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DECOMPOSABLE BRAIDS AND LINKAGES(1)

BY

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ABSTRACT. An n-braid is called k-decomposable if and only if the removal of k arbitrary strands results in a trivial $(n-k)$-braid. k-decomposable n-linkages are similarly defined. All k-decomposable n-braids are generated by an explicit geometric process, and so are all k-decomposable n-linkages. The latter are not always closures of k-decomposable n-braids. Many examples are given.

Introduction. This paper results from the investigation of linkages with the property that the removal of any arbitrary link results in the remaining links falling apart from each other. Such linkages have been termed "decomposable", and in the course of examining them, it has been necessary to weaken this property as follows. If the removal of any arbitrarily chosen k links from a linkage on n links results in the remaining $n-k$ links falling apart from each other, the original linkage on n links is termed "k-decomposable".

§I consists of a sequence of lemmas resulting in theorems giving various constructive enumerations of families of linkages, including a construction for the family of all decomposable linkages. §II consists of examples of families of decomposable linkages of interest.

I wish to thank Professor Wilhelm Magnus, who posed problems treated in this paper, and helped kindly and patiently in their solutions.

I. The geometric construction of decomposable braids and linkages.

Definition 1. A braid [2] on n strands shall be called *decomposable* iff whenever a single arbitrary strand is removed, the remaining braid on $n-1$ strands is deformable into the identity braid on $n-1$ strands.

For any braid to be decomposable, it is necessary for each strand to end in the same position in which it began. We may therefore restrict our attention to the subgroup, I_n, of elements of the braid group, B_n, on n strands, which leave strand positions the same at their ends as at their beginnings.

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First we consider the case of decomposable braids on three strands. Let b be an arbitrary braid in I_3. Let $b_{1,2}$ be the braid obtained from b by removing the third strand from b and stretching it alongside the intertwining, if any, of strands one and two, in a position of noninvolvement. Similarly define $b_{1,3}$ and $b_{2,3}$ (see Figure 1).

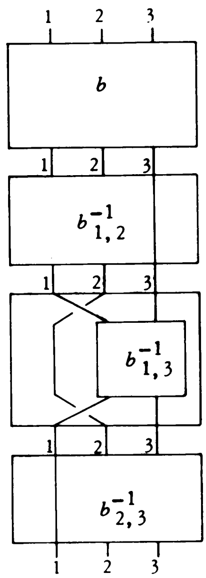


Figure 1

Lemma 1. *The 3-braid* $bb^{-1}_{1,2}b^{-1}_{1,3}b^{-1}_{2,3}$ *is a decomposable 3-braid.*

Proof. Any 2-braid which leaves the strand positions invariant is decomposable, trivially. Therefore the removal of any single strand has the effect of decomposing two of the b^{-1}_{ij}, and leaving a third untouched. This third undecomposed 2-braid is precisely the inverse of the fragment of b remaining after the strand was removed. Since they undo each other, the result of removing any single strand is the trivial 2-braid. Q.E.D.

Lemma 2. *The set of all 3-braids of the form* $bb^{-1}_{1,2}b^{-1}_{1,3}b^{-1}_{2,3}$ *is precisely the set of all decomposable 3-braids.*

Proof. Consider the mapping $d: I_3 \to I_3$ defined by $d(b) = bb^{-1}_{1,2}b^{-1}_{1,3}b^{-1}_{2,3}$. The range of d is a subset of the set of all decomposable 3-braids by Lemma 1. By the decomposing property of decomposable braids, and the construction of the $b_{i,j}$'s, if b were decomposable then each $b_{i,j}$ would be trivial and $d(b) = b$. Thus the range of d includes all decomposable 3-braids. Q.E.D.

This construction may, with suitable modification, be used to construct all decomposable n-braids. At various stages in the processes, one obtains n-braids which decompose if any k arbitrary strands $(k < n)$ are removed.

Definition 2. An n-braid will be called *k-decomposable* iff the removal of any k arbitrary strands results in a trivial $(n - k)$-braid.

Thus 1-decomposable is decomposable and conversely. We now proceed with the construction of decomposable n-braids.

Lemma 3. *The inverse of any decomposable n-braid is also decomposable.*

Proof. Let b be a decomposable n-braid. Then bb^{-1} is also decomposable since it is the trivial braid. Removing any strand of bb^{-1} decomposes the "b-section" of bb^{-1}, and thus it must decompose the "b-section" of bb^{-1} as well to decompose all of bb^{-1}. Q.E.D.

Let b be a braid in I_n, and consider the $\binom{n}{2}$ braids, $b_{i,j}$, obtained from b by removing $n - 2$ strands of b and then replacing them beside in their original positions, but not interwoven with, the remaining ith and jth strands of b which may or may not be intertwined.

Lemma 4. *The braids*

$$\beta = b \prod_{j=2;\, i<j}^{n} b_{i,j}^{-1}, \qquad \forall b \in I_n,$$

are exactly the set of all $(n - 2)$-decomposable n-braids.

