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REGULARLY VARYING FUNCTIONS AND CONVOLUTIONS WITH REAL KERNELS

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ABSTRACT. Let ϕ be a positive, measurable function and k a real-valued function on $(0, \infty)$, $k \in L^1(dt/t)$. We give conditions on ϕ and k sufficient to deduce the regular variation of ϕ from the assumption that

$$\alpha = \lim_{x\to\infty} \frac{1}{\phi(x)} \int_0^\infty \phi(t) k\left(\frac{x}{t}\right) \frac{dt}{t} \quad \text{exists } (\alpha \neq 0, \infty).$$

The general theorems extend in certain ways results of other authors and yield a new theorem on the relation between the radial growth and zero-distribution of those entire functions which are canonical products of nonintegral order with negative zeros.

1. **Introduction.** Let $\phi(x)$ be nonnegative and measurable, and let $k(x)$ be real-valued on $(0, \infty)$. Assume that $\phi * k(x) = \int_0^\infty \phi(t)k(x/t)\,dt/t$ exists as a Lebesgue integral for all $x > 0$. We find further conditions on ϕ and k which allow us to conclude from the hypothesis

$$\phi * k(x) \sim \alpha\phi(x) \qquad (x \to \infty), \tag{1.1}$$

where α is finite and nonzero, that

$$\phi(x) = x^\rho L(x) \qquad (0 < x < \infty), \tag{1.2}$$

where ρ is finite and L is slowly varying in the sense of Karamata [10], i.e., for each $\sigma > 0$

$$L(\sigma x)/L(x) \to 1 \qquad (x \to \infty). \tag{1.3}$$

A function ϕ which satisfies (1.2), (1.3) is said to be regularly varying of order ρ.

For nonnegative kernels, similar problems have been considered by Edrei and Fuchs [7], Drasin [4], Shea [14], and most recently, Drasin and Shea [5]. The most general result is that of Drasin and Shea (cf. §2, Theorem A) who, with only a weak tauberian condition on ϕ, obtained (1.2) from (1.1) for nonnegative kernels $k \in L^1(dt/t)$.

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Ganelius [8] considered the broader problem of complex-valued ϕ and k and used standard Fourier transform methods and a theorem of Poincaré concerning solutions of difference equations to deduce from (1.1) that

$$\lim_{x\to\infty} \frac{\phi(\sigma x)}{\phi(x)} \quad \text{exists} \tag{1.4}$$

for each $\sigma > 0$. Here $k \in L^1(dt/t)$ and the Fourier transform of k satisfies some relatively strong conditions–k must be of "meromorphic type" (cf. §3)–which, however, do hold for many of the standard kernels of analysis. For positive ϕ it is easy to see that (1.4) is equivalent to (1.2), (1.3) provided the limit in (1.4) is finite and nonzero.

The basic argument we use is that of [14]; the same argument was adapted in [5] to solve a convolution inequality problem as well as to obtain (1.2) from (1.1) for $k \geq 0$. The properties of "Pólya peaks" (cf. §6) and related inequalities established in [6] also play an important role in the present paper, as they did in [5]. The adaptation here is complicated by the fact that k may change sign, but still is quite similar to both [14] and [5]. The usefulness of the basic argument is further established since our results apply directly to certain kernels which are not of meromorphic type and, hence, do not satisfy the hypotheses of Ganelius' theorem (cf. §3).

In §2 we state our general theorems and discuss their relation to Drasin and Shea's theorem for nonnegative kernels. An application of our results to a problem in function theory appears in §3, and in §4 we give an example of a kernel k which satisfies all but one of our hypotheses and for which our theorems fail. The remaining sections are devoted to proofs.

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2. Statement and discussion of results. Let $k(x)$ be real-valued and $\phi(x) \geq 0$ for $0 < x < \infty$, and assume that the convolution $\phi * k(x)$ exists on $(0, \infty)$. The kernel $k(t)$ has a Mellin transform given by

$$K(z) = \int_0^\infty k(t) \frac{dt}{t^{1+z}};$$

the transform converges absolutely for Re z in an interval (τ_1, τ_2), possibly including one or both endpoints, and it is analytic in $\tau_1 < \operatorname{Re} z < \tau_2$.

We shall require $\tau_1 < 0 < \tau_2$; then $k(t) \in L^1(dt/t)$, and, in fact, $t^{-\gamma}k(t) \in L^1(dt/t)$ for each γ, $\tau_1 < \gamma < \tau_2$. The last observation is true even if $\tau_1 > 0$ or $\tau_2 < 0$ and enables us to apply our results to convolutions $\psi * h(x)$ in which $h \notin L^1(dt/t)$ but ψ vanishes near the origin (cf. §3).

The order of ϕ is given by

$$\rho = \rho(\phi) = \limsup_{x\to\infty} \frac{\log \phi(x)}{\log x};$$

it is usually defined for nondecreasing functions, but the concept is still useful if ϕ satisfies a weaker tauberian condition such as

$$\liminf_{1\leq t\leq a;\, x\to\infty} \frac{\phi(tx)}{\phi(x)} > 0 \qquad \text{for some } a > 1. \tag{2.1}$$

This is the same condition which yields the very general result in [5]. The author originally imposed a stronger tauberian condition on ϕ; he thanks Professor David Drasin for the proof in §9 of the sufficiency of (2.1).

In addition to (2.1) we shall assume that $\tau_1 < \rho < \tau_2$. This hypothesis can be motivated by the fact that if (1.1) holds with k nonnegative and suitably well behaved near 0 and ∞ and with ϕ nondecreasing, then $\tau_1 \leq \rho \leq \tau_2$ with strict inequality at τ_j if $K(\tau_j) = \infty\, (j = 1,2)$ (cf. [12], [6]).

We must also impose certain restrictions on k and K which hold automatically when $k \geq 0$. The first assumption on K is that in an appropriate subinterval of (τ_1, τ_2)

$$K(\gamma) \text{ is monotonic.} \tag{2.2}$$

In addition, we require

$$|K'(\rho)| + |K''(\rho)| > 0, \tag{2.3}$$

$$K(\rho) \neq K(\rho + i\sigma) \qquad (0 < \sigma < \infty); \tag{2.4}$$

here, of course, $\rho = \rho(\phi)$. Ganelius [8] made the assumption (2.4) and observed that his theorem need not hold without it; we show in §4 that our theorems also may fail if (2.4) does not hold. The hypotheses (2.2)–(2.4) are discussed further at the end of this section.

Finally, the methods we use force us to restrict the behavior of $k(t)$ for either large or small t. In Theorem 1 we assume that either

$$k(t) = 0 \qquad (0 < t < 1) \tag{2.5}$$

or $k(t)$ satisfies the following three conditions: for some δ, $0 < \delta < 1$,

$$k(t) \geq 0 \quad \text{or} \quad k(t) \leq 0 \qquad (0 < t < \delta), \tag{2.6}$$

$$\int_0^\delta |k(t)| \frac{dt}{t} > 0, \tag{2.7}$$

and, if $\tau_2 < \infty$,

$$|K(\tau_2 -)| = \infty. \tag{2.8}$$

In Theorem 1a, $k(1/t)$ is assumed to satisfy either (2.5) or (2.6), (2.7).

