

FATOU THEOREMS FOR EIGENFUNCTIONS OF THE INVARIANT DIFFERENTIAL OPERATORS ON SYMMETRIC SPACES

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ABSTRACT. On a Riemannian symmetric space of noncompact type we introduce a generalization of the Poisson kernel which may be used to generate simultaneous eigenfunctions of the invariant differential operators with eigenvalues not necessarily zero. We investigate the boundary behavior of our generalized Poisson integrals, extending to them many of the Fatou-type theorems known for harmonic functions.

Introduction. A classical theorem of Fatou states that a bounded measurable function on the circle is equal to the nontangential limit of its Poisson integral almost everywhere. Korányi in [12] and [13] (cf. also [11]) has defined several types of admissible convergence of a function on a Riemannian symmetric space of noncompact type to a function on any Furstenberg-Satake boundary of that symmetric space and has proven the analogous theorem for semirestricted admissible convergence, extending a result of Helgason and himself [8] for radial convergence. In the case of a symmetric space of rank one, the result had also been proven in a more complicated manner by Knapp [10]; in this case the theorem extends easily to all integrable functions on the (in this case unique) boundary and even to finite regular signed measures on the boundary. (By pointwise convergence to a measure almost everywhere we understand pointwise convergence almost everywhere to its Radon-Nikodym derivative.) Some extension of the theorem for arbitrary noncompact symmetric spaces to function spaces on the boundary which are larger than L^∞ has been carried out recently by Lindahl [14].

The Poisson kernel, whose integrals against functions (or, more generally, measures, distributions, etc.) on the maximal Furstenberg-Satake boundary generate

Received by the editors February 9, 1972.

AMS (MOS) subject classifications (1970). Primary 53C35, 43A85.

Key words and phrases. Generalized Poisson integrals, unrestricted admissible convergence, products of rank-one spaces, restricted admissible convergence.

(¹) Some of the results of this paper were included in the author's Ph. D. thesis at Yeshiva University, where the author was supported by a university assistantship. The author is now a C. L. E. Moore instructor at the Massachusetts Institute of Technology, partially supported by NSF grant P22930. The author wishes to express his gratitude to God and to Dr. A. Korányi of Yeshiva University and Dr. S. Helgason of the Massachusetts Institute of Technology for invaluable assistance at all stages of this work.

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harmonic functions on the symmetric space, is only one example of a type of exponential kernel which can be used to generate simultaneous eigenfunctions of the invariant differential operators on the symmetric space; namely, kernels of the form $e^{-(i\lambda + \rho)(H(g^{-1}k))}$, where H denotes the projection on the abelian part of the Iwasawa decomposition, ρ denotes the half-sum of the positive restricted roots, and λ is any complex linear function on the Cartan subspace \mathfrak{a} (precise definitions to be given in §1). (The Poisson kernel corresponds to the case $\lambda = -i\rho$.) Karpelevič [9] (as, in a less precise way, also Furstenberg [2]) has proven that every nonnegative eigenfunction of all invariant differential operators is the integral with respect to some finite positive regular measure on the maximal boundary of such an exponential kernel with $\lambda \in -i\bar{\alpha}_+^*$, where $\bar{\alpha}_+^*$ is the closed positive Weyl chamber. (λ determines the eigenvalues.) In the rank-one case Helgason [6] has proven that every eigenfunction of all invariant differential operators is the integral of such an exponential kernel with $\lambda \in \alpha^* - i\bar{\alpha}_+^*$ with respect to some "functional" on the boundary, although he has characterized the boundary functionals which occur only in the case of the noneuclidean disc. (In that case they are the analytic functionals.)

One is prompted to study the boundary behavior of the eigenfunctions of the invariant differential operators obtained from these more general integrals. Since the integral gotten from the function constant 1 on the boundary is some spherical function, which we denote by ϕ_λ , it is natural to divide the eigenfunction by ϕ_λ and to ask whether this ratio converges in some sense to its boundary values. This question was first asked by Helgason. In [7] he states that in the case of the noneuclidean disc the Fourier coefficients of the above-mentioned ratio on any circle centered at the origin approach the Fourier coefficients of the boundary analytic functional as the noneuclidean radius of the circle approaches infinity, where $\lambda = 0$ (in Helgason's notation, $\mu = \frac{1}{2}$) or $\lambda \in \alpha^* - i\bar{\alpha}_+^*$ (in his notation, $\text{Re } \mu > \frac{1}{2}$). Of course, a stronger form of convergence is desirable, at least for the integrals of elements of some "nice" function spaces on the boundary. Helgason, in an unpublished manuscript, does prove radial convergence of the ratio under consideration to its boundary value at a point of continuity of a bounded measurable function on the maximal boundary of any noncompact symmetric space for $\lambda \in \alpha^* - i\bar{\alpha}_+^*$ (in the rank-one case, also for $\lambda = 0$). The proof is contained essentially in that of Lemmas 1.2 and 1.3 of Chapter IV of [6].

In this paper we shall extend the above result of Helgason for radial convergence at points of continuity to unrestricted admissible convergence at Lebesgue points (in particular, almost everywhere). In the case of spaces of rank one and their products, and for some values of λ even in the general case,

we shall also extend Helgason's result for L^∞ to larger function spaces.

In §1 we define the generalized Poisson kernel, develop some of its properties, and use them to prove some "soft" Fatou-type theorems (for continuous functions, and for convergence in function-space topologies). In §2 we establish those results for pointwise unrestricted admissible convergence almost everywhere which are known for arbitrary symmetric spaces of noncompact type. In §3 we discuss the additional results known for products of spaces of rank one.

1. Generalized Poisson integrals. Let X be a Riemannian symmetric space of noncompact type; G , the connected isometry group of X ; K , the isotropy group of the point $o \in X$. Letting $\mathfrak{g}, \mathfrak{k}$ denote the Lie algebras of G and K , respectively, we can form a Cartan decomposition of \mathfrak{g} : $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let α be a maximal abelian subspace of \mathfrak{p} . Introduce an ordering in α^* , the dual of α , and let \mathfrak{n} and $\bar{\mathfrak{n}}$, respectively, denote the sums of the eigenspaces of $\text{ad}_{\mathfrak{g}}(\alpha)$ corresponding to the positive and to the negative eigenvalues. We can form the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \alpha + \mathfrak{n}$, $G = KAN$, where A and N are the analytic subgroups of G corresponding to the subalgebras α and \mathfrak{n} , respectively, of \mathfrak{g} . For $g \in G$, we shall write $k(g), H(g), n(g)$ to denote the elements of K, α , and N , respectively, such that $g = k(g)\exp H(g)n(g)$. If M is the centralizer of α in K , then $K/M \cong G/MAN$ is the maximal Furstenberg-Satake boundary of X . There is a unique K -invariant probability measure on K/M ; we denote it by dk_M . We denote the set of positive eigenvalues of $\text{ad}_{\mathfrak{g}}(\alpha)$ by Q^+ and their half-sum (with multiplicities) by ρ . We let $P^+ = \{\alpha \in Q^+ \mid \frac{1}{2}\alpha \notin Q^+\}$. We let $\alpha_+^* = \{\mu \in \alpha^* \mid (\forall \alpha \in P^+)(\alpha, \mu) > 0\}$, where $\langle \cdot, \cdot \rangle$ is the bilinear form on α^* dual to the Killing form of \mathfrak{g} on α . (We also write α_+ for $\{H \in \alpha \mid (\forall \alpha \in P^+)\alpha(H) > 0\}$.)

