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INTERSECTIONS OF QUASI-LOCAL DOMAINS

BY

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ABSTRACT. Let $R = \bigcap V_i$ be an intersection of quasi-local domains with a common quotient field K. Our goal is to find conditions on the V_i's in order to get some or all of V_i's to be localizations of R. We show for example that if V_1 is a 1-dimensional valuation domain and if $V_1 \not\supseteq V_2$, then both V_1 and V_2 are localizations of $R = V_1 \cap V_2$.

It is well known that for any domain R we have $R = \bigcap R_{M_\alpha}$ where M_α ranges over the maximal ideals of R. However, if a domain R is represented as an intersection of quasi-local domains we cannot conclude that the domains occurring in the intersection are actually localizations of R. Our purpose will be to find some conditions sufficient to ensure that some or all of the quasi-local domains occurring in the intersection are indeed localizations of R.

1. Preliminary remarks. Suppose $R \subseteq V$ where V is a quasi-local domain with maximal ideal M. Then $M \cap R$ is called the *center of* V *on* R. We observe that if P is the center of V on R, then $R_P \subseteq V$. In fact, V is a localization of R if and only if $V = R_P$.

We also note that if $R \subseteq V$ are domains with the same quotient field, then the nonzero ideals of V contract to nonzero ideals of R.

All rings under consideration are commutative domains with identity. Our terminology will be pretty much that of [1, Chapter 2, §4].

2. Finite intersections. A well-known result (see [1, p. 78] or [2, p. 38]) states that if $R = V_1 \cap \cdots \cap V_n$ where the V_i's are all valuation domains with the same quotient field, and if P_i is the center of V_i on R, then $V_i = R_{P_i}$ for all i. Our first result is a slight generalization of this theorem. The proof, although basically the same as that in [1], will be repeated in slightly more generality. First we will need a preliminary lemma, the proof of which is found in [1].

Lemma A. *Let x be a unit in a quasi-local domain V. Then there exists an*

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integer k *(depending on* x*) such that for any integer* m *prime to* k, $1 + x + \cdots x^{m-1}$ *is a unit in* V.

Proof. See [1, p. 77].

Theorem 1. *Suppose* $R = V_1 \cap \cdots \cap V_n \cap W$ *where the* V_i*'s are valuation domains and* W *is a domain with a finite number of maximal ideals. Suppose that the* V_i*'s and* W *all have the same quotient field. Let* $N_1, \cdots, N_r$ *be the maximal ideals of* W *and let* $Q_j = N_j \cap R$. *Then* $W = \bigcap_{j=1}^r R_{Q_j}$.

Proof. Let $x \in W$. If x is a unit of W_{N_j} choose an integer h_j as in Lemma A. Similarly if x is a unit of V_i, choose an integer k_i. Pick $m > 1$ prime to all the k's and h's. Consider $s = (1+x+\cdots+x^{m-1})^{-1}$. It follows that if $x \in V_i$ (or W_{N_j}) then s is a unit of V_i (or W_{N_j}). If $x \notin V_i$, then $y = x^{-1}$ is a nonunit of V_i and $s = y^{m-1}(1+y+\cdots+y^{m-1})^{-1}$ is a nonunit of V_i. Hence $s \in R - Q_j$ for all j. If $x \notin V_i$, then $sx = y^{m-2}(1+y+\cdots+y^{m-1})^{-1} \in V_i$. If $x \in V_i$ (or W_{N_j}), then $sx \in V_i$ (or W_{N_j}). Thus $sx \in R$, and hence $x = sx/s \in \bigcap_{j=1}^r R_{Q_j}$. So $W \subseteq \bigcap_{j=1}^r R_{Q_j}$. The reverse inclusion is immediate since $R_{Q_j} \subseteq W_{N_j}$ for all j.

As a corollary we get the aforementioned result.

Corollary 2. *Suppose* $R = V_1 \cap \cdots \cap V_n$ *where the* V_i*'s are all valuation domains with the same quotient field. Let* P_i *be the center of* V_i *on* R. *Then* $V_i = R_{P_i}$ *for all* i.

We would naturally like to know under what additional hypotheses to Theorem 1 can we conclude that $V_i = R_{P_i}$ for all i. The following result tells us that if the V_i's are all 1-dimensional and if the intersection is irredundant, then we have the desired conclusion.

