x) = G(|x|)$, $x \in R^N$, is a radial positive definite function in R^N for $N = 1, 2, \dots$. See [19, Theorem 2, p. 817] and $b_N(x) = H(|x|)$, $x \in S_N(a)$, is an exponentially convex function in $S_N(a)$ for $N = 1, 2, 3, \dots$. See [15, Theorem 2]. Therefore

$$f_N(x) = F(|x|) = g_N(x) + b_N(x), \quad x \in S_N(a),$$

is a positive definite function relative to the orthogonal group $O(N)$ in $S_N(a)$ for every $N = 1, 2, \dots$, by 4 of Theorem 3.

Conversely, if $f_N(x) = F(|x|)$, $x \in S_N(a)$, is positive definite relative to the orthogonal group $O(N)$ for $N = 1, 2, 3, \dots$, then there exists by Theorem 2 for every N a bounded positive Radon measure γ_N on R such that

$$(33) \quad f_N(x) = \int_{-\infty}^{\infty} \Omega_N(|x|\lambda^{1/2}) d\gamma_N(\lambda)$$

for all $x \in S_N(a)$. The proof now proceeds mutatis mutandis as in [15].

Equation (33) may be rewritten in the form

$$f_N(x) = \int_0^{\infty} \Omega_N(|x|(2Nt)^{1/2}) d\alpha_N(t) + \int_0^{\infty} \Omega_N(ix|(2Nt)^{1/2}) d\beta_N(t)$$

for all $x \in S_N(a)$, or

$$(34) \quad F(r) = \int_0^{\infty} \Omega_N(r(2Nt)^{1/2}) d\alpha_N(t) + \int_0^{\infty} \Omega_N(ir(2Nt)^{1/2}) d\beta_N(t)$$

for all $0 \leq r < a$, where α_N and β_N are bounded positive Radon measures on $[0, \infty)$ and β_N is such that $\int_0^{\infty} \Omega_N(ir(2Nt)^{1/2}) d\beta_N(t) < \infty$ for all $0 \leq r < a$.

From (34) follows that

$$(35) \quad F(0) = \|\alpha_N\| + \|\beta_N\|, \quad \text{whence } \|\alpha_N\| \leq F(0), \quad \|\beta_N\| < F(0)$$

for all N . Hence there exists a subsequence $(\alpha_{N_k})_{k \geq 1}$ $((\beta_{N_k})_{k \geq 1})$ of (α_N) $((\beta_N))$ which converges to a bounded positive Radon measure α (β) on $[0, \infty)$ in the w^* -topology of the conjugate space of $C_{\infty}([0, \infty))$ (the space of complex valued continuous functions on $[0, \infty)$ vanishing at infinity).

Let $0 \leq r < a$, then for any $r < r' < a$ and every positive number B ,

$$\begin{aligned} & \left| F(r) - \int_0^{\infty} \Omega_N(r(2Nt)^{1/2}) d\alpha_N(t) - \int_0^B \Omega_N(ir(2Nt)^{1/2}) d\beta_N(t) \right| \\ & \leq \int_B \Omega_N(ir(2Nt)^{1/2}) d\beta_N(t) \\ & \leq \frac{\Omega_N(ir(2Nt)^{1/2})}{\Omega_N(ir'(2Nt)^{1/2})} \int_B \Omega_N(ir'(2Nt)^{1/2}) d\beta_N(t) \quad \text{by [15, Lemma 1]} \\ & \leq \frac{\Omega_N(ir(2Nt)^{1/2})}{\Omega_N(ir'(2Nt)^{1/2})} \int_0^{\infty} \Omega_N(ir'(2Nt)^{1/2}) d\beta_N(t) \\ & \leq \frac{\Omega_N(ir(2Nt)^{1/2})}{\Omega_N(ir'(2Nt)^{1/2})} (F(r') + \|\alpha_N\|) \leq \frac{\Omega_N(ir(2Nt)^{1/2})}{\Omega_N(ir'(2Nt)^{1/2})} (F(r') + F(0)). \end{aligned}$$

Hence, if we choose B to be a point at which β is continuous (i.e. $\beta(\{B\}) = 0$), let