The purpose of (2.5)–(2.7) is to exclude kernels $k(t)$ which change sign infinitely often near the origin or which are identically zero in some interval $(0, \eta)$ and change sign infinitely often in every interval $(\eta, \eta + \varepsilon)$, $\varepsilon > 0$. Note, however, that (2.6) does not preclude the vanishing of k near the origin. Since few kernels of normal interest are such that both $k(t)$ and $k(1/t)$ fail to satisfy (2.5) or (2.6), (2.7), these hypotheses do not seem significantly restrictive; however, the author doubts that they are necessary.

Theorem 1. *Let $\phi(x)$ be a nonnegative, measurable function and $k(x)$ a real-valued function on* $(0, \infty)$, *and assume*

$$\Phi(x) \equiv \phi * k(x) \sim \alpha\phi(x) \qquad (x \to \infty) \tag{2.9}$$

for some finite $\alpha \neq 0$. Suppose $\tau_1 < 0 < \tau_2$ and that there exists ξ_1, $\tau_1 \leq \xi_1 < \tau_2$, such that (2.2) *holds for $\xi_1 \leq \gamma \leq \tau_2$, $\gamma \neq \tau_1$. Suppose, in addition, that k satisfies either* (2.5) *or* (2.6), (2.7), *and* (2.8).

Assume ϕ satisfies the tauberian condition (2.1),

$$\phi \in L^\infty(0, T) \quad \text{for all } T < \infty, \tag{2.10}$$

and ϕ has order $\rho(\neq \tau_1)$, $\xi_1 \leq \rho < \tau_2$.

If (2.3) *and* (2.4) *hold, then*

$$\phi(x) = x^\rho L(x) \qquad (0 < x < \infty) \tag{2.11}$$

where L is slowly varying and

$$\alpha = K(\rho). \tag{2.12}$$

By restricting the behavior of $k(t)$ for $t > 1$ instead of for $t < 1$, we can prove

Theorem 1a. *Assume* (2.9) *holds, $\tau_1 < 0 < \tau_2$, and that there exists ξ_2, $\tau_1 < \xi_2 \leq \tau_2$, such that* (2.2) *holds for $\tau_1 < \gamma \leq \xi_2$, $\gamma \neq \tau_2$. Suppose, in addition, that $k(1/t)$ satisfies either* (2.5) *or* (2.6), (2.7), *and if $\tau_1 > -\infty$,*

$$|K(\tau_1 +)| = \infty. \tag{2.8'}$$

Let $\phi(\geq 0)$ be as in Theorem 1, *but assume that $\rho(\neq \tau_2)$ satisfies $\tau_1 < \rho \leq \xi_2$. If* (2.3) *and* (2.4) *hold, then* (2.11) *and* (2.12) *also hold.*

Theorems 1 and 1a basically concern kernels which change sign; however, it is interesting to compare these statements to Drasin and Shea's theorem for nonnegative kernels [5], or Drasin's earlier result in [4]. The theorem from [5] is

Theorem A. *Let $\phi(\geq 0)$ be measurable and satisfy* (2.1), (2.9), *and* (2.10) *where $k \geq 0$, $\tau_1 < 0 < \tau_2$, and $\tau_1 < \rho < \tau_2$.*

Then (2.11) *and* (2.12) *hold.*

The major differences between Theorem A and Theorems 1 and 1a (other than the fact that $k \geq 0$ in the former) are the restrictions (2.2), (2.3), and (2.4) on K in Theorems 1 and 1a. However, these conditions are necessarily satisfied by any nonnegative kernel $k(\not\equiv 0)$; to see this, simply note that

$$K''(\gamma) = \int_0^\infty (\log t)^2 k(t) \frac{dt}{t^{1+\gamma}} > 0 \qquad (\tau_1 < \gamma < \tau_2),$$

so that $K(\gamma)$ is convex. It follows that K satisfies (2.3) and is monotonic on either $\tau_1 < \gamma < \tau_2$ or on each of the intervals $\tau_1 < \gamma \leq \gamma_0$ and $\gamma_0 \leq \gamma < \tau_2$, where γ_0 is determined by

$$K(\gamma_0) = \min_{\tau_1 < \gamma < \tau_2} K(\gamma).$$

Finally, (2.4) holds since

$$\operatorname{Re}\{K(\gamma) - K(\gamma + i\sigma)\} = \int_0^\infty k(t)\{1 - \cos(\sigma \log t)\} \frac{dt}{t^{1+\gamma}} > 0$$

whenever $\tau_1 < \gamma < \tau_2$ and $0 < \sigma < \infty$.

3. **An application to function theory.** Let $f(z)$ be a nonconstant entire function and let $n(r)$ denote the number of zeros of $f(z)$ in $|z| \leqslant r$. Also, let $M(r) = \max_\theta |f(re^{i\theta})|$ and define the order of f to be the order of $\log M(r)$.

For each nonnegative integer q we consider canonical products of genus q and order λ, $q < \lambda < q + 1$, with negative zeros, i.e., functions of the form

$$f(z) = \prod_{n=1}^{\infty} E\left(-\frac{z}{a_n}, q\right) \qquad (0 < a_n \leq a_{n+1}), \tag{3.1}$$

where the Weierstrass primary factor $E(z, q)$ is defined by

$$E(z, q) = (1 - z)\exp\{z + (1/2)z^2 + \cdots + (1/q)z^q\}. \tag{3.2}$$

(The exponential factor does not appear when $q = 0$.) Such functions are extremal for a wide variety of problems.

We are interested in the relation between the radial growth and zero-distribution of these functions. In 1913 Valiron [17] established the abelian

Theorem B. *Let $f(z)$ be a canonical product of genus q with negative zeros. If*

$$n(r) \sim r^\lambda L(r) \qquad (r \to \infty) \tag{3.3}$$

where $L(r)$ is slowly varying and $q < \lambda < q + 1$, then

$$\log|f(re^{i\theta})| = (\pi \csc \pi\lambda \cos \theta\lambda + o(1)) r^\lambda L(r) \qquad (r \to \infty) \tag{3.4}$$

for each $\theta \neq \pi$.

Bowen and Macintyre [3] essentially obtained the following tauberian converse to Theorem B in 1951.

Theorem C. *Let $f(z)$ be a canonical product of genus q with negative zeros. Then* (3.4), *with $q < \lambda < q + 1$, implies* (3.3) *provided*

(i) $0 \leq |\theta| < \pi/2\lambda$ *and* $n(r) = o(r^{\pi/2|\theta|})\ (r \to \infty)$, *or*

(ii) $(2p - 1)\pi/2\lambda < |\theta| < (2p + 1)\pi/2\lambda$ *and* $n(r) = o(r^{(2p+1)\pi/2|\theta|})\ (r \to \infty)$ *where p is a positive integer.*

We prove the following theorem complementary to Theorems B and C.