For $\lambda \in \alpha_+^*$, the complexification of α^* , we write

$$\begin{aligned} \phi_\lambda(g \cdot o) &= \int_{K/M} e^{-(i\lambda + \rho)(H(g^{-1}k))} dk_M, \\ P_\lambda(g \cdot o, kM) &= [\phi_\lambda(g \cdot o)]^{-1} e^{-(i\lambda + \rho)(H(g^{-1}k))} \end{aligned}$$

whenever $\phi_\lambda(g \cdot o) \neq 0$. It is clear that ϕ_λ and P_λ are well-defined functions on X and $X_\lambda \times K/M$, respectively, where $X_\lambda = \{x \in X \mid \phi_\lambda(x) \neq 0\}$. If f is an integrable function on K/M , we shall write

$$F_\lambda(x) = \int_{K/M} P_\lambda(x, kM) f(kM) dk_M,$$

for $x \in X_\lambda$. Sometimes we shall, abusing the notation slightly, indicate a measure, or even a distribution, on K/M also by $f(kM)dk_M$, in which case the integral of $P_\lambda(x, kM)$ with respect to $f(kM)dk_M$ will still be denoted by $F_\lambda(x)$.

For $\lambda \in \alpha^* - i\alpha_+^*$, Harish-Chandra [4] has derived an asymptotic formula for ϕ_λ , as follows:

$$(1.1) \quad \phi_\lambda(ka \cdot o) \sim c(\lambda)e^{(i\lambda - \rho)(\log a)} \quad \text{as } a \rightarrow +\infty,$$

provided that $|c(\lambda)| < \infty$, where c is some meromorphic function on α_C^* . (Here and in the sequel $k \in K$, $a \in A$, \log denotes the inverse of $\exp_{\mathfrak{g}}|_A$, and the notation $a \rightarrow +\infty$ means $(\forall \alpha \in P^+) \alpha(\log a) \rightarrow +\infty$. Note that ϕ_λ is K -invariant.) The function $c(\lambda)$ has been computed by Gindikin and Karpelevič [3] and is given by the formula

$$(1.2) \quad c(\lambda) = \int_{\bar{N}} e^{-(i\lambda + \rho)(H(\bar{n}))} d\bar{n} \\ = c_0 \prod_{\alpha \in P^+} \frac{\Gamma(\langle i\lambda, \alpha_0 \rangle) 2^{-\langle i\lambda, \alpha_0 \rangle}}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \langle i\lambda, \alpha_0 \rangle)) \Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda, \alpha_0 \rangle))},$$

where \bar{N} is the analytic subgroup of G corresponding to $\bar{\mathfrak{n}}$, c_0 is some constant, $\alpha_0 = \alpha / \langle \alpha, \alpha \rangle$, and m_α denotes the multiplicity of α . Since $\langle i\lambda, \alpha \rangle > 0$, we have in fact that $0 < |c(\lambda)| < \infty$. The integral in (1.2) is absolutely convergent.

In the case $\text{rank } X \equiv \dim \mathfrak{a} = 1$, Harish-Chandra [4, p. 302] gives an explicit formula for ϕ_λ in terms of the hypergeometric function F :

$$\phi_\lambda(ka \cdot o) = F(\frac{1}{4}(m_\alpha + 2m_{2\alpha} - 2i\lambda(H_0)), \frac{1}{4}(m_\alpha + 2m_{2\alpha} + 2i\lambda(H_0)); \\ \frac{1}{2}(m_\alpha + m_{2\alpha} + 1); -(\sinh \alpha(\log a))^2)$$

where α is the unique element of P^+ , $H_0 \in \mathfrak{a}$ such that $\alpha(H_0) = 1$. This formula permits us to determine the asymptotic behavior of ϕ_λ as $a \rightarrow +\infty$ also for $\lambda = 0$. Namely, an application of formula (18) or (19) on p. 63 of [1] yields

$$(1.3) \quad \phi_0(ka \cdot o) \sim \frac{\Gamma(\frac{1}{2}(m_\alpha + m_{2\alpha} + 1)) 2^{\frac{1}{2}(m_\alpha + m_{2\alpha})}}{\Gamma(\frac{1}{4}(m_\alpha + 2m_{2\alpha})) \Gamma(\frac{1}{4}(m_\alpha + 2))} \alpha(\log a) e^{-\rho(\log a)}.$$

By substituting formula (1.1) or (1.3) into the definition of P_λ , we obtain

$$(1.4) \quad P_\lambda(a \cdot o, kM) \sim \begin{cases} C_\lambda e^{(\rho - i\lambda)(\log a)} e^{-(i\lambda + \rho)(H(a^{-1}k))}, & \lambda \in \alpha^* - i\alpha_+^*, \\ C_0 [\alpha(\log a)]^{-1} e^{\rho(\log a)} e^{-\rho(H(a^{-1}k))}, & \lambda = 0, \text{ rank } X = 1, \end{cases}$$

as $a \rightarrow +\infty$, where C_λ is a nonzero constant depending on λ . (In particular, $a \cdot o \in X_\lambda$ whenever $\alpha(\log a)$ is big enough for each $\alpha \in P^+$.)

Now we shall prove a lemma which will enable us to replace o in (1.4) by an arbitrary $x \in X$.

1.1 Lemma. *If we write $a \cdot x = k'a' \cdot o$, $k' \in K$, $a' \in \exp \bar{\mathfrak{a}}_+$, then as $a \rightarrow +\infty$, $k'M \rightarrow eM$ and $a' \rightarrow +\infty$.*

Proof. By an Iwasawa decomposition we may write $x = \bar{n}_1 a \cdot o$, $\bar{n}_1 \in \bar{N}$, $a_1 \in A$. Then $a \cdot x = a\bar{n}_1 a_1 \cdot o = \bar{n}_1^a a a_1 \cdot o$. (Here and in the sequel we use the notation $g^b = bgb^{-1}$.) Lemma 50 of [5] states that, for $\bar{n}_0 \in \bar{N}$, $a_0 \in A$, if

$\bar{n}_0 a_0 \cdot o \in Ka'_0 K$, $a'_0 \in \exp \bar{\alpha}_+$, then $\lim_{a_0 \rightarrow +\infty} (\log a'_0 - \log a_0) = H(\bar{n}_0)$, and it follows from the proof (e.g., by applying Dini's theorem to the convergent net on the bottom of p. 604, which is obviously monotonically decreasing) that the convergence is uniform for \bar{n}_0 in a compact subset of \bar{N} . Now take $\bar{n}_0 = \bar{n}_1^a$, $a_0 = aa_1$. As $a \rightarrow +\infty$, $\bar{n}_0 \rightarrow e$ and $a_0 \rightarrow +\infty$. By uniformity (since \bar{n}_0 is eventually in a compact subset of \bar{N}), we may interchange limits and obtain that $a' \rightarrow +\infty$ as $a \rightarrow +\infty$. Now, by compactness of K/M , $k'M$ has an accumulation point as $a \rightarrow +\infty$. We shall show that the only possible accumulation point of $k'M$ is eM , and hence that $k'M \rightarrow eM$. Indeed, for $k_0 M \neq eM$,

$$d(a \cdot x, k_0 a' \cdot o) \geq d(a \cdot o, k_0 a' \cdot o) - d(a \cdot x, a \cdot o) = d(a \cdot o, k_0 a' \cdot o) - d(x, o),$$

which is eventually greater than some positive quantity as $a \rightarrow +\infty$ because X has nonpositive curvature.