$N = N_k$ and let $k \rightarrow \infty$, we obtain the inequality

$$(36) \quad \left| F(r) - \int_0^\infty e^{-r^2 t} d\alpha(t) - \int_0^B e^{r^2 t} d\beta(t) \right| \leq e^{(r^2 - r'^2)B} (F(r') + F(0))$$

because $\lim_{N \rightarrow \infty} \Omega_N(z(2N)^{1/2}) = e^{-z^2}$ uniformly on any compact subset of the complex plane (hence $\lim_{N \rightarrow \infty} \Omega_N(ir(2Nt)^{1/2}) = e^{r^2 t}$ uniformly on $0 \leq t \leq B$ for fixed r) and $\lim_{N \rightarrow \infty} \Omega_N(r(2Nt)^{1/2}) = e^{r^2 t}$ uniformly on $0 \leq t < \infty$. See [15] and [19, Lemma 3, p. 820].

If we now let $B \rightarrow \infty$ in equation (36) through a sequence of points at which β is continuous we obtain the equation $F(r) = \int_0^\infty e^{-r^2 t} d\alpha(t) + \int_0^\infty e^{r^2 t} d\beta(t)$ for every $0 \leq r < a$. This equation may be rewritten in the form (32). This completes the proof.

Since a continuous function F on $0 \leq r < a$ ($0 < a \leq \infty$) has an integral representation of the form

$$F(r) = \int_{-\infty}^\infty e^{rt} d\mu(t) \quad \text{for all } 0 \leq r < a,$$

where μ is a bounded positive Radon measure on R , if and only if F is exponentially convex on $0 < r < a$ (see [15]), we obtain the following corollary.

Corollary 2. *A continuous function F on $0 \leq r < a$ ($0 < a \leq \infty$) is exponentially convex on $0 < r < a$ if and only if $f_N(x) = F(|x|^2)$, $x \in S_N(a^{1/2})$, is positive definite relative to the orthogonal group $O(N)$ for every $N = 1, 2, 3, \dots$.*

Corollary 2 is an extension of a theorem by I. J. Schoenberg [19, Theorem 3, p. 821] which states that a real valued function F on $0 \leq r < \infty$ is completely monotonic (cf. loc. cit.) if and only if $f_N(x) = F(|x|^2)$, $x \in R^N$, is radial positive definite for every $N = 1, 2, 3, \dots$.

To obtain Schoenberg's theorem we combine Corollary 1 with Corollary 2. We get

A continuous function F on $0 \leq r < \infty$ is bounded and exponentially convex on $0 < r < \infty$ if and only if $f_N(x) = F(|x|^2)$, $x \in R^N$, is positive definite for every $N = 1, 2, 3, \dots$.

But a continuous exponentially convex function F on $0 < r < \infty$,

$$F(r) = \int_{-\infty}^\infty e^{rt} d\mu(t) \quad \text{for all } 0 < r < \infty,$$

where μ is a bounded positive Radon measure on R , is bounded if and only if $\text{supp } \mu \subset (-\infty, 0]$. From this follows that *a continuous function F on $0 \leq r < \infty$ is bounded and exponentially convex on $0 < r < \infty$ if and only if F has an integral representation of the form*

$$F(r) = \int_0^\infty e^{-rt} d\mu_1(t) \quad \text{for all } 0 \leq r < \infty,$$

where μ_1 is a bounded positive Radon measure on $[0, \infty)$. Functions which have such an integral representation are precisely (by the classical theorem of Hausdorff-Bernstein and Widder, cf. [19]) functions which are completely monotonic on $0 \leq r < \infty$.

In summary the following holds for a continuous function F on an interval $0 \leq r < a$ ($0 < a \leq \infty$). For every $N = 1, 2, 3, \dots$ let $f_N(x) = F(|x|^2)$, $x \in S_N(a^{1/2})$. Then F has a representation of the form

1. $F(r) = \int_0^\infty e^{-rt} d\mu(t)$ if and only if $f_N(x)$ is positive definite for every N . (For $a = \infty$ this was proved by Schoenberg [19, Theorem 2] and extended to $0 < a \leq \infty$ by Nussbaum [15, Theorem 1].)