Theorem 2. *Let $f(z)$ be a canonical product of genus q and order λ, $q < \lambda < q + 1$, with negative zeros. If for some fixed $\theta \neq \pi$*

$$\log|f(re^{i\theta})| \sim \alpha n(r) \qquad (r \to \infty)$$

where α is finite and nonzero, then (3.3) *holds and*

$$\alpha = \pi \csc \pi\lambda \cos \theta\lambda. \tag{3.5}$$

For $\lambda = q$ or $q + 1$, $n(r)/\log f(r)$ can tend to zero but (3.3) fails. Examples of such behavior for $q = 0$ appear in [1, pp. 101–104]; obvious modifications of those examples yield the general case. We also remark that the case $q = 0, \theta = 0$ of Theorem 2 appears in [4]; other special cases (involving kernels $k \geq 0$) have also been known.

Proof of Theorem 2. In the classical formula [17] (cf. [2, p. 55] for $q = 0$),

$$\log f(z) = (-1)^q \int_0^\infty n(t)\left(\frac{z}{t}\right)^q \frac{z}{t+z}\frac{dt}{t} \qquad (|\arg z| < \pi),$$

put $z = re^{i\theta}$ (θ fixed, $|\theta| < \pi$) and take real parts to obtain

$$\log|f(re^{i\theta})| = \int_0^\infty n(t)h\left(q,\theta;\frac{r}{t}\right)\frac{dt}{t} \qquad (0 < r < \infty), \tag{3.6}$$

where

$$h(q,\theta;t) = (-1)^q t^{q+1}\{\cos(q+1)\theta + t\cos q\theta\}\{1 + 2t\cos\theta + t^2\}^{-1} \qquad (0 < t < \infty).$$

Since $h \notin L^1(dt/t)$ for any q and θ, we multiply both sides of (3.6) by $r^{-q-(1/2)}$ to obtain

$$\Phi(r) \equiv r^{-q-(1/2)}\log|f(re^{i\theta})| = \int_0^\infty \phi(t)k\left(q,\theta;\frac{r}{t}\right)\frac{dt}{t} \qquad (0 < r < \infty), \tag{3.7}$$

where $\phi(t) = t^{-q-(1/2)}n(t)$, $k(q,\theta;t) = t^{-q-(1/2)}h(q,\theta;t)$ $(0 < t < \infty)$. Note that the new kernel is integrable for all q and θ. Moreover, $\Phi(r) \sim \alpha\phi(r)\ (r \to \infty)$, and it is this relation to which we wish to apply Theorems 1 and 1a.

Clearly, since $n(t)$ is a nondecreasing function vanishing near the origin, $\phi(t)$

satisfies (2.10) and the tauberian condition (2.1). Also, $\phi(t)$ has order $\rho = \lambda - q - (1/2)$, so that $-1/2 < \rho < 1/2$ for all values of q.

The transforms $K(q,\theta;z)$ and $H(q,\theta;z)$ of $k(q,\theta;t)$ and $h(q,\theta;t)$, respectively, are related by the equation

$$(3.8) \qquad K(q,\theta;z) = H(q,\theta;z + q + (1/2)) \qquad (\tau_1 < \operatorname{Re} z < \tau_2)$$

where τ_1 and τ_2 are the endpoints of the interval of absolute convergence of $K(q,\theta;z)$. Thus, to establish (2.2), (2.3), and (2.4) for $K(q,\theta;z)$, it is sufficient to establish the same properties for $H(q,\theta;z)$ on its interval of absolute convergence. This approach not only simplifies our discussion, but also yields directly properties of the more frequently encountered kernels $h(q,\theta;t)$.

It is clear from (3.1) and (3.2) that $|f(re^{i\theta})| = |f(re^{-i\theta})|$ so that we may assume $0 \leqslant \theta < \pi$. If q is any nonnegative integer and if $0 \leqslant \theta < \pi$, then $H(q, \theta; z)$ converges absolutely for $\tau_1^* < \operatorname{Re} z < \tau_2^*$ where $\tau_1^* = q - 1$ or q accordinq as $\cos q\theta$ does or does not vanish, and $\tau_2^* = q + 2$ or $q + 1$ according as $\cos(q + 1)\theta$ does or does not vanish. Of course, for each q, $\tau_j = \tau_j^* - q - (1/2) (j = 1,2)$ so that $\tau_1 < \rho < \tau_2$ for all values of q. Furthermore, a direct computation yields

$$\begin{aligned} H(q,\theta;z) &= (-1)^q \pi \csc \pi(q + 1 - z) \cos \theta z \\ &= \pi \csc \pi z \cos \theta z \qquad (\tau_1^* < \operatorname{Re} z < \tau_2^*). \end{aligned}$$

Thus, $|H(q,\theta;\tau_j^*)| = \infty \, (j = 1,2)$ so that $K(q,\theta;\gamma)$ satisfies (2.8) and (2.8′).

Observe next that, for each integer $q \geqslant 0$, $h(q,\theta;t)$ assumes both positive and negative values only if $\cos q\theta$ and $\cos(q + 1)\theta$ have opposite signs, i.e., if θ lies in one of the following intervals:

(3.9)

(i) $(1/(q + 1))(\pi/2 + 2j\pi) < \theta < (1/q)(\pi/2 + 2j\pi)$ for $j = 0, 1, \ldots, p(q)$ where $p(q) = [(2q - 3)/4]$ if q is even and $p(q) = [(2q - 3)/4] + 1$ if q is odd,

(ii) $(1/(q + 1))(3\pi/2 + 2j\pi) < \theta < (1/q)(3\pi/2 + 2j\pi)$ for $j = 0, 1, \ldots, p'(q)$ where $p'(q) = [(2q - 1)/4]$ if q is even and $p'(q) = [(2q - 1)/4] - 1$ if q is odd,

(iii) $(1/(q + 1))(\pi/2 + q\pi) < \theta < \pi$.

If θ is not in an interval in (3.9), then $h(q,\theta;t)$ is of one sign on $(0, \infty)$, and (2.2)–(2.4) follow as in the discussion at the end of §2. In this case the result can also be proved using the methods of [4], [14], or [5].