Theorem 3. *Let* $R = V \cap W$ *where* V *is a* 1*-dimensional valuation domain and* W *is a domain with a finite number of maximal ideals. Suppose that* V *and* W *have the same quotient field* K, *and that* $W \not\subseteq V$. *Let* $N_1, \cdots, N_r$ *be the maximal ideals of* W *and let* $Q_j = N_j \cap R$. *Let* P *be the center of* V *on* R. *Then* $V = R_P$ *and* $W = \bigcap_{j=1}^r R_{Q_j}$.

Proof. The conclusion is trivial in the case $W = K$. So now assume that $W \neq K$. That $W = \bigcap_{j=1}^r R_{Q_j}$ is just Theorem 1. Hence the quotient field of R is K. Of course $R_P \subseteq V$ is immediate, and so we need only show that $V \subseteq R_P$. Let $x \in V$. So $x = a/b$ where $a, b \in R$. Now since K is the quotient field of R it follows that $P, Q_1, \cdots, Q_r$ are all nonzero primes of R. So clearly we can choose a and b so that $b \in P \cap Q_1 \cap \cdots \cap Q_r$. Now choose $y \in W - V$. By the same argument as that used in the proof of Theorem 1, for a suitable integer $m > 1$, $u = (1+y+\cdots+y^{m-1})^{-1}$ is a unit of $W_{N_1}, \cdots, W_{N_r}$ and u is a nonunit of V. Hence $u \in P - (Q_1 \cup \cdots \cup Q_r)$. If γ is a valuation associated with V, then the value group of γ has an Archimedean order since V is 1-dimensional [3, Chapter

2]. So there exists an integer $k > 0$ such that $k\gamma(u) > \gamma(b)$. Now $b + u^k \notin Q_1 \cup \cdots \cup Q_r$. Thus $b/(b + u^k) \in R_{Q_1} \cap \cdots \cap R_{Q_r} = W$. Since $\gamma(b + u^k) = \gamma(b)$, $b/(b + u^k)$ is a unit of V. Thus $b/(b + u^k) \in R - P$. Now $xb/(b + u^k) = a/(b + u^k) \in V$ since both factors are in V. Since $b + u^k \notin Q_1 \cup \cdots \cup Q_r$, $a/(b + u^k) \in W$. Thus $a/(b + u^k) \in V \cap W = R$ and hence $x \in R_P$.

Ohm's example [4, p. 330] shows that the hypothesis that W has only finitely many maximal ideals cannot be dropped from Theorem 3. Heinzer [5] has recently generalized Theorem 3 by replacing the requirement that W have only finitely many maximal ideals with the hypothesis that W has a nonzero Jacobson radical. Ohm and Heinzer show in [6] that if V is a rational valuation domain, then the conclusion $V = R_P$ holds with no conditions on the Jacobson radical of W.

Remark. Suppose $R = V_1 \cap \cdots \cap V_n$ where the V_i's are quasi-local domains. Let P_i be the center of V_i on R. Then $P_1 \cup \cdots \cup P_n$ = the set of nonunits of R. Thus any maximal ideal of R must be equal to one of the P_i's, and hence R has at most n maximal ideals.

Let $R = \bigcap_I V_i$. We say the intersection is *irredundant* if for all $j \in I$ we have $\bigcap_{I - \{j\}} V_i \not\subseteq V_j$.

Corollary 4. *Suppose $R = V_1 \cap \cdots \cap V_n \cap W$ is an irredundant intersection where the V_i's are 1-dimensional valuation domains and W is a domain with a finite number of maximal ideals. Suppose that the V_i's and W all have the same quotient field. Let P_i be the center of V_i on R. Let $N_1, \cdots, N_r$ be the maximal ideals of W and let $Q_j = N_j \cap R$. Then $V_i = R_{P_i}$ for $i = 1, 2, \cdots, n$ and $W = \bigcap_{j=1}^r R_{Q_j}$.*

Proof. $W = \bigcap_{j=1}^r R_{Q_j}$ follows from Theorem 1. $R = V_i \cap (\bigcap_{j \neq i} V_j \cap W)$ is an irredundant intersection and $\bigcap_{j \neq i} V_j \cap W = \bigcap_{j \neq i} V_j \cap W_{N_1} \cap \cdots \cap W_{N_r}$ has only finitely many maximal ideals. Thus $V_i = R_{P_i}$ follows from Theorem 3.

3. **General intersections.** A number of results in [1] come under the assumption that certain quasi-local domains occurring in a particular intersection are either valuation or 1-dimensional. In an attempt to unify and generalize some of these results, we make the following definition. A domain V is called an LS *domain* (linear spectrum) if the prime ideals of V are linearly ordered by inclusion. We note the following characterization of LS-domains.

