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FINITE GROUPS WITH NICELY SUPPLEMENTED SYLOW NORMALIZERS

BY

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ABSTRACT. This paper considers finite groups G whose Sylow normalizers are supplemented by groups D having a cyclic Hall $2'$-subgroup. G is solvable and all odd order composition factors of G are cyclic. If $S \in \mathrm{Syl}_2(D)$ is cyclic, dihedral, semidihedral, or generalized quaternion, then G is almost supersolvable.

Let $\mathfrak{D}$ denote the class of finite groups D which satisfy:

(*) $D = ST$, where $S \in \mathrm{Syl}_2(D)$ and T is cyclic group of odd order.

We say G is $\mathfrak{D}$-supplemented if G is finite and every Sylow normalizer in G has a supplement $D \in \mathfrak{D}$.

Theorem 1. *$\mathfrak{D}$-supplemented groups are solvable.*

Proof. Assume the theorem is false, and let G be a counterexample of minimal order. Since any homomorphic image of G is $\mathfrak{D}$-supplemented, G/N is solvable for any $1 \neq N \lhd G$. Thus, G has a unique minimal normal subgroup M. M is nonsolvable, and so 2 divides $|M|$ by the Feit-Thompson Theorem. Choose $P \in \mathrm{Syl}_2(M)$ and $Q \in \mathrm{Syl}_2(G)$, $P \leq Q$. By the Frattini argument $G = MN(P)$. Let $D \in \mathfrak{D}$ be a supplement for $N(Q)$. Since $Q \in \mathrm{Syl}_2(G)$, we can assume D is cyclic of odd order. Choose a subgroup $H \geq N(P)$ which is maximal in G. Since $N(P) \geq N(Q)$, D is a supplement for H. $(D \cap H)^G = (D \cap H)^H \leq H$. If $D \cap H \neq 1$, then $M \leq (D \cap H)^G \leq H$, a contradiction. Consequently, $N(P) = N(Q)$ is maximal in G, and D is a complement for $N(P)$. G has a faithful primitive representation on the $d = |D|$ cosets of $N(P)$, and D is regularly represented. If d is not prime, then D is a B-group [8, 25.2], and so G is 2-transitive. Otherwise d is prime, and G is 2-transitive by a theorem of Burnside.

Recent results of Shult and O'Nan classify 2-transitive groups H in which H_α is a 2-local subgroup. If $T = O_2(H_\alpha)$ is semiregular on $\Omega - \{\alpha\}$, then Shult's Fusion Theorem (see [5]) implies that H has a regular normal subgroup, or $N \trianglelefteq H \leq \mathrm{Aut}(N)$, where N is isomorphic to $\mathrm{PSL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$, or $\mathrm{Sz}(2^{2a+1})$ in its standard 2-transitive permutation representation. (We need Shult's result only in the case $O_2(G_\alpha) \in \mathrm{Syl}_2(G)$. This special case follows from Suzuki's work

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on finite groups with independent Sylow 2-subgroups [7].) If T is not semiregular, then work of O'Nan [6] implies that H has a regular normal subgroup or $N \trianglelefteq H \leq \mathrm{Aut}(N)$, where $N \simeq \mathrm{PSL}_n(2^a)$. Since G has no regular normal subgroup and $O_2(G_\alpha)$ is a Sylow 2-subgroup of G, the only possibility is $N \trianglelefteq G \leq \mathrm{Aut}(N)$, where $N \simeq \mathrm{PSL}_2(2^a)$, $\mathrm{PSU}_3(2^a)$, or $\mathrm{Sz}(2^{2a+1})$. In these cases one easily finds a prime p and $S \in \mathrm{Syl}_p(G)$ so that $N(S)$ has no supplement $D \in \mathfrak{D}$. For example, if $G \simeq \mathrm{PSL}_2(4)$ take $p = 3$, and if $G \simeq \mathrm{P\Gamma L}_2(4)$ take $p = 2$.

Remark. If $G \simeq \mathrm{PSL}_2(2^a)$ and $S \in \mathrm{Syl}_2(G)$, then $N(S)$ has a cyclic complement of odd order.

Theorem 2. *If G is $\mathfrak{D}$-supplemented then every chief factor of G of odd order is cyclic.*

Proof. Let G be a counterexample of minimal order. A result of Huppert [4, VI. 8.6] implies that $\Phi(G) = 1$. G has a unique minimal normal subgroup M. Since G is solvable, M is an elementary abelian p-group. p is odd. Set $P = O_p(G)$. P is elementary abelian since $\Phi(P) \leq \Phi(G) = 1$.