If θ does belong to one of the intervals in (3.9), then neither $\cos q\theta$ nor $\cos(q + 1)\theta$ vanishes, so that $\tau_1^* = q$, $\tau_2^* = q + 1$. To see that $H(q,\theta;\gamma)$ is monotonic on the entire interval $(q, q + 1)$, note that

$$H'(q,\theta;\gamma) = -\pi(\sin \pi\gamma)^{-2} g(\theta;\gamma) \qquad (q < \gamma < q + 1)$$

where $g(\theta;\gamma) = \theta \sin \pi\gamma \sin \theta\gamma + \pi \cos \pi\gamma \cos \theta\gamma \, (q \leq \gamma \leq q + 1)$. The function $g(\theta;\gamma)$ has extreme values only at the endpoints of $[q, q + 1]$ and at one interior point γ_0 given by

$$\begin{aligned}\gamma_0 &= (1+4j)\pi/2\theta, &&\left(\frac{1}{q+1}\left(\frac{\pi}{2}+2j\pi\right)<\theta<\frac{1}{q}\left(\frac{\pi}{2}+2j\pi\right)\right),\\ &= (3+4j)\pi/2\theta, &&\left(\frac{1}{q+1}\left(\frac{3\pi}{2}+2j\pi\right)<\theta<\frac{1}{q}\left(\frac{3\pi}{2}+2j\pi\right)\right),\\ &= (2q+1)\pi/2\theta, &&\left(\frac{1}{q+1}\left(\frac{\pi}{2}+q\pi\right)<\theta<\pi\right).\end{aligned}$$

If we let sgn $(\cdot)$ denote the usual signum function, then $\operatorname{sgn}(g(q)) = \operatorname{sgn}(g(\gamma_0)) = \operatorname{sgn}(g(q+1)) = (-1)^s$ where $s = q$ if θ lies in an interval in (3.9)(i) or (iii) and $s = q+1$ if θ lies in an interval in (3.9)(ii). Thus, $g(\theta;\gamma)$ has only one sign for $q \le \gamma \le q+1$, and $H(q,\theta;\gamma)$ satisfies both (2.2) and (2.3) on (τ_1^*, τ_2^*).

Finally, suppose that for some q and θ there exist γ, $\tau_1^* < \gamma < \tau_2^*$, and σ, $0 < \sigma < \infty$, such that $H(q,\theta;\gamma) = H(q,\theta;\gamma + i\sigma)$, or equivalently,

$$\sin \pi\gamma \cos \theta(\gamma + i\sigma) = \sin \pi(\gamma + i\sigma)\cos \theta\gamma.$$

Substituting the exponential representations for $\sin \pi z$ and $\cos \pi z$ and taking imaginary parts, we obtain $e^{\pi\sigma} + e^{-\pi\sigma} = e^{\theta\sigma} + e^{-\theta\sigma}$ where θ is fixed, $0 \le \theta < \pi$. Since $\sigma = 0$ is the only real solution of this equation,

$$H(q,\theta;\gamma) \neq H(q,\theta;\gamma + i\sigma) \qquad (\tau_1^* < \gamma < \tau_2^*, 0 < \sigma < \infty),$$

and $K(q,\theta;\gamma)$ satisfies (2.4) for any ρ, $\tau_1 < \rho < \tau_2$.

It now follows from the properties of $H(q,\theta;z)$ and from (3.8) that if θ lies in one of the intervals in (3.9), then either Theorem 1 or Theorem 1a, with $\xi_1 = \tau_1$, $\xi_2 = \tau_2$, yields

$$\alpha = K(q,\theta;\rho) = \pi \csc \pi\lambda \cos \theta\lambda \tag{3.10}$$

and

$$\phi(r) = r^\rho L(r) \qquad (0 < r < \rho) \tag{3.11}$$

where L is slowly varying. If θ does not lie in an interval in (3.9), then $k(q,\theta;t)$ has only one sign on $(0, \infty)$, and (3.10) and (3.11) can be obtained, as mentioned previously, from the results of [4], [14], or [5] as well as from Theorems 1 and 1a with $\xi_1 = \xi_2 = \gamma^*$. Here, of course, γ^* is determined by $|K(q,\theta;\gamma^*)| = \min_\gamma |K(q,\theta;\gamma)|$. In either case, (3.3) and (3.5) follow immediately from the definition of ϕ and the value of ρ. The proof of Theorem 2 is now complete.

The theorem of Ganelius [8], as stated, requires the Fourier transform $\hat{k}$ of a kernel k to agree with the reciprocal of an entire function g in its domain of existence (g must satisfy additional conditions, as well). However, if $k(t) = k(q,\theta;t)$, then $\hat{k}(\sigma) = H(q,\theta;q + (1/2) + i\sigma)$, and for θ in an interval in (3.9), $H(q,\theta;z)$ is analytic in $\tau_1^* < \operatorname{Re} z < \tau_2^*$ with precisely one zero there. Thus, for these θ a suitable g cannot exist. (Ganelius mentions [8, p. 16] that his restriction to kernels of meromorphic type is not essential, but I have not been able to derive Theorem 2 for θ in an interval in (3.9) with his methods.)

4. A counterexample. We give an example which shows that Theorems 1 and 1a may fail if condition (2.4) is omitted. Recall that Ganelius [8] observed that his theorem need not hold without (2.4); however, he did not actually construct a kernel for which (2.4) fails. Drasin [4], on the other hand, did give an example which shows that if k is not of one sign then even a continuous increasing function $\phi(\geq 0)$ can satisfy $\phi * k(x) = \alpha\phi(x)\,(0 < x < \infty)$, but not vary regularly. His kernels, however, change sign infinitely often near 0 and ∞ and, hence, do not satisfy the hypotheses of Theorems 1 and 1a.

We construct kernels k having only two changes of sign and satisfying the following:

(i) $\tau_1 = (\pi/4) - 1$, $K(\tau_1 +) = \infty$, $\tau_2 = \infty$, $K(\tau_2 -) = -\infty$,

(ii) $K'(\gamma) < 0\,(\tau_1 < \gamma < \tau_2)$,

(iii) except for (2.4), k satisfies the hypotheses of Theorems 1 and 1a with $\xi_1 = \tau_1$, $\xi_2 = \tau_2$,

(iv) $K((\pi/4) + i\sigma) = K(\pi/4)$ for $\sigma = \pm\pi$, and

(v) there exists a positive, continuous,increasing function ϕ on $(0, \infty)$ of order $\rho = \pi/4$ such that, for some finite $\alpha \neq 0$, $\phi * k(x) = \alpha\phi(x)\,(0 < x < \infty)$, but ϕ is *not* regularly varying.

First, fix $a > 0$ such that

$$\int_a^\infty e^{-x} \sin \pi x\, dx = (1 + \pi^2)^{-1}(\sin \pi a + \pi \cos \pi a)e^{-a} = 0$$

and $\cos \pi a > 0$. Then

$$\int_a^\infty (1 - \cos \pi x)e^{-x} dx = (1 - \cos \pi a)e^{-a},$$

and $h(x)$, defined by

$$\begin{aligned} h(x) &= 0, & -\infty < x < -2,\\ &= -\tfrac{1}{2}(1 - \cos \pi a)e^{-a}, & -2 \leq x \leq 0,\\ &= 0, & 0 < x < a,\\ &= e^{-x}, & a \leq x < \infty, \end{aligned}$$

satisfies

$$\int_{-\infty}^\infty h(x) \sin \pi x\, dx = \int_{-\infty}^\infty h(x)(1 - \cos \pi x)\, dx = 0 \tag{4.1}$$

and

$$\int_{-\infty}^\infty e^{\pm i\pi x} h(x)\, dx = \int_{-\infty}^\infty h(x)\, dx = e^{-a} \cos \pi a > 0. \tag{4.2}$$

Now fix $A > 2$ and set

$$k(t) = t^{\rho}h(\log t), \qquad \phi(t) = t^{\rho}(A + \cos \pi \log t) \qquad (\rho = \pi/4, 0 < t < \infty).$$

A simple calculation using (4.1) yields $\phi * k(x) = K(\pi/4)\phi(x)\,(0 < x < \infty)$, where, by (4.2), $K(\pi/4) > 0$.