We are now prepared to prove the main lemma of this section.

1.2 Lemma. *Let $\lambda \in \alpha^* - i\alpha_+^*$. The kernel P_λ has the following properties:*

- (i) For $x \in X_\lambda$, $k_0 \in K$; $k_0 \cdot x \in X_\lambda$ and $P_\lambda(k_0 \cdot x, k_0 kM) = P_\lambda(x, kM)$.
- (ii) For $x \in X_\lambda$, $\int_{K/M} P_\lambda(x, kM) dk_M = 1$.
- (iii) There exists a constant M_λ such that, for each $x \in X$, as $a \rightarrow +\infty$ we have eventually $\int_{K/M} |P_\lambda(a \cdot x, kM)| dk_M \leq M_\lambda$.
- (iv) For any neighborhood U of eM in K/M , $x \in X$,

$$\lim_{a \rightarrow +\infty} \int_{K/M-U} |P_\lambda(a \cdot x, kM)| dk_M = 0.$$

If $\text{rank } x = 1$ these properties hold also for $\lambda = 0$.

Proof. (i) and (ii) follow immediately from the definition of P_λ .

To prove (iii), we let $\eta \in \alpha^*$ such that $\lambda \in \alpha^* - i\eta$. Then

$$\begin{aligned} |P_\lambda(a \cdot x, kM)| &= |\phi_\lambda(a \cdot x)|^{-1} \phi_{-i\eta}(a \cdot x, kM) \\ &= |\phi_\lambda(a' \cdot o)|^{-1} \phi_{-i\eta}(a' \cdot o) P_{-i\eta}(a \cdot x, kM) \\ &\sim |c(\lambda)|^{-1} c(-i\eta) P_{-i\eta}(a \cdot x, kM) \text{ as } a \rightarrow +\infty \end{aligned}$$

by (1.1) and Lemma 1.1 (where a' is as in Lemma 1.1). Thus by (ii) we may take M_λ to be any number greater than $|c(\lambda)|^{-1} c(-i\eta)$.

By Lemma 1.1, (i), and the K -invariance of dk_M , it is enough to prove (iv) for $x = o$. To do so we transfer the integration from K/M to \bar{N} . By a lemma of Bruhat and Harish-Chandra, the map $\bar{n} \rightarrow k(\bar{n})M$ is a diffeomorphism from \bar{N} onto an open dense subset of K/M . According to Lemma 44 of [4], this map has Jacobian equal to $e^{-2\rho(H(\bar{n}))}$, if the Haar measure on \bar{N} is normalized suitably. Let $V = \{\bar{n} \in \bar{N} \mid k(\bar{n}) \in U\}$. Then, for $\lambda \in \alpha^* - i\alpha_+^*$, by (1.4),

$$\begin{aligned} \lim_{a \rightarrow +\infty} \int_{K/M-U} |P_\lambda(a \cdot o, kM)| dk_M &= \lim_{a \rightarrow +\infty} \int |P_\lambda(a \cdot o, k(\bar{n})M)| e^{-2\rho(H(\bar{n}))} d\bar{n} \\ &= \lim_{a \rightarrow +\infty} |C_\lambda| e^{(\rho-\eta)(\log a)} \int_{\bar{N}-V} e^{-(\rho+\eta)(H(a^{-1}k(\bar{n})))} e^{-2\rho(H(\bar{n}))} d\bar{n} \\ &= \lim_{a \rightarrow +\infty} |C_\lambda| e^{2\rho(\log a)} \int_{\bar{N}-V} e^{-(\rho+\eta)(H(\bar{n}^{a^{-1}}))} e^{(\eta-\rho)(H(\bar{n}))} d\bar{n}, \end{aligned}$$

since clearly $H(gh) = H(gk(b)) + H(b)$ and $H(ga) = H(g) + \log a$. Now we choose δ such that $0 < \delta < 1$, $\rho - \delta\eta \in \alpha^*$. We have that

$$\begin{aligned} \lim_{a \rightarrow +\infty} \int_{K/M-U} |P_\lambda(a \cdot o, kM)| dk_M &\leq |C_\lambda| e^{2\rho(\log a)} \int_{\bar{N}-V} e^{-(\rho+\delta\eta)(H(\bar{n}^{a^{-1}}))} d\bar{n} \\ &= |C_\lambda| \int_{\bar{N}-V} a^{-1} e^{-(\rho+\delta\eta)(H(\bar{n}))} d\bar{n}. \end{aligned}$$

(We have made the change of variable $\bar{n} \rightarrow \bar{n}^a$, which has Jacobian $e^{-2\rho(\log a)}$ (cf. [12, Theorem 2.6]).)

Since $V a^{-1} \rightarrow \bar{N}$ as $a \rightarrow +\infty$, and $e^{-(\rho+\delta\eta)(H(\bar{n}))}$ is integrable by the absolute integrability in (1.2), we have $\lim_{a \rightarrow +\infty} \int_{K/M-U} |P_\lambda(a \cdot o, kM)| dk_M = 0$.

For rank $X = 1$, $\lambda = 0$, an explicit formula of Helgason [7, Chapter III, Corollary 1.18] yields

$$\begin{aligned} &e^{-\rho(H(a^{-1}k(\bar{n})))} \\ (1.5) \quad &= e^{\rho(\log a)} \left[\frac{1 + r|X|^2 + 4r|Y|^2}{(1 + re^{2\alpha(\log a)}|X|^2)^2 + 4re^{4\alpha(\log a)}|Y|^2} \right]^{\frac{1}{4}(m_\alpha + 2m_{2\alpha})}, \end{aligned}$$

where $\bar{n} = \exp X \exp Y$, $X \in \mathfrak{g}_{-\alpha}$, $Y \in \mathfrak{g}_{-2\alpha}$ (where \mathfrak{g}_β denotes the β eigenspace of $\text{ad}_\mathfrak{g}(a)$), $|Z| = B(Z, \theta Z)^{1/2}$ for $Z \in \bar{\mathfrak{n}}$ (where B is the Killing form of \mathfrak{g} and θ is the symmetry), $r = [4(m_\alpha + 4m_{2\alpha})]^{-1}$. It follows from (1.4) that

$$\begin{aligned} &P_0(a \cdot o, k(\bar{n})M) \\ (1.6) \quad &\sim C_0 [\alpha(\log a)]^{-1} e^{2\rho(\log a)} \\ &\cdot \left[\frac{1 + r|X|^2 + 4r|Y|^2}{(1 + re^{2\alpha(\log a)}|X|^2)^2 + 4re^{4\alpha(\log a)}|Y|^2} \right]^{\frac{1}{4}(m_\alpha + 2m_{2\alpha})} \end{aligned}$$

