TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 210, 1975

TWO WEIGHT FUNCTION NORM INEQUALITIES FOR THE POISSON INTEGRAL

BY

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ABSTRACT. Let $f(x)$ denote a complex valued function with period 2π, let

$$P_r(f, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)f(y)dy}{1-2r\cos(x-y)+r^2}$$

be the Poisson integral of $f(x)$ and let $|I|$ denote the length of an interval I. For $1 < p < \infty$ and nonnegative $U(x)$ and $V(x)$ with period 2π it is shown that there is a C, independent of f, such that

$$\sup_{0\leq r<1} \int_{-\pi}^{\pi} |P_r(f, x)|^p\, U(x)\, dx \leq C\int_{-\pi}^{\pi} |f(x)|^p\, V(x)\, dx$$

if and only if there is a B such that for all intervals I

$$\left[\frac{1}{|I|}\int_I U(x)\, dx\right]\left[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\, dx\right]^{p-1} \leq B.$$

Similar results are obtained for the nonperiodic case and in the case where $U(x)dx$ and $V(x)dx$ are replaced by measures.

1. **Introduction.** In [4] Rosenblum showed that if $1 \leq p < \infty$, $0 \leq r < 1$, $f(x)$ has period 2π and μ is a finite Borel measure with period 2π, then there is a C, independent of r and f, such that

$$\int_{-\pi}^{\pi} |P_r(f, x)|^p\, d\mu(x) \leq C\int_{-\pi}^{\pi} |f(x)|^p\, d\mu(x) \tag{1.1}$$

if and only if $d\mu(x) = V(x)dx$ is absolutely continuous and there is a constant K, independent of x and h, such that

$$\frac{1}{h}\int_{x-h}^{x+h}\left[\frac{1}{hV(t)}\int_{t-h}^{t+h} V(s)\, ds\right]^{1/(p-1)} dt \leq K \tag{1.2}$$

for all $h > 0$ and all x. It is easy to see that a necessary and sufficient condition for (1.2) to hold is that there is a B such that for every interval I

$$\left[\frac{1}{|I|}\int_I V(x)\, dx\right]\left[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\, dx\right]^{p-1} \leq B; \tag{1.3}$$

Received by the editors July 23, 1974.

AMS (MOS) subject classifications (1970). Primary 26A86, 44A15; Secondary 40G10, 42A24.

[1]Supported in part by NSF grant GP 38540.

this follows from the fact that the left side of (1.2) is bounded below by

$$\frac{1}{h}\int_{x-h/2}^{x+h/2}\left[\frac{1}{hV(t)}\int_{x-h/2}^{x+h/2} V(s)\,ds\right]^{1/(p-1)} dt$$

and above by

$$\frac{1}{h}\int_{x-2h}^{x+2h}\left[\frac{1}{hV(t)}\int_{x-2h}^{x+2h} V(s)\,ds\right]^{1/(p-1)} dt.$$

Rosenblum also considered (1.1) when $p = 1$; he proved the same result with (1.2) replaced by

$$\frac{1}{h}\int_{x-h}^{x+h} V(t)\,dt \leqslant KV(x).$$

The purpose of this paper is to give a simpler proof than Rosenblum's of a more general result. Specifically, the following will be proved in §§2 and 3.

THEOREM 1. *If* $1 \leqslant p < \infty$, $0 \leqslant r < 1$, μ *and* ν *are Borel measures of period* 2π *and* $f(x)$ *has period* 2π, *then there is a* C, *independent of* f *and* r, *such that*

$$\int_{-\pi}^{\pi} |P_r(f, x)|^p\,d\mu(x) \leqslant C\int_{-\pi}^{\pi} |f(x)|^p\,d\nu(x) \tag{1.4}$$

if and only if for every interval I

$$\left[\frac{\mu(I)}{|I|}\right]\left[\frac{1}{|I|}\int_I\left[\frac{d\nu_a(x)}{dx}\right]^{-1/(p-1)} dx\right]^{p-1} \leqslant B, \tag{1.5}$$

where B *is independent of* I *and* ν_a *denotes the absolutely continuous part of* ν.

In Theorem 1 and throughout this paper

$$\left[\frac{1}{|I|}\int_I\left[\frac{d\nu_a(x)}{dx}\right]^{-1/(p-1)} dx\right]^{p-1}$$

is to be interpreted as $\operatorname{ess\,sup}_{x\in I}[d\nu_a(x)/dx]^{-1}$ if $p = 1$ and $0 \cdot \infty$ is to be interpreted as 0.