To see that properties (ii) and (iv) hold, note that

$$(4.3) \qquad K(\gamma + i\sigma) = \int_{-\infty}^{\infty} e^{-i\sigma x} e^{(\rho-\gamma)x} h(x)\,dx \qquad (\tau_1 < \gamma < \tau_2, |\sigma| < \infty).$$

Property (ii) follows from the resulting relation

$$K'(\gamma) = \int_{-\infty}^{\infty} x e^{(\rho-\gamma)x} h(x)\,dx \qquad (\tau_1 < \gamma < \tau_2),$$

since the integrand is nonnegative; property (iv) follows directly from (4.3) and (4.2). The remaining properties are easily verified.

5. **Preliminaries for Theorem 1; proof of (2.12).** Following [5], we fix $\lambda \neq 0$, $\tau_1 < \lambda < \rho$, and define $\psi(x)$ by

$$(5.1) \qquad \psi(x) = \int_1^x \phi(t)\frac{dt}{t^{1+\lambda}} \qquad (0 < x < \infty).$$

Further restrictions on λ will be imposed later. It follows from (5.1), (2.1), and the definition of order that ψ has order $\rho(\psi) = \rho - \lambda$. In addition, since $\phi \geq 0$ and $\lambda < \rho$, $\psi(x)$ is nondecreasing and unbounded for $x > 1$; thus, all bounded terms are necessarily $o(\psi(x))$ as $x \to \infty$. Finally, by (2.9)

$$(5.2) \qquad \int_1^x \Phi(t)\frac{dt}{t^{1+\lambda}} \sim \alpha\psi(x) \qquad (x \to \infty).$$

Using the relation $\phi * k(x) = k * \phi(x)$ and interchanging the order of integration yield

$$(5.3) \quad \int_1^x \Phi(t)\frac{dt}{t^{1+\lambda}} = \int_0^{\infty} k(t)\left\{\psi\left(\frac{x}{t}\right) - \psi\left(\frac{1}{t}\right)\right\}\frac{dt}{t^{1+\lambda}} \qquad (0 < x < \infty).$$

From (5.1), (2.10), the absolute convergence of $K(z)$ for $\tau_1 < \operatorname{Re} z < \tau_2$, and $\tau_1 < \rho < \tau_2$, we have that $\int_0^\infty k(t)\psi(1/t)\,dt/t^{1+\lambda}$ exists and, hence, is $o(\psi(x))$ as $x \to \infty$; similarly,

$$\int_0^1 k\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{-\lambda}\psi(t)\frac{dt}{t} = \int_x^{\infty} k(t)\psi\left(\frac{x}{t}\right)\frac{dt}{t^{1+\lambda}} = o(\psi(x)) \qquad (x \to \infty).$$

Combining these observations with (5.2) and (5.3), we obtain

$$(5.4) \qquad \Psi(x) \equiv \int_0^{\infty} k_\lambda(t)\psi\left(\frac{x}{t}\right)\frac{dt}{t} \sim \alpha\psi(x) \qquad (x \to \infty)$$

where

$$k_\lambda(t) = t^{-\lambda}k(t) \qquad (0 < t < \infty) \tag{5.5}$$

and, without loss of generality, ψ vanishes in (0, 1).

We shall eventually deduce the regular variation of ψ from (5.4); then (2.11) will follow by a tauberian argument. We begin by establishing (2.12) via an argument developed in [9, pp. 224–227] and depending upon a lemma of Pólya [13], [16, pp. 275–276] concerning the singularities of certain integral transforms.

First, note that the transform $K_\lambda(z)$ of $k_\lambda(t)$ converges absolutely and satisfies $K_\lambda(z) = K(z+\lambda)$ for $\tau_1 - \lambda < \operatorname{Re} z < \tau_2 - \lambda$. Thus, we may integrate $\Psi = \psi * k_\lambda$ and use Fubini's theorem to obtain

$$\int_0^\infty \{\Psi(t) - K_\lambda(\gamma)\psi(t)\}\frac{dt}{t^{1+\gamma}} = 0 \qquad (\rho - \lambda < \gamma < \tau_2 - \lambda). \tag{5.6}$$

This equation is analogous to (3.5) in [9]. Since $K_\lambda(z)$ is regular in a neighborhood of $z = \rho - \lambda$ and the above integral of ψ diverges for $\gamma < \rho(\psi) = \rho - \lambda$, the argument in [9] yields $\alpha \le K_\lambda(\rho - \lambda) = K(\rho) \le \alpha$, which establishes (2.12).

6. The growth of ψ. We first obtain an estimate on the growth of ψ corresponding to the estimate (1.2) in [14]; the same estimate is established in [5]. It is here that we need restrictions on the behavior of our kernel near the origin; in addition, we make our first use of (2.12).

Lemma 1. *Let k satisfy either* (2.5) *or* (2.6) *and* (2.7), *and assume $K(z)$ converges absolutely for $\tau_1 < \operatorname{Re} z < \tau_2$ where $\tau_1 < 0 < \tau_2$. Let $\phi(\ge 0)$ satisfy* (2.10) *and have order ρ, $\tau_1 < \rho < \tau_2$. Define k_λ and ψ by* (5.3) *and* (5.1), *respectively, where $\tau_1 < \lambda < \rho$, $\lambda \ne 0$.*

If

$$\Psi(x) \equiv k_\lambda * \psi(x) \sim K(\rho)\psi(x) \qquad (x \to \infty) \tag{6.1}$$

and if λ is sufficiently close to ρ, then there exist constants m and M such that

$$\limsup_{x\to\infty} \frac{\psi(\sigma x)}{\psi(x)} \le M\sigma^m \tag{6.2}$$

for each $\sigma > 1$.