Proposition 5. *The following are equivalent for a domain R:*

(i) *R is an LS-domain.*

(ii) *Every radical ideal of R is prime.*

(iii) *If $a, b \in R$, then there exists an integer $m > 0$ such that a divides b^m or b divides a^m.*

Proof. (i) $\Longrightarrow$ (ii). Let I be a radical ideal of R. So $I = \bigcap P_\lambda$ where P_λ ranges over the primes of R which contain I. But since $\{P_\lambda\}$ is a chain of primes, $\bigcap P_\lambda$ is prime.

(ii) $\Longrightarrow$ (iii). $\sqrt{(a)} \cap \sqrt{(b)} = P$ is a radical ideal of R and is therefore prime. But then $\sqrt{(a)}$ and $\sqrt{(b)}$ must be comparable. Suppose $\sqrt{(a)} \subseteq \sqrt{(b)}$. Then $a \in \sqrt{(b)}$ and so there exists an integer $m > 0$ such that $a^m \in (b)$.

(iii) $\Longrightarrow$ (i). Suppose that the primes of R are not totally ordered. So there exists P and Q primes of R such that $P \not\subseteq Q$ and $Q \not\subseteq P$. Pick $x \in P - Q$ and $y \in Q - P$. By (iii) there exists $m > 0$ such that $x^m \in (y) \subseteq Q$ or $y^m \in (x) \subseteq P$. So $x \in Q$ or $y \in P$, a contradiction.

Suppose $R = \bigcap V_i$ is an intersection of quasi-local domains all with the same quotient field K. We say the intersection is *locally finite* if, for any $0 \neq x \in R$, x is a unit in all but finitely many of the V's. We say the intersection is *strongly locally finite* if, for any $0 \neq x \in K$, x is a unit in all but finitely many of the V's.

Observation. Suppose $R = \bigcap V_i$ is a locally finite intersection of quasi-local domains lying between R and its quotient field K. Then if $0 \neq x \in K$, we can write $x = a/b$ where $a, b \in R - \{0\}$. Then the only V_i's in which x is not a unit are those in which either a or b is a nonunit. There are only finitely many such V_i's by our local finiteness assumption. Hence the intersection is strongly locally finite. Thus we see that if R has the same quotient field as the V_i's then local finiteness is equivalent to strong local finiteness. We also note that trivially any finite intersection is strongly locally finite.

The following theorem generalizes a theorem in [1, p. 80] and a lemma of Griffin [7, p. 721].

Theorem 6. *Suppose $R = \bigcap V_i$ is a strongly locally finite intersection of LS-domains all with the same quotient field K. Let S be a multiplicatively closed subset of $R - \{0\}$. In V_i let N_i be the prime maximal with respect to exclusion of S. Then $R_S = \bigcap V_{i_{N_i}}$, and this is a strongly locally finite intersection of LS-domains all with quotient field K.*

Proof. That $V_{i_{N_i}}$ is an LS-domain with quotient field K is obvious. Since $N_i \cap S = \emptyset$, S consists of units in $V_{i_{N_i}}$. Hence $R_S \subseteq \bigcap V_{i_{N_i}}$. To show the reverse inclusion, let $x \in \bigcap V_{i_{N_i}}$. Since $x \in K$, x is in all but finitely many of the V_i's, say $x \notin V_1, \cdots, V_n$. Since $x \in V_{1_{N_1}}$, $x = a/b$ where $a, b \in V_1$, $b \notin N_1$. Of course b is a nonunit of V_1 since $x \notin V_1$. Let $P = \sqrt{(b)}$ = the intersection of those primes of V_1 which contain b. P is prime by Proposition 5. Claim $P \cap S \neq \emptyset$. If not, then $P \subseteq N_1$ which is maximal with respect to exclusion of S. But then $b \in P \subseteq N_1$ contrary to our condition $b \notin N_1$. So let

$t_1 \in P \cap S \subseteq \sqrt{(b)}$. Now there exists an integer $n > 0$ such that $t_1^n/b \in V_1$. Let $s_1 = t_1^n \in S$. So $s_1 x = t_1^n/b \cdot a \in V_1$. In like manner choose $s_2, \cdots, s_n \in S$ such that $s_j x \in V_j$. Then $s = s_1 s_2 \cdots s_n \in S$ and we have $sx \in V_1 \cap \cdots \cap V_n$. But since $x \in \bigcap_{i>n} V_i$ we get $sx \in \bigcap_i V_i = R$. So $x = sx/s \in R_S$. That the new intersection is strongly locally finite is obvious since $V_i \subseteq V_{iN_i}$ for all i and since $R = \bigcap V_i$ is strongly locally finite.