2. $F(r) = \int_0^\infty e^{rt} d\mu(t)$ if and only if $f_N(x)$ is exponentially convex for every N . (Nussbaum [15, Theorem 2].)

3. $F(r) = \int_{-\infty}^\infty e^{rt} d\mu(t)$ if and only if $f_N(x)$ is positive definite relative to the orthogonal group $O(N)$ for every N . This is Theorem 5. (The measures μ in 1–3 are all bounded positive Radon measures on the interval of integration indicated, respectively.)

6. Distributions positive definite relative to the orthogonal group.

Definition 2. Let T be a distribution in a ball $S_N(a) = \{x \in R^N \mid |x| < a\}$, $0 < a \leq \infty$, in R^N and $G = S_N(a/2)$.

(a) T is said to be radial positive definite if it is rotation invariant and if $\langle T, \phi * \phi^* \rangle \geq 0$ for all $\phi \in \mathcal{D}(G)$.

(b) T is said to be positive definite relative to the orthogonal group $O(N)$ if it is rotation invariant and if $\langle T, \phi * \phi^* \rangle \geq 0$ for all radial $\phi \in \mathcal{D}(G)$.

Here $\phi^*(x) = \overline{\phi(-x)}$ ($= \overline{\phi(x)}$ if ϕ is radial) and $*$ denotes convolution.

The following lemma has essentially been proved in the course of the proof of Theorem 2.

Lemma 1. The set of all elements $\{\phi * \psi\}$, where ϕ and ψ are in $\mathcal{D}^\#(S_N(a/2))$, is total in $\mathcal{D}^\#(S_N(a))$.

Proof. It is sufficient to show by the Hahn-Banach theorem that if $T \in \mathcal{D}^\#(S_N(a))'$ and $\langle T, \phi * \psi \rangle = 0$ for all ϕ and ψ in $\mathcal{D}^\#(S_N(a/2))$ then $T = 0$. By Proposition 1 this is equivalent to showing that if $T \in \mathcal{D}'^\#(S_N(a))$ and $\langle T, \phi * \psi \rangle = 0$ for all ϕ and ψ in $\mathcal{D}^\#(S_N(a/2))$, then $\langle T, \phi \rangle = 0$ for all $\phi \in \mathcal{D}^\#(S_N(a))$.

Let T be a rotation invariant distribution in $S_N(a)$ such that $\langle T, \phi * \psi \rangle = 0$ for all ϕ and ψ in $\mathcal{D}^\#(S_N(a/2))$.

Fix $0 < \delta < a/2$ and for every $0 < \epsilon < \delta$ let ρ_ϵ be a function in $\mathcal{D}^\#(S_N(a/2))$ such that $\rho_\epsilon \geq 0$, $\int \rho_\epsilon dx = 1$ and $\text{supp } \rho_\epsilon \subset \{x \in R^N \mid |x| < \epsilon\}$. Then

$$(37) \quad \langle T, (\phi * \rho_\epsilon) * (\psi * \rho_\epsilon) \rangle = 0$$

for all ϕ and ψ in $\mathcal{D}^\#(S_N((a/2) - \delta))$ and $0 < \epsilon < \delta$. (37) may be rewritten in the form

$$(38) \quad \langle T * \rho_\epsilon * \rho_\epsilon, \phi * \psi \rangle = 0$$

for all ϕ and ψ in $\mathcal{D}^\#(S_N((a/2) - \delta))$. Now $T * \rho_\epsilon * \rho_\epsilon$ is a rotation invariant C^∞ -function when restricted to $S_N(a - 2\delta)$. (38) corresponds to equation (24). It follows as in the proof of Theorem 2 that

$$T * \rho_\epsilon * \rho_\epsilon = 0 \quad \text{in } S_N(a - 2\delta).$$

Thus if $\phi \in \mathcal{D}^\#(S_N(a))$ and $\text{supp } \phi \subset S_N(a - 2\delta)$ then

$$\langle T, \phi * \rho_\epsilon * \rho_\epsilon \rangle = \langle T * \rho_\epsilon * \rho_\epsilon, \phi \rangle = 0 \quad \text{for all } 0 < \epsilon < \delta.$$

From this follows since $\phi * \rho_\epsilon * \rho_\epsilon \rightarrow \phi$ in $\mathcal{D}^\#(S_N(a))$ as $\epsilon \rightarrow 0$ that

$$\langle T, \phi \rangle = 0 \quad \text{for all } \phi \in \mathcal{D}^\#(S_N(a))$$

with $\text{supp } \phi \subset S_N(a - 2\delta)$. Since $0 < \delta < a/2$ was arbitrary it follows that $\langle T, \phi \rangle = 0$ for all $\phi \in \mathcal{D}^\#(S_N(a))$.