Proof. We first establish (6.2) for some $\sigma_0 > 1$ and then iterate to obtain (6.2) for all $\sigma > 1$. We use the usual notation for the positive and negative parts of a real function f:

$$f^+(t) = \max\{f(t), 0\}, \qquad f^-(t) = -\min\{f(t), 0\}.$$

Assume (2.5) holds and take λ sufficiently close to ρ to ensure that

$$|K(\lambda)| = \left|\int_1^\infty k_\lambda(t)\frac{dt}{t}\right| > \frac{2|K(\rho)|}{3}. \tag{6.3}$$

If $K(\rho) > 0$, we choose $\sigma_0 > 1$ such that

$$0 < \int_1^{\sigma_0} k_\lambda^+(t)\frac{dt}{t} < \frac{K(\rho)}{3}. \tag{6.4}$$

Then, for $x \geq x_0$, we have from (6.1) that

$$\begin{aligned}\frac{K(\rho)\psi(\sigma_0 x)}{2} &< \Psi(\sigma_0 x) = \left(\int_0^x + \int_x^{\sigma_0 x}\right)\psi(t)k_\lambda\left(\frac{\sigma_0 x}{t}\right)\frac{dt}{t} \\ &\leq \psi(x)\int_{\sigma_0}^\infty k_\lambda^+(t)\frac{dt}{t} + \psi(\sigma_0 x)\int_1^{\sigma_0} k_\lambda^+(t)\frac{dt}{t},\end{aligned}$$

or, using (6.4), $\psi(x)\int_{\sigma_0}^\infty k_\lambda^+(t)\,dt/t \geq K(\rho)\psi(\sigma_0 x)/6$. Thus, when k satisfies (2.5) and $K(\rho) > 0$, we have

$$\limsup_{x\to\infty} \frac{\psi(\sigma_0 x)}{\psi(x)} \leq M \tag{6.5}$$

where M is a finite constant. If $K(\rho) < 0$, we multiply (6.1) by -1 and deduce (6.5) by the same argument.

To establish (6.5) when (2.6) and (2.7) hold, assume $k(t) \geq 0\,(0 < t < \delta)$, where δ is given in (2.6), and choose $\eta > 1$ such that $\sigma_0 \equiv \eta\delta > 1$ and $\int_0^{1/\eta} k_\lambda(t)\,dt/t > 0$. It follows that

$$\begin{aligned}\Psi(x) &= \left(\int_0^{x/\delta} + \int_{x/\delta}^\infty\right)\psi(t)k_\lambda\left(\frac{x}{t}\right)\frac{dt}{t} \\ &\geq -\psi\left(\frac{x}{\delta}\right)\int_\delta^\infty k_\lambda^-(t)\frac{dt}{t} + \psi(\eta x)\int_0^{1/\eta} k_\lambda^+(t)\frac{dt}{t}.\end{aligned} \tag{6.6}$$

If $K(\rho) > 0$, then $\Psi(x)$ is asymptotic to a positive nondecreasing function and, therefore, since $\delta < 1$, $\Psi(x) < C\psi(x/\delta)\,(x > x_0)$ for any constant $C > 1$. Using this inequality in (6.6) we obtain

$$\psi(\eta x)\int_0^{1/\eta} k_\lambda^+(t)\frac{dt}{t} \leq \left(C + \int_\delta^\infty k_\lambda^-(t)\frac{dt}{t}\right)\psi\left(\frac{x}{\delta}\right) \qquad (x > x_0). \tag{6.7}$$

On the other hand, if $K(\rho) < 0$, then $\Psi(x) < 0$ for $x > x_0$, and we immediately obtain (6.7), with $C = 0$, from (6.6). In either case setting $x = \delta y$ in (6.7) yields (6.5) for $\sigma_0 = \eta\delta$.

For arbitrary $\sigma > 1$ determine the positive integer N by $\sigma_0^{N-1} < \sigma \leq \sigma_0^N$; then (6.2) follows from (6.5) and the inequalities

$$\frac{\psi(\sigma t)}{\psi(t)} \leq \prod_{j=1}^N \frac{\psi(\sigma_0^j t)}{\psi(\sigma_0^{j-1} t)} \leq M^N \qquad (t > 1).$$

This completes the proof of Lemma 1.

To estimate the "tails" of certain convolutions we shall need information on the "Pólya peaks" of ψ. Points $x_n \to \infty$ such that

$$\psi(t)/\psi(x_n) \geq (t/x_n)^p\{1 - o(1)\} \qquad (a_n^{-1}x_n \leq t \leq a_n x_n) \tag{6.8}$$

holds for some $a_n \to \infty$ are called Pólya peaks of order p of the second kind for ψ. Pólya peaks of the first kind are defined in terms of the reverse inequality.

It follows from the main result in [6] that ψ has Pólya peaks of each kind of order $p(< \infty)$ if and only if $\mu_*(\psi) \leq p \leq \rho_*(\psi)$ where

$$\rho_*(\psi) = \sup\left\{p\colon \ \limsup_{x,\sigma\to\infty} \frac{\psi(\sigma x)}{\sigma^p\psi(x)} = \infty\right\},$$

$$\mu_*(\psi) = \inf\left\{p\colon \ \liminf_{x,\sigma\to\infty} \frac{\psi(\sigma x)}{\sigma^p\psi(x)} = 0\right\}.$$

The next lemma provides bounds on $\rho_*(\psi)$ and another estimate on the growth of ψ.

Lemma 2. *Let k, ϕ, λ, k_λ, ψ, and Ψ be as in Lemma* 1. *If* (6.1) *holds, then*

$$\tau_1 - \lambda < \rho_*(\psi) < \tau_2 - \lambda \tag{6.9}$$

and

$$\begin{gathered}\psi(t) < C\psi(x)(t/x)^\beta \\ (\rho_*(\psi) < \beta < \infty, C = C(\beta); x_0(\beta) \leq x < t < \infty).\end{gathered} \tag{6.10}$$

Proof. The left-hand inequality in (6.9) follows from $\rho(\psi) = \rho - \lambda > \tau_1 - \lambda$ and from $\rho_*(\psi) \geq \rho(\psi)$, a relation observed in [6]. It follows from (6.2) that $\rho_*(\psi) \leq m < \infty$; clearly, then, $\rho_*(\psi) < \tau_2 - \lambda$ whenever $\tau_2 = \infty$. In particular, (6.9) holds if k satisfies (2.5), since in this case $\tau_2 = \infty$.

When (2.6) and (2.7) hold and $\tau_2 < \infty$, so that we are assuming (2.8), we prove (6.9) by contradiction. Assume $k(t) \geq 0 (0 < t < \delta)$, and suppose $\rho_*(\psi) \geq \tau_2 - \lambda$. Then for each finite p, $\tau_2 - \lambda \leq p \leq \rho_*(\psi)$, let x_n be a sequence of Pólya peaks of the second kind of order p for ψ, i.e., let (6.8) hold with $x_n = x_n(p)$.

Using (6.8) in the first integral of the inequality

$$\Psi(x_n) \geq \int_{x_n/\delta}^{\infty} \psi(t)k_\lambda\left(\frac{x_n}{t}\right)\frac{dt}{t} - \int_0^{x_n/\delta} \psi(t)k_\lambda^-\left(\frac{x_n}{t}\right)\frac{dt}{t},$$

we obtain, for $n \geq n_0$,

$$\{1 - o(1)\}\psi(x_n)\int_{1/a_n}^{\delta} k_\lambda(t)\frac{dt}{t^{1+p}} \leq \Psi(x_n) + \psi\left(\frac{x_n}{\delta}\right)\int_\delta^\infty k_\lambda^-(t)\frac{dt}{t} \leq C|\Psi(x_n)|$$

where the last inequality follows from (6.1) and (6.2). Dividing by $\psi(x_n)$ and letting $n \to \infty$ yield

$$\int_0^\delta k_\lambda(t)\frac{dt}{t^{1+p}} \leq C|K(p)|;$$

since $|K_\lambda(\tau_2 - \lambda)| = \infty$ and $p \geq \tau_2 - \lambda$, the above integral diverges and we have a contradiction. Thus, $\rho_*(\psi) < \tau_2 - \lambda$ and (6.9) holds.

