TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 189, 1974

QUASI-BOUNDED AND SINGULAR FUNCTIONS(1)

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ABSTRACT. A general formulation is given for the concepts of quasi-bounded and singular functions, thereby extending to a much broader class of functions the concepts initially formulated by Parreau in the harmonic case. Let Ω be a bounded Euclidean region. With the underlying space taken as the class $\mathcal{M}$ of all nonnegative functions u on Ω admitting superharmonic majorants, an operator S is introduced by setting Su equal to the regularization of the infimum over $\lambda \geq 0$ of the regularized reduced functions for $(u - \lambda)^+$. Quasi-bounded and singular functions are then defined as those u for which $Su = 0$ and $Su = u$, respectively. A development based on properties of the operator S leads to a unified theory of quasi-bounded and singular functions, correlating earlier work of Parreau (1951), Brelot (1967), Yamashita(1968), Heins (1969), and others. It is shown, for example, that a nonnegative function u on Ω is quasi-bounded if and only if there exists a nonnegative, increasing, convex function φ on $[0,\infty]$ such that $\varphi(x)/x \to +\infty$ as $x \to \infty$ and $\varphi \circ u$ admits a superharmonic majorant. Extensions of the theory are made to the vector lattice generated by the positive cone of functions u in $\mathcal{M}$ satisfying $Su \leq u$.

1. **Introduction.** Quasi-bounded and singular harmonic functions were first defined and studied by M. Parreau [7] in 1951, and the subject has since attracted considerable interest in connection with its applications to complex function theory and potential theory. Some recent developments and extensions, including applications to Hardy spaces, are detailed in the monograph of M. H. Heins [5]. The concept of a quasi-bounded function has been further generalized by the present authors so as to avoid the harmonicity requirement. This generalization, which arose naturally in the process of extending the Phragmén-Lindelöf maximum principle in [2], serves as a backdrop for the present work.

We start by refining the notions of quasi-bounded and singular functions and proceed to develop basic elements of the theory of such functions. Our approach involves introduction of an operator S on a class $\mathcal{M}$ of functions. The quasi-bounded and singular functions are then those functions u in $\mathcal{M}$ for which $Su = 0$ and $Su = u$, respectively. Many of the classical results for the harmonic case are shown to hold in this general setting or to admit appropriate generalizations. Moreover, a very concise treatment of the harmonic case can be given in terms of the operator S, and we include this for the sake of completeness.

Although the theory adapts at once to hyperbolic Riemann surfaces, the present treatment will be restricted to the case of bounded regions in Euclidean

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Presented to the Society, March 27, 1971 and April 10, 1971 under the titles *Quasibounded and singular functions* and *Projections onto spaces of quasibounded and singular functions*; received by the editors August 17, 1972.

AMS (MOS) subject classifications (1970). Primary 31C05.

Key words and phrases. Quasi-bounded functions, singular functions, superharmonic functions, harmonic functions, potentials, projection operators.

(1) Research supported in part by the National Science Foundation.

space. For purposes of notation and examples we shall visualize these regions as lying in the plane. Throughout the sequel, the underlying bounded region will be denoted by Ω, and the class of all nonnegative functions on Ω admitting superharmonic majorants will be denoted by $\mathscr{M}$. Suppose that u is a function in the class $\mathscr{M}$. Then for each real number $\lambda \geq 0$ the function $(u - \lambda)^+$ will likewise belong to the class $\mathscr{M}$, and we can form the reduced function

$$R_\lambda u = R(u - \lambda)^+. \tag{1.1}$$

This is defined as the infimum of the class of all superharmonic majorants of $(u - \lambda)^+$. (See M. Brelot [3] or L. L. Helms [6] for a discussion of reduced functions.)

As is well known, the reduced functions $R_\lambda u$ are quasi superharmonic. That is, each $R_\lambda u$ coincides quasi everywhere (i.e. with the possible exception of a polar set) with a function superharmonic on Ω. Moreover, this superharmonic function is unique, since it appears as the lower regularization (pointwise limit inferior) of the given quasi superharmonic function. The lower regularization of a quasi superharmonic function v will be denoted by $\hat{v}$. We put

$$S_\lambda u = \hat{R}_\lambda u. \tag{1.2}$$

We next note that the family of functions $S_\lambda u \quad (\lambda \geq 0)$ is decreasing, and we write

$$S_\infty u = \lim_{\lambda \to \infty} S_\lambda u. \tag{1.3}$$

Of course, in view of the monotonicity in λ we could just as well take $\lambda = n$ $(n = 1, 2, \ldots)$ and define $S_\infty u$ as the limit of the sequence $\{S_n u\}$. The function $S_\infty u$ is quasi superharmonic on Ω, and we denote by Su the lower regularization of $S_\infty u$, i.e.

$$Su = \hat{S}_\infty u. \tag{1.4}$$

The resulting operator S assigns to each function u in $\mathscr{M}$ a corresponding nonnegative superharmonic function Su on Ω. It is clear also that the operator S is conformally invariant.

According to the classical definition, a harmonic function h on Ω is called *quasi-bounded* if it is the limit of an increasing sequence of nonnegative bounded harmonic functions. Similarly, a nonnegative harmonic function h on Ω is called *singular* if the only nonnegative bounded harmonic function on Ω majorized by h is the function identically zero. We shall show that the quasi-bounded and singular harmonic functions on Ω can be characterized as those nonnegative harmonic functions h on Ω for which $Sh = 0$ and $Sh = h$, respectively.

On these grounds we are led to introduce general quasi-bounded and singular functions based on analogous definitions. That is, we shall call a function u on Ω *quasi-bounded* if u is a function in $\mathcal{M}$ for which

$$Su = 0, \tag{1.5}$$

and we shall call u *singular* if u is a function in $\mathcal{M}$ for which

$$Su = u. \tag{1.6}$$

General quasi-bounded functions are shown to admit the following characterization, extending results of M. Parreau [7] and S. Yamashita [8] for the harmonic case. A function $u \geq 0$ on Ω is quasi-bounded if and only if there exists a nonnegative, increasing, convex function φ on $[-\infty, +\infty]$ such that

$$\lim_{t \to \infty} \frac{\varphi(t)}{t} = +\infty$$

and the composite function $\varphi \circ u$ belongs to the class $\mathcal{M}$. Various closure properties and convergence theorems are shown to hold for quasi-bounded and singular functions.

With regard to the general behavior of Su, the most significant results occur when the functions u are superharmonic. An explicit formula is at hand for Su in the case of u the Green's potential of a positive mass distribution p, namely

$$Su(z) = \int_{E_\infty} G_z(\zeta)\, dp(\zeta) \qquad (z \in \Omega),$$

where $G_z(\zeta)$ is the Green's function and E_∞ is the set on which u is $+\infty$. This leads to the decomposition of a nonnegative superharmonic function u as $u = q + s$, where q is quasi-bounded and s singular, thus extending the classical Parreau decomposition theorem for harmonic functions. Monotone convergence theorems are also established for S applied to sequences of nonnegative superharmonic functions.

We conclude by considering the positive cone $\mathcal{P}$ of functions u $(\in \mathcal{M})$ satisfying $Su \leq u$. It is shown that $\mathcal{P}$ consists precisely of those u which can be decomposed as a sum of quasi-bounded and singular functions. Moreover, the linear space $\mathcal{Q} = \mathcal{P} - \mathcal{P}$ generated by $\mathcal{P}$ is a vector lattice, and S admits an extension as a positive linear operator on $\mathcal{Q}$.

As might be expected, the concepts of quasi-bounded and singular functions, as well as the general theory of the operators S_λ and S, can be carried over to an axiomatic setting. This has been done by the present authors for a particularly simple set of axioms, and the results will appear in a forthcoming paper dealing at greater length with the operators S_λ.

It should be noted that quasi-bounded potentials are the same as the semi-bounded potentials defined by M. Brelot in 1965 (see [4]). Let χ_E denote the characteristic function of a set E. Brelot calls a potential u on Ω "semibounded" if

$$\inf_{\lambda \geq 0} \hat{R}(u\chi_{[u \geq \lambda]}) = 0 \tag{1.7}$$

quasi everywhere on Ω. This approach can be taken as the basis for defining operators B_λ and B analogous to S_λ and S, respectively. That is, for u any function in $\mathcal{M}$ and λ any nonnegative real number, we put

$$B_\lambda u = \hat{R}(u\chi_{[u \geq \lambda]}) \tag{1.8}$$

and define Bu as the lower regularization of

$$B_\infty u = \lim_{\lambda \to \infty} B_\lambda u. \tag{1.9}$$

(That $B_\infty u$ is quasi superharmonic follows by monotonicity, as in the case of S_∞.) Although the operators B_λ are, in general, distinct from S_λ, we show that $B = S$ (see the end of §2). In particular, this implies that quasi-bounded potentials are the same as semibounded potentials. The latter result is also implicit in recent work of B. Fuglede (see the remarks appended in §10 of the present paper).

2. **Basic properties of the operator** S. In studying the operator S, repeated use is made of certain elementary properties, which we proceed to collect.