Proof. As in Lemmas 1 and 2, removing any $n - 2$ strands of β decomposes each $b_{i,j}^{-1}$ for which either of the subscripts is the number of the position of a removed strand. The only remaining $b_{i,j}^{-1}$ exactly cancels the part of b still left intact and the result is a trivial 2-braid. Furthermore, if b were an $(n - 2)$-decomposable n-braid, each $b_{i,j}$ would be trivial. Thus it is seen that the range of a function $d: I_n \to I_n$ defined by $d(b) = \beta$, both contains and is contained in the set of all $(n - 2)$-decomposable n-braids. Q.E.D.

Call the set of all k-decomposable n-braids $D_{k,n}$, and let $b_{i_1 i_2 \ldots i_{n-k+1}}$ be the braid obtained from b by removing $k - 1$ strands and replacing them by straight strands in their former positions, but not interwoven with the $(n - k + 1)$-braid left as a vestige of b.

Lemma 5. *Each of the $\binom{n}{n-k+1}$ $b_{i_1 i_2 \ldots i_{n-k+1}}$, as constructed above, decompose if any of the strands numbered i_1 through i_{n-k+1} are removed.*

Proof. Since b was assumed k-decomposable, the removal of any k strands of b results in a trivial braid. Therefore the removal of $k - 1$ strands of b results in the nontrivial strip of $b_{i_1 i_2 \ldots i_{n-k+1}}$ and the removal of any other strand, numbered i_1 through i_{n-k+1}, results in "complete destruction" of b, and therefore of the nontrivial component of $b_{i_1 i_2 \ldots i_{n-k+1}}$ too. Q.E.D.

Lemma 6. *The set of all n-braids,*

$$\beta = b \prod_{i_1 = 1;\, i_1 < i_2 \cdots < i_{n-k+1}}^{n-k} b^{-1}_{i_1 i_2 \ldots i_{n-k+1}}, \quad \forall b \in D_{k,n},$$

is exactly the set $D_{k-1,n}$

Proof. By Lemmas 4 and 5, each of the $b^{-1}_{i_1 i_2 \ldots i_{n-k+1}}$ are decomposable in strands numbered i_1 through i_{n-k+1}. If $k-1$ strands are removed from β, then all that is left of its b part is the (possibly) intertwined sections of $b_{i_1 i_2 \ldots i_{n-k+1}}$ on strands numbered i_1 through i_{n-k+1}. Since the strands removed appear in the (possibly) intertwined sections of all other $b^{-1}_{j_1 j_2 \ldots j_{n-k+1}}$ on strands numbered j_1 through j_{n-k+1}, all the $b^{-1}_{j_1 j_2 \ldots j_{n-k+1}}$ are decomposed by the removal of the $k-1$ strands, except for the fragment of $b^{-1}_{i_1 i_2 \ldots i_{n-k+1}}$ which annihilates the fragment left of b by its very construction. Thus the removal of $k-1$ arbitrary strands from β results in a trivial $(n-k+1)$-braid. If b itself were $(k-1)$-decomposable, each of the $b^{-1}_{i_1 i_2 \ldots i_{n-k+1}}$ would have been trivial braids. The rest of the proof is identical with the ends of the proofs of Lemmas 2 and 4. Q.E.D.

Thus the process described for constructing $D_{k-1,n}$ from $D_{k,n}$ may be interated $n-2$ times, starting with $I_n = D_{n-1,n}$ to obtain $D_{1,n}$, and during the process, all $D_{k,n}$ for $1 < k < n-1$. The process of taking the closure of an n-braid by identifying the initial point of the strand starting in position number i with the terminal point of the strand ending in position i, for all i between 1 and n, clearly results in a knot or in a linkage of knots. J. W. Alexander [1] showed that any knot or linkage of knots may be represented as the closure of some braid. It is clear that the closure of any braid in I_n results in a linkage consisting of n trivial knots linked together (perhaps trivially, perhaps not).

Definition 3. A *simple linkage* is a linkage in which each link is a trivial knot.

Definition 4. A *k-decomposable n-linkage* is a linkage on n links with the property that the removal of any arbitrary k of those links results in a trivial $(n-k)$-linkage. 1-decomposable linkages will be called *decomposable.*

Although the closure of any braid in I_n results in a simple n-linkage, not every simple n-linkage may be presented as the closure of an n-braid. In fact, not every decomposable simple n-linkage may be presented as a closed braid on n strands.