The following theorem extends Theorem 2 to distributions positive definite relative to the orthogonal group.

Theorem 6. *A distribution T on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N is positive definite relative to the orthogonal group $O(N)$ if and only if there exists a tempered positive Radon measure γ on R such that*

$$(39) \quad \langle T, \phi \rangle = \int_{-\infty}^{\infty} \tilde{\phi}(\lambda^{1/2}) d\gamma(\lambda) \quad \text{for all } \phi \in \mathcal{D}(S_N(a)).$$

Here

$$\tilde{\phi}(z) = \int_{S_N(a)} \Omega_N(z|x|) \phi(x) dx.$$

The measure γ may be chosen such that its support is contained in $[0, \infty) \cup (-\infty, 0]$ if in addition to $\langle T, \phi * \phi^* \rangle \geq 0$ for all $\phi \in \mathcal{D}^\#(S_N(a/2))$

$$(40) \quad \langle T, (-\Delta\phi) * \phi^* \rangle \geq 0 \quad (\langle T, (\Delta\phi) * \phi \rangle \geq 0)$$

for all $\phi \in \mathcal{D}^\#(S_N(a/2))$.

Proof. On $E = \mathcal{D}^\#(G)$, $G = S_N(a/2)$, we define a pseudo-inner product by setting

$$(\phi, \psi) = \langle T, \phi * \psi^* \rangle \quad \text{for } \phi \text{ and } \psi \text{ in } E.$$

Proceeding exactly as in the proof of Theorem 2 we conclude that there exists

a positive Radon measure γ on R such that

$$\langle T, \phi * \psi^* \rangle = \int_{-\infty}^{\infty} \overline{\tilde{\phi}(\lambda^{1/2})} \tilde{\psi}(\lambda^{1/2}) d\gamma(\lambda)$$

for all ϕ and ψ in E and that we may choose γ in such a way that $\text{supp } \gamma \subset [0, \infty)$ ($\text{supp } \phi \subset (-\infty, 0]$) if (40) holds.

First we show that

$$f_2(x) = \int_{\lambda < 0} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) < \infty \quad \text{for all } |x| < a.$$

This is seen as follows: If $\phi \in \mathcal{D}^\#(G)$, $\phi \geq 0$ and $\phi \neq 0$, then

$$\int_{\lambda < 0} \int_G \int_G \Omega_N(|x|\lambda^{1/2}) \Omega_N(|y|\lambda^{1/2}) \phi(x) \phi(y) dx dy d\gamma(\lambda) \leq \int_{-\infty}^{\infty} |\tilde{\phi}(\lambda^{1/2})|^2 d\gamma(\lambda) < \infty$$

and hence by Fubini's theorem (and the fact that $\int_G \Omega_N(|x|\lambda^{1/2}) \phi(x) dx \geq 0$ for $\lambda < 0$)

$$\int_{\lambda < 0} \Omega_N(|x|\lambda^{1/2}) \Omega_N(|y|\lambda^{1/2}) d\gamma(\lambda) < \infty$$

for almost all (x, y) in $G \times G$. From this follows since $t \rightarrow \Omega_N(t\lambda^{1/2})$, $t \geq 0$, is an increasing function for $\lambda < 0$, that

$$\int_{\lambda < 0} \Omega_N(|x|\lambda^{1/2}) \Omega_N(|y|\lambda^{1/2}) d\gamma(\lambda) < \infty$$

for all x and y in G . Hence (if we argue as in Theorem 2 following (21))

$\int_{\lambda < 0} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda) < \infty$ for all $|x| < a$. It follows—using the Lebesgue dominated convergence theorem—that

$$(41) \quad f_2(x) = \int_{\lambda < 0} \Omega_N(|x|\lambda^{1/2}) d\gamma(\lambda)$$

is continuous on $|x| < a$. From this integral representation of f_2 it is seen that f_2 is radial exponentially convex on $S_N(a)$ and $\gamma((-\infty, 0)) = f_2(0) < \infty$.