The nonperiodic version of Theorem 1 will follow from the same reasoning; this is stated as Theorem 2 in §4. The fact that for nonnegative $U(x)$ and $V(x)$

$$\sup_{0\leqslant r<1}\int_{-\pi}^{\pi} |P_r(f, x)|^p\,U(x)\,dx \leqslant C\int_{-\pi}^{\pi} |f(x)|^p\,V(x)\,dx \tag{1.6}$$

holds if and only if for every interval I

$$\left[\frac{1}{|I|}\int_I U(x)\,dx\right]\left[\frac{1}{|I|}\int_I [V(x)]^{-1/(p-1)}\,dx\right]^{p-1} \leqslant B \tag{1.7}$$

is an immediate corollary of Theorem 1.

In the light of recent results concerning other operators, Theorem 1 and this corollary are not as natural an extension of Rosenblum's result as they may seem. For example, if $1 < p < \infty$, $U(x)$ and $V(x)$ are nonnegative and $U(x) = V(x)$, then there is a C, independent of f, such that

$$\int_{-\pi}^{\pi} \left(\sup_{0 \leq r < 1} |P_r(f, x)| \right)^p U(x)\, dx \leq C \int_{-\pi}^{\pi} |f(x)|^p V(x)\, dx \tag{1.8}$$

if and only if (1.3) is true; this strengthened version of Rosenblum's result follows from [2, Theorem 2, p. 215] since $\sup_{0 \leq r < 1} |P_r(f, x)|$ is equivalent to the Hardy-Littlewood maximal function for nonnegative f. If the assumption $U(x) = V(x)$ is dropped, however, (1.7) does not imply (1.8); an example showing this is in §5 of [2]. In fact, the problem of characterizing the weight functions for which (1.8) is true is unsolved and evidently quite difficult. Similarly, it is shown in [1] that if $1 < p < \infty$, $U(x)$ and $V(x)$ are nonnegative and $U(x) = V(x)$, then (1.3) is necessary and sufficient for

$$\int_{-\infty}^{\infty} |\tilde{f}(x)|^p U(x)\, dx \leq C \int_{-\infty}^{\infty} |f(x)|^p V(x)\, dx \tag{1.9}$$

where $\tilde{f}(x) = \lim_{\epsilon \to 0^+} \int_{|t| > \epsilon} (f(x - t)/t)\, dt$ is the Hilbert transform of $f(x)$. Again, (1.7) does not imply (1.9) if the assumption $U(x) = V(x)$ is dropped; for an example see [3]. It is rather surprising, therefore, that (1.7) is necessary and sufficient for (1.6) whether it is assumed that $U(x) = V(x)$ or not.

It should be noted that Theorem 1 does imply all of Rosenblum's result with his assumptions that $\mu = \nu$ and $\mu([-\pi, \pi]) < \infty$. The only problem is to show that (1.5) implies that μ is absolutely continuous since (1.3) is obviously equivalent to (1.5) once absolute continuity is proved. To prove that μ is absolutely continuous, observe that since $\mu([-\pi, \pi]) < \infty$, $d\mu_a(x)/dx < \infty$ almost everywhere. If $d\mu_a(x)/dx = 0$ on a set of positive measure, then (1.5) implies that $\mu([-\pi, \pi]) = 0$ and μ is absolutely continuous. Therefore, assume that $0 < d\mu_a(x)/dx < \infty$ almost everywhere. Then, for any interval I,

$$1 \leq \left(\frac{1}{|I|} \int_I \left[\frac{d\mu_a(x)}{dx} \right]^{1/p} \left[\frac{d\mu_a(x)}{dx} \right]^{-1/p} dx \right)^p. \tag{1.10}$$

Applying Hölder's inequality to the right side of (1.10) then shows that

$$1 \leq \left(\frac{\mu_a(I)}{|I|} \right) \left(\frac{1}{|I|} \int_I \left[\frac{d\mu_a(x)}{dx} \right]^{-1/(p-1)} dx \right)^{p-1}. \tag{1.11}$$

Multiplying (1.11) by $\mu(I)$ and using (1.5) shows that $\mu(I) \leq B\mu_a(I)$; since this is true for every interval I, μ is absolutely continuous.

2. **Proof that (1.5) implies (1.4).** The proof of this part of Theorem 1 will be done by proving two simple lemmas and then combining them. In this section and §4 the notation, $f_h(x) = h^{-1} \int_{x-h}^{x+h} |f(t)|\, dt$, will be used.