Since $2\rho = (m_\alpha + 2m_{2\alpha})\alpha$,

$$e^{2\rho(\log a)} \left[\frac{(1 + r|X|^2)^2 + 4|Y|^2}{(1 + re^{2\alpha(\log a)}|Y|^2)^2 + 4re^{4\alpha(\log a)}|Y|^2} \right]^{\frac{1}{4}(m_\alpha + 2m_{2\alpha})}$$

remains bounded as $a \rightarrow +\infty$, uniformly for $\bar{n} \in \bar{N} - V$. Thus $P_0(a \cdot o, k(\bar{n})M)$ approaches 0 as $a \rightarrow +\infty$, uniformly for $\bar{n} \in \bar{N} - V$. In particular,

$$\lim_{a \rightarrow +\infty} \int_{\bar{N}-V} |P_0(a \cdot o, k(\bar{n})M)| e^{-2\rho(H(\bar{n}))} d\bar{n} = 0.$$

Since X_λ is K -invariant (Lemma 1.2(i)), we can consider, for a function F on X_λ and a point $x \in X_\lambda$, the function F^x on K defined by $F^x(k) = F(k \cdot x)$. If $f(kM)dk_M$ is a distribution on K/M , we can define a distribution $\tilde{f}(k)dk$ on K by

$$\int_K \tilde{g}(k)\tilde{f}(k)dk = \int_{K/M} \left[\int_M \tilde{g}(km)dm \right] f(kM)dk_M,$$

for \tilde{g} any test function on K . It is clear that if f is a measure (respectively: an L^p function, a continuous function) so is \tilde{f} . Using this notation we can now state a "soft" Fatou-type theorem:

1.3 Theorem. *Let $\lambda \in \alpha^* - i\alpha_+^*$ or $\text{rank } X = 1, \lambda = 0$. For any $x \in X$,*

$$\lim_{a \rightarrow +\infty} F^{a \cdot x} = \tilde{f}$$

- (i) *uniformly, $f \in C(K/M)$,*
- (ii) *in $L^p(K)$, $f \in L^p(K/M)$, $1 \leq p < \infty$,*
- (iii) *weak* against $L^1(K)$, $f \in L^\infty(K/M)$,*
- (iv) *weak* against $C(K)$, f a finite regular signed measure on K/M ,*
- (v) *weak* against $C^\infty(K)$, f a distribution on K/M .*

Proof. We omit the details of the proof, since the theorem follows from Lemma 1.2 by standard methods of harmonic analysis (cf. [16, Chapter XVII, Theorems (1.20), (1.23)]).

2. Admissible convergence. We now cite, for the case of the maximal boundary K/M only, Korányi's definition of unrestricted admissible convergence. (In this case unrestricted and semirestricted admissible convergence as defined in [13] coincide.)

For $T \in \alpha$ and C a compact subset of X , a *truncated unrestricted admissible domain* at the boundary point kM is a set of the form

$$\mathcal{Q}_C^T(kM) = \{ka \cdot x \mid \log a - T \in \bar{\alpha}_+, x \in C\}.$$

A function F on X is said to *converge admissibly and unrestrictedly* to a value L at kM if, for every C compact in X and $\epsilon > 0$, there exists a T (depending on C and ϵ) such that, for all $x \in \mathcal{Q}_C^T(kM)$, $|F(x) - L| < \epsilon$.

It is clear that unrestricted admissible convergence at kM does not depend on the choice of k in the coset kM . A group-invariance property for unrestricted admissible convergence is proven in [12, Proposition 4.1].

Given a function f on K/M we may define a function f_1 on \bar{N} by $f_1(\bar{n}) = f(k(\bar{n})M)$. We say that \bar{n}_0 is a *Lebesgue point* of f_1 if

$$\lim_{a \rightarrow +\infty} \frac{1}{\text{mes}(V^a)} \int_{V^a} |f_1(\bar{n}_0\bar{n}) - f_1(\bar{n}_0)| d\bar{n} = 0$$

for some (hence any) compact neighborhood V of e in \bar{N} . According to [13, §3, Corollary 2], if $f_1 \in L^p(\bar{N})$, $1 < p \leq \infty$, then almost all points of \bar{N} are Lebesgue points of f_1 . Since $k(\bar{N})M$ is of full measure in K/M , and since the property of being a Lebesgue point is a local property, if $f \in L^p(K/M)$, $1 < p \leq \infty$, almost all points of K/M are of the form $k(\bar{n})M$ with \bar{n} a Lebesgue point of f_1 . We shall call such points Lebesgue points of f . For $f \in L^\infty(K/M)$ we shall prove a Fatou-type theorem at Lebesgue points of f .

First, however, we prove a lemma which will be useful throughout this paper to extend results from radial or orispherical radial convergence to admissible convergence. This lemma appears to simplify all proofs of admissible convergence, even for the harmonic case (e.g., the results of [12], [13], and [14]), and to give some insight into the reason for considering this type of convergence.

2.1 Lemma. *Let C be a compact subset of X , $\lambda \in \alpha^* - i\alpha_+^*$ or $\text{rank } X = 1$, $\lambda = 0$. Then there exists a constant $C_{\lambda, C}$ (depending on λ and C alone) such that for all $k \in K$, $x \in C$,*

$$|P_\lambda(g \cdot x, kM)| \leq C_{\lambda, C} |P_\lambda(g \cdot o, kM)|$$

whenever $g \in G$ is of the form $g = k_1 a k_2$; $k_1, k_2 \in K$; $\log a - T \in \alpha_+$, for some $T \in \alpha$ (depending on λ and C).

Proof.

$$(2.1) \quad |P_\lambda(g \cdot x, kM)| |P_\lambda(g \cdot o, kM)|^{-1} \\ = (|\phi_\lambda(g \cdot o)| |\phi_\lambda(g \cdot x)|^{-1}) |e^{-(i\lambda + \rho)(H(g'^{-1}k) - H(g^{-1}k))}|,$$

where g' is such that $g' \cdot o = g \cdot x$. We proceed to estimate each factor of the right-hand side of (2.1).