Finally, as observed in [6], the conclusion (6.10) is a simple consequence of $\rho_*(\psi) < \infty$.

7. **An integral equation.** Continuing to follow the arguments used in [14] and [5], we fix $\sigma > 0$ and choose any sequence $t_n \to \infty$ such that $c = \lim_{n\to\infty}(\psi(\sigma t_n)/\psi(t_n))$ exists. To establish the regular variation of ψ, we must show that $c = \sigma^{\rho-\lambda}$; the first step is to reduce the problem to one of solving an integral equation.

The functions $g_n(u) = \psi(ut_n)/\psi(t_n)$ are nondecreasing and uniformly bounded on every finite interval. Helly's "selection principle" [18], applied to the sequence $g_n(u)$, yields a subsequence (which we may assume to be the full sequence) converging for all $u > 0$ to a positive nondecreasing function $g(u)$ such that $g(1) = 1$ and $g(\sigma) = c$. Moreover, by (6.10), g has order $\rho(g)$ satisfying

$$\rho(g) \leq \rho_*(\psi) < \tau_2 - \lambda. \tag{7.1}$$

A change of variable in (5.4) leads to

$$\left|\frac{\Psi(rt_n)}{\psi(rt_n)}\frac{\psi(rt_n)}{\psi(t_n)} - \int_0^{sr}\frac{\psi(ut_n)}{\psi(t_n)}k_\lambda\left(\frac{r}{u}\right)\frac{du}{u}\right| = \left|\int_{sr}^\infty \psi(ut_n)k_\lambda\left(\frac{r}{u}\right)\frac{du}{u}\right|\frac{1}{\psi(t_n)}$$

where $r > 0$ and $s > 0$. In the right-hand side of this equation, make the change of variable $u = t/t_n$ and use the inequality (6.10). Then, by (5.4) and dominated convergence, we have

$$\left|\alpha g(r) - \int_0^{sr} g(u)k_\lambda\left(\frac{r}{u}\right)\frac{du}{u}\right| \leq Cg(sr)s^{-\beta}\int_0^{1/s}|k_\lambda(u)|\frac{du}{u^{1+\beta}}$$

for each β, $\rho_*(\psi) < \beta < \tau_2 - \lambda$. Since $\rho(g) < \beta$ and since $K_\lambda(\beta)$ exists, we can let $s \to \infty$ and obtain

$$g(r) = \frac{1}{\alpha}\int_0^\infty g(t)k_\lambda\left(\frac{r}{t}\right)\frac{dt}{t} \qquad (0 < r < \infty). \tag{7.2}$$

In the next section we show that the only nonnegative, nondecreasing solution of (7.2) satisfying $g(1) = 1$ is $g(u) = u^{\rho-\lambda}$. Thus, $c = g(\sigma) = \sigma^{\rho-\lambda}$ is independent of the particular sequence t_n, and the regular variation of ψ follows at once:

$$\psi(t) = t^{\rho-\lambda}L_1(t) \qquad (0 < t < \infty) \tag{7.3}$$

where L_1 is slowly varying.

8. **The only admissible solution of (7.2).** Clearly, $g(u) = u^{\rho-\lambda}$ is a solution of (7.2). That it is the only nonnegative, nondecreasing solution satisfying $g(1) = 1$

will follow from the properties of $K_\lambda(z)$ and from a standard result in Fourier analysis [15, pp. 305–307] (cf. [14], [5]). To use that result effectively, we must impose additional restrictions on λ and find bounds for g at 0 and ∞.

Recall that our original choice of $\lambda \neq 0$ required $\tau_1 < \lambda < \rho$, and that in the proof of Lemma 1 when $k(t) \equiv 0 (0 < t < 1)$ we made the restriction (6.3). Observe now that in any strip $|\mathrm{Re}\, z| \leq a$ in which $K_\lambda(z)$ exists, $K_\lambda(z) \to 0$ uniformly as $\mathrm{Im}\, z \to \pm\infty$. Thus $K_\lambda(z) - \alpha$ has only finitely many zeros in such a strip, and we may choose β and λ such that λ satisfies the conditions above and, in addition,

$$\tau_1 < \lambda - \beta < \lambda < \rho < \lambda + \beta < \tau_2,$$

$$K_\lambda(z) \neq \alpha \qquad (|\mathrm{Re}\, z| \leq \beta, z \neq \rho - \lambda).$$

Note that the last condition also involves the hypothesis (2.4).

We can now determine the order of g. First, since (7.1) holds and g is nondecreasing, we have $\tau_1 - \lambda < 0 \leq \rho(g) < \tau_2 - \lambda$. If we let $g_1(t)$ be $0 (0 < t < 1)$ and agree with $g(t) (1 \leq t < \infty)$, then $\rho(g_1) = \rho(g)$ and, by (7.2),

$$\alpha G_1(r) \equiv g_1 * k_\lambda(r) \sim \alpha g_1(r) \qquad (r \to \infty).$$

We may integrate $\alpha G_1 = g_1 * k_\lambda$ to obtain an equation like (5.6) and again use Pólya's lemma [13] as in [9] to deduce $\alpha = K_\lambda(\rho(g))$. But by (2.12), the choice of λ above, and the hypothesis that $K(\gamma)$ is monotonic for $\xi_1 \leq \gamma < \tau_2$, $\gamma \neq \tau_1$, the only α-value of $K_\lambda(\gamma)$ for $\gamma \geq 0$ is $\rho - \lambda$. Thus, $\rho(g) = \rho - \lambda$ and, for any b, $\rho - \lambda < b < \beta$,

$$g(u) = O(u^b) \qquad (u \to \infty), \qquad g(u) = O(u^{-b}) \qquad (u \to 0+).$$

It follows now from [15, pp. 305–307] and the hypothesis (2.3), which ensures that $K_\lambda(z) - \alpha$ has at most a double zero at $z = \rho - \lambda$, that for some constants A and B

$$g(u) = Au^{\rho-\lambda} + Bu^{\rho-\lambda} \log u \qquad (0 < u < \infty).$$

Because g is nonnegative and $g(1) = 1$, we must have $g(u) = u^{\rho-\lambda}$.

9. **Proof of (2.11).** To deduce the regular variation of ϕ from (7.3), we first prove a lemma, due to Professor David Drasin (private communication), which shows that ϕ satisfies a stronger tauberian condition than (2.1).