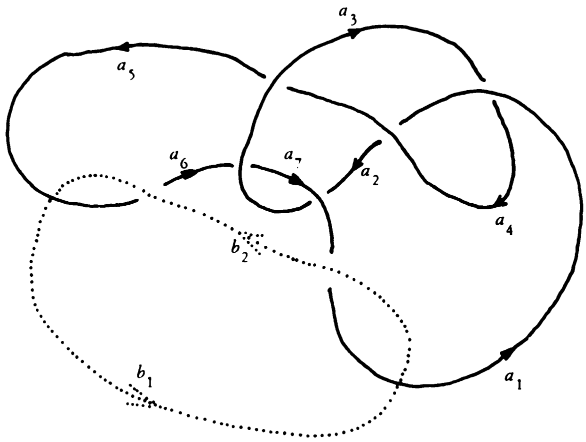


Figure 2

Lemma 7. *The simple 2-linkage depicted in Figure 2 may not be represented as the closure of a 2-braid.*

Proof. We first compute the fundamental group, G, of three-space after the linkage in Figure 2 has been removed from it. The generators are: $a_1, a_2, \cdots, a_7, b_1, b_2$.

The defining relators are

(1) $a_1 a_3 a_1^{-1} a_4^{-1}$ or $a_4 = a_1 a_3 a_1^{-1}$,

(2) $a_1 a_4^{-1} a_2^{-1} a_4$ or $a_2 = a_4 a_1 a_4^{-1} = a_1 a_3 a_1 a_3^{-1} a_1^{-1}$ from (1),

(3) $a_2 a_7 a_3^{-1} a_7^{-1}$ or $a_2 = a_7 a_3 a_7^{-1}$,

(4) $a_3 a_7^{-1} a_3^{-1} a_6$ or $a_6 = a_3 a_7 a_3^{-1}$,

(5) $a_3 a_4 a_3^{-1} a_5^{-1}$ or $a_5 = a_3 a_4 a_3^{-1} = a_3 a_1 a_3 a_1^{-1} a_3^{-1}$ from (1),

(6) $b_2 a_6^{-1} b_2^{-1} a_5$ or $b_2^{-1} a_5 b_2 = a_6 = a_3 a_7 a_3^{-1}$ from (4),

(7) $b_2 a_7 b_2^{-1} a_1^{-1}$ or $a_1 = b_2 a_7 b_2^{-1}$,

(8) $b_2 a_5^{-1} b_1^{-1} a_5$ or $b_1 = a_5^{-1} b_2 a_5 = a_1^{-1} b_2 a_1$ from (9),

(9) $b_1 a_1^{-1} b_2^{-1} a_1$ or $b_1 = a_1^{-1} b_2 a_1$.

We use these relations to express all generators in terms of a_1, a_3, a_5, a_7, and b_2, eliminating a_2, a_4, a_6, and b_1 by using (2), (1), (4), and (9) respectively. The result is a presentation of G on the five generators: a_1, a_3, a_5, a_7, and b_2, with the defining relations:

(1*) $a_7 a_3 a_7^{-1} = a_1 a_3 a_1 a_3^{-1} a_1^{-1}$,

(2*) $b_2^{-1} a_5 b_2 = a_3 a_7 a_3^{-1}$,

(3*) $a_5 = a_3 a_1 a_3 a_1^{-1} a_3^{-1}$,

(4*) $a_1 = b_2 a_7 b_2^{-1}$,

(5*) $a_5^{-1} b_2 a_5 = a_1^{-1} b_2 a_1$.

Suppose now that the linkage depicted in Figure 2 could be represented as the closure of a 2-braid. We know the fundamental groups of the complements in 3-space of all closures of braids in I_2; they can be presented as

$$T_k = \langle x_1, x_2; (x_1x_2)^k x_1 (x_1x_2)^{-k} = x_1, (x_1x_2)^k x_2 (x_1x_2)^{-k} = x_2 \rangle,$$

where k is a fixed integer, $0, \pm 1, \pm 2, \cdots$, which is the linking number of the two closed curves. An inspection of Figure 2 shows that $k = 0$ in this case. Therefore, if our linkage could be presented as the closure of a 2-braid, the fundamental group of its complement in 3-space would be F_2, the free group of rank two. We shall show that this is impossible. Towards this end we observe first, by abelianizing G, that

(i) none of the generators a_1, a_3, a_5, a_7, b_2 is in the commutator subgroup, G', of G, and none of them can be equal to an mth power of another element of G without $m = \pm 1$, and

(ii) the elements $a_\nu a_\mu^{-1}$; $\nu, \mu = 1, 3, 5, 7$ belong to G'.

Next we use the following result about free groups. If two elements u, v of a free group commute, then there exists another element, w, such that both u and v are powers of w.

Looking now at relation (5*), we see that $a_1 a_5^{-1}$ (an element in G') commutes with $b_2 \notin G'$. Therefore (since $b_2 \neq 1 \bmod G'$), we have that $a_1 a_5^{-1}$ is a 0th power, or

(6*) $a_1 = a_5$.

Replacing a_1 in (4*) by a_5, we obtain

(7*) $b_2^{-1} a_5 b_2 = a_7$

and combining this with (2*), we see that a_3 and a_7 commute. Since both of them are $\neq 1 \bmod G'$, and because of (i), we find

(8*) $a_3 = a_7^{\pm 1}$.