If $g = k_1 a k_2$, $k_2 \cdot x = \bar{n}_1 a_1 \cdot o$,

$$|\phi_\lambda(g \cdot o)| |\phi_\lambda(g \cdot x)|^{-1} = |\phi_\lambda(a \cdot o)| |\phi_\lambda(a \bar{n}_1 a_1 \cdot o)|^{-1} \\ = |\phi_\lambda(a \cdot o)| |\phi_\lambda(\bar{n}_1^a a a_1 \cdot o)|^{-1} = |\phi_\lambda(a \cdot o)| |\phi_\lambda(a' \cdot o)|^{-1},$$

where $a' \in \exp \bar{\alpha}_+$ is such that

$$\bar{n}_1^a a a_1 \cdot o \in K a' \cdot o.$$

By Lemma 50 of [5] and the remark on uniformity in the proof of Lemma 1.1, since \bar{n}_1 and a_1 are confined to compact sets when x is, and by (1.1) or (1.3) as appropriate, $|\phi_\lambda(g \cdot o)| |\phi_\lambda(g \cdot x)|^{-1}$ is bounded, for $\log a - T \in \alpha_+$ for suitable $T \in \alpha_+$, by any constant exceeding

$$\text{Max}_{\{(\bar{n}_1, a_1) | \bar{n}_2 a_1 \cdot o \in K \cdot C \text{ for some } \bar{n}_2 \in \bar{n}_1^{-\exp \alpha_+}\}} |e^{(i\lambda - \rho)(H(\bar{n}_1) + \log a_1)}|.$$

Now, $g'^{-1}g$ belongs to the compact set $D = \{g \in G \mid o \in g \cdot C\}$.

$$H(g'^{-1}k) - H(g^{-1}k) = H(g'^{-1}gk(g^{-1}k)) \in H(DK).$$

Thus $|e^{-(i\lambda + \rho)(H(g'^{-1}k) - H(g^{-1}k))}|$ is bounded by $\text{Max}_{b \in DK} |e^{-(i\lambda + \rho)(H(b))}|$.

We are now prepared to prove the principal result of this section.

2.2 Theorem. (i) Let $f \in L^\infty(K/M)$, k_0M be a Lebesgue point of f , $\lambda \in \alpha^* - i\alpha_+^*$. Then F_λ converges admissibly and unrestrictedly to f at k_0M . (In particular, F_λ converges admissibly and unrestrictedly to f at almost all points of K/M .)

(ii) If $\text{rank } X = 1$ this result holds also for $\lambda = 0$ and holds for all finite regular signed measures on K/M . (By a Lebesgue point of a measure ν , we mean a point $k(\bar{n}_0)M$ for which

$$\lim_{a \rightarrow +\infty} \frac{1}{\text{mes}(V^a)} \int_{\bar{n}_0} V^a |d\nu(k(\bar{n})) - Ld\bar{n}| = 0$$

for some L ; and in that case the convergence is to the limit L .)

Proof. According to [12, Proposition 4.2], F_λ converges admissibly and unrestrictedly to $f(k(\bar{n}_0)M)$ at $k(\bar{n}_0)M$ if (and, in fact, only if) for every $\epsilon > 0$ and every compact subset U of \bar{N} there exists a $T \in \alpha$ (depending on ϵ and U) such that $|F_\lambda(x) - f(k(\bar{n}_0)M)| < \epsilon$ whenever

$$x \in \Gamma_U^T(\bar{n}_0) = \{x \in \bar{n}_0 a U \cdot o \mid a \in A, \log a - T \in \bar{\alpha}_+\}.$$

For $x \in \bar{n}_0 a U \cdot o$, $\log a - T \in \bar{\alpha}_+$ for a suitable T , we have, by Lemma 2.1,

$$\begin{aligned} |F_\lambda(x) - f(k(\bar{n}_0)M)| &\leq \int_{K/M} |f(kM) - f(k(\bar{n}_0)M)| |P_\lambda(x, kM)| dk_M \\ &\leq C_{\lambda, U \cdot o} \int_{K/M} |f(kM) - f(k(\bar{n}_0)M)| |P_\lambda(\bar{n}_0 a \cdot o, kM)| dk_M \\ &= C_{\lambda, U \cdot o} |\phi_\lambda(\bar{n}_0 a \cdot o)|^{-1} \int_{K/M} |f(kM) - f(k(\bar{n}_0)M)| |e^{-(i\lambda + \rho)(H(a^{-1}\bar{n}_0^{-1}k))}| dk_M \end{aligned}$$

By transferring the integration to \bar{N} and applying the properties of the H -function as in the proof of Lemma 1.2, we find, after changing the variable \bar{n} to $\bar{n}_0 \bar{n}$, that this is equal to

$$\begin{aligned} C_{\lambda, U \cdot o} |e^{(\rho + i\lambda)(\log a)}| |\phi_\lambda(\bar{n}_0 a \cdot o)|^{-1} \\ \cdot \int_{\bar{N}} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| |e^{(i\lambda - \rho)(H(\bar{n}_0 k(\bar{n})))}| |e^{-(i\lambda + \rho)(H(\bar{n}^{-1} a^{-1}))}| |e^{(i\lambda - \rho)(H(\bar{n}))}| d\bar{n}. \end{aligned}$$

To prove (i) we note that, for $X \in \alpha^* - i\alpha_+^*$, by (1.1) and Lemma 50 of [5] (mentioned in the proof of Lemma 1.1), this is bounded, for a suitable choice of T , by

$$M_{\lambda, U} e^{2\rho(\log a)} \int_{\bar{N}} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| |e^{-(i\lambda + \rho)(H(\bar{n}^{a^{-1}}))} | |e^{(i\lambda - \rho)(H(\bar{n}))} | d\bar{n},$$

where $M_{\lambda, U}$ is any constant exceeding

$$C_{\lambda, U} |c(\lambda)|^{-1} |e^{(\rho - i\lambda)(H(\bar{n}_0))} | \text{Max}_{g \in \bar{n}_0^K} |e^{(i\lambda - \rho)(H(g))} |.$$

As in the proof of Lemma 1.2 (iv),

$$(2.2) \quad |F_\lambda(x) - f(k(\bar{n}_0)M)| \leq M_{\lambda, U} e^{2\rho(\log a)} \int_{\bar{N}} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| e^{-(\rho + \delta\eta)(H(\bar{n}^{a^{-1}}))} d\bar{n}.$$

We decompose the integral in (2.2) into an integral over V^a and an integral over the rest of \bar{N} , where V is a compact neighborhood of e in \bar{N} .

$$e^{2\rho(\log a)} \int_{\bar{N} - V^a} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| e^{-(\rho + \delta\eta)(H(\bar{n}^{a^{-1}}))} d\bar{n} \leq 2 \|f\|_\infty \int_{\bar{N} - V} e^{-(\rho + \delta\eta)(H(\bar{n}))} d\bar{n} < \frac{\epsilon}{2M_{\lambda, U}}$$

for a suitable choice of V because the integral converges.