Theorem 7. *Suppose that $R = \bigcap V_i$ is a locally finite intersection of LS-domains between R and its quotient field K. Let P_i be the center of V_i on R and let $\{M_{i\gamma}\}$ be the chain of nonzero primes of V_i. Let $P_{i\gamma} = M_{i\gamma} \cap R$. Suppose that for some fixed j we have $P_{i\gamma} \not\subseteq P_j$ for all i_γ such that $i \neq j$. Then $V_j = R_{P_j}$.*

Proof. By Theorem 6, $R_{P_j} = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Eliminating obviously redundant copies of K from this intersection we have $R_{P_j} = \bigcap_\Gamma V_{kN_k}$ where k ranges over $\Gamma = \{i \mid N_i \neq (0)\}$. Now if $k \in \Gamma$, we have $N_k V_{kN_k} \cap R_{P_j} = QR_{P_j}$ where Q is some prime of R contained in P_j. Since $N_k \neq (0)$ we have $N_k \cap R =$ some $P_{k\gamma}$. So $[(N_k V_{kN_k} \cap V_k) \cap R] = N_k \cap R =$ some $P_{k\gamma} = QR_{P_j} \cap R = Q \subseteq P_j$. Consequently some $P_{k\gamma} \subseteq P_j$ and hence $k = j$ by hypothesis. So $\Gamma = \{j\}$ and $R_{P_j} = V_{jN_j} = V_j$.

Corollary 8. *Suppose $R = \bigcap V_i$ is a locally finite intersection of* 1*-dimensional quasi-local domains between R and its quotient field K. Let P_i be the center of V_i on R. Suppose $P_i \not\subseteq P_j$ for all pairs i, j with $i \neq j$. Then $V_i = R_{P_i}$ for all i.*

An example by Graham Evans [1, p. 78] shows that the incomparability assumption in Corollary 8 cannot be dropped.

Proposition 9. *Suppose $R = \bigcap V_i$ is an intersection of quasi-local domains all with quotient field K. Let P_i be the center of V_i on R. If for some j, R_{P_j} is a valuation domain with quotient field K, then $V_j = R_{P_j}$.*

Proof. Of course $R_{P_j} \subseteq V_j \subseteq K$ and since R_{P_j} is a valuation domain it follows that V_j is a localization of R_{P_j}. Hence V_j is a localization of R and so $V_j = R_{P_j}$.

Theorem 10. *Suppose $R = \bigcap V_i$ is a locally finite intersection of valuation domains lying between R and its quotient field K. Let P_i be the center of V_i on R. If, for some j, $\operatorname{rk}(P_j) = 1$, then $V_j = R_{P_j}$.*

Proof. By Theorem 6, $R_{P_j} = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Eliminating the obviously redundant copies of K

we have $R_{P_j} = \bigcap_\Gamma V_{iN_i}$ where $\Gamma = \{i \mid N_i \neq (0)\}$. If $k \in \Gamma$, then $N_k V_{kN_k} \cap R_{P_j} = QR_{P_j}$ for some prime of R, $Q \subseteq P_j$. But $Q = N_k \cap R \neq (0)$ since $N_k \neq (0)$. Hence $Q = P_j$. Now if $0 \neq x \in P_j$, then x is a nonunit of V_{kN_k} for all $k \in \Gamma$ and thus Γ must be finite. So $R_{P_j} = V_{1N_1} \cap \cdots \cap V_{nN_n}$. By Corollary 2, $R_{P_j} = V_{kN_k}$ holds for each $k \in \Gamma$. Hence R_{P_j} is a valuation domain and thus $V_j = R_{P_j}$ by Proposition 9.

If under the hypotheses of Theorem 10, V_j is a rational valuation domain and is irredundant in the intersection, then by the aforementioned result of Heinzer and Ohm [6], $V_j = R_{P_j}$. This is essentially a result due to Krull [1, p. 81], at least when all of the V_i's are rational valuation domains. In Ohm's example [4, p. 330] we see that under the hypotheses of Theorem 10, if $\operatorname{rk}(P_j) = 2$, then $V_j = R_{P_j}$ need not hold even if V_j is irredundant in the intersection.