Lemma 2.1. *For u and v functions in $\mathcal{M}$ and α any nonnegative real number*
(1) $S(\alpha u) = \alpha Su$,
(2) $S(u + v) \leq Su + Sv$,
(3) $u \leq v$ *implies* $Su \leq Sv$,
(4) $S(u \wedge v) \leq (Su) \wedge (Sv)$, *where* $\wedge$ *denotes the lower envelope.*

Proof. Conclusion (3) is immediate, and (4) follows from (3). To prove (1), we start with a nonnegative superharmonic function φ on Ω such that, for given $\lambda \geq 0$, $u \leq \varphi + \lambda$. Then $\alpha u \leq \alpha\varphi + \alpha\lambda$, so that $S_{\alpha\lambda}(\alpha u) \leq \alpha\varphi$. Taking the infimum over φ leads to $S_{\alpha\lambda}(\alpha u) \leq \alpha S_\lambda u$, and in the limit as $\lambda \to \infty$ there results

$$S(\alpha u) \leq \alpha Su.$$

To establish equality, all that remains is to set aside the trivial case of $\alpha = 0$ and observe that $Su = S(\alpha u/\alpha) \leq S(\alpha u)/\alpha$.

We apply a similar argument to prove (2). Given $\lambda \geq 0$, let φ and ψ be nonnegative superharmonic functions on Ω such that $u \leq \varphi + \lambda$ and $v \leq \psi + \lambda$. Then, $u + v \leq \varphi + \psi + 2\lambda$, and this inequality shows that

$$S_{2\lambda}(u + v) \leq S_\lambda u + S_\lambda v.$$

Conclusion (2) follows by letting $\lambda \to \infty$.

It is a trivial matter to exhibit quasi-bounded functions, since all nonnegative bounded functions are obviously quasi-bounded. We note also that inequality (2) of Lemma 2.1 implies that the sum of two quasi-bounded functions is quasi-bounded. To exhibit a class of singular functions, we make use of Green's potentials.

Lemma 2.2. *Let u be the Green's potential of a positive mass distribution p on Ω. If the support of p has capacity zero, then u is singular.*

Proof. With $\lambda \geq 0$ fixed, let φ be a nonnegative superharmonic function on Ω such that $u \leq \varphi + \lambda$. Then, denoting the support of p by F, we take $\{F_n\}$ as an exhaustion of F by compact sets and define u_n as the Green's potential of $p \mid F_n$ $(n = 1, 2, \ldots)$. Since, for each n, $u_n - \varphi$ is a subharmonic function on $\Omega - F_n$ bounded above by λ and u_n tends to zero at the regular boundary points of Ω, the Phragmén-Lindelöf maximum principle ensures that $u_n \leq \varphi$ $(n = 1, 2, \ldots)$ on $\Omega - F_n$ and hence on Ω. This results in $u_n \leq Su$ and consequently $u \leq Su$. On the other hand, since u is superharmonic and nonnegative, the obvious inequality $u \leq u + \lambda$ implies $Su \leq u$, establishing $Su = u$ as claimed.

Strict inequality can actually hold in (2) and (4) of Lemma 2.1. The example which we shall give to show this for (2) uses the following domination property.

Lemma 2.3. *Let Ω be the unit disc and u any function in $\mathscr{M}$ such that, for all z on some open segment σ issuing from the origin,*

$$u(z) \geq \log(1/|z|).$$

Then, for all z on Ω,

$$Su(z) \geq \log(1/|z|).$$

Proof. Again we fix $\lambda \geq 0$ and take Ω as any nonnegative superharmonic function on Ω such that $u \leq \varphi + \lambda$. It is convenient next to introduce the subharmonic function v on $\Omega - \{0\}$ defined by

$$v(z) = \log(1/|z|) - \varphi(z) - \lambda$$

and to consider the region $\omega = \Omega - \bar{\sigma}$. Then, for z on ω we have

$$\limsup_{z \to \zeta} v(z) \leq 0 \qquad (\zeta \in \partial\omega),$$

except perhaps at the end points of σ, and also

$$v(z) \leq \log(1/|z|). \tag{2.1}$$

These two inequalities allow us to apply the generalized Phragmén-Lindelöf maximum principle given in Theorem 2.1 of [2], it being noted that quasi-boundedness of the right-hand member in (2.1) follows from Theorem 3.4 of [2]. There results

$$\log(1/|z|) \leq \varphi(z) + \lambda$$

for z on ω and hence for z on Ω. The lemma follows.

That strict inequality can occur in (4) of Lemma 2.1 is easily seen, for instance, by taking u and v as Green's potentials of unit masses concentrated at distinct points of Ω. Indeed, Lemma 2.2 yields $Su = u$ and $Sv = v$, whereas the evident boundedness of $u \wedge v$ results in $S(u \wedge v) = 0$.

To arrive at strict inequality in (2) of Lemma 2.1, we take Ω as the unit disc, put

$$w(z) = \log(1/|z|) \qquad (z \in \Omega), \tag{2.2}$$

and define u and v as the functions on Ω equal to w on the open right half-disc and open left half-disc, respectively, and to zero elsewhere. Then $u + v \leq w$, so that $S(u + v) \leq Sw = w$ by Lemma 2.2. Since Lemma 2.3 shows that $Su = Sv = w$, we conclude that $S(u + v) < Su + Sv$ on $\Omega - \{0\}$.

An argument based on Lemma 2.3 shows also that $Su > 0$ can hold even when $u = 0$ almost everywhere. For this we have only to take u equal to the function w in (2.2) on some open segment σ issuing from the origin and to zero elsewhere on the disc Ω.

We note, however, that the condition $u = 0$ quasi everywhere on Ω forces $Su = 0$, in view of the following general property.

Lemma 2.4. *If u and v are functions in $\mathcal{M}$ such that $u = v$ $[u \leq v]$ quasi everywhere on Ω, then $Su = Sv$ $[Su \leq Sv]$ on Ω.*

Proof. It suffices to prove the inequality case, and we thus suppose that $u \leq v$ holds except on a polar subset E of Ω. By the nature of E, there exists a nonnegative superharmonic function ψ on Ω assuming the value $+\infty$ at all points of E. For each $\varepsilon > 0$, the inequality $u \leq v + \varepsilon\psi$ holds throughout Ω, so that

$$Su \leq Sv + \varepsilon S\psi.$$

Letting $\varepsilon \to 0$ yields $Su \leq Sv$ quasi everywhere on Ω, and superharmonicity ensures that this inequality actually holds everywhere on Ω.

There are certain conditions under which equality can be shown to hold in (2) of Lemma 2.1. This turns out to be the case for harmonic functions, and we shall prove, more generally, in §7 that equality holds in the case of superharmonic functions. For the moment let us simply observe that equality holds whenever one of the functions is quasi-bounded.

Lemma 2.5. *If u is any function in $\mathcal{M}$ and q is quasi-bounded on Ω, then $S(u + q) = Su$.*

Proof. Immediate from the fact that $u \leq u + q$ implies $Su \leq S(u+q) \leq Su + Sq = Su$.

Although the behavior of Su for superharmonic functions u will be treated systematically in §7, one important aspect of this behavior is worth citing at this stage.

Lemma 2.6. *If u is any nonnegative superharmonic function on Ω, then $Su \leq u$.*

Proof. As already noted in the proof of Lemma 2.2, the inequality $u \leq u + \lambda$ leads to $S_\lambda u \leq u$ and hence $Su \leq u$.

In turn, Lemma 2.6 shows that the inequality

$$S^2 u \leq Su \tag{2.3}$$

holds for all u in $\mathcal{M}$.

Finally, we show that the operator B (introduced at the end of §1) coincides with S. For any function u in $\mathcal{M}$, the inequality $Bu \geq Su$ follows at once from the evident inequality

$$u\chi_{[u \geq \lambda]} \geq (u - \lambda)^+.$$

To proceed in the reverse direction, we first observe that the inequality

$$u\chi_{[u \geq \lambda]} \leq (1 + \varepsilon)(u - \varepsilon\lambda/(1+\varepsilon))^+$$

holds for all positive numbers λ and ε. There results

$$B_\lambda u \leq (1 + \varepsilon) S_{\varepsilon\lambda/(1+\varepsilon)} u,$$

and, in the limit as $\lambda \to \infty$ and $\varepsilon \to 0$, this yields $Bu \leq Su$. Hence, $Bu = Su$ for all u in $\mathcal{M}$.

3. **The harmonic case.** It is essential to correlate the present definitions of quasi-bounded and singular functions with those given by Parreau in the harmonic case. Although this can be done by using lattice methods, as is implicit in S. Yamashita [8], we shall give arguments based directly on the operators S_λ and S.

The following preliminary observations are useful. Suppose that h is a nonnegative harmonic function on Ω. Then, for all $\lambda \geq 0$, $S_\lambda h$ is likewise a nonnegative harmonic function. It is, in fact, the least harmonic majorant of the subharmonic function $(h - \lambda)^+$. Applying the Harnack theorem, we infer that Sh is a nonnegative harmonic function on Ω. Moreover, the function

$$h - S_\lambda h \qquad (\lambda \geq 0) \tag{3.1}$$

is a bounded nonnegative harmonic function on Ω, in view of the inequalities $h \leq S_\lambda h + \lambda$ and $S_\lambda h \leq h$.