But

$$e^{2\rho(\log a)} \int_{V^a} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| e^{-(\rho + \delta\eta)(H(\bar{n}^{a^{-1}}))} d\bar{n} \leq e^{2\rho(\log a)} \int_{V^a} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| d\bar{n} = \frac{\text{mes}(V)}{\text{mes}(V^a)} \int_{V^a} |f_1(\bar{n}_0 \bar{n}) - f_1(\bar{n}_0)| d\bar{n} < \frac{\epsilon}{2M_{\lambda, U}}$$

for a suitable choice of T , by the definition of Lebesgue point. Therefore

$$|F_\lambda(x) - f(k(n_0)M)| < \epsilon.$$

To prove (ii), we let rank $X = 1$, V be a compact neighborhood of e in \bar{N} , and, for suitable $a_0 \in A$, we estimate the integral

$$\int_{k(\bar{n}_0 V^{a_0})M} |f(kM) dk_M - Ldk_M| |P_\lambda(x, kM)| = \int_{V^{a_0}} |f(\bar{n}_0 \bar{n}) d\bar{n} - Ld\bar{n}| |P_\lambda(x, k(\bar{n}_0 \bar{n})M)| e^{-2\rho(H(\bar{n}_0 \bar{n}))}$$

as in the proof of (i), using the fact that \bar{n}_0 is a Lebesgue point of f_1 , where L is as in the definition of Lebesgue point. (For f a function, $L = f(k(n_0)M)$ a.e.) (In this case, since $\dim \alpha = 1$, an easy generalization of the classical Hardy-Littlewood maximal theorem (cf. [16, Chapter XVII, (3.5) and (2.1)]) yields that

almost all points are Lebesgue points even for L^1 functions and for finite regular signed measures.)

We estimate the integral over the rest of K/M as follows:

$$(2.3) \quad \int_{K/M - k(\bar{n}_0 V^{a_0})M} |f(kM)dk_M - Ldk_M| |P_\lambda(x, kM)| \\ \leq C_{\lambda, U \cdot o} \int_{K/M - k(\bar{n}_0 V^{a_0})M} |f(kM)dk_M - Ldk_M| |P_\lambda(\bar{n}_0 a \cdot o, kM)|.$$

By a suitable choice of T we can make $\bar{n}_0 a \cdot o = k' a' \cdot o$ with $k' \in W$, where $\bar{W} \subset k(\bar{n}_0 V^{a_0})M$. Then the expression (2.3) is bounded by

$$(2.4) \quad C_{\lambda, U \cdot o} \left\{ \sup_{\log a' - T \in \mathfrak{a}_+; l \in W^{-1}(K/M - k(\bar{n}_0 V^{a_0})M)} |P_\lambda(a' \cdot o, lM)| \right\} \\ \cdot (L + \|f(kM)dk_M\|_v),$$

where $\|\cdot\|_v$ denotes the variation norm. But, for $\lambda = 0$ by (1.6), and for $\lambda = (u - iv)\alpha$, $v > 0$, by a similar application of (1.4) and (1.5) which yields

$$(2.5) \quad P_\lambda(a \cdot o, k(\bar{n})M) \\ \sim C_\lambda e^{2\rho(\log a)} \left| \frac{(1 + r|X|^2)^2 + 4r|Y|^2}{(1 + re^{2\alpha(\log a)}|X|^2)^2 + 4re^{4\alpha(\log a)}|Y|^2} \right|^{\frac{1}{4}(m_\alpha + 2m_{2\alpha} + 2i[u - iv])},$$

where notations are as in (1.5), we see that the supremum in (2.4) may be made as small as desired.

While the local result of Theorem 2.2(i) for Lebesgue points cannot be generalized for arbitrary-rank spaces to $L^p(K/M)$ for any finite p (cf. [16, at the top of p. 305]), there is some hope for a global result asserting unrestricted admissible convergence on some unspecified set of full measure. A result in this direction, but with severe restrictions on both λ and p , follows. A more complete result for products of rank-one spaces will be shown in the next section.

2.3 Theorem. *Let H_0 be the element of \mathfrak{a}_+ for which $\beta(H_0) = 1$ for every simple restricted root β , $p_0 = 2\rho(H_0) - 1$, $f \in L^p(K/M) \cap L^1(K/M)$ for some $p > p_0$. Let $\lambda \in -i\rho + \mathfrak{a}^* - i\bar{\alpha}_+^*$. Then F_λ converges admissibly and unrestrictedly to f almost everywhere. If $X = \text{SL}(n, \mathbb{R})/\text{SO}(n)$, this result holds for $p > [n^2/4] - 1$. (By a simple restricted root we mean a positive eigenvalue of $\text{ad}_{\mathfrak{g}}(\alpha)$ which is not the sum of two positive eigenvalues of $\text{ad}_{\mathfrak{g}}(\alpha)$.)*

Proof. The theorem for the harmonic case ($\lambda = -i\rho$) is proven by Lindahl [14], using maximal function estimates. From (1.4) we have, for $\lambda \in \mathfrak{a}^* - i\bar{\alpha}_+^*$,

$$P_\lambda(a \cdot o, k(\bar{n})M) \sim C_\lambda e^{2\rho(\log a)} e^{-(i\lambda + \rho)(H(\bar{n}a^{-1}) - H(\bar{n}))}.$$

Since $H(\bar{n}a^{-1}) - H(\bar{n}) = H(a^{-1}k(\bar{n})) + \log a \in (\alpha_+^*)^*$ for $a \in \exp \alpha_+$ by Lemma 35 of [4], it follows immediately that, for $\lambda_0 \in \alpha^* - i\alpha_+^*$, $\lambda \in \lambda_0 + \alpha^* - i\bar{\alpha}_+^*$, as $a \rightarrow +\infty$ we have eventually

$$|P_\lambda(a \cdot o, k(\bar{n})M)| \leq C_{\lambda_0, \lambda} |P_{\lambda_0}(a \cdot o, k(\bar{n})M)|,$$

where $C_{\lambda_0, \lambda}$ is any constant exceeding $|C_\lambda| |C_{\lambda_0}|^{-1}$. Setting $\lambda_0 = -i\rho$ and, of course, using Theorem 1.3(i) in place of the corresponding theorem for $\lambda = -i\rho$ we can now extend the proof of Lindahl to our more general case.

3. Products of spaces of rank one. In the case of a symmetric space of rank one, the asymptotic behavior of the kernel P_λ is known quite explicitly for $\lambda \in \alpha^* - i\alpha_+^*$ (2.5) and for $\lambda = 0$ (1.6). Also available in the rank-one case but not in general is a maximal theorem of Knapp [10, Theorem 5.1], which we shall state shortly after introducing some required notation.

For $\bar{n} \in \bar{N}$ we set $|\bar{n}| = \text{Max}\{|X|, |Y|^{1/2}\}$, where $X, Y, | \cdot |$, are as in (1.5). We set $B(R) = \{\bar{n} \in \bar{N} \mid |\bar{n}| \leq R\}$. For $k_0M \in K/M$ we set $B_R(k_0M) = k_0k(B(R))M$, $0 < R < \infty$; $B_\infty(k_0M) = K/M$. One verifies easily that this definition does not depend upon the representative k_0 of the coset k_0M .

For $f \in L^1(K/M)$ we define the maximal function

$$f^*(k_0M) = \sup_{0 < R \leq \infty} \frac{1}{\text{mes}(B_R(k_0M))} \int_{B_R(k_0M)} |f(kM)| dk_M.$$

The following lemma is due to Knapp [10, Theorem 5.1].