LEMMA 1. *If $f(x)$ has period 2π and μ and ν are Borel measures with period 2π that satisfy* (1.5), *then for every $h > 0$,*

$$\int_{-\pi}^{\pi} [f_h(x)]^p \, d\mu(x) \leqslant 6^{p+1} B \int_{-\pi}^{\pi} |f(t)|^p \, d\nu(x). \tag{2.1}$$

Assume first that $h \leqslant \pi$ and let N be the least integer such that $Nh \geqslant \pi$. Then the left side of (2.1) is bounded by

$$\sum_{k=-N}^{N-1} \int_{kh}^{(k+1)h} h^{-p} \left[\int_{x-h}^{x+h} |f(t)| \, dt \right]^p d\mu(x),$$

and this is bounded by

$$\sum_{k=-N}^{N-1} \left[h^{-p} \int_{kh}^{(k+1)h} d\mu(x) \right] \left[\int_{(k-1)h}^{(k+2)h} |f(t)| \, dt \right]^p. \tag{2.2}$$

Using Hölder's inequality on the second integral in (2.2) shows that (2.2) is bounded by

$$\sum_{k=-N}^{N-1} \left(\frac{\mu([kh, (k+1)h])}{h} \right) \left(\frac{1}{h} \int_{(k-1)h}^{(k+2)h} \left[\frac{d\nu_a}{dx} \right]^{-1/(p-1)} dx \right)^{p-1}$$
$$\cdot \left(\int_{(k-1)h}^{(k+2)h} |f(t)|^p \, d\nu_a(t) \right);$$

note that if $d\nu_a/dx$ is 0 on a set of positive measure, $\mu([-\pi, \pi]) = 0$ and (2.2) is still bounded by this. Now (1.5) with $I = [(k-1)h, (k+2)h]$ shows that this is bounded by

$$3^p B \sum_{k=-N}^{N-1} \int_{(k-1)h}^{(k+2)h} |f(t)|^p \, d\nu_a(t). \tag{2.3}$$

Since all the intervals $[(k-1)h, (k+2)h]$ are subsets of $[-2\pi, 2\pi]$ and no point is in more than three of these intervals, (2.3) is bounded by

$$3^{p+1} 2B \int_{-\pi}^{\pi} |f(t)|^p \, d\nu(t).$$

This completes the proof if $h \leqslant \pi$. If $h > \pi$, $f_h(x) \leqslant 2f_\pi(x)$ and the result follows from the first part.

LEMMA 2. *If $f(x)$ has period 2π and $0 \leqslant r < 1$, then there is a constant K, independent of f and r, such that*

$$|P_r(f, x)| \leqslant K \int_{1-r}^{2\pi} (1 - r) h^{-2} f_h(x) \, dh. \tag{2.4}$$

By reversing the order of integration, the integral on the right side of (2.4) is greater than or equal to

$$\int_{x-\pi}^{x+\pi} |f(t)| \left[\int_{|x-t|\vee(1-r)}^{2\pi} (1-r)h^{-3}\,dh\right] dt$$

where $|x-t|\vee(1-r)$ denotes the larger of $|x-t|$ and $1-r$. The inner integral can be calculated and is clearly greater than a positive constant times

$$\frac{1}{2\pi}\left(\frac{1-r^2}{(1-r)^2+2r[1-\cos(x-t)]}\right)$$

since $|x-t| \leqslant \pi$. This proves Lemma 2.

To show that (1.5) implies (1.4), use Lemma 2 to show that the left side of (1.4) is bounded above by

$$K^p \int_{-\pi}^{\pi} \left[\int_{1-r}^{2\pi} (1-r)h^{-2} f_h(x)\,dx\right]^p d\mu(x). \tag{2.5}$$

By Minkowski's integral inequality, (2.5) is bounded by

$$K^p\left(\int_{1-r}^{2\pi}\left[\int_{-\pi}^{\pi}[f_h(x)]^p\,d\mu(x)\right]^{1/p}(1-r)h^{-2}\,dh\right)^p \tag{2.6}$$

By Lemma 1, (2.6) is bounded by

$$6^{p+1}BK^p\left[\int_{-\pi}^{\pi}|f(x)|^p\,d\nu(x)\right]\left[\int_{1-r}^{2\pi}(1-r)h^{-2}\,dh\right]^p$$

Since the last integral is less than 1, (1.4) follows.

3. **Proof that (1.4) implies (1.5).** This proof need only be done for intervals I with $|I| \leqslant 2\pi$, since if $|I| > 2\pi$ the left side of (1.5) is bounded by 2^p times its value for the interval $[-\pi, \pi]$. Given any $f(x)$, let $f_1(x) = 0$ on the support of the singular part of ν and let $f_1(x) = f(x)$ elsewhere. Then $P_r(f_1, x) = P_r(f, x)$ and $\int_{-\pi}^{\pi}|f_1(x)|^p\,d\nu(x) = \int_{-\pi}^{\pi}|f(x)|^p\,d\nu_a(x)$. Therefore, if $d\nu_a(x)/dx$ is written as $V(x)$, (1.4) with f replaced by f_1 implies that

$$\int_{-\pi}^{\pi}|P_r(f,x)|^p\,d\mu(x) \leqslant C\int_{-\pi}^{\pi}|f(x)|^p\,V(x)\,dx. \tag{3.1}$$

Consequently, the proof that (1.4) implies (1.5) can be completed by showing that (3.1) implies (1.5) for intervals I with $|I| \leqslant 2\pi$.