If we weaken the hypotheses of Theorem 10 so as to only require that the V_i's be LS-domains, we get a similar result for those V_i's which are not redundant in the intersection. First we note a proposition which allows us to refine a redundant intersection to an irredundant one.

Proposition 11. *Suppose $R = \bigcap_{i \in \Gamma} V_i$ is a locally finite intersection of quasi-local domains between R and its quotient field K. Then $R = \bigcap_{i \in \Gamma^*} V_i$ is an irredundant intersection for some $\Gamma^* \subseteq \Gamma$.*

Proof. *Let* $\mathcal{S} = \{\Gamma_\nu \subseteq \Gamma \mid R = \bigcap_{i \in \Gamma_\nu} V_i\}$. $\mathcal{S}$ is partially ordered by inclusion. Suppose $\{\Gamma_{\nu_j}\}$ is a chain in $\mathcal{S}$. Claim: $\overline{\Gamma} = \bigcap \Gamma_{\nu_j} \in \mathcal{S}$. Then Zorn's lemma is applicable and a minimal element Γ^* in $\mathcal{S}$ will clearly satisfy our requirements. To verify our claim, we must show that $\bigcap_{\overline{\Gamma}} V_i = R$. Clearly $R \subseteq \bigcap_{\overline{\Gamma}} V_i$. Suppose $x \in K$, $x \notin R$. Since the intersection is actually strongly locally finite it follows that x is in all but finitely many of the V_i's. Say $V_1, \cdots, V_n$ are exactly those V_i's which exclude x. Clearly for some $1 \leq k \leq n$ we have $k \in \Gamma_{\nu_j}$ for all j. If not, there exists $\Gamma_{\nu_{j(k)}}$ not containing k for each $1 \leq k \leq n$. Then $\Gamma_{\nu_{j0}} = \min\{\Gamma_{\nu_{j(k)}}\}$ does not contain any k such that $1 \leq k \leq n$. So we have $x \in \bigcap_{\Gamma_{\nu_{j0}}} V_i = R$, a contradiction. Hence for some k such that $1 \leq k \leq n$ we have $k \in \overline{\Gamma} = \bigcap_j \Gamma_{\nu_j}$, and so $x \notin \bigcap_\Gamma V_i$. Thus $\bigcap_{\overline{\Gamma}} V_i \subseteq R$.

Theorem 12. *Suppose $R = \bigcap_\Gamma V_i$ is a locally finite intersection of LS-domains between R and its quotient field. Suppose V_j is a 1-dimensional valuation domain such that $\bigcap_{\Gamma - \{j\}} V_i \not\subseteq V_j$. Let P_j be the center of V_j on R. If P_j has rank $= 1$, then $V_j = R_{P_j}$.*

Proof. By Theorem 6, $R_{P_j} = \bigcap_\Gamma V_{iN_i}$ where N_i is maximal in V_i with respect to exclusion of $R - P_j$. Of course $V_{jN_j} = V_j$. By hypothesis $V_j \not\supseteq$

$\bigcap_{\Gamma-\{j\}} V_i$ and so $V_{jN_j} = V_j \not\supseteq \bigcap_{\Gamma-\{j\}} V_{iN_i}$. By Proposition 11 we can find $\Gamma^* \subseteq \Gamma$ such that $R_{P_j} = \bigcap_{\Gamma^*} V_{iN_i}$ is an irredundant, locally finite intersection; of course $j \in \Gamma^*$. Since $\mathrm{rk}(P_j) = 1$ we have $N_i V_{iN_i} \cap R_{P_j} = P_j R_{P_j}$ for all $i \in \Gamma^*$. (This contraction is $\neq (0)$ since R_{P_j} has quotient field K and $N_i \neq (0)$.) By local finiteness, there can be only finitely many elements in Γ^*. So we can write $R_{P_j} = V_j \cap (V_{1N_1} \cap \cdots \cap V_{tN_t})$ where $V_{1N_1} \cap \cdots \cap V_{tN_t} \not\subseteq V_j$. But $V_{1N_1} \cap \cdots \cap V_{tN_t}$ has only finitely many maximal ideals, and so by Theorem 3, $V_j = R_{P_j}$.