Also we obtain from (1*) that

$$a_7 a_3 a_7^{-1} a_3^{-1} = a_1 a_3 a_1 a_3^{-1} a_1^{-1} a_3^{-1} = 1.$$

We can now use (6*), (7*), and (8*) to eliminate a_5, a_7, and a_3, and we are left with a presentation of G in terms of two generators a_1, and b_2, with the single defining relator,

$$a_1 b_2^{-1} a_1^{\pm 1} b_2 a_1 = b_2^{-1} a_1^{\pm 1} b_2 a_1 b_2^{-1} a_1^{\pm 1} b_2.$$

This relation is not trivial. Therefore our group G cannot be isomorphic with F_2, since G is isomorphic with a proper quotient group of the free group on a_1 and b_2, and free groups of finite rank are hopfian. Q.E.D.

The linkage depicted in Figure 2 is decomposable trivially, by virtue of its being a 2-linkage. Whereas it cannot be presented as the closure of a 2-braid, it is the closure of the 3-braid in Figure 3. It should be noted that our example also proves that two circles may be linked with linking number 0 in a manner defying separation.

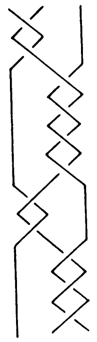


Figure 3

Lemma 7 establishes the necessity of dealing with braids other than those found in I_n if we are to generate all decomposable linkages as the closures of (decomposable) braids. An arbitrary n-braid, b, permutes the initial positions of its strands. Denote the cycles of the permutation of strand positions induced by b, $C_1, C_2, \cdots, C_m$; $1 \leq m \leq n$.

Definition 5. An n-braid, b, on m cycles $C_1, C_2, \cdots, C_m$ shall be called *cycle-trivial* iff b is conjugate to an n-braid c, on m cycles such that

(i) each cycle permutes blocks of initially adjacent strands, and

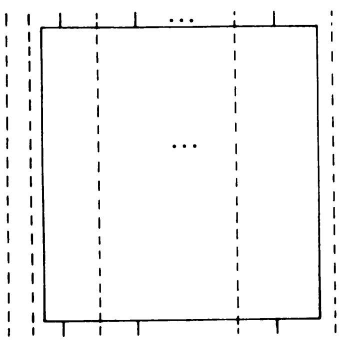
(ii) if strands i and j of c belong to different cycles, then they do not cross or otherwise tangle.

Thus a cycle-trivial braid on m cycles may be pictured as the conjunction, without interweaving, of m braids, each of which consists of a single cycle. Due to the fact that none of the m cycles interweaves with any other in a cycle-trivial braid, the m-linkage obtained by taking the closure of a cycle-trivial braid on m cycles is a trivial linkage, although not necessarily simple, in that it consists of m unlinked component knots, each of which may, or may not, be a trivial knot. The fundamental group of the complement in 3-space of such a linkage is, of course, the free product, without amalgamation, of the knot groups of its links. In the event that all the cycles are of length 1, cycle-triviality becomes triviality in the usual sense.

Definition 6. A braid, b, on m cycles, $C_1, C_2, \cdots, C_m$, shall be called *k-cycle-decomposable* iff the removal of all strands associated with any arbitrary k cycles results in a cycle-trivial braid on $m - k$ cycles.

The process by which all k-cycle-decomposable braids on n strands are constructed is similar to that by which $D_{k,n}$ was constructed. (Note that when $m = n$, a k-cycle-decomposable n-braid is k-(strand)-decomposable.)

Let b be any arbitrary braid on 3 cycles, C_1, C_2, and C_3. Let $b_{1,2}$ be the braid on cycles C_1 and C_2 which remains from b after removing all strands from b in cycle C_3 and stretching the strands alongside, or over, the strands of cycles C_1 and C_2 (see Figure 4). Note that $b_{1,2}$ now consists of cycles C_1, C_2, and as many more *trivial cycles* as there were strands in C_3. Construct $b_{1,3}$ and $b_{2,3}$ similarly. Also construct b_1 by removing all strands in b associated with cycles C_2 and C_3 and restretching them tautly as in the $b_{i,j}$. Similarly construct b_2 and b_3.



Dashed strands were formerly in C_3.

Solid strands within rectangle are in C_1, C_2 and are still intertwined.

Figure 4

Lemma 8. *The braid*

$$b \cdot b_{1,2}^{-1} \cdot b_1 b_2 \cdot b_{1,3}^{-1} \cdot b_1 b_3 \cdot b_{2,3}^{-1} \cdot b_2 b_3,$$

where b is any braid on three cycles, is 3-cycle-decomposable.

Proof. Note that the removal of all strands of cycle C_i from braids $b_{i,j}$ or $b_{i,j}$ and restretching them tautly results in braid b_j. Thus the removal of all strands in C_3 results in the braid: $\overset{*}{b}\overset{*}{b}{}_{1,2}^{-1}\overset{*}{b}_1\overset{*}{b}_2\overset{*}{b}{}_1^{-1}\overset{*}{b}_1 1\overset{*}{b}{}_2^{-1}\overset{*}{b}_2 1$, where the * denotes removal of the straight strands formerly associated with cycle C_3. This braid freely reduces to $\overset{*}{b}_1\overset{*}{b}_2$, which is clearly cycle-trivial in the two cycles C_1 and C_2. Similar results are obtained from the removal of all strands associated either with cycle C_1 or C_2. Q.E.D.