Given I with $|I| \leqslant 2\pi$, let $Q = \int_I [V(x)]^{-1/(p-1)}\,dx$ and let $p' = p/(p-1)$. If $Q = 0$, (1.5) follows because of the convention $0 \cdot \infty = 0$. If $Q = \infty$, $[V(x)]^{-1/p}$ is not in $L^{p'}$ on I so there is a function $g(x)$ in L^p on I such that $g(x)[V(x)]^{-1/p}$ is not integrable on I. Let $f(x) = g(x)[V(x)]^{-1/p}$ on I and 0 elsewhere. Then $P_r(f, x) = \infty$ for all x and the right side of (3.1) is finite since $g(x)$ is in L^p on I. Therefore, $\mu(I) = 0$ and (1.5) is true.

If $0 < Q < \infty$ and $p > 1$, let $f(x) = [V(x)]^{-1/(p-1)}$ if $x + 2n\pi \in I$ for some integer n and 0 elsewhere. Let r be the larger of $1 - |I|$ and 0. Then for x in I,

$$P_r(f, x) \geqslant \frac{A}{|I|} \int_I [V(y)]^{-1/(p-1)}\, dy$$

where A is a positive constant independent of x, V, p and I. Then (3.1) implies that

$$\int_I \left[\frac{A}{|I|} \int_I [V(y)]^{-1/(p-1)}\, dy\right]^p d\mu(x) \leqslant C \int_I [V(x)]^{-1/(p-1)}\, dx. \tag{3.2}$$

Dividing by the integral on the right side of (3.2) then gives (1.5).

If $0 < Q < \infty$ and $p = 1$, choose $\epsilon > 0$ and let E be the subset of I where $V(x) < \epsilon + \operatorname{ess\,inf}_{y\in I} V(y)$. Let $f(x)$ equal 1 if $x + 2n\pi \in E$ for some integer n and 0 otherwise, and let r be the larger of $1 - |I|$ and 0. Then for x in I, $P_r(f, x) \geqslant A|E|/|I|$ where A is a positive constant independent of x, V and I, and $|E|$ denotes the Lebesgue measure of E. Then (3.1) implies that

$$\frac{A|E|\mu(I)}{|I|} \leqslant C|E| \left[\epsilon + \operatorname*{ess\,inf}_{y\in I} V(y)\right].$$

Dividing by $A|E|$ and using the fact that ϵ was arbitrary gives

$$\frac{\mu(I)}{|I|} \leqslant \frac{C}{A} \operatorname*{ess\,inf}_{y\in I} V(y);$$

this is equivalent to (1.5) with the appropriate interpretation for $p = 1$.

4. The nonperiodic case. Given $f(x)$ defined on $(-\infty, \infty)$, let

$$f(t, x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{tf(y)dy}{t^2 + (x-y)^2}$$

be the usual Poisson integral. The nonperiodic theorem is the following.

THEOREM 2. *If $1 \leqslant p < \infty$, $t > 0$ and μ and ν are Borel measures, then there is a C, independent of f, such that*

$$\int_{-\infty}^{\infty} |f(t, x)|^p\, d\mu(x) \leqslant C \int_{-\infty}^{\infty} |f(x)|^p\, d\nu(x) \tag{4.1}$$

if and only if for every interval I (1.5) *holds where B is independent of I and ν_a denotes the absolutely continuous part of ν.*

The proof that (1.5) implies (4.1) uses the following analogues of Lemmas 1 and 2.

LEMMA 3. *If μ and ν are Borel measures that satisfy* (1.5), *then for every $h > 0$*

$$\int_{-\infty}^{\infty} [f_h(x)]^p\, d\mu(x) \leqslant 3^{p+1} B \int_{-\infty}^{\infty} |f(t)|^p\, d\nu(x).$$

LEMMA 4. *There is a constant K, independent of f and t, such that $|f(t, x)| \leqslant K \int_t^\infty th^{-2} f_h(x)\, dh$.*

The proof of Lemma 3 is the same as that of Lemma 1 except that the sum is taken from $-\infty$ to ∞ and the initial restriction on the length of I is not needed. Lemma 4 is proved in the same way that Lemma 2 was. The rest of the proof that (1.5) implies (4.1) is the same as the proof that (1.5) implies (1.4) except that Lemmas 3 and 4 are used in place of Lemmas 1 and 2.

The proof that (4.1) implies (1.5) is essentially the same as the proof in §3; the reduction to intervals I with $|I| \leq 2\pi$ is not needed and in the last two cases t should be chosen equal to $|I|$ instead of r being the larger of $1 - |I|$ and 0.

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