3.1 Lemma. For $f \in L^1(K/M)$, $\xi > 0$,

$$\text{mes}(\{kM \mid f^*(kM) \geq \xi\}) \leq C'\xi^{-1} \|f\|_1,$$

where C' is a constant (independent of f, ξ).

Knapp's lemma has the following corollary, which can be proven by a standard interpolation method (as in [15, Chapter 1, Theorem (13.15)(i)]).

3.2 Corollary. If $f \in L^p(K/M)$, $1 < p \leq \infty$, then

$$f^* \in L^p(K/M) \text{ and } \|f^*\|_p \leq A_p \|f\|_p,$$

where A_p is a constant (independent of f).

We now use Knapp's maximal function f^* to estimate the generalized Poisson integrals F_λ of f , as follows:

3.3 Lemma. *Let rank $X = 1$, $\lambda = 0$ or $\lambda \in \alpha^* - i\alpha_+^*$, $f \in L^1(K/M)$, C be compact in X . Then for a suitable choice of T (depending on λ and C but not on k and f) there exists a constant $A_{\lambda, C}$ (independent of k, f) such that $|F_\lambda(x)| \leq A_{\lambda, C} f^*(kM)$ for all $x \in \mathcal{Q}_C^T(kM)$.*

Proof. Since the operator $*$ commutes with translations by K , it suffices to prove the lemma for $k = e$. We write B_R for $B_R(eM)$.

For $x \in \mathcal{Q}_C^T(eM)$, $x = a \cdot y$, $y \in C$, $\log a - T \in \bar{\alpha}_+$. For a suitable choice of T we have, by Lemma 2.1,

$$(3.1) \quad \begin{aligned} |F_\lambda(x)| &\leq \int_{K/M} |f(kM)| |P_\lambda(a \cdot y, kM)| dk_M \\ &\leq C_{\lambda, C} \int_{K/M} |f(kM)| |P_\lambda(a \cdot o, kM)| dk_M. \end{aligned}$$

To simplify notation we shall set $t = \alpha(\log a)$. (Hence $a = \exp tH_0$, H_0 as in Theorem 2.3.) We shall also set $\lambda = (u - iv)\alpha$, $u = v = 0$ or $v > 0$. From (2.5) it follows immediately that, for sufficiently large t , for $v > 0$, for some constant A_λ ,

$$(3.2) \quad |P_\lambda(\exp tH_0 \cdot o, k(\bar{n})M)| \leq \begin{cases} A_\lambda e^{(m_\alpha + 2m_{2\alpha})t}, & \bar{n} \text{ arbitrary,} \\ A_\lambda e^{-2vt} |\bar{n}|^{-(m_\alpha + 2m_{2\alpha} + 2v)}, & |\bar{n}| \leq 1, \\ A_\lambda e^{-2vt}, & |\bar{n}| > 1. \end{cases}$$

From (1.6) it follows that, for sufficiently large t , for some constant A_0 ,

$$(3.3) \quad |P_0(\exp tH_0 \cdot o, k(\bar{n})M)| \leq \begin{cases} A_0 t^{-1} e^{(m_\alpha + 2m_{2\alpha})t}, & \bar{n} \text{ arbitrary,} \\ A_0 t^{-1} |\bar{n}|^{-(m_\alpha + 2m_{2\alpha})}, & |\bar{n}| \leq 1, \\ A_0 t^{-1}, & |\bar{n}| > 1. \end{cases}$$

In the sequel we shall always assume t large enough for estimate (3.2) or (3.3) to hold and $t \geq 1$.

We can now break up the integral on the right-hand side of (3.1) into three integrals—over the sets $B_{e^{-t}}$, $B_1 - B_{e^{-t}}$, and $B_\infty - B_1$.

We estimate the integral over $B_{e^{-t}}$ by the first inequality in (3.2) or (3.3).

$$\begin{aligned}
 & \int_{B_{e^{-t}}} |f(kM)| |P_\lambda(\exp tH_0 \cdot o, kM)| dk_M \\
 (3.4) \quad & \leq A_\lambda e^{(m_\alpha + 2m_{2\alpha})t} \int_{B_{e^{-t}}} |f(kM)| dk_M \\
 & = A_\lambda \text{mes}(B(1)) [\text{mes}(B(e^{-t}))]^{-1} \int_{B_{e^{-t}}} |f(kM)| dk_M \\
 & \leq A_\lambda \text{mes}(B(1)) [\text{mes}(B_{e^{-t}})]^{-1} \int_{B_{e^{-t}}} |f(kM)| dk_M \leq A_\lambda \text{mes}(B(1)) f^*(eM).
 \end{aligned}$$

Next we estimate the integral over $B_\infty - B_1$ by the last inequality in (3.2) or (3.3):

$$\begin{aligned}
 (3.5) \quad & \int_{B - B_1} |f(kM)| |P_\lambda(\exp tH_0 \cdot o, kM)| dk_M \leq A_\lambda e^{-2\nu} \int_{B_\infty} |f(kM)| dk_M \\
 & \leq A_\lambda e^{-2\nu} f^*(eM).
 \end{aligned}$$

To estimate the integral over $B_1 - B_{e^{-t}}$ by the second inequality in (3.2) or (3.3), we must consider separately the cases $\nu > 0$ and $u = \nu = 0$ and must further subdivide, as follows.

For $\nu > 0$,

$$\begin{aligned}
 & \int_{B_1 - B_{e^{-t}}} |f(kM)| |P_\lambda(\exp tH_0 \cdot o, kM)| dk_M \\
 & = \sum_{j=0}^{\infty} \int_{B_1 B_{2^{j+1}e^{-t}} - B_{2^j e^{-t}}} |f(kM)| |P_\lambda(\exp tH_0 \cdot o, kM)| dk_M \\
 & \leq \sum_{j=0}^{\infty} A_\lambda e^{(m_\alpha + 2m_{2\alpha})t} 2^{-(m_\alpha + 2m_{2\alpha} + 2\nu)j} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 (3.6) \quad & = \sum_{j=0}^{\infty} A_\lambda 2^{(m_\alpha + 2m_{2\alpha} - 2\nu)j} \text{mes}(B(1)) \\
 & \quad \cdot [\text{mes}(B(2^{j+1}e^{-t}))]^{-1} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 & \leq \sum_{j=0}^{\infty} A_\lambda 2^{m_\alpha + 2m_{2\alpha} - 2\nu j} \text{mes}(B(1)) [\text{mes}(B_{2^{j+1}e^{-t}})]^{-1} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 & \leq \sum_{j=0}^{\infty} A_\lambda 2^{m_\alpha + 2m_{2\alpha} - 2\nu j} \text{mes}(B(1)) f^*(eM) = A \frac{2^{m_\alpha + 2m_{2\alpha}}}{1 - 2^{-2\nu}} \text{mes}(B(1)) f^*(eM).
 \end{aligned}$$