It is now an easy matter to correlate our definition of quasi-boundedness with the classical one for harmonic functions.

Theorem 3.1. *Let h be a nonnegative harmonic function on Ω. Then $Sh = 0$ if and only if h is the limit of an increasing sequence of bounded nonnegative harmonic functions on Ω.*

Proof. If $Sh = 0$, then the functions $h_n = h - S_n h$ $(n = 1, 2, \ldots)$ are bounded nonnegative harmonic functions on Ω with $h_n \uparrow h$. Conversely, let $\{h_n\}$ be any sequence of bounded, nonnegative, harmonic functions on Ω with $h_n \uparrow h$. By Lemmas 2.5 and 2.6, we have $Sh = S(h - h_n) \leq h - h_n$, forcing $Sh = 0$, and the proof is complete.

We next start with the functions (3.1) and let λ tend to ∞ through integral values to obtain

Theorem 3.2. *If h is a nonnegative harmonic function on Ω, then $h - Sh$ is a quasi-bounded harmonic function on Ω.*

To correlate our definition of singular functions with the classical one for the harmonic case, we proceed as follows.

Theorem 3.3. *Let h be a nonnegative harmonic function on Ω. Then $Sh = h$ if and only if the only bounded nonnegative harmonic function on Ω majorized by h is the function identically zero.*

Proof. Suppose first that $Sh = h$ and that u is a bounded nonnegative harmonic function on Ω with $u \leq h$. Then, by Lemmas 2.5 and 2.6 we have

$$Sh = S(h - u) \leq h - u.$$

Hence, $u \leq h - Sh = 0$, proving that u must be identically zero. Conversely, if the only bounded nonnegative harmonic function u on Ω majorized by h is the function identically zero, then taking u as (3.1) yields $S_\lambda h = h$ $(\lambda \geq 0)$ and therefore $Sh = h$. This completes the proof.

The Parreau decomposition theorem falls directly out of the above considerations.

Theorem 3.4. *Every nonnegative harmonic function h on Ω admits a unique decomposition as*

$$h = q + s, \tag{3.2}$$

where q and s are, respectively, quasi-bounded and singular harmonic functions on Ω. These functions are determined by h according to the formulas

$$s = Sh \quad \text{and} \quad q = h - Sh. \tag{3.3}$$

Proof. First, let s and q be defined by (3.3), so that (3.2) is immediate. Theorem 3.2 guarantees that q is quasi-bounded, and an application of Lemma 2.5 to (3.2) shows that $s = Ss$, i.e. that s is singular. Formulas (3.3) follow generally from the decomposition (3.2) by a similar argument, and this proves uniqueness of s and q.

In addition to the characterization of singular harmonic functions given in (3.3), a characterization in terms of majorization by functions Su can be formulated as follows.

Theorem 3.5. *A nonnegative harmonic function h on Ω is singular if and only if $h \leq Su$ holds for some u in $\mathcal{M}$.*

Proof. Assuming $h \leq Su$, we take $\lambda \geq 0$ and let φ be any nonnegative superharmonic function on Ω such that $u \leq \varphi + \lambda$. Consequently, $Su \leq \varphi$ and $\varphi - h \geq 0$. From the inequality

$$(u - h)^+ \leq \varphi - h + \lambda$$

we then infer that $S_\lambda(u - h)^+ \leq S_\lambda u - h$ and hence that $S(u - h)^+ \leq Su - h$. Moreover, $u \leq (u - h)^+ + h$ implies $Su \leq S(u - h)^+ + Sh$, which leads at once to $h \leq Sh$. Since the reverse inequality is contained in Lemma 2.6, equality must hold, proving that h is singular. The converse holds trivially with $u = h$, and the proof is complete.

An immediate corollary of Theorem 3.5 is that a nonnegative harmonic function on Ω is singular if and only if it is majorized by a singular function. This statement actually turns out to be equivalent to that of Theorem 3.5, since we show later (in §7) that all functions of the form Su ($u \in \mathcal{M}$) are singular. We note further that Theorem 3.5 leads easily to the fact, already established in Theorem 3.3, that the nonnegative harmonic functions h for which $Sh = h$ are singular in the classical sense. Indeed, every bounded harmonic function h_1 satisfying $0 \leq h_1 \leq h$ then satisfies $h_1 \leq Sh$ and therefore $h_1 = Sh_1 = 0$.

The following monotonicity property holds in the harmonic case.

Theorem 3.6. *If $\{h_n\}$ is a monotone decreasing sequence of harmonic functions on Ω with limit h, then $Sh_n \downarrow Sh$.*

Proof. Obvious from the inequalities

$$Sh_n \leq Sh + S(h_n - h) \leq Sh + h_n - h \qquad (n = 1, 2, \ldots).$$

4. **Quasi-boundedness conditions.** Combined results of Yamashita [8] and Parreau [7] serve to characterize quasi-bounded harmonic functions as those nonnegative harmonic functions for which the composites with certain nonnegative increasing convex functions admit harmonic majorants. Alternative derivations, using only internal arguments, are presented in Chapter II of Heins [5].

Our central result here is the following extension of this quasi-boundedness criterion to the general case of functions which are not assumed harmonic.

Theorem 4.1. *Let u be a function on Ω. Then u^+ is quasi-bounded if and only if there exists a nonnegative increasing convex function φ on $[-\infty, +\infty]$ such that*

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = +\infty \tag{4.1}$$

and

$$\varphi \circ u \quad \text{admits a superharmonic majorant.} \tag{4.2}$$

Before we proceed to the proof, some comments on the content of the theorem are in order. It obviously characterizes quasi-bounded functions as those nonnegative functions u which satisfy (4.1) and (4.2). More generally, however, it serves to characterize those functions u admitting quasi-bounded majorants (since such majorants exist if and only if u^+ is quasi-bounded). By *convexity* of φ on an infinite interval we mean convexity in the usual sense on the finite part of the interval, together with continuity in the extended sense on the infinite interval. Since we need to form composites of φ with extended real-valued functions u, it is, of course, essential to have φ defined on the extended real axis in Theorem 4.1.

Let us denote by Φ the class of all nonnegative increasing convex functions φ on $[-\infty, +\infty]$ satisfying (4.1). Suppose now that u is a subharmonic function on Ω and that φ is in Φ. Under these conditions, the composite function $\varphi \circ u$ is subharmonic, and hence the existence of a superharmonic majorant is equivalent to the existence of a harmonic majorant. These observations furnish the following specialization of Theorem 4.1 to the case of subharmonic functions.

Corollary 4.2. *Let u be a subharmonic function on Ω. Then u^+ is quasi-bounded if and only if there exists a function φ in Φ such that $\varphi \circ u$ admits a harmonic majorant.*

It is easily seen that Corollary 4.2 contains results of Yamashita [8, p. 61], Parreau [7], and Heins [5, p.17]. Note, however, that one important part of Theorem 2, p. 17, of Heins [5] does not extend to the general setting of Theorem 4.1. This is the conclusion that $\varphi \circ u^+$ is quasi-bounded whenever u is subharmonic and $\varphi \circ u$ admits a harmonic majorant. The following simple example shows that this conclusion does not even extend to superharmonic functions. Let Ω be the unit disc, and put

$$u(z) = [P_z(e^{i\theta})]^{1/2} \qquad (z \in \Omega) \tag{4.3}$$

for fixed θ, where $P_z(e^{i\theta})$ denotes the Poisson kernel. Superharmonicity of u is immediate from the harmonicity of the Poisson kernel. Moreover, the function φ defined by

$$\varphi(t) = t^2 \quad \text{for } t \geq 0, \tag{4.4}$$
$$= 0 \quad \text{for } t < 0,$$

belongs to the class Φ, but the composite function

$$(\varphi \circ u)(z) = P_z(e^{i\theta}) \qquad (z \in \Omega)$$

is not quasi-bounded.

Turning now to the proof of Theorem 4.1, we adopt Heins' viewpoint of relying on internal arguments. Thanks to the elementary properties already at hand for the operator S, these arguments can be made rather concise, and we split the derivation into the "if" and "only if" parts. Proof of the former can be strengthened by noting that it does not depend on monotonicity or convexity of φ, and we state this generalized result explicitly as

Theorem 4.3. *Let φ be a nonnegative function on $[-\infty, +\infty]$ satisfying* (4.1) *and the condition $\varphi(+\infty) = +\infty$. If u is any function on Ω such that $\varphi \circ u$ admits a superharmonic majorant, then u^+ is quasi-bounded.*

Proof. Condition (4.1) guarantees that, for each $\varepsilon > 0$, there exists a positive real number N such that $t \leq \varepsilon\varphi(t)$ for $t > N$. Thus, $t \leq \varepsilon\varphi(t) + N$ holds for all t. Replacing t by $u(z)$ leads at once to

$$u^+ \leq \varepsilon\varphi \circ u + N.$$

Since $\varphi \circ u$ is in $\mathscr{M}$, we can apply the operator S to obtain $Su^+ \leq \varepsilon S(\varphi \circ u)$, and the theorem follows by letting $\varepsilon \to 0$.