Next we show that there exists an integer $m \geq 0$ such that

$$(42) \quad \int_0^\infty \frac{d\gamma(\lambda)}{(1 + \lambda)^{N+m+1}} < \infty.$$

This shows that γ is tempered (since $\gamma((-\infty, 0)) < \infty$).

To prove this we construct a sequence of functions $(\omega_n)_{n \geq 1}$ which has the following properties:

1. $\omega_n = \phi_n * \psi_n$ where ϕ_n and $\psi_n \in \mathcal{D}^\#(G)$.
2. $\hat{\omega}_n(t) = \hat{\phi}_n(t) \hat{\psi}_n(t) \geq 0$ for all $t \in R^N$.
3. $\lim_{n \rightarrow \infty} \hat{\omega}_n(t) = \hat{\omega}(t)$ exists for all $t \in R^N$ and $\hat{\omega}(t) \geq A/(1 + |t|^2)^{N+m+1}$ for all $t \in R^N$, where $A > 0$ and m is a nonnegative integer.
4. $|\langle T, \omega_n \rangle| \leq 1$ for all n .

(We follow the construction of a "barrier sequence" in [8, pp. 162–164]. The proof in loc. cit. contains some minor mistakes.)

Let $K = \{x \in R^N \mid |x| \leq r\}$, where r is a fixed number $0 < r < a$. Since T is a continuous linear functional on $\mathcal{D}(S_N(a))$ the restriction of T to $\mathcal{D}_K(S_N(a))$ —the subspace of $\mathcal{D}(S_N(a))$ consisting of all elements $\phi \in \mathcal{D}(S_N(a))$ with $\text{supp } \phi \subset K$ —is continuous. Hence there exists an integer $m \geq 0$ and a number $\delta > 0$ such that $|\langle T, \phi \rangle| \leq 1$ for all $\phi \in \mathcal{D}_K(S_N(a))$ such that $\sup_{|p| \leq m; x \in K} |D^p \phi(x)| \leq \delta$. Here $p = (p_1, p_2, \dots, p_N)$, p_i positive or zero integers, $|p| = p_1 + p_2 + \dots + p_N$ and $D^p = (\partial/\partial x_i)^{p_1} \dots (\partial/\partial x_N)^{p_N}$.

Let $\eta(x)$ be the function in $L^1(R^N)$ whose Fourier transform $\hat{\eta}(t) = 1/(1 + |t|^2)^{N+m+1}$;

$$(43) \quad \eta(x) = \frac{1}{(2\pi)^N} \int_{R^N} \frac{e^{ix \cdot t}}{(1 + |t|^2)^{N+m+1}} dt.$$

Choose any function $\chi \in \mathcal{D}^\#(S_N(a))$ with $\text{supp } \chi \subset K/8 = \{x \in R^N \mid |x| \leq r/8\}$ and let

$$(44) \quad \omega = B\eta(\chi * \chi^*),$$

where $B > 0$ is a constant to be determined later. Then $\text{supp } \omega \subset K/4$ and

$$\begin{aligned} \hat{\omega}(t) &= B(\hat{\eta} * |\hat{\chi}|^2)(t) = B \int_{R^N} |\hat{\chi}(u)|^2 \hat{\eta}(t - u) du \\ &\geq B \int_{|u| \leq 1} |\hat{\chi}(u)|^2 \frac{1}{(1 + |t - u|^2)^{N+m+1}} du \\ &\geq \frac{B}{(1 + (|t| + 1)^2)^{N+m+1}} \int_{|u| \leq 1} |\hat{\chi}(u)|^2 du \\ &= \frac{BC}{(1 + (|t| + 1)^2)^{N+m+1}} = BC \left(\frac{(1 + |t|^2)}{1 + (|t| + 1)^2} \right)^{N+m+1} \frac{1}{(1 + |t|^2)^{N+m+1}} \\ &\geq \frac{BC'}{(1 + |t|^2)^{N+m+1}} \end{aligned}$$

for all $t \in R^N$, where $C' = \inf_{t \in R^N} ((1 + |t|^2)/(1 + (|t| + 1)^2))^{N+m+1} C$. Thus

$$(45) \quad \hat{\omega}(t) \geq A/(1 + |t|^2)^{N+m+1} \text{ for all } t \in R^N$$

where $A = BC'$.