Lemma 9. *Let b be a braid on three cycles, C_i; $i = 1, 2, 3$. Let c_i be a braid on cycle C_i and as many more straight uninvolved strands as are necessary so that b and c_i have the same number of strands. Further, let c_i, when closed, result in a trivial linkage of trivial knots only. Then the braid*

$$bb_{1,2}^{-1}b_1b_2b_{1,3}^{-1}b_1b_3b_{2,3}^{-1}b_2b_3b_1^{-1}b_2^{-1}b_3^{-1}c_1c_2c_3,$$

when closed, results in a simple decomposable 3-linkage.

Proof. $c_1c_2c_3$ is cycle-trivial, and by Lemma 8, the removal of all strands associated with any of the cycles of b will reduce the segment in the b's to the trivial braid, since the b_i commute with each other. All that is left is $\overset{*}{c}_i\overset{*}{c}_j$, whose closure is a simple trivial linkage, by construction of the c's. Q.E.D.

The fact that the b_i commute with each other may be generalized as follows. Let a cycle-trivial braid be factored into segments such that all the strands associated with any one cycle interact in one segment in which no strands of any other cycle interact. Then all those segments commute with each other.

Lemma 10. *The set of all braids on three cycles, of the form $bb_{1,2}^{-1}b_1b_2b_{1,3}^{-1}b_1b_3b_{2,3}^{-1}b_2b_3$, is precisely the set of all cycle-decomposable braids on three cycles.*

Proof. The same technique used in the proof of Lemma 2 yields the desired result. Q.E.D.

Lemma 11. *The set of all braids on three cycles, of the form*

$$bb_{1,2}^{-1}b_1b_2b_{1,3}^{-1}b_1b_3b_{2,3}^{-1}b_2b_3b_1^{-1}b_2^{-1}b_3^{-1}c_1c_2c_3,$$

is precisely the set of all braids on three cycles which, when closed, result in simple decomposable 3-linkages.

Proof. The same technique used in the proof of Lemma 2 yields the desired result. Q.E.D.

Lemma 12. *The inverse of any k-cycle-decomposable braid is also k-cycle-decomposable.*

Proof. The inverse of any cycle-trivial braid is also cycle-trivial since the only interactions which need be undone are between strands belonging to a given cycle. Thus if b is k-cycle-decomposable, the removal of all strands associated with any cycle of $b \cdot b^{-1}$ leaves the b-section cycle-trivial, and must also leave the b^{-1} section cycle-trivial in order to undo the cycle-trivial remnant of b. Thus b^{-1} is cycle-trivial. Q.E.D.

Lemma 13. *The braids*

$$\beta = b \prod_{j=2;\, i<j}^{n} b_{i,j}^{-1}b_ib_j,$$

for all braids b on n cycles, are exactly the set of all $(n-2)$-cycle-decomposable braids on n cycles.

Proof. As in Lemma 8, the removal of all strands associated with any $n-2$ cycles in β leaves each $b_{i,j}^{-1}b_ib_j$ for which either or both of the subscripts is the cycle number of a cycle all of whose strands haven been removed trivial, or empty. The sole remaining $b_{i,j}^{-1}$ precisely cancels out the part of b left intact, resulting in the cycle-trivial braid $b_i b_j$. If b were itself $(n-2)$-cycle-decomposable, each $b_{i,j}^{-1}$ would be cycle-trivial, hence $b_{i,j}^{-1} = b_i^{-1} b_j^{-1}$, cancelling the $b_i b_j$ term. As in the proof of Lemma 4, the function $d(b) = \beta$ maps each $(n-2)$-cycle-decomposable braid onto itself, and each other braid onto an $(n-2)$-cycle-decomposable braid. Thus the range of d contains and is contained in the set of all $(n-2)$-cycle-decomposable braids. Q.E.D.

Lemma 14. *The set of all braids*

$$\beta = b \prod_{j=2;\, i<j}^{n} b_{i,j}^{-1} b_i b_j \prod_{k=1}^{n} b_k^{-1} \prod_{l=1}^{n} c_l;$$

for all braids b on n cycles, is precisely the set of all braids which, when closed, result in simple $(n-2)$-decomposable n-linkages.

Proof. The same technique used in the proof of the preceding lemma yields the desired result. Q.E.D.

Let b be any braid on n cycles. Generalizing the constructions prior to Lemmas 5 and 8, call the braid obtained from b by removing all strands associated with $k-1$ cycles and stretching them tautly over the remnant left of b, $b_{i_1 i_2 \ldots i_{n-k+1}}$, where the subscripts are those of the cycles whose strands were left undisturbed.