There is a prime $q \neq p$ and a q-group $1 \neq Q < G$ so that $PQ \trianglelefteq G$. $P = [P, Q] \times C_P(Q)$. Since $[P, Q] \neq 1$ and $C_P(Q)$ and $[P, Q]$ are normal in G, $C_P(Q) = 1$. G is a split extension of P by $N(Q)$. If $Q \leq Q_1 \in \mathrm{Syl}_q(G)$, then $N(Q) \geq N(Q_1)$. Consequently, $N(Q)$ has a supplement $D \in \mathfrak{D}$. D contains an element x of order $p^m = |P|$. The image $\bar{x}$ of x in $\bar{G} = G/P$ has order at least p^{m-1}. Since $\bar{G}$ is isomorphic to a subgroup of $\mathrm{GL}_m(p)$, $pm > p^{m-1}$. Hence, $m = 2$. G contains an element of order p^2, and so p divides $|\bar{G}|$. But $O_p(\bar{G}) = 1$ and $\bar{G}$ is solvable. The only possibility is $p = 3$ and $\bar{G} \simeq \mathrm{SL}_2(3)$ or $\mathrm{GL}_2(3)$. Then the normalizer of $S \in \mathrm{Syl}_2(G)$ has index 9 or 27 in G. However, G contains no elements of order 9, a final contradiction.

Let $\mathfrak{D}^*$ denote the class of finite groups D which are the product of a cyclic group T of odd order and a cyclic, dihedral, semidihedral, or generalized quaternion 2-group S. T is a Hall $2'$-subgroup of D and $S \in \mathrm{Syl}_2(D)$. $D \in \mathfrak{D}^*$ implies $D \in \mathfrak{D}$, so that $\mathfrak{D}^*$-supplemented groups are solvable. Buchthal [1] has show that certain solvable $\mathfrak{D}^*$-supplemented groups are either supersolvable or have Σ_4 as a homomorphic image.

Theorem 3. *If G is $\mathfrak{D}^*$-supplemented, then G contains a normal subgroup N such that every G-composition factor of N is cyclic and G/N is isomorphic to 1, A_4, Σ_4, or one of the groups $\Gamma_1, \Gamma_2, \Gamma_3$ defined below.*

The group Γ_1 is defined as follows. Let W be an elementary abelian group of order 16. Choose $g \in \mathrm{Aut}(W)$ so that $|g| = 3$ and $C_W(g) = 1$. Let S be a Sylow 2-subgroup of $N_{\mathrm{Aut}(W)}(\langle g \rangle) \simeq \Gamma\mathrm{L}_2(4)$. S and g generate a group X of order 24.

Define Γ_1 to be the split extension of W by X. The normalizer $N(R)$ of $R \in \mathrm{Syl}_3(\Gamma_1)$ has index 16 in Γ_1. The only supplements $D \in \mathfrak{D}^*$ for $N(R)$ are semidihedral or generalized quaternion groups of order 16. (These facts are established in the proof of Theorem 3.)

Suppose $W \simeq Z_4 \times Z_4$. Let a and b be generators of W. Define automorphisms g, x, z, and s of W as follows.

1. $a^g = b^{-1}$, $b^g = ab^{-1}$,
2. $a^z = a^{-1}$, $b^z = b^{-1}$,
3. $a^x = ab^2$, $b^x = a^2b^{-1}$,
4. $a^s = b$, $b^s = a$.

The element $g \in \mathrm{Aut}(W)$ has order 3, while x, z, and s are involutions. $C_{\mathrm{Aut}(W)}(g) = \langle g, x, z\rangle$ and $N_{\mathrm{Aut}(W)}(\langle g\rangle) = \langle g, x, z, s\rangle = X$. Γ_2 is the split extension of W by X. $S = \langle a, b, x, z, s\rangle$ is a Sylow 2-subgroup of Γ_2. S contains no elements of order 16, and every element of order 8 in S is conjugate to sa. $N_S(\langle sa\rangle)$ is a split extension of $\langle sa\rangle$ by the 4-group $\langle zb, a^2\rangle$. $\langle sa, zb\rangle$ and $\langle sa, a^2\rangle$ are complements for $N_S(\langle g\rangle)$ in S, while $\langle sa, zba^2\rangle \cap N_S(\langle g\rangle) = \langle sz\rangle$. Also, $\langle sa, zb\rangle$ is semidihedral, and $\langle sa, a^2\rangle$ is neither dihedral nor semidihedral. These facts yield the following result.

Lemma 1. *Γ_2 is $\mathfrak{D}^*$-supplemented. Any proper subgroup of Γ_2 which contains $\langle a, b, g\rangle$ and is $\mathfrak{D}^*$-supplemented is conjugate in Γ_2 to $\Gamma_3 = \langle a, b, g, z, s\rangle$. Moreover, if $\Gamma = \Gamma_2$ or Γ_3 and $R \in \mathrm{Syl}_3(\Gamma)$, then $N(R)$ has index* 16 *in Γ and the only supplements $D \in \mathfrak{D}^*$ for $N(R)$ are semidihedral groups of order* 16.