Suppose that $R = V_1 \cap \cdots \cap V_n$ where the V_i's are quasi-local domains. Let P_i be the center of V_i on R and suppose that $V_i = R_{P_i}$ for all i. If N is prime in R, then $N \subseteq P_1 \cup \cdots \cup P_n$; and hence $N \subseteq P_k$ for some k. So NR_{P_k} is prime in V_k and $NR_{P_k} \cap R = N$. Consequently any prime of R is the contraction of a prime of one of the V_i's. The following theorem gives us a condition sufficient to ensure that this will be the case even if $V_i = R_{P_i}$ does not hold for all i.

Theorem 13. *Suppose $R = V_1 \cap \cdots \cap V_n$ where the V_i's are LS-domains all with quotient field K. Let P be prime in R. Then $P = N \cap R$ for N prime in some V_i.*

Proof. By Theorem 6, $R_P = V_{1N_1} \cap \cdots \cap V_{nN_n}$ where N_i is maximal in V_i with respect to disjointness from $R - P$. Let $Q_i = N_i \cap R \subseteq P$. Now $PR_P = (N_1 V_{1N_1} \cap R_P) \cup \cdots \cup (N_n V_{nN_n} \cap R_P) =$ the set of nonunits of R_P. Intersecting both sides with R yields $P = Q_1 \cup \cdots \cup Q_n$. Hence $P = Q_i$ for some i.

Theorem 14. *Suppose $R = \bigcap V_i$ is a strongly locally finite intersection of LS-domains all with the same quotient field K. Let P be a rank 1 prime of R. Then $P = N \cap R$ for N prime in some V_i.*

Proof. Once again Theorem 6 is applicable and we have $R_P = \bigcap V_{iN_i}$ where N_i is maximal in V_i with respect to excluding $R - P$. So for each i we have $(0) \subseteq N_i \subseteq P$. Suppose $N_i \cap R = (0)$ for all i. Then Theorem 6 tells us that $R_{(0)} = \bigcap V_{iN_i} = R_P$ a contradiction, since $\mathrm{rk}(P) = 1$. Thus for some i we have $(0) \neq N_i \cap R \subseteq P$ and so $N_i \cap R = P$.

4. **Some counterexamples.** Suppose we are given $R = V \cap W$ where V and W are quasi-local domains with the same quotient field. Let P and Q be the centers of V and W on R respectively. If V is a valuation domain, then $W = R_Q$ by Theorem 1. If the intersection is irredundant and if V is a 1-dimensional valuation domain, we also have $V = R_P$ by Theorem 3. In the following example V is a 2-dimensional valuation domain and W is a 3-dimensional regular local

ring. Here we have $V \neq R_P$; and thus we see that the hypothesis of 1-dimensionality in Theorem 3 is essential.

Our first example is a generalization due to Stephen McAdam of one of my examples. As an application, I will give the details of my original example.

Example 1. Let T be a domain with prime ideals $\overline{Q}$, $\overline{M}$, $\overline{N}$ such that $\overline{Q} \subseteq \overline{M} \cap \overline{N}$, $\overline{M} \not\subseteq \overline{N}$, $\overline{N} \not\subseteq \overline{M}$, and such that $T_{\overline{Q}}$ is not a valuation domain. Let V be a valuation overring of T with primes M and Q lying over $\overline{M}$ and $\overline{Q}$ respectively and with M maximal in V [2, p. 37]. Let $R = V \cap T_{\overline{N}}$ and let $P = M \cap R$. We will show that $V \neq R_P$.

Claim $\overline{Q}T_{\overline{N}} \subseteq P$: Let $x \in \overline{Q}T_{\overline{N}}$. So $x = q/s$ where $q \in \overline{Q}$, $s \in T - \overline{N}$. Suppose $s/q \in V$. Then $s \in qV \cap T \subseteq Q \cap T = \overline{Q} \subseteq \overline{N}$, a contradiction. Thus $s/q \notin V$ and so $q/s \in M$ since V is a valuation domain. Hence $\overline{Q}T_{\overline{N}} \subseteq M \cap T_{\overline{N}} = M \cap R = P$. In particular we may view $\overline{Q}T_{\overline{N}}$ as a prime ideal in R as well as in $T_{\overline{N}}$.

Now by Theorem 1, $T_{\overline{N}} = T_{(\overline{N}T_{\overline{N}} \cap R)}$. Thus

$$T_{\overline{Q}} = (T_{\overline{N}})_{(\overline{Q}T_{\overline{N}})} = [R_{(\overline{N}T_{\overline{N}} \cap R)}]_{(\overline{Q}T_{\overline{N}})} = R_{(\overline{Q}T_{\overline{N}})} \supseteq R_P.$$

But $T_{\overline{Q}}$ is not a valuation domain and hence neither is R_P. Consequently $R_P \neq V$.