From (43) we infer

$$(46) \quad |D^p \eta(x)| \leq \frac{1}{(2\pi)^N} \int_{R^N} \frac{|t|^{|p|} dt}{(1 + |t|^2)^{N+m+1}} \leq M \text{ for all } |p| \leq m \text{ and } x \in R^N,$$

where M is a constant.

Now choose a function $\rho \in \mathcal{D}^\#(S_N(a))$ with $\text{supp } \rho \subset K/8$, $\int_{RN} \rho \, dx = 1$ and set $\rho_n(x) = n^N \rho(nx)$.

Let

$$\phi_n = \rho_n * \rho_n^*, \quad \psi_n = \rho_n * \rho_n^* * \omega$$

and

$$\omega_n = \phi_n * \psi_n, \quad \text{for } n = 1, 2, 3, \dots$$

Then conditions 1 and 2 are clearly satisfied. 3 is also satisfied because of (45) and $\widehat{\omega}_n(t) = \widehat{\omega}(t) |\widehat{\rho}_n(t)|^4 = \omega(t) |\widehat{\rho}(t/n)|^4 \rightarrow \widehat{\omega}(t) |\widehat{\rho}(0)|^4 = \widehat{\omega}(t)$ as $n \rightarrow \infty$ since $\widehat{\rho}(0) = \int \rho(x) \, dx = 1$. Finally, $\omega_n \in \mathcal{D}_K(S_N(a))$ and

$$\begin{aligned} \sup_{|p| \leq m; x \in K} |D^p \omega_n(x)| &= \sup_{|p| \leq m; x \in K} |((D^p \omega) * \phi_n * \phi_n)(x)| \\ &\leq \sup_{|p| \leq m; x \in K} |D^p \omega(x)| \int_{RN} |(\phi_n * \phi_n)(x)| \, dx \\ &\leq \sup_{|p| \leq m; x \in K} |D^p \omega(x)| = B \sup_{|p| \leq m; x \in K} |D^p \eta(x) (\chi * \chi^*)(x)| \end{aligned}$$

by (44). Hence—by (46)—if we choose B sufficiently small

$$\sup_{|p| \leq m; x \in K} |D^p \omega_n(x)| \leq \delta$$

for all n and $|\langle T, \omega_n \rangle| \leq 1$ for all $n = 1, 2, \dots$. Thus 4 is also satisfied.

Now we are ready to prove (42).

First we note that if $\phi \in \mathcal{D}(S_N(a))$, then

$$\begin{aligned} \langle f_2, \phi \rangle &= \int_{RN} f_2(x) \phi(x) \, dx = \int_{\lambda < 0} \int_{RN} \Omega_N(|x| \lambda^{1/2}) \phi(x) \, dx \, d\gamma(\lambda) \\ (47) \quad &= \int_{\lambda < 0} \widetilde{\phi}(\lambda^{1/2}) \, d\gamma(\lambda) \end{aligned}$$

by (41) and Fubini's theorem. Therefore

$$\begin{aligned} \int_{\lambda < 0} \widetilde{\omega}_n(\lambda^{1/2}) \, d\gamma(\lambda) &= \langle f_2, \omega_n \rangle = \langle f_2, \phi_n * \phi_n * \omega \rangle = \langle f_2 * \check{\omega}, \phi_n * \phi_n \rangle \\ (48) \quad &= \langle f_2 * \omega, \phi_n * \phi_n \rangle \rightarrow \langle f_2 * \omega \rangle(0) = \langle f_2, \omega \rangle \text{ as } n \rightarrow \infty. \end{aligned}$$