Lemma 3. *Assume k and ϕ satisfy the hypotheses of Theorem* 1. *Then*

$$\liminf_{\sigma \to 1+; x \to \infty} \frac{\phi(\sigma x)}{\phi(x)} \geq 1. \tag{9.1}$$

Proof. Let k_λ, ψ, and Ψ be as above. Then ψ satisfies (7.3), and, as noted in [5], a standard tauberian argument due to Landau [11], [2, pp. 58–59], but with (2.1) in place of (9.1), yields for some M

$$M^{-1} < \phi(x)/x^{\lambda}\psi(x) < M \qquad (x > x_0). \tag{9.2}$$

We now show that for each $\varepsilon > 0$ there exists A_ε such that if $A > A_\varepsilon$ and $x > x_0(\varepsilon)$, then

$$\int_0^{x/A} \phi(u)\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u} + \int_{Ax}^{\infty} \phi(u)\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u} < \varepsilon\phi(x). \tag{9.3}$$

First, by (9.2) and (6.9), we have for sufficiently large A and x

$$\begin{aligned}\int_{Ax}^{\infty} \phi(u)\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u} &\le Mx^{\lambda}\int_{Ax}^{\infty} \psi(u)\left(\frac{u}{x}\right)^{\lambda}\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u}\\ &\le CM\psi(x)x^{\lambda}\int_{Ax}^{\infty}\left(\frac{u}{x}\right)^{\beta}\left|k_{\lambda}\left(\frac{x}{u}\right)\right| \frac{du}{u}\\ &\le CM^2\phi(x)\int_0^{1/A} |k_{\lambda}(u)|\frac{du}{u^{1+\beta}}\\ &< (\varepsilon/3)\phi(x) \qquad (\rho_*(\psi) < \beta < \tau_2 - \lambda).\end{aligned}$$

The same argument, with x_0 as in (9.2), yields

$$\int_{x_0}^{x/A} \phi(u)\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u} < (\varepsilon/3)\phi(x)$$

provided A and x are sufficiently large.

Finally, it follows from (9.2) that $x^{-\gamma}\phi(x) \to \infty$ $(x \to \infty)$ for each $\gamma < \lambda$. Also, since $K(\gamma)$ converges absolutely for $\tau_1 < \gamma < 0$,

$$\int_x^{\infty} |K(u)|\frac{du}{u} = o(x^{\gamma}) \qquad (\tau_1 < \gamma < 0, x \to \infty).$$

Combining these observations with (2.10), we obtain, for sufficiently large x,

$$\int_0^{x_0} \phi(u)\left|k\left(\frac{x}{u}\right)\right| \frac{du}{u} < (\varepsilon/3)\phi(x).$$

Thus, (9.3) is established.

We can now prove (9.1). Let $\varepsilon > 0$, choose A_ε such that (9.3) holds, and let $A > A_\varepsilon$. Since for any slowly varying function L the limit in (1.3) holds uniformly for $a \le \sigma \le b$, $0 < a < b < \infty$, it follows from (7.3) and (9.2) that the functions $\phi(x/u)/\phi(x)\,(A^{-1} < u < A, x \ge x_1)$ are bounded above and below by absolute positive constants. Thus,

$$\left|\int_{1/A}^{A} \frac{\phi(x/u)}{\phi(x)} k(\sigma u)\frac{du}{u}\right| > \left|\int_{1/A}^{A} \frac{\phi(x/u)}{\phi(x)} k(u)\frac{du}{u}\right| - \varepsilon \qquad (1 \le \sigma \le \sigma_0(\varepsilon)) \tag{9.4}$$

uniformly in $x \geq x_1$, for by dominated convergence as $\sigma \to 1+$ the first integral converges uniformly in $x \geq x_1$ to the second integral.

Using (2.9), (9.3), and (9.5) we have for all large x and $\sigma - 1$ small, $\sigma > 1$,

$$\begin{aligned}(|\alpha| + 3\varepsilon)\phi(\sigma x) &\geq |\Phi(\sigma x)| + 2\varepsilon\phi(\sigma x)\\ &\geq \left|\int_{x/A}^{Ax} \phi(t)k\left(\frac{\sigma x}{t}\right)\frac{dt}{t}\right|\\ &= \left|\int_{1/A}^{A} \frac{\phi(x/u)}{\phi(x)}k(\sigma u)\frac{du}{u}\right|\phi(x)\\ &> \left|\int_{x/A}^{Ax} \phi(t)k\left(\frac{x}{t}\right)\frac{dt}{t}\right| - \varepsilon\phi(x)\\ &\geq |\Phi(x)| - 3\varepsilon\phi(x) \geq (|\alpha| - 4\varepsilon)\phi(x).\end{aligned}$$

Since ε is arbitrary, ϕ clearly satisfies (9.1).

To complete the proof of Theorem 1 we again apply Landau's argument to (7.3), but with the stronger relation (9.1), and deduce (2.11) in the standard manner.

10. **Proof of Theorem 1a.** Choose $\lambda\,(\neq 0)$ and $\beta > 0$ to satisfy $\tau_1 < \lambda - \beta < \rho < \lambda < \lambda + \beta < \tau_2$ and $K_\lambda(z) \neq \alpha\,(|\mathrm{Re}\, z| \leq \beta, z \neq \rho - \lambda)$. As before, $K_\lambda(z)$ is the transform of $k_\lambda(t) = t^{-\lambda}k(t)\,(0 < t < \infty)$. Now, however, we define ψ_1 and Ψ_1 by

$$\psi_1(x) = \int_x^\infty \phi(t)\frac{dt}{t^{1+\lambda}}, \qquad \Psi_1(x) = \int_x^\infty \Phi(t)\frac{dt}{t^{1+\lambda}} \qquad (0 < x < \infty).$$

By Fubini's theorem and (2.9) we have

$$\Psi_1(x) = \psi_1 * k_\lambda(x) \sim \alpha\psi_1(x) \qquad (x \to \infty). \tag{10.1}$$

Since ψ_1 has order $\rho - \lambda$, the conclusion (2.12) can be obtained from (10.1) as in §5.

If we now set

$$H(x) = \Psi_1(1/x), \qquad \theta(x) = \psi_1(1/x), \qquad J(x) = k_\lambda(1/x) \qquad (0 < x < \infty),$$

then $\theta(x)$ is an increasing function, $J(x)$ satisfies the hypotheses of Theorem 1, and

$$H(x) = \theta * J(x) \sim \alpha\theta(x) \qquad (x \to 0). \tag{10.2}$$

The representation $\theta(x) = x^{\lambda-\rho}L_1(x)\,(0 < x < \infty)$, where $L_1(1/x)$ is slowly varying as $x \to \infty$, follows easily from (10.2) by obvious variations of the arguments in §§6–8. Thus,

$$\psi_1(x) = x^{\rho-\lambda}L_1(1/x) \qquad (0 < x < \infty),$$

and (2.11) can be obtained as in §9. This completes the proof of Theorem 1a.

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