Let k be a field and let x, y, z be algebraically independent over k. Let $\mathbf{R}$ denote the additive group of reals. Consider the valuation γ on $k(x, y, z)$ given by

(i) $\gamma(\alpha) = (0, 0)$ for all $0 \neq \alpha \in k$,

(ii) $\gamma(x) = (\pi, 0)$,

(iii) $\gamma(y) = (1, 0)$,

(iv) $\gamma(z) = (0, 1)$,

where the value group is a subgroup of $\mathbf{R} \times \mathbf{R}$ lexicographically ordered, i.e., $(a, b) > (c, d)$ if and only if either $a > c$ or $a = c$ and $b > d$. Since the values of x, y, and z are linearly independent over $\mathbf{Z}$, our valuation is well defined on $k(x, y, z)$. Now in Example 1, let $V = \{r \in k(x, y, z) \mid \gamma(r) \geq (0, 0)\}$. Let $T = k[x, y, z]$, $\overline{Q} = (x, y)T$, $\overline{M} = (x, y, z)T$, $\overline{N} = (x, y, z + 1)T$. So V is a 2-dimensional valuation domain and $W = T_{\overline{N}}$ is a 3-dimensional regular local ring. Since $x/y \in V - W$ and $1/z \in W - V$, $R = V \cap W$ is an irredundant intersection.

Suppose that $R = V \cap W$ is an irredundant intersection of quasi-local domains, and that we have $V = R_P$, $W = R_N$ where P and N are the centers of V and W on R respectively. Then certainly P and N are not comparable. In Example 1, P is the center of V on R and $N = \overline{N}T_{\overline{N}} \cap R$ is the center of $W = T_{\overline{N}}$ on R. But $z \in P - N$ and $z + 1 \in N - P$ and hence P and N are not comparable. Thus we also see by Example 1 that incomparability of P and N is not sufficient to ensure that $V = R_P$ and $W = R_N$.

Theorems 1 and 7 both give one sided results and can be combined to give the following:

Corollary 15. *Suppose $R = V \cap W$ is an irredundant intersection of quasi-local domains with the same quotient field. Let P and Q be the centers of V and W on R respectively. Suppose that V is a valuation domain, W is one-dimensional, and $Q \not\subseteq P$. Then $V = R_P$ and $W = R_Q$.*

In Example 2 we will show that the hypothesis $Q \not\subseteq P$ is essential in this corollary. In this example V is a 2-dimensional valuation domain and W is a one-dimensional quasi-local domain. $V \cap W = R$ is irredundant but $Q \subseteq P$ and $V \neq R_P$.

Before giving the details of this example we will define and verify some known facts about a certain class of domains.

Suppose R is a domain contained in a field K. We define the *composite power series ring* $[[R, K))$ as: $[[R, K)) = \{f \in K[[x]] \mid$ the constant term of f lies in $R\}$. $[[R, K))$ is a domain with quotient field $K((x))$. If J is an ideal in R, by a slight abuse of notation we let $[[J, K)) = \{f \in K[[x]] \mid$ the constant term of f lies in $J\}$. Clearly $[[J, K))$ is an ideal of $[[R, K))$. Furthermore, if P is prime in R, then $[[P, K))$ is prime in $[[R, K))$. We will show that all of the nonzero primes of $[[R, K))$ are gotten in this way.

Proposition 16. *Let $\mathfrak{R} = [[R, K))$ be a composite power series ring in the variable x and let $(0) \neq \mathfrak{P}$ be prime in $\mathfrak{R}$. Then*

(a) $x \in \mathfrak{P}$,

(b) $ax \in \mathfrak{P}$ *for all* $a \in K$,

(c) $[[(0), K)) \subseteq \mathfrak{P}$,

(d) $[[P, K)) = \mathfrak{P}$ *where* $P = \mathfrak{P} \cap R$.

Proof. (a) Let $0 \neq f \in \mathfrak{P}$. So $f = x^n(a_0 + a_1x + a_2x^2 + \cdots)$ where $a_i \in K$, $a_0 \neq 0$, and n is a nonnegative integer. Now $a_0^{-1}x \in \mathfrak{R}$ and so $(a_0^{-1}x)f = x^{n+1}(1 + a_0^{-1}a_1x + \cdots) \in \mathfrak{P}$. Now $(1 + a_0^{-1}a_1x + \cdots)$ is a unit of $\mathfrak{R}$. So $x^{n+1} \in \mathfrak{P}$ and hence $x \in \mathfrak{P}$.