There remains the "only if" part of the proof of Theorem 4.1, and this is established as follows. Let u be a function on Ω with the property that u^+ is quasi-bounded. Using the notation of (1.1), we see that, quasi everywhere on Ω,

$$Su^+ = \lim_{n\to\infty} S_n u^+ = \lim_{n\to\infty} R_n u^+ = 0. \tag{4.5}$$

Hence, there exists a point of Ω at which all of these equalities hold, and we shall fix z_0 as such a point. We can then find a strictly increasing sequence $\{n_k\}$ of positive integers such that

$$R_{n_k} u^+(z_0) < 1/2^k \qquad (k = 1, 2, \ldots). \tag{4.6}$$

By definition of the reduced functions $R_{n_k} u^+$, this implies the existence of a sequence $\{v_k\}$ of nonnegative superharmonic functions on Ω satisfying

$$(u - n_k)^+ \leq v_k \quad \text{and} \quad v_k(z_0) < 1/2^k \tag{4.7}$$

for all k.

The function φ will now be defined by setting

$$\varphi(t) = \sum_{k=1}^{\infty} (t - n_k)^+ \qquad (-\infty \le t \le +\infty).$$

Here we note that the infinite series actually reduces to a finite sum whenever $t < +\infty$ and that $\varphi(t) = 0$ for $t \le n_1$. It is apparent, moreover, that φ is a piecewise linear function with slope k on the interval $[n_k, n_{k+1}]$, so that φ is an increasing convex function on $[-\infty, +\infty]$. For $n_m < t < +\infty$ we have

$$\varphi(t) > \sum_{k=1}^{m} (t - n_k) = mt - \sum_{k=1}^{m} n_k,$$

and hence

$$\liminf_{t\to\infty} \frac{\varphi(t)}{t} \ge m.$$

This proves (4.1). Since the composite function $\varphi \circ u$ is given by $\varphi \circ u = \sum_{k=1}^{\infty} (u - n_k)^+$, we see from (4.7) that $\varphi \circ u$ has $\sum v_k$ as a superharmonic majorant, and the proof is complete.

Whenever φ admits an inverse function, an evident variant of Theorem 4.3 can be formulated in terms of $\psi = \varphi^{-1}$. This leads to the following result, which we proceed to establish by direct methods, not requiring the existence of φ^{-1}.

Theorem 4.4. *Let ψ be a locally bounded, nonnegative function on* $[0,+\infty]$ *satisfying the condition*

(4.8) $$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0.$$

If u is any function in $\mathcal{M}$, then $\psi \circ u$ is quasi-bounded.

Proof. To each $\varepsilon > 0$ there corresponds a positive real number N such that $N < t < +\infty$ implies $\psi(t) < \varepsilon t$. Since, by hypothesis, ψ has a finite upper bound M on $[0,N]$, we see that $\psi(t) \le \varepsilon t + M$ holds for all t on $[0,+\infty]$. Thus, $\psi \circ u \le \varepsilon u + M$ on Ω. Application of the operator S results in $S(\psi \circ u) \le \varepsilon Su$, and the theorem follows by letting $\varepsilon \to 0$.

Corollary 4.5. *If u is any function in $\mathcal{M}$ and α any real number satisfying* $0 < \alpha < 1$, *then the functions* u^α, $\log^+ u$, *and* $u^\alpha \log^+ u$ *are quasi-bounded.*

5. **Further properties of the operator** S. The general behavior of S, as an operator on $\mathcal{M}$, will be examined more closely, particularly as applied to sequences and series of functions. Discussion of Su for the case of u superharmonic will, however, be deferred until §7, since it is based on the results for Green's potentials developed in §6. It turns out that the superharmonic case is central to the entire theory. The reason for this, as noted in the following theorem, is that Su, for any function u in $\mathcal{M}$, can always be calculated by applying S to the superharmonic function $\hat{R}u$.

Theorem 5.1. *For any function u in $\mathcal{M}$, the reduced function Ru and its lower regularization $\hat{R}u$ satisfy*

$$SRu = S\hat{R}u = Su.$$

Proof. Inasmuch as $\hat{R}u = Ru$ quasi everywhere, Lemma 2.4 guarantees that $S\hat{R}u = SRu$. The proof thus reduces to showing that $SRu = Su$, and for this we proceed as follows. Let λ be any nonnegative real number and φ any nonnegative superharmonic function on Ω such that $u \leq \varphi + \lambda$. Clearly, $Ru \leq \varphi + \lambda$, so that $S_\lambda Ru \leq \varphi$. This yields $S_\lambda Ru \leq S_\lambda u$ and hence $SRu \leq Su$. On the other hand, it is evident from $u \leq Ru$ that $Su \leq SRu$, proving that $SRu = Su$.

Since $\hat{R}u = S_0 u$, we can state the essential content of Theorem 5.1 as $SS_0 u = Su$. A stronger result can actually be established, namely that $SS_\lambda u = Su$ for all $\lambda \geq 0$, but we shall not have use for it here.

Turning our attention now to the application of S to sequences of functions in $\mathcal{M}$, we have the following restricted convergence property.

Theorem 5.2. *Let u and u_n $(n = 1,2,\ldots)$ be functions in $\mathcal{M}$ such that $u_n \geq u$ $(n = 1,2,\ldots)$ and $u_n \to u$ quasi everywhere on Ω. If the differences $u_n - u$ $(n = 1,2,\ldots)$ are quasi superharmonic, then $Su_n \to Su$ quasi everywhere on Ω.*

Proof. In view of the quasi superharmonicity of $u_n - u$, we have

$$Su_n \leq Su + S(u_n - u) \leq Su + u_n - u \to Su$$

quasi everywhere on Ω. The inequality $Su_n \geq Su$ $(n = 1,2,\ldots)$ then forces $Su_n \to Su$ quasi everywhere on Ω.

Note that the conclusion is false, in general, if the functions $u_n - u$ are not assumed quasi superharmonic. For example, in the case of the unit disc we can take $u_n(z)$ as the product of $\log(1/|z|)$ and the characteristic function of the open sector from angle 0 to angle $1/n$. Then $u_n \downarrow 0$ but, by Lemma 2.3, we have $Su_n(z) \geq \log(1/|z|)$ for all n.

As an application of Theorem 5.2, a simple derivation shows that the present definition of quasi-boundedness is equivalent to that given in [2].

Corollary 5.3. *Let u be a nonnegative function on Ω. Then u is quasi-bounded if and only if there exist sequences $\{v_n\}$ and $\{M_n\}$, in which each v_n is a nonnegative superharmonic function on Ω and each M_n is a real number, such that*

$$u \leq v_n + M_n \qquad (n = 1,2,\ldots) \tag{5.1}$$

and $v_n \to 0$ quasi everywhere on Ω.

Proof. If (5.1) holds, then clearly $Su \leq Sv_n$ for all n, and Theorem 5.2 ensures that $Sv_n \to 0$ quasi everywhere on Ω. This forces $Su = 0$ quasi everywhere, hence everywhere, on Ω, proving that u must be quasi-bounded. Conversely, if u is quasi-bounded, then $S_n u \to 0$ quasi everywhere on Ω as $n \to \infty$, and we note that the inequalities $u \leq S_n u + n$ $(n = 1, 2, \ldots)$ hold quasi everywhere on Ω. Taking ψ as a nonnegative superharmonic function on Ω assuming the value $+\infty$ on the union of the exceptional polar sets, we see that (5.1) holds with $v_n = S_n u + \psi/n$ and $M_n = n$. This completes the proof.

From these concluding remarks it is plain that Corollary 5.3 remains in force if the inequalities (5.1) are only required to hold quasi everywhere on Ω. Also, as pointed out in Lemma 3.1 of [2], a nonnegative subharmonic function on Ω is quasi-bounded if and only if it admits a quasi-bounded harmonic majorant. This fact appears as a special case of Theorem 5.1, since when u is subharmonic, the function Ru is just the least harmonic majorant of u.

Of course, Theorem 5.2 has direct applications to sequences $\{u_n\}$ which are monotone decreasing. In looking at what can happen in the case of increasing sequences, we perceive at once that $u_n \uparrow u$ does not generally imply $Su_n \uparrow Su$. This is obvious, for example, by taking u_n $(n = 1, 2, \ldots)$ as the truncate at n of any nonzero singular function u, so that $Su_n = 0$ and $Su = u$. Thus, the convergence property fails for increasing sequences even when u and all u_n are assumed superharmonic.

Nevertheless, we again have access to a restricted convergence theorem for increasing sequences, and it turns out that this is most conveniently formulated in terms of infinite series. Although the actual convergence property depends on properties of Su for the case of u superharmonic, and will therefore be dealt with in §7, we proceed to show that a corresponding inequality holds in the general case.