Lemma 15. *If b is a k-cycle-decomposable braid on n cycles, then each of its $\binom{n}{n-k+1}$ $\overset{*}{b}_{i_1 i_2 \ldots i_{n-k+1}}$ are 1-cycle-decomposable braids on $n-k+1$ cycles, where the * refers to the removal of those strands restretched tautly in the construction of $b_{i_1 i_2 \ldots i_{n-k+1}}$.*

Proof. Since b was assumed k-cycle-decomposable and $\overset{*}{b}_{i_1 i_2 \ldots i_{n-k+1}}$ is the result of deleting all strands of b associated with $k-1$ cycles of b, the removal of all strands in any additional cycle would yield a cycle-trivial braid. Q.E.D.

Let $\Delta_{k,m}$ be the set of all k-cycle-decomposable braids on m cycles.

Lemma 16. *The set of all braids of the form*

$$\beta = b \prod_{1 \le i_1 < i_2 < \cdots < i_{m-k+1} \le m} b_{i_1 i_2 \cdots i_{m-k+1}}^{-1} b_{i_1} b_{i_2} \cdots b_{i_{m-k+1}}, \quad \textit{for all } b \textit{ in } \Delta_{k,m}$$

is precisely the set $\Delta_{k-1,m}$.

Proof. The removal of all strands associated with an arbitrary $k-1$ cycles of β leaves its b section a braid in $\Delta_{1,m-k+1}$. All factors in the product collapse except for one which is the inverse of the remnant of b and a cycle-trivial braid to its right. Free cancellation yields a cycle-trivial braid.

If b were $(k-1)$-cycle-decomposable, all factors in the product section of β would be trivial due to the nature and commutativity of sections of cycle-trivial braids and their inverses.

Thus the range of the function $\delta(b)=\beta$ is seen to contain and be contained in $\Delta_{k-1,m}$ for b in $\Delta_{k,m}$. Q.E.D.

Lemma 17. *The set of all braids of the form*

$$\gamma=\beta\cdot\prod_{l=1}^{m} b_l^{-1}\prod_{p=1}^{m} c_p,$$

where β ranges over all k-cycle-decomposable braids, is precisely the set of all braids whose closures are simple $(k-1)$-decomposable m-linkages.

Proof. The same technique used in the proof of Lemma 16 yields the desired result. Q.E.D.

Theorem 1. *The set of all k-decomposable n-linkages may be constructively enumerated.*

Proof. The closure of any k-cycle-decomposable braid on n cycles is a k-decomposable n-linkage. By the result of J.W. Alexander [1] that any linkage or knot is representable as a closed braid, all k-decomposable n-linkages are closures of braids. Inspection of Definition 5 reveals that if a braid is not cycle-trivial, then its closure is not a trivial linkage. Further, if a braid on n cycles is not k-cycle-decomposable, then removing all strands associated with any arbitrary k cycles does not result in a cycle-trivial braid. Hence all k-decomposable n-linkages are the closures of k-cycle-decomposable braids on n cycles and conversely. Since the process in Lemma 16 may be iterated from $k=n-1$ to $k=1$, all k-cycle-decomposable braids on n cycles may be constructively enumerated. Q. E. D.

Corollary. *The set of all decomposable n-linkages may be constructively enumerated.*

Theorem 2. *The set of all simple k-decomposable n-linkages may be constructively enumerated.*

Proof. Substitution of the process in Lemma 17 into the proof of Theorem 1, where the process in Lemma 16 was used yields the desired result. Q.E.D.

Corollary. *The set of all simple decomposable n-linkages may be constructively enumerated.*

II. Examples of families of decomposable braids and linkages. The shortest decomposable 3-braid which, upon closing, results in a nontrivial decomposable 3-linkage is: $(\sigma_1\sigma_2^{-1})^3$, where the σ_i's are defined in Artin [2]. (See Figure 5 for the braid and its familar closure.)

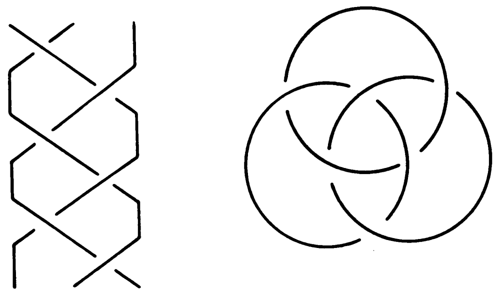


Figure 5

The fact that this braid is the shortest nontrivial decomposable 3-braid follows from an enumeration of all words in σ_1 and σ_2 of length six or less, and application of the algorithm of Chapter I, to determine decomposability. The nontriviality of the linkage resulting from its closure follows from the lemma below.

Lemma 1. *The closures of all nonnegative powers of* $(\sigma_1\sigma_2^{-1})^3$ *result in topologically distinct 3-linkages.*

Proof. The Alexander polynomials of the linkages resulting from the closures of different powers of $(\sigma_1\sigma_2^{-1})^3$ will be shown to be of different degrees. Magnus and Peluso [3] give the following method of computing the Alexander polynomial, in v, of a closed 3-braid written as a word in σ_1 and σ_2.