Proof of Theorem 3. In the following discussion, Γ denotes any one of the groups Γ_1, Γ_2, or Γ_3.

Let G be a counterexample of minimal order. Choose $N \trianglelefteq G$ of minimal order so that $G/N \simeq 1$, A_4, Σ_4, or Γ. (E.g., if G has both Σ_4 and Γ as homomorphic images, choose N such that $G/N \simeq \Gamma$.) N contains a unique minimal normal subgroup M of G. M is not cyclic. Theorem 2 implies that M is a 2-group. Set $P = O_2(N)$. $C_N(P) \leq P$ and $O_{2'}(G) = 1$. Suppose $\Phi(P) \neq 1$. Then by induction each G-composition factor of $P/\Phi(P)$ is cyclic, and so $G/C_G(P/\Phi(P))$ is a 2-group. Hence, $G/C_G(P)$ is a 2-group [3, 5.1.4], in which case $P \cap Z(G) \neq 1$. This contradiction implies that P is elementary abelian.

Assume $P \neq N$. Then there is a prime $q \neq 2$ and a q-group $1 \neq Q < N$ so that QP is normal in G. $C_P(Q) = 1$. Let $|P| = 2^m$. If $C_G(P) = P$, then the proof of Theorem 2 shows that $2^{m-2} < 2m$, or $m \leq 5$. If $m < 4$, there is no choice for q. If $m = 5$, then $q = 31$. But the normalizer in $\mathrm{GL}_5(2)$ of a group of order 31 has order $31 \cdot 5$. It follows that $P \in \mathrm{Syl}_2(G)$, and so $N(Q)$ is not $\mathfrak{D}^*$-supplemented. Thus, the only possibility is $m = 4$ and $q = 3$ or 5. $N(Q)$ has a supplement D

which is cyclic, dihedral, semihedral, or generalized quaternion of order at least $|P| = 16$. D has no normal elementary abelian subgroup of order 4, and so $|D \cap P| \leq 2$. Thus, DP/P has order at least 8. The normalizer in $GL_4(2)$ of a cyclic group of order 5 is metacyclic group of order 60. Consequently, $q = 3$. Since $C_P(Q) = 1$, the normalizer in $GL_4(2)$ of Q is $\Gamma L_2(4)$. Since G is solvable, the only possibility is that $N(Q)$ is a split extension of Q by D_8 or Σ_4. In either case $O_2(G/P) \simeq Z_2 \times Z_2 \simeq C_P(O_2(G/P))$. By induction G/P acts reducibly on $P/C_P(O_2(G/P))$, which is not the case. Therefore, $C_G(P)$ properly contains P, whence $N \neq G$. There is a group $P < K \trianglelefteq G$ so that $K/P \simeq Z_2 \times Z_2$. $C_P(Q) = 1$ and $[K, Q] \leq P$ imply $C_K(Q) \simeq Z_2 \times Z_2$ [3, 5.3.15]. $C_K(Q) = C_K(PQ)$ is normal in G. Then $K \leq C(P)$ implies $K = P \times C_K(Q)$, and so K is elementary abelian. If $X \leq G$ let $\tilde{X}$ denote the image of X in $\tilde{G} = G/C_K(Q)$. By induction G has a normal subgroup $H \geq C_K(Q)$ so that $\tilde{G}/\tilde{H} \simeq 1$, A_4, Σ_4 or Γ, and each $\tilde{G}$ composition factor of $\tilde{H}$ is cyclic. From the facts that M is noncyclic, $M \cap C_K(Q) = 1$, and M is the only minimal normal subgroup of G contained in N, it follows that N is isomorphic to a subgroup of $\tilde{G}/\tilde{H}$. Thus, $Q \simeq Z_3$. Choose $S \in \mathrm{Syl}_3(G)$. Suppose $G/N \simeq \Gamma$. Then $G : N(S) \geq 64$. Consequently, there is a cyclic, dihedral, semidihedral, or generalized quaternion group D of order at least 64 which is a supplement for $N(S)$. For $X \leq G$ let $\bar{X}$ denote the image of X in $\bar{G} = G/N \simeq \Gamma$. Then $\bar{S} \in \mathrm{Syl}_3(\bar{G})$ and $N_{\bar{G}}(\bar{S}) = \overline{N_G(S)}$. Thus, $\bar{D} \simeq D/D \cap N$ is a supplement for $N_{\bar{G}}(\bar{S})$. But $D \cap N \neq 1$ since Γ has exponent 24. Hence, $\bar{D}$ is cyclic or dihedral, whereas a $\mathfrak{D}^*$-supplement for $N_{\bar{G}}(\bar{S})$ in $\bar{G} \simeq \Gamma$ must be semidihedral or generalized quaternion of order 16. This contradiction implies $G/N \simeq A_4$ or Σ_4, whence $N \simeq A_4$, or Σ_4. Then $S \simeq Z_3 \times Z_3$, $G : N(S) = 16$, and $|G/K|$ divides 36. A Sylow 2-subgroup of G/K is not cyclic of order 4, and so the exponent of G divides 12. Hence, $N(S)$ does not have a supplement $D \in \mathfrak{D}^*$.