Theorem 5.4. *Let $\{u_n\}$ be a sequence of functions in $\mathcal{M}$. If there exists a sequence $\{v_n\}$ of superharmonic functions on Ω such that $u_n \leq v_n$ $(n = 1, 2, \ldots)$ and $\sum v_n$ is not identically $+\infty$, then*

$$(5.2) \qquad S\left(\sum_{n=1}^{\infty} u_n\right) \leq \sum_{n=1}^{\infty} Su_n.$$

Proof. Let us put $u = \sum_{n=1}^{\infty} u_n$ and recall the notation introduced in (1.3). Since the conditions $S_\infty u_n = Su_n$ $(n = 1, 2, \ldots)$, $S_\infty u = Su$, and $\sum v_n < +\infty$ all hold quasi everywhere, these conditions hold simultaneously except on some polar set E $(\subset \Omega)$. Let us fix z_0 as any point of $\Omega - E$ and take ε as any positive number. There then exists a sequence $\{\lambda_n\}$ of positive integers such that

$$S_{\lambda_n} u_n(z_0) \leq Su_n(z_0) + \varepsilon/2^n \qquad (n = 1, 2, \ldots).$$

Since the inequalities

$$u_n \leq S_{\lambda_n} u_n + \lambda_n \qquad (n = 1, 2, \ldots)$$

hold quasi everywhere on Ω, we see that for each m $(= 1, 2, \ldots)$

$$u = \sum_{n=1}^{m} u_n + \sum_{n=m+1}^{\infty} u_n \leq \sum_{n=1}^{m} (S_{\lambda_n} u_n + \lambda_n) + \sum_{n=m+1}^{\infty} v_n$$

quasi everywhere on Ω. Hence, setting $\sigma_m = \sum_{n=1}^{m} \lambda_n$ $(m = 1, 2, \ldots)$ results in

$$S_{\sigma_m} u \leq \sum_{n=1}^{m} S_{\lambda_n} u_n + \sum_{n=m+1}^{\infty} v_n,$$

and this holds everywhere on Ω. (Note that the hypothesis on $\sum v_n$ ensures superharmonicity of the series and its remainders.) In the limit as $m \to \infty$ this yields

$$Su(z_0) = S_\infty u(z_0) \leq \sum_{n=1}^{\infty} S_{\lambda_n} u_n(z_0) \leq \sum_{n=1}^{\infty} Su_n(z_0) + \varepsilon,$$

hence

$$Su(z_0) \leq \sum_{n=1}^{\infty} Su_n(z_0).$$

Consequently, (5.2) holds quasi everywhere on Ω. That it must actually hold throughout Ω then follows by superharmonicity of $\sum Su_n$ (which is dominated by $\sum v_n$). This completes the proof.

Obviously, strict inequality can occur in (5.2), since we have already observed that it can occur for finite sums. We remark also that (5.2) fails, in general, when the domination hypotheses are dropped. To see this, one has only to take u as a singular function and define u_n as the function equal to u on the set $[n - 1 < u \leq n]$ $(n = 1, 2, \ldots)$ and to zero elsewhere on Ω. Indeed, it is then apparent that $u = \sum u_n$ and $Su = u > 0 = \sum Su_n$.

When the functions u_n in Theorem 5.4 are all quasi-bounded, so that $Su_n = 0$, equality plainly holds in (5.2). We thus have the following convergence theorem for quasi-bounded functions.

Corollary 5.5. *Suppose that $\{u_n\}$ is a sequence of quasi-bounded functions on Ω for which there exists a sequence $\{v_n\}$ of corresponding superharmonic majorants with $\sum v_n$ not identically $+\infty$. Then $\sum u_n$ is quasi-bounded on Ω.*

6. **Characterization of Su for Green's potentials.** It will be shown that the operator S takes Green's potentials into Green's potentials and that this mapping admits a very simple representation. Basically, when u is the Green's potential of a positive mass distribution p, then Su is just the Green's potential of the mass distribution obtained by restricting p to the G_δ set $[u = +\infty]$. This, and similar results, enable us to deal effectively with superharmonic functions.

A familiar result in potential theory asserts that if u and v are Green's potentials satisfying $u \leq v$, then the concentrated mass of u at any point z_0 cannot exceed that of v at the same point. This can be proved, for example, by using the fact that the concentrated mass at z_0 is the limit as $r \to 0$ of the circumferential mean of the potential over $|z - z_0| = r$ divided by $\log(1/r)$. Essential use will be made of the following generalization of this domination principle.

Lemma 6.1. *Let w_1 and w_2 be δ-subharmonic functions on Ω with mass distributions m_1 and m_2, respectively. If $w_1 \leq w_2$ quasi everywhere on Ω, then $m_1(E) \leq m_2(E)$ holds for every polar set E ($\subset \Omega$).*

Proof. Since we are dealing here with differences of subharmonic functions, there is no loss of generality in assuming that w_2 is identically zero. We shall make this assumption and discard the subscripts on w_1 and m_1. Thus, w is taken as any δ-subharmonic function satisfying $w \leq 0$ quasi everywhere on Ω, and the object is to show that its mass distribution m satisfies $m(E) \leq 0$ for every polar set E. This will be done by showing that $m(K) \leq 0$ holds for all compact subsets K of E. Let us therefore fix K as a compact polar set and consider regions ω containing K and having closure in Ω. Without loss of generality we shall presume w to be a Green's potential.

The key step is to sweep out the mass m, in the classical sense, from the region $\omega - K$. This yields a swept-out mass distribution m^* on Ω, for which the Green's potential w^* is harmonic on $\omega - K$. Since w^* is given on $\omega - K$ as the Wiener function obtained by using w as the boundary function, the hypothesis $w \leq 0$ quasi everywhere on Ω ensures that $w^* \leq 0$ holds on $\omega - K$. The fact that K is a polar set then allows us to extend w^* from a harmonic function on $\omega - K$ to a subharmonic function on ω. Moreover, the mass distribution m^* for w^* coincides on ω with that for the extended subharmonic function, so that, in particular, $m^*(K) \leq 0$.

Recalling that the swept-out mass $m^*(K)$ consists of the original mass $m(K)$ together with that fraction of $m(\omega - K)$ swept onto K, we infer that

$$m(K) \leq |m|(\omega - K).$$

A suitable choice of ω renders the right-hand member arbitrarily small, and this forces $m(K) \leq 0$, completing the proof.

We are now in a position to prove the fundamental representation theorem for Su as a Green's potential when u is a Green's potential.

Theorem 6.2. *If u is the Green's potential of a positive mass distribution p on Ω, then Su is the Green's potential of p confined to $E_\infty = [u = +\infty]$, i.e.*

$$(6.1) \qquad Su(z) = \int_{E_\infty} G_z(\zeta)\, dp(\zeta) \qquad (z \in \Omega).$$

Proof. We first establish two special cases:

(i) if p does not charge E_∞, then $Su = 0$; and

(ii) if p is supported by a polar set, then $Su = u$.

Assertion (i) is immediate from Corollary (5.5), since here $u = \sum_{n=1}^{\infty} u_n$, where u_n is the bounded Green's potential obtained by confining p to the set $[n - 1 \leq u < n]$ $(n = 1, 2, \dots)$.

To prove (ii), we take λ as any positive number and φ as any nonnegative superharmonic function on Ω such that $u \leq \varphi + \lambda$. An application of Lemma 6.1 shows that p is dominated by the mass distribution for φ, and this implies that $u \leq \varphi$ on Ω. It follows that $u \leq S_\lambda u$ for all $\lambda > 0$, so that $u \leq Su$. Since the reverse inequality automatically holds for nonnegative superharmonic functions, this shows that $Su = u$, as asserted.

Having verified (i) and (ii), we turn to the general case and decompose u as $u = u_0 + u_1$, where u_0 is the Green's potential of p confined to E_∞ and u_1 is the Green's potential of p confined to $\Omega - E_\infty$. Property (i) ensures that u_1 is quasi-bounded, and property (ii) ensures that u_0 is singular. Hence, $Su = Su_0 = u_0$, completing the proof.

It is now an easy matter to characterize quasi-bounded and singular Green's potentials.

Corollary 6.3. *Let u be the Green's potential of a positive mass distribution p on Ω. Then*

(1) *u is quasi-bounded if and only if p does not charge the set $[u = +\infty]$;*

(2) *u is singular if and only if p is supported by the set $[u = +\infty]$ (or, equivalently, by any set of capacity zero).*

The parenthetical portion of (2) uses the following elementary property of mass distributions on sets of capacity zero.

Lemma 6.4. *Let u be the potential of a positive mass distribution p supported by a Borel set B. If B has capacity zero, then p is, in fact, supported by $B \cap [u = +\infty]$.*

Proof. It suffices to observe that the restriction of p to each of the sets $B \cap [n \leq u < n + 1]$ $(n = 1, 2, \dots)$ yields a bounded potential. Since these sets have capacity zero, the mass on each one must vanish, leaving all of the mass on $B \cap [u = +\infty]$ as asserted.

It should be noted that property (1) of Corollary 6.3 has been stated by Brelot for the case of semibounded potentials (see [4, p. 41]), and we have shown in §2 that these are the same as quasi-bounded potentials.

A fundamental consequence of Theorem 6.2 is that it ensures the following decomposition of Green's potentials into quasi-bounded and singular parts.