Here $\check{\omega}(x) = \omega(-x)$. Now

$$\begin{aligned} \int_0^\infty \widetilde{\omega}_n(\lambda^{1/2}) \, d\gamma(\lambda) &= \int_{-\infty}^\infty \widetilde{\omega}_n(\lambda^{1/2}) \, d\gamma(\lambda) - \langle f_2, \omega_n \rangle \\ &= \langle T, \omega_n \rangle - \langle f_2, \omega_n \rangle \leq 1 + |\langle f_2, \omega_n \rangle|; \end{aligned}$$

i.e.,

$$\int_0^\infty \widetilde{\omega}_n(\lambda^{1/2}) \, d\gamma(\lambda) \leq 1 + |\langle f_2, \omega_n \rangle| \quad \text{for all } n.$$

Hence, if we let $n \rightarrow \infty$ it follows using Fatou's Lemma and 3 and (48), that $\int_0^\infty (A/(1+\lambda)^{N+m+1}) d\gamma(\lambda) \leq 1 + |\langle f_2, \omega \rangle| < \infty$. This proves (42). From (42) follows that $\int_0^\infty |\tilde{\phi}(\lambda^{1/2})| d\gamma(\lambda) < \infty$ for all $\phi \in \mathcal{D}(R^N)$ and that $\langle S, \phi \rangle = \int_0^\infty \tilde{\phi}(\lambda^{1/2}) d\gamma(\lambda)$ for $\phi \in \mathcal{D}(R^N)$ defines a tempered distribution S in $\mathcal{D}(R^N)$. Furthermore S is obviously rotation invariant and

$$\langle S, \phi * \phi^* \rangle = \int_0^\infty |\tilde{\phi}(\lambda^{1/2})|^2 d\gamma(\lambda) \geq 0 \quad \text{for all } \phi \in \mathcal{D}(R^N).$$

Thus S is a rotation invariant positive definite distribution in R^N .

Finally, if ϕ and ψ are elements in $\mathcal{D}^\#(G)$ then

$$\langle T, \phi * \psi^* \rangle = \int_{-\infty}^\infty \tilde{\phi}(\lambda^{1/2}) \overline{\tilde{\psi}(\lambda^{1/2})} d\gamma(\lambda) = \langle S, \phi * \psi^* \rangle + \langle f_2, \phi * \psi^* \rangle$$

and therefore by continuity and Lemma 1

$$\langle T, \phi \rangle = \langle S, \phi \rangle + \langle f_2, \phi \rangle = \int_{-\infty}^\infty \tilde{\phi}(\lambda^{1/2}) d\gamma(\lambda)$$

for all $\phi \in \mathcal{D}^\#(S_N(a))$. It follows from Proposition 1 that

$$\langle T, \phi \rangle = \langle T, \phi^\# \rangle = \langle S, \phi^\# \rangle + \langle f_2, \phi^\# \rangle = \langle S, \phi \rangle + \langle f_2, \phi \rangle = \int_{-\infty}^\infty \tilde{\phi}(\lambda^{1/2}) d\gamma(\lambda)$$

for all $\phi \in \mathcal{D}(S_N(a))$. The "if" part of Theorem 6 is obvious.

The following theorems are immediate corollaries of Theorem 6 and are analogous to Theorems 3, 4 and 4'. In fact they are extensions of these theorems, respectively.

Theorem 7. For a distribution T on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:

1. T is positive definite relative to the orthogonal group.
2. There exists a radial positive definite distribution S on R^N and a radial exponentially convex function f_2 on $S_N(a)$ such that $T = S + f_2$ in $S_N(a)$.

Theorem 8. For a distribution T on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:

1. T is radial positive definite in $S_N(a)$.
2. T and $-\Delta T$ are both positive definite relative to the orthogonal group $O(N)$.
3. There exists a radial positive definite distribution S on R^N such that $T = S$ in $S_N(a)$.

The equivalence of 1 and 3 of the preceding theorem is an extension of Rudin's theorem [18] to distributions.

Theorem 8'. For a distribution T on a ball $S_N(a)$, $0 < a \leq \infty$, in R^N the following statements are equivalent:

1. T is a radial exponentially convex function on $S_N(a)$.
2. T and ΔT are both positive definite relative to the orthogonal group $O(N)$.

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