The only remaining case is $P = N$. Then $G \neq N$ and there is an element $g \in G$ of order 3. Assume $G/N \simeq \Gamma$. $C_G(N)$ does not contain g, for otherwise $G/C_G(N)$ is a 2-group, and $N \cap Z(G) \neq 1$. Hence, $N : C_N(g) \geq 4$, and so $G : N(\langle g \rangle) \geq 64$. Since G has exponent 24 or 48, $N(\langle g \rangle)$ has no supplement $D \in \mathfrak{D}^*$. Thus, $G/N \simeq A_4$ or Σ_4. Set $K = O_2(G)$. $K/N \simeq Z_2 \times Z_2$. Suppose $[N, K] = 1$. Then $C_N(g) = 1$, and by induction $|N| = 4$. A supplement $D \in \mathfrak{D}^*$ for $N(\langle g \rangle)$ has order at least 16, whence $|D \cap K| \geq 8$. Since K has exponent 2 or 4, $D \cap K$ is dihedral or quaternion. Thus, K is a nonabelian group of order 16 and exponent 4. According to Burnside [2, p. 146] $|K'| = 2$, and so N contains a subgroup of order 2 which is normal in G. This contradiction implies $[N, K] = U \neq 1$.

By induction each G-composition factor of N/M is cyclic of order 2. Consequently, g centralizes N/M. Then $K = N[K, g]$ also centralizes N/M, so $U \leq M$. Thus, $U = M$ and K centralizes U. Then $C_U(g) = 1$. By induction

$|U| = 4$. Set $V = C_N(g)$, so that $N = V \times U$. Choose $H < K$ such that $H:N = 2$. $C_N(H)$ contains U and therefore is normalized by g. Then $C_N(H) = C_N(H^g) = C_N(HH^g) = C_N(K)$. Since $C_N(K) \cap C_N(g) = 1$, $C_N(H) = U$. Consequently, $|N| = 8$ or 16.

Suppose there is an element $x \in K$ so that the image $\bar{x}$ of x in $\bar{K} = K/U$ has order 4. $x^2 \in N$ since $K/N \simeq Z_2 \times Z_2$, but $x \notin N$ since N is elementary. Hence, $C_K(x^2)$ properly contains N, which is not the case. Consequently, K/U is elementary abelian, and so $K/U = C_{K/U}(g) \times [K/U, g]$. Since $U = [U, g] < W = [K, g]$ and $C_U(g) = 1$, K is a split extension of W by V. W has order 16 and is normal in G. Since W has exponent at most 4 and $|W'| \neq 2$, W is abelian.

Let $R = \langle g \rangle$. G is a split extension of W by $N(R)$. $N(R)$ acts faithfully on W. $N(R)$ has a supplement D which is cyclic, dihedral, semidihedral, or generalized quaternion. Since K has exponent 4, the only possibility is $G/N \simeq \Sigma_4$, D is a complement for $N(R)$, and D is dihedral, semidihedral, or generalized quaternion of order 16. Suppose $W \simeq Z_4 \times Z_4$. Then Lemma 1 yields $G \simeq \Gamma_2$ or Γ_3. This contradiction implies that W is elementary abelian of order 16, and G is isomorphic to a subgroup of Γ_1. The nonidentity elements of $N \cup W$ have order 2, while all elements in $K - (N \cup W)$ have order 4. Since D has no normal 4-group, $D \cap N = D \cap W \simeq Z_2$. Hence, $D \cap K$ is a quaternion group, and so D is semidihedral or generalized quaternion of order 16. Moreover, $|DW| = |D||W|/|D \cap W| = 2^7$, so that $|G| = 3 \cdot 2^7 = |\Gamma_1|$. Then $G \simeq \Gamma_1$, a final contradiction.

Thus, G has a normal subgroup N so that every G-composition factor of N is cyclic and $G/N \simeq 1$, A_4, Σ_4, Γ_1, Γ_2, or Γ_3. N is the join of all groups $H \trianglelefteq G$ which are supersolvably embedded in G, and so N is unique.

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