Theorem 6.5. *Let u be the Green's potential of a positive mass distribution p on Ω. Then u admits a unique decomposition as $u = q + s$, where q and s are, respectively, quasi-bounded and singular Green's potentials on Ω. Moreover, the mass distributions for q and s are obtained by restricting p to the sets $[u < +\infty]$ and $[u = +\infty]$, respectively.*

Proof. The existence of the decomposition is evident by defining q and s as the respective Green's potentials and applying Corollary 6.3. To verify uniqueness, we then start with the decomposition $u = q + s$ and apply the operator S to conclude that $s = Su$ and hence that $q = u - Su$ quasi everywhere.

Note that if we discard the requirement that q be a Green's potential, then q is determined uniquely only to within a polar set. For example, the decomposition would still hold when q is assigned values arbitrarily at points where s is $+\infty$.

7. **The superharmonic case.** Information obtained from the above representation of Su for Green's potentials can be combined with the classical results for harmonic functions to yield general results for the case of nonnegative superharmonic functions. A preliminary lemma is needed here to show how S operates on functions of the form $u + h$, where u and h are in $\mathcal{M}$ and h is harmonic, and we have

Lemma 7.1. *If u is a function in $\mathcal{M}$ and h any nonnegative harmonic function on Ω, then $S(u + h) = Su + Sh$.*

Proof. Drawing on the classical Parreau decomposition theorem for harmonic functions (Theorem 3.4), we have $h = q + s$ with q and s nonnegative quasi-bounded and singular harmonic functions, respectively. Since plainly

$$S(u + h) = S(u + s + q) = S(u + s),$$

the conclusion of the lemma amounts just to the assertion that

$$S(u + s) = Su + s \tag{7.1}$$

for all singular harmonic functions s on Ω.

To prove (7.1), we fix s and take λ as any positive number and φ as any nonnegative superharmonic function on Ω satisfying $u + s \leq \varphi + \lambda$. In particular, $s \leq \varphi + \lambda$ and therefore $s = Ss \leq \varphi$, so that $\varphi - s$ is a nonnegative superharmonic function on Ω. The inequality $u \leq \varphi - s + \lambda$ then implies $S_\lambda u \leq \varphi - s$ and hence $Su + s \leq \varphi$. Taking the infimum over the class of all admitted φ yields $Su + s \leq S_\lambda(u + s)$ and finally $Su + s \leq S(u + s)$. The desired equality (7.1) then follows from the fact that

$$S(u + s) \leq Su + Ss = Su + s.$$

This completes the proof.

In particular, Lemma 7.1 shows that S is additive over the class of nonnegative harmonic functions on Ω. It is easy to see also from the discussion in §6 that S is additive over the class of Green's potentials of positive mass distributions. Indeed, suppose that u and v are Green's potentials of positive mass distributions p and q, respectively. Then the integral for $S(u + v)$, expressed according to formula (6.1), can be written as the sum of integrals over $[u + v = +\infty]$ taken with respect to p and q. By Lemma 6.4 the latter integrals reduce to integrals over $[u = +\infty]$ and $[v = +\infty]$, respectively, and it follows from Theorem 6.2 that $S(u + v) = Su + Sv$.

Starting with the Riesz decomposition theorem and applying Lemma 7.1, we can combine the additivity properties of S noted above for harmonic functions and Green's potentials to get

Lemma 7.2. *If u and v are nonnegative superharmonic functions on Ω, then $S(u + v) = Su + Sv$.*

We recall that the sum of any two quasi-bounded functions is again quasi-bounded, in view of the subadditivity of S. Lemma 7.2 now shows that the sum of any two singular functions is singular. Indeed, if s_1 and s_2 are singular functions on Ω, then they are obviously nonnegative superharmonic functions satisfying $S(s_1 + s_2) = s_1 + s_2$.

The finite additivity property expressed in Lemma 7.2 extends at once to the following general additivity property.

Theorem 7.3. *Let u_n $(n = 1, 2, \ldots)$ be nonnegative superharmonic functions on Ω. If $\sum u_n$ is not identically $+\infty$, then*

$$S\left(\sum_{n=1}^{\infty} u_n\right) = \sum_{n=1}^{\infty} Su_n.$$

Proof. From Lemma 7.2 we have

$$\sum_{n=1}^{m} Su_n = S\left(\sum_{n=1}^{m} u_n\right) \leq S\left(\sum_{n=1}^{\infty} u_n\right),$$

and in the limit as $m \to \infty$ this yields $\sum Su_n \leq S \sum u_n$. Since the reverse inequality is already at hand in Theorem 5.4, equality must hold.

Supplementing the convergence property of quasi-bounded functions given in Corollary 5.5 we now have the following counterpart for singular functions.

Corollary 7.4. *If s_n $(n = 1, 2, \ldots)$ are singular functions on Ω for which $\sum s_n$ is not identically $+\infty$, then $\sum s_n$ is singular.*

These convergence results for general quasi-bounded and singular functions are evident direct generalizations of classical convergence properties of quasi-bounded and singular harmonic functions (see Heins [5, p. 8]). They can, of course, be stated also as convergence properties of monotone increasing sequences. From this viewpoint we are led to investigate corresponding convergence properties for monotone decreasing sequences. Some information in this direction has been noted for harmonic functions in Theorem 3.6 and for more general functions in Theorem 5.2. More can be said about the monotone decreasing case, however, and we shall return to this presently with more powerful tools at our disposal.

A fundamental property, which is now easily established, is that the theorem on decomposition into quasi-bounded and singular functions is valid for arbitrary nonnegative superharmonic functions. Specifically, we have

Theorem 7.5. *Let u be a nonnegative superharmonic function on Ω. Then u admits a unique decomposition as*

$$u = q + s, \tag{7.2}$$

where q and s are, respectively, quasi-bounded and singular superharmonic functions on Ω. These functions are given by

$$s = Su \quad \text{and} \quad q = u - Su, \tag{7.3}$$

with the latter formula holding quasi everywhere on Ω.

Further, if $u = u_1 + u_2$ is the Riesz decomposition of u as the sum of a Green's potential u_1 and a harmonic function u_2, then

$$q = q_1 + q_2 \quad \text{and} \quad s = s_1 + s_2, \tag{7.4}$$

where q_1 and q_2 are the quasi-bounded parts of u_1 and u_2, respectively, and s_1 and s_2 are their singular parts.

Proof. By Lemma 7.1 we see from the Riesz decomposition of u that

$$Su = S(u_1 + u_2) = s_1 + s_2$$

and hence that

$$u - Su = u_1 - Su_1 + u_2 - Su_2 = q_1 + q_2$$

quasi everywhere. Thus, the functions s and q defined by (7.3) (with q extended by removing its removable singularities) are nonnegative superharmonic functions on Ω. It is obvious also that s is singular and q quasi-bounded, and that together they yield the desired decomposition (7.2). Uniqueness is then apparent from the fact that an application of S to (7.2) yields (7.3). This completes the proof.

We now know that, for any function u in $\mathscr{M}$, the function Su $(= S\hat{R}u)$ is singular. Equivalently, a function s on Ω is singular if and only if $s = Su$ for some u in $\mathscr{M}$. As a property of the operator S, this result can be phrased as

Corollary 7.6. *The operator S is idempotent, i.e. $S^2u = Su$ for all u in $\mathscr{M}$.*

We turn our attention next to some observations concerning singular functions.

Lemma 7.7. *If u and v are nonnegative superharmonic functions on Ω such that $u + v$ is singular, then both u and v must be singular.*

Proof. Although this follows easily by decomposing u and v into their quasi-bounded and singular parts, we give a simple argument independent of the decomposition theorem. All that is needed is to combine the inequality $u + v = S(u + v) \leq Su + Sv$ with the inequalities $Su \leq u$ and $Sv \leq v$. This forces $Su = u$ and $Sv = v$, and the lemma is proved.

Suppose now that s is any singular function on Ω. Since s is superharmonic, it can be decomposed according to the Riesz theorem as the sum of a Green's potential and a nonnegative harmonic function. As is evident from Lemma 7.7, both the potential part and the harmonic part of s must be singular. Applying Corollary 6.3 to the potential part results in

Lemma 7.8. *If s is any singular function on Ω, then the mass distribution for s is supported by the polar set $[s = +\infty]$.*

Combining this with Lemma 6.1 yields

Lemma 7.9. *If s and σ are singular functions on Ω with $s \geq \sigma$, then there exists a singular function φ on Ω such that $s = \sigma + \varphi$.*

Proof. The role of Lemma 6.1 is to ensure that there exists a nonnegative superharmonic function φ on Ω for which $s = \sigma + \varphi$, and the singularity of φ is then immediate from Lemma 7.7.

With this information in hand, we return to the question of convergence properties of monotone sequences of singular functions.

Theorem 7.10. *Let $\{s_n\}$ be a sequence of singular functions on Ω. Then*

(1) if $\{s_n\}$ is increasing to a function s not identically $+\infty$, the function s must be singular, and

(2) if $\{s_n\}$ is decreasing to a function s, the function s must coincide quasi everywhere on Ω with a singular function.

Proof. In the case of $\{s_n\}$ increasing, we see from Lemma 7.9 that there exists a sequence $\{\varphi_n\}$ of singular functions on Ω such that $s_{n+1} = s_n + \varphi_n (n = 1,2, \ldots)$. There results $s = s_1 + \sum_{n=1}^{\infty} \varphi_n$, and Corollary 7.4 guarantees that s must be singular. This proves conclusion (1).