(b) Since $x \in \mathfrak{P}$ and $a^2x \in \mathfrak{R}$ we have $x(a^2x) = (ax)^2 \in \mathfrak{P}$. Thus $ax \in \mathfrak{P}$.

(c) Let $f = a_1x + a_2x^2 + \cdots + a_ix^i + \cdots \in [[(0), K))$. So $f = a_1x + x(a_2x + a_3x^2 + \cdots)$. The first term is in $\mathfrak{P}$ by (b) and the second term is in $\mathfrak{P}$ by (a).

(d) Let $f = a_0 + a_1x + \cdots \in \mathfrak{P}$. So $f - a_0 \in [[(0), K)) \subseteq \mathfrak{P}$ and hence $a_0 \in \mathfrak{P} \cap R = P$. Therefore $\mathfrak{P} \subseteq [[P, K))$. Conversely if $f = a_0 + a_1x + \cdots \in [[P, K))$, then $f - a_0 \in [[(0), K)) \subseteq \mathfrak{P}$ and $a_0 \in P \subseteq \mathfrak{P}$. Thus $f = (f - a_0) + a_0 \in \mathfrak{P}$ and so $[[P, K)) \subseteq \mathfrak{P}$.

Consequently the primes of $\mathfrak{R}$ are (0) and the $[[P, K))$ where P ranges over the primes of R.

Proposition 17. *If R is an n-dimensional LS-domain and K is any field containing R, then $[[R, K))$ is an $(n+1)$-dimensional LS-domain.*

Proof. If $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_n$ are the primes of R, then $(0) \subset [[(0), K)) \subset [[P_1, K)) \subset \cdots \subset [[P_n, K))$ are the primes of $[[R, K))$.

Proposition 18. $[[R, K))$ *is a valuation domain if and only if R is a valuation domain with quotient field K.*

Proof. Suppose $[[R, K))$ is a valuation domain. Let the quotient field of R be $k \subseteq K$. Suppose $0 \neq y \in K \subseteq K((x))$ = quotient field of $[[R, K))$. Then either y or y^{-1} lies in $[[R, K))$. So y or y^{-1} lies in $[[R, K)) \cap K = R$. Thus R is a valuation domain with quotient field K.

Conversely suppose that R is a valuation domain with quotient field K. Consider $0 \neq f \in K((x))$. So $f = x^n(a_0 + a_1x + \cdots + a_ix^i + \cdots)$ where $a_i \in K$, $a_0 \neq 0$, $n \in \mathbf{Z}$. Now $f^{-1} = x^{-n}(b_0 + b_1x + \cdots + b_ix^i + \cdots)$ where $b_i \in K$.

(i) If $n > 0$, then $f \in [[R, K))$.

(ii) If $n < 0$, then $f^{-1} \in [[R, K))$.

(iii) If $n = 0$, then either $a_0 \in R$ in which case $f \in [[R, K))$ or $b_0 = a_0^{-1} \in R$ in which case $f^{-1} \in [[R, K))$. Hence in any case we have f or f^{-1} lies in $[[R, K))$ and thus $[[R, K))$ is a valuation domain.

Example 2. Let k be a field and let y be an indeterminant. So $T = k[[y]]$ is a discrete valuation ring with quotient field $L = k((y))$. Let $K = k(y)$. Consider the following composite power series rings in the variable x: $V = [[T, L))$ and $W = [[K, L))$. By our last two propositions, we see that V is a 2-dimensional valuation domain with quotient field $L((x))$, and that W is a one-dimensional quasi-local domain with quotient field $L((x))$. Since T and K are incomparable rings, it follows that V and W are also incomparable. Hence $R = V \cap W$ is an irredundant intersection. Let $\overline{M} = yT$ = the maximal ideal of T. Then $M = [[\overline{M}, L))$ is the maximal ideal of V. $N = [[(0), L))$ is the maximal ideal of W. So $P = M \cap R = [[\overline{M} \cap K, L))$ and $Q = N \cap R = [[(0), L))$. Since $Q \subseteq P$ it follows that $R_P \subseteq R_Q = W$ by Theorem 1. Since $V \nsubseteq W$ we have $V \neq R_P$.

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