Let

$$T(\sigma_1) = \begin{bmatrix} -v^{-1} & 0 \\ v^{-1} & 1 \end{bmatrix}, \quad T(\sigma_2) = \begin{bmatrix} 1 & 1 \\ 0 & -v^{-1} \end{bmatrix},$$

$T(\alpha \cdot \beta) = T(\alpha)T(\beta)$, and $T(\alpha^{-1}) = (T(\alpha))^{-1}$, for any braids α, and β written as words in σ_1 and σ_2. Then the Alexander polynomial in v of a braid β is given by

$$A_\beta(v) = \frac{1 - v^{-1}}{1 - v^{-3}} \det (T(\beta) - I).$$

For $\beta = (\sigma_1\sigma_2^{-1})^{3n}$,

$$T(\beta) = \left[\begin{pmatrix} -v^{-1} & 0 \\ v^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ 0 & -v \end{pmatrix}\right]^{3n} = \begin{bmatrix} -v^{-1} & -1 \\ v^{-1} & 1-v \end{bmatrix}^{3n}.$$

Setting

$$\det\begin{bmatrix}\lambda + v^{-1} & 1\\ -v^{-1} & \lambda - 1 + v\end{bmatrix} = 0$$

to determine the eigenvalues of

$$\begin{bmatrix}-v^{-1} & -1\\ v^{-1} & 1-v\end{bmatrix}, \qquad \lambda^2 + (v^{-1} + v - 1)\lambda + 1 = 0,$$

$$\lambda = \frac{1 - v - v^{-1}}{2} \pm \frac{(v + v^{-2} - 2v - 2v^{-1} - 1)^{1/2}}{2}.$$

Under the substitution $\cos\phi = (1 - v - v^{-1})/2$, $\lambda = \cos\phi \pm i\sin\phi$.

$$\det\left[\begin{pmatrix}-v^{-1} & -1\\ v^{-1} & 1-v\end{pmatrix}^{3n} - I\right] = [(\cos\phi + i\sin\phi)^{3n} - 1][(\cos\phi - 1\sin\phi)^{3n} - 1]$$

$$= 2(1 - \cos 3n\phi),$$

$$\frac{1 - v^{-1}}{1 - v^{-3}} = \frac{v^2}{v^2 + v + 1} = \frac{v}{2(1 - \cos\phi)}.$$

Thus

$$A_\beta(v) = \frac{1 - \cos 3n\phi}{1 - \cos\phi}\cdot v.$$

Letting $w^2 = e^{i\phi}$,

$$\frac{1 - \cos 3n\phi}{1 - \cos\phi} = \frac{2 - w^{6n} - w^{-6n}}{2 - w^2 - w^{-2}} = \left(\frac{w^{3n} - w^{-3n}}{w - w^{-1}}\right)^2$$

$$= (w^{3n-1} + w^{3n-3} + w^{3n-5} + \cdots + w^{1-3n})^2.$$

The result of back-substitutions will be seen to yield a polynomial whose degree is dependent on n. Q.E.D.

The 2-linkage not representable as a closed 2-braid presented in the proof of Lemma 7 is one of a family of linkages constructed via the following processes:

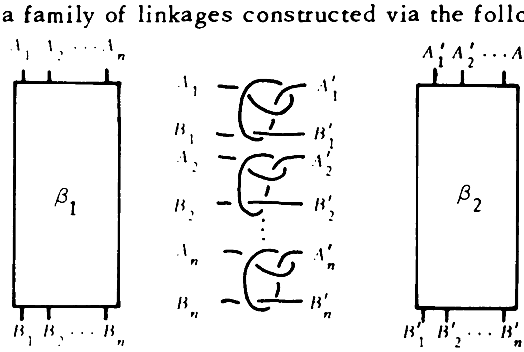


Figure 6

Take n copies of the slip-knot part of the linkage in Figure 2 and cut each in two places as pictured in Figure 6. Also, take two decomposable n-braids β_1 and β_2 and join the beginning and end of one strand of β_1 with the ends of one cut in a slip-knot. Join the other ends of the other cut in the same slip-knot to the beginning and end of another strand in β_2. Continue until all slip-knots and strands in β_1 and β_2 are exhausted. The result is a simple decomposable n-linkage which probably cannot be expressed as the closure of an n-braid.

A generalization of the above process proceeds from taking n knot products of slip-knots instead of n single slip-knots. For instance, the 2-linkage formed by taking the knot product of k slip-knots as one of the knot components in the above process, a circle as the other, and the two 2-braids σ_1^{2p}, and σ_1^{2q} is a simple decomposable 2-linkage with linking number $\pm(p \pm q)$, depending upon the orientation assigned the two links (see Figure 7). This linkage may be represented as the closure of a $(2k+1)$-braid, but probably not as the closure of a braid on fewer strands.