In the case of $\{s_n\}$ decreasing, we again draw on Lemma 7.9, but this time to obtain a sequence $\{\varphi_n\}$ of singular functions on Ω such that $s_1 = s_n + \varphi_n$ $(n = 1,2,\ldots)$. Here the sequence $\{\varphi_n\}$ is clearly increasing to some nonnegative superharmonic (in fact, singular) function φ, and we have $s_1 = s + \varphi$ on Ω. By the convergence theorem for decreasing sequences of superharmonic functions (see e.g. Brelot [3, p. 77]) there exists a superharmonic function $\hat{s}$ on Ω such that $s = \hat{s}$ quasi everywhere. Hence, $s_1 = \hat{s} + \varphi$ on Ω, and singularity of $\hat{s}$ is immediate from Lemma 7.7. Conclusion (2) is thus established, and the proof is complete.

Let us look, finally, at the behavior of $\{Su_n\}$ for monotone sequences $\{u_n\}$ of nonnegative superharmonic functions. As noted earlier, the use of truncates makes it plain that $\{Su_n\}$ does not generally converge to Su anywhere on Ω in the case when $\{u_n\}$ increases to u. However, we do have a convergence property for increasing sequences in which the successive differences $u_n - u_{n-1}$ are superharmonic. Roughly analogous to Theorem 5.2, but for increasing sequences, this result can be stated as

Theorem 7.11. *Let $\{u_n\}$ be an increasing sequence of superharmonic functions on Ω for which the limit function u is not identically $+\infty$. If the successive differences $u_n - u_{n-1}$ $(n = 2, 3, \dots)$ are quasi superharmonic on Ω, then $Su_n \to Su$ on Ω.*

Proof. As a convenience, we introduce the function $u_0 \equiv 0$ on Ω. There is no loss of generality in assuming, as we shall, that the functions $u_n - u_{n-1}$ $(n = 1, 2, \dots)$ are actually superharmonic on Ω, since the exceptional polar sets which would otherwise be present are ignored by the operator S. Since u then appears as

$$u = \sum_{n=1}^{\infty} (u_n - u_{n-1}),$$

Theorem 7.3 yields

$$Su = \sum_{n=1}^{\infty} S(u_n - u_{n-1}). \tag{7.5}$$

Superharmonicity permits us to write

$$Su_n = Su_{n-1} + S(u_n - u_{n-1}) \qquad (n = 1, 2, \dots),$$

so that the right-hand member of (7.5) is just the limit quasi everywhere of $\{Su_n\}$. Inasmuch as Su and all Su_n are superharmonic, the fact that $\{Su_n\}$ is monotone increasing forces it to converge to Su everywhere on Ω. This completes the proof.

Turning next to the case of monotone decreasing sequences of nonnegative superharmonic functions, we show that the result implicit in Theorem 5.2 can be considerably strengthened for such sequences.

Theorem 7.12. *If $\{u_n\}$ is a sequence of nonnegative superharmonic functions on Ω with $u_n \downarrow u$ quasi everywhere, then $Su_n \downarrow Su$ quasi everywhere.*

Proof. Here we make use of the decomposition of u_n as $u_n = q_n + s_n$ $(n = 1, 2, \dots)$, where q_n is quasi-bounded and s_n singular. From the expression for s_n as $s_n = Su_n$ $(n = 1, 2, \dots)$ it is clear that the sequence $\{s_n\}$ is decreasing. Thus, by Theorem 7.10 there exists a singular function s on Ω such that $s_n \downarrow s$ quasi everywhere on Ω. Invoking the hypothesis that $u_n \to u$ quasi everywhere and using the fact that the functions q_n are nonnegative, we conclude that there exists a nonnegative function q on Ω such that $u = q + s$ quasi everywhere on Ω. There results $Su \geq Ss = s$. On the other hand, the inequalities $u \leq u_n$ quasi everywhere $(n = 1, 2, \dots)$ imply $Su \leq Su_n = s_n$ and consequently $Su \leq s$. This establishes $Su = s$, and the convergence property $s_n \downarrow s$ quasi everywhere can be rewritten as $Su_n \downarrow Su$ quasi everywhere, completing the proof.

Even when $u_n \downarrow u$ everywhere on Ω, the conclusion that $Su_n \downarrow Su$ only quasi everywhere is best possible. This is obvious, for instance, by considering the functions $u_n = q + s/n$, where s is the potential of a positive unit mass at the origin and q is a quasi-bounded potential with $q(0) = +\infty$.

8. **The positive cone $\mathcal{P}$.** Let $\mathcal{P}$ be the class of all functions u in $\mathcal{M}$ satisfying the inequality

$$Su \leq u. \tag{8.1}$$

This class is evidently closed under the operations of addition and multiplication by nonnegative constants, and hence forms a positive cone. The cone $\mathcal{P}$ has a number of interesting properties, which we proceed to examine.

For example, we know that $\mathcal{P}$ contains all nonnegative superharmonic functions and, in fact, all functions u of the form $u = \varphi + q$, where φ is a nonnegative superharmonic function and q a quasi-bounded function. (Functions of the latter form will be shown subsequently to exhaust $\mathcal{P}$.) Further, $\mathcal{P}$ has the lattice closure property

$$u, v \in \mathcal{P} \Rightarrow u \wedge v \in \mathcal{P} \tag{8.2}$$

(immediate from (4) of Lemma 2.1), and, more generally, for any nonempty family of functions u_α $(\alpha \in A)$

$$u_\alpha \in \mathcal{P} \Rightarrow \wedge_{\alpha \in A} u_\alpha \in \mathcal{P} \tag{8.3}$$

(immediate from $Su \leq Su_\alpha \leq u_\alpha$).

We turn now to the fundamental characterization of $\mathcal{P}$ as the largest subclass of $\mathcal{M}$ for which the decomposition theorem holds.

Theorem 8.1. *The class $\mathcal{P}$ consists of precisely those functions u on Ω which can be expressed as*

$$u = q + s, \tag{8.4}$$

where q is quasi-bounded and s singular. Here s is uniquely determined by u as $s = Su$, and q is determined uniquely quasi everywhere as $q = u - Su$.

Proof. Functions u of the form (8.4) are obviously in $\mathcal{P}$, since $Su = s \leq u$. For the converse we start with an arbitrary function u in $\mathcal{P}$ and put $s = Su$, so that $u - s \geq 0$ quasi everywhere on Ω. From the relations $s = S\hat{R}u$ and $u \leq \hat{R}u$ quasi everywhere, there results

$$u - s \leq \hat{R}u - S\hat{R}u \tag{8.5}$$

quasi everywhere on Ω. Since $\hat{R}u$ is a nonnegative superharmonic function, Theorem 7.5 ensures that the right-hand member in (8.5) coincides quasi everywhere with a quasi-bounded function. The same must therefore be true of the left-hand member, and the representation (8.4) follows. An application of S to (8.4) yields the asserted uniqueness properties, completing the proof.

In view of the decomposition theorem for nonnegative superharmonic functions, the class $\mathcal{P}$ can also be characterized in terms of these functions.

Corollary 8.2. *The class $\mathcal{P}$ consists of precisely those functions u on Ω which can be expressed as*

$$u = \varphi + q, \tag{8.6}$$

where φ is a nonnegative superharmonic function and q is quasi-bounded.

The operator S was shown, in Lemma 7.2, to be additive over the class of nonnegative superharmonic functions. On the basis of what happens in the harmonic case (Lemma 7.1), it is natural to expect that additivity will still hold when only one of the functions is required to be superharmonic. We prove the stronger result that additivity holds when one of the functions belongs to the class $\mathcal{P}$.

Theorem 8.3. *If u is a function in $\mathcal{M}$ and v any function in $\mathcal{P}$, then $S(u + v) = Su + Sv$.*

Although the proof can be carried out along the lines of that already given for Lemma 7.1, by bringing into play Theorem 7.5 and Lemma 6.1, we prefer the following slightly different approach. The key lies in an additivity property of reduced functions, namely

Lemma 8.4. *If u is a function in $\mathcal{M}$ and s any singular function on Ω, then $\hat{R}(u + s) = \hat{R}u + s$.*

Proof. Since $\hat{R}$ is subadditive, so that $\hat{R}(u + s) \leq \hat{R}u + s$, our task reduces to proving the reverse inequality. Thus, let φ be a superharmonic function on Ω satisfying $\varphi \geq u + s$. Using the fact that φ dominates s, we infer from Lemma 6.1 that there exists a nonnegative superharmonic function ψ on Ω such that $\varphi = \psi + s$. Then $\psi \geq u$ holds quasi everywhere on Ω, so that $\varphi \geq \hat{R}u + s$ and hence $\hat{R}(u + s) \geq \hat{R}u + s$. This completes the proof.

Returning now to Theorem 8.3, we begin by decomposing v according to Theorem 8.1 as $v = q + s$, where q is quasi-bounded and s singular. Lemmas 8.4 and 7.2 can then be used to arrive at the chain of equalities

$$\begin{aligned} S(u + v) &= S(u + s) = S\hat{R}(u + s) \\ &= S(\hat{R}u + s) = S\hat{R}u + s = Su + Sv, \end{aligned}$$

completing the proof.