$$\begin{aligned}
 & \int_{B_1 - B} e^{-t} |f(kM)| |P_0(\exp tH_0 \cdot o, kM)| dk_M \\
 &= \sum_{j=0}^{\infty} \int_{B_1 \cap B_{2^{j+1}e^{-t}} - B_{2^j e^{-t}}} |f(kM)| |P_0(\exp tH_0 \cdot o, kM)| dk_M \\
 &\leq \sum_{j=0}^{[t/\log 2]+1} A_0 t^{-1} e^{(m_\alpha + 2m_{2\alpha})t} 2^{-(m_\alpha + 2m_{2\alpha})j} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 &= \sum_{j=0}^{[t/\log 2]+1} A_0 t^{-1} 2^{m_\alpha + 2m_{2\alpha} j} \text{mes}(B(1)) \\
 (3.7) \quad & \cdot [\text{mes}(B(2^{j+1}e^{-t}))]^{-1} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 &\leq \sum_{j=0}^{[t/\log 2]+1} A_0 t^{-1} 2^{m_\alpha + 2m_{2\alpha} j} \text{mes}(B(1)) \\
 & \quad \cdot [\text{mes}(B_{2^{j+1}e^{-t}})]^{-1} \int_{B_{2^{j+1}e^{-t}}} |f(kM)| dk_M \\
 &\leq \{[t/\log 2] + 1\} A_0 t^{-1} 2^{m_\alpha + 2m_{2\alpha} j} \text{mes}(B(1)) f^*(eM) \\
 &\leq \{1/\log 2 + 1\} A_0 2^{m_\alpha + 2m_{2\alpha} j} \text{mes}(B(1)) f^*(eM).
 \end{aligned}$$

By combination of the estimates in (3.1), (3.4), (3.5), and (3.6) or (3.7), the lemma follows.

The preceding lemmas enable us to prove the principal result of this section.

3.4 Theorem. *Let $X = \prod_{i=1}^m X_i$, rank $X_i = 1$. Let $f \in L^p(K/M)$, $1 < p \leq \infty$. Let $\lambda = \sum_{i=1}^m (u_i - iv_i)\alpha_i$, where α_i is the (smaller) positive eigenvalue of $\text{ad}_{\mathfrak{g}}(\alpha_i)$, where α_i is the Cartan subspace of the space X_i , $u_i = v_i = 0$ or $v_i > 0$. Then F_λ converges admissibly and unrestrictedly to f almost everywhere.*

Proof. Since $L^\infty(K/M)$ is contained in every $L^p(K/M)$, we may assume $p < \infty$. Then we may fix $\epsilon > 0$ and set $f = f^1 + f^2$, $f^1 \in C(K/M)$, $\|f^2\|_p < \epsilon^{1+1/p}$.

Fix C compact in X , and let $x_{kM} \in \mathcal{Q}_C^T(kM)$.

For suitable T , independent of kM , $|F_\lambda^1(x_{kM}) - f^{-1}(kM)| < \epsilon$ by Theorem 1.3(i) and Lemma 1.1.

$$\{kM \mid |F_\lambda^2(x_{kM}) - f^2(kM)| \geq 2\epsilon\} \subset \{kM \mid |f^2(kM)| \geq \epsilon\} \cup \{kM \mid |F_\lambda^2(x_{kM})| \geq \epsilon\}.$$

$$\text{mes} (\{kM \mid |f^2(kM)| \geq \epsilon\}) < \epsilon.$$

Since we may assume that C is a product $C = \prod_{i=1}^m C_i$, C_i compact in X_i ; by iteration of Lemma 3.3, applied to each X_i , for suitable choice of T ,

$$\|F_\lambda^2(x_{kM})\|_{L^p(K/M)} \leq A_{\lambda, C} f_{1\dots m}^{2*} (kM),$$

for some constant $A_{\lambda, C}$, where f_i^* indicates the Knapp maximal operator of the space X_i acting on f regarded as a function on K_i/M_i , the maximal boundary of X_i , other coordinates of kM being held fixed. By iteration of Lemma 3.2, for some constant A_p ,

$$\|f_{1\dots m}^{2*} \|_p \leq A_p \|f^2\|_p,$$

so that

$$\|F_\lambda^2(x_{kM})\|_{L^p(K/M)} \leq A_{\lambda, C} A_p \|f^2\|_p < A_{\lambda, C} A_p \epsilon^{1+1/p}.$$

$$\text{mes} (\{kM \mid |F_\lambda^2(x_{kM})| \geq \epsilon\}) < (A_{\lambda, C} A_p)^p \epsilon.$$

Therefore $|F_\lambda(x_{kM}) - f(kM)| < 3\epsilon$ except on a set of measure less than $(1 + (A_{\lambda, C} A_p)^p)\epsilon$.

Since ϵ is arbitrary and x_{kM} is arbitrary in $\mathcal{Q}_C^T(kM)$, the theorem follows.

In the case $v_i > 0$ the result of Theorem 3.4 cannot be extended to $f \in L^1(K/M)$. A counterexample can be constructed by the method of [16, Chapter XVII, Theorem (2.14)(ii)]. For a more restricted type of convergence, however, a positive result will be stated below (Theorem 3.5).

In the case $u_i = v_i = 0$ for some i , the boundary behavior of F_λ for $f \in L^1(K/M)$ remains an open question.

We now give some definitions meaningful in any symmetric space of noncompact type. These definitions in their present form are due to Korányi [12, §2].

A *truncated restricted admissible domain* at a boundary point kM with respect to an element $H \in \mathfrak{a}_+$ is a set of the form

$$\mathcal{R}_{H, C}^\tau(kM) = \{k \exp tH \cdot x \mid t \geq \tau, x \in C\}$$

for some $\tau \in \mathbb{R}$, C compact in X .

A function F on X converges *admissibly and restrictedly with respect to H* to a number L at kM if for every C compact in X , $\epsilon > 0$, there exists a number τ such that $|F(x) - L| < \epsilon$ for all $x \in \mathcal{R}_{H, C}^\tau(kM)$. One verifies easily that this definition does not depend upon the choice of the representative k of the coset kM .

By iteration of the estimates (3.2), one can prove by standard methods of harmonic analysis the following theorem.

3.5 Theorem. Let $X = \prod_{i=1}^m X_i$, $\text{rank } X_i = 1$, f be a finite regular signed measure on K/M , $\lambda \in \alpha^* - i\alpha_+^*$, $H \in \alpha_+$. Then F_λ converges admissibly and restrictedly to f with respect to H almost everywhere.

Remark. Dr. Korányi has recently called to my attention a letter of Lindahl dated September 6, 1971, which proves Lemma 2.1 for the harmonic case and suggests the use of this lemma to extend results on orispherical radial convergence to admissible convergence. Although Lindahl's proof for this case has priority, mine is independent. The same letter of Lindahl also mentions results of Knapp and E. M. Stein improving the conditions on p in Theorem 2.3 for certain symmetric spaces. In $\text{SL}(n, \mathbf{R})/\text{SO}(n)$ the theorem holds for $p > n - 2$, and in $\text{Sp}(2, \mathbf{R})/U(2)$ it holds for $p > 2$.

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