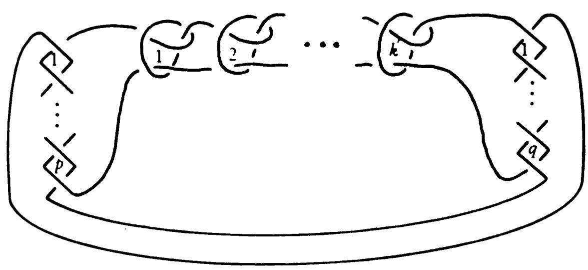


Figure 7

If, instead of taking a circle as the second trivial knot, one were to take a single slip-knot, the resulting linkage is representable as the closure of a $2(k+1)$-braid, but probably not as the closure of any braid on fewer strands.

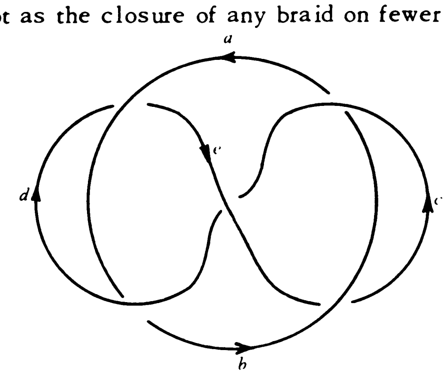


Figure 8

The minimal example of a simple 2-linkage which is not representable as the closure of a 2-braid is given in Figure 8. The directed segments of this projection may be labeled, and the linkage group presented upon generators corresponding to these labels with relators obtained from crossings in the usual manner. In this case, it is

$$\langle a, b, c, d, e;\ bca^{-1}c^{-1} = ebc^{-1}b^{-1} = dad^{-1}b^{-1} = ada^{-1}e^{-1} = ced^{-1}e^{-1} = 1\rangle.$$

Using the first, second and fourth relators, we obtain $b = cac^{-1}$, $e = bcb^{-1} = caca^{-1}c^{-1}$, $d = a^{-1}ea = a^{-1}caca^{-1}c^{-1}a$.

Under these substitutions, the third and fifth relators become

$$a^{-1}caca^{-1}c^{-1}acac^{-1}a^{-1}c^{-1}aca^{-1}c^{-1}$$

and

$$caca^{-1}c^{-1}a^{-1}cac^{-1}a^{-1}c^{-1}acac^{-1}a^{-1}$$

which may be seen to be inverses of conjugates of each other. Since the exponent sum for each generator in the relator remaining after one has been eliminated is 0, it is a (nontrivial) commutator. Thus the linkage group in question cannot be free on 2 generators, as would be the case if this linkage were the closure of a 2-braid. Its winding number is 0.

We conclude with two examples of decomposable 4-linkages. The 4-linkage depicted in Figure 9 is a simple decomposable 4-linkage.

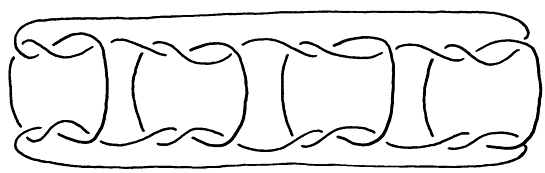


Figure 9

The linkage in Figure 9 may be represented as the closure of the 8-braid in Figure 10, but probably not as the closure of a braid on fewer strands.

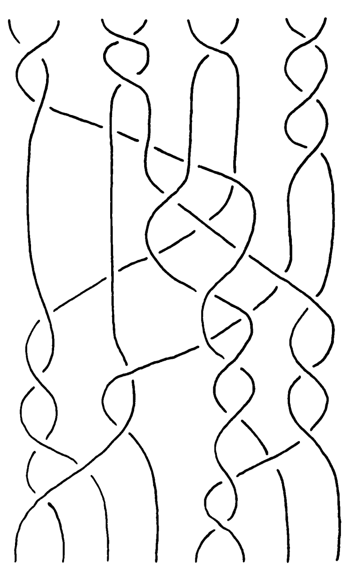


Figure 10

An n-linkage similar to the 4-linkage in Figure 9 may be represented as the closure of a $2n$-braid, for even n, and a $(2n+1)$-braid, for odd n. If the uppermost and lower most horizontal strands in the above construction were crossed, then the resulting n-linkage would be representable as a closed $(2n+1)$-braid for even n, and as a closed $2n$-braid for odd n.

The difficulty of proving the number of strands in a braid decomposition of a linkage to be minimal is incommensurate with the value of such a result at this time for any but decompositions into 3-braids of 2-linkages which are simple.

The shortest nontrivial 4-braid found which is decomposable is depicted in Figure 11. It is

$$\sigma_1^2\sigma_2^2\sigma_3^{-2}\sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_3^2\sigma_2^{-2}.$$

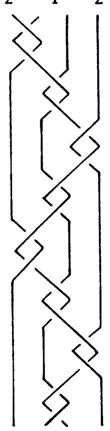


Figure 11

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