Theorem 8.3, of course, implies that S is finitely additive over $\mathcal{P}$. The question as to whether S is actually countably additive, i.e. whether Theorem 7.3 has an analogue for functions in $\mathcal{P}$, can be dealt with as follows.

Theorem 8.5. *Let u_n $(n = 1,2,\dots)$ be functions in $\mathcal{P}$ for which there exist superharmonic majorants v_n $(n = 1,2,\dots)$ with $\sum v_n$ not identically $+\infty$. Then the function*

$$u = \sum_{n=1}^{\infty} u_n \tag{8.7}$$

is also in $\mathcal{P}$ and satisfies

$$Su = \sum_{n=1}^{\infty} Su_n. \tag{8.8}$$

Proof. To deduce that u is in $\mathcal{P}$, we can use either (i) the decomposition of the functions u_n into quasi-bounded and singular parts, or (ii) the relations

$$Su \leq \sum_{n=1}^{\infty} Su_n \leq \sum_{n=1}^{\infty} u_n = u.$$

The proof is then completed just as in the superharmonic case.

A further result which generalizes at once to the class $\mathcal{P}$ is Theorem 7.12. Indeed, superharmonicity enters the original proof merely as a device for ensuring the decompositions $u_n = q_n + s_n$. The theorem therefore holds when the functions u_n are only assumed to belong to $\mathcal{P}$, and in this case the limit function is likewise in $\mathcal{P}$, in view of (8.3).

One important application of positive cones is in defining orderings. In particular, the cone $\mathcal{P}$ generates an ordering $\prec$, which we shall refer to as the *$\mathcal{P}$-ordering*. In this ordering, the inequality $u \prec v$ means simply that there exists a function ψ in $\mathcal{P}$ such that $v = u + \psi$. In certain respects the $\mathcal{P}$-ordering resembles the *specific ordering*, which is defined analogously but with ψ allowed to lie in the cone of nonnegative superharmonic functions. For singular functions, both the specific ordering and the $\mathcal{P}$-ordering are equivalent to the natural ordering.

Specific ordering is implicit in Theorems 5.2 and 7.11, which could have been stated somewhat more concisely in this language. We note also that the argument used in establishing Theorem 5.2 proves the following counterpart for the $\mathcal{P}$-ordering.

Theorem 8.6. *If u and u_n $(n = 1,2,\dots)$ are functions in $\mathcal{P}$ with $u \prec u_n$ $(n = 1,2,\dots)$ and $u_n \to u$ quasi everywhere on Ω, then $Su_n \to Su$ quasi everywhere on Ω.*

9. **Linear extension of the operator S.** Although the domain of definition of S is the class $\mathcal{M}$, Theorem 5.1 makes evident the fact that S is completely

determined by its behavior on the class of nonnegative superharmonic functions. There is thus no loss of information in restricting attention to the latter class. It is just as easy, however, to deal with the slightly larger class $\mathcal{P}$, and we shall take this as our starting point. We then form the linear space $\mathcal{Q} = \mathcal{P} - \mathcal{P}$ and observe that S can be extended to a linear operator on $\mathcal{Q}$.

More specifically, $\mathcal{Q}$ consists of all functions u on Ω which can be expressed quasi everywhere as

$$u = u_1 - u_2 \tag{9.1}$$

for a suitable choice of functions u_1 and u_2 in $\mathcal{P}$. Since such differences are only defined quasi everywhere, in general, we shall interpret equality of functions in $\mathcal{Q}$ as meaning equality quasi everywhere on Ω. To extend S from $\mathcal{P}$ to $\mathcal{Q}$, we put

$$Su = Su_1 - Su_2. \tag{9.2}$$

Of course, we are obligated to show that the resulting function Su is independent of the choice of u_1 and u_2 admitted in (9.1), but this is an easy consequence of the additivity of S over $\mathcal{P}$. Indeed, if $u = u_1' - u_2'$ is another admitted decomposition, so that $u_1' + u_2 = u_1 + u_2'$ quasi everywhere on Ω, then $Su_1' + Su_2 = Su_1 + Su_2'$ and hence $Su_1' - Su_2' = Su_1 - Su_2$ quasi everywhere.

We shall make the convention in what follows that S is extended to $\mathcal{Q}$ according to (9.2). The resulting operator is readily seen to be linear. That is, for all functions u and v in $\mathcal{Q}$ and real numbers α, we have

$$S(\alpha u) = \alpha Su \quad \text{and} \quad S(u + v) = Su + Sv. \tag{9.3}$$

Under the natural ordering on $\mathcal{Q}$, but with the inequality $u \leq v$ interpreted as meaning that the pointwise inequality holds quasi everywhere, the operator S is easily seen also to be positive. Indeed, from the inequality $u = u_1 - u_2 \geq 0$ quasi everywhere there results $Su_1 \geq Su_2$ and hence $Su = Su_1 - Su_2 \geq 0$.

We note, further, that the ordered space $\mathcal{Q}$ is actually a vector lattice. For this it is enough to know that u in $\mathcal{Q}$ implies u^+ in $\mathcal{Q}$, and this is evident at once from the identity $u^+ = u_1 - u_1 \wedge u_2$ coupled with the lattice closure property (8.2). In view of (4) of Lemma 2.1, this expression for u^+ also leads to the inequalities

$$(Su)^+ \leq Su^+ \quad \text{and} \quad (Su)^- \leq Su^-. \tag{9.4}$$

From the above observations we infer

Theorem 9.1. *Under the natural ordering, $\mathcal{Q}$ is a vector lattice and S (extended) a positive linear operator on $\mathcal{Q}$ satisfying* (9.4) *and hence* $|Su| \leq S|u|$.

At this point it is convenient to extend the earlier terminology by calling a function in $\mathcal{Q}$ *quasi-bounded* if it can be expressed quasi everywhere as the difference of two quasi-bounded functions in $\mathcal{M}$, and *singular* if it can be expressed quasi everywhere as the difference of two singular functions in $\mathcal{M}$. In the case of functions in $\mathcal{Q}$ which are already in $\mathcal{M}$ these definitions differ slightly from the earlier ones by allowing exceptional polar sets. This is not a matter of great concern, since the operator S ignores polar sets anyway, but, if necessary, we can refer to the earlier functions as quasi-bounded and singular *in the strict sense.*

Let u be a function in $\mathcal{Q}$, represented according to (9.1). Writing u_1 and u_2 in terms of their quasi-bounded and singular parts as $u_1 = q_1 + s_1$ and $u_2 = q_2 + s_2$, we have

$$u = q + s, \tag{9.5}$$

where $q = q_1 - q_2$ and $s = s_1 - s_2$ quasi everywhere. Moreover, the functions q and s are uniquely determined quasi everywhere as

$$s = Su \quad \text{and} \quad q = u - Su. \tag{9.6}$$

Formula (9.5) thus furnishes the decomposition property for functions in $\mathcal{Q}$, extending the result given for superharmonic functions in Theorem 7.5. It is clear also from (9.6) that S is a projection of $\mathcal{Q}$ onto the space of general singular functions, and that the operator $Q = I - S$, where I is the identity mapping, is a projection of $\mathcal{Q}$ onto the space of general quasi-bounded functions.

We conclude with a few remarks concerning general quasi-bounded and singular functions. First, as a direct corollary of Lemma 7.9, we have

(9.7) *the condition $s \geq 0$ on Ω is necessary and sufficient for a singular function s in $\mathcal{Q}$ to be singular in the strict sense.*

In particular, the nonnegative singular functions in $\mathcal{Q}$ must be superharmonic. It follows that S is a positive operator relative to the specific ordering in the range space; i.e.

(9.8) *if u is a nonnegative function in $\mathcal{Q}$, then Su is a nonnegative superharmonic function on Ω, singular in the strict sense.*

For any function u in $\mathcal{Q}$,

(9.9) *u is quasi-bounded if and only if $|u|$ is quasi-bounded.*

Indeed, the inequality $|Su| \leq S|u|$ shows that quasi-boundedness of u follows from quasi-boundedness of $|u|$, and, for the converse, we use the inequality $|u| \leq q_1 + q_2$, where q_1 and q_2 are quasi-bounded functions in $\mathcal{M}$ for which $u = q_1 - q_2$. On the other hand, given that a function u in $\mathcal{Q}$ is singular, one cannot assert that $|u|$ must be singular. A counterexample is afforded by setting $u = s_1 - s_2$ on the unit disc, where s_1 and s_2 are Poisson kernels corresponding to distinct boundary points. Plainly, $|u|$ is subharmonic, but not harmonic. It therefore cannot be singular, since it would then have to be superharmonic, by (9.7).

10. **Added in proof.** Attention should be called to the recent work of B. Fuglede on semibounded potentials in his monograph *Finely harmonic functions* (Lecture Notes in Math., no. 289, Springer-Verlag, Berlin, 1972). In particular, as noted at the end of §1 of the present paper, Fuglede's work contains an alternative proof that quasi-bounded potentials are the same as the semibounded potentials of Brelot. (See Lemma 6.4, p. 50, of Fuglede's monograph.)

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