

THE RESIDUE CALCULUS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Let W be a complex manifold and V an analytic variety. Then homology classes in $W - V$ which bound in V , called the geometric residues, are studied. In fact, a long exact sequence analogous to the Thom-Gysin sequence for nonsingular V is formed by a geometric construction. A geometric interpretation of the Leray spectral sequence of the inclusion of $W - V \subset V$ is also given.

If the complex codimension of V is one, then one shows that each cohomology class of $W - V$ can be represented by a differential form of the type $\theta \wedge \lambda + \eta$ where λ is the kernel associated to V and $\theta|_V$ is the Poincaré residue of this class.

1. **Introduction.** Let W be a complex manifold and V a subvariety of codimension q in W . Then let

$$R_p(V) = \text{kernel} \{H_p(W - V) \rightarrow H_p(W)\}$$

and $R_p(V)$ are called the geometric q -residues of V .

In the classical case when W is a Riemann surface $R_1(V)$ is generated by small circles γ_i about the points $P_i \in V$ and the relation is given by $\sum \gamma_i \sim 0$ in case W is compact. This is a special case of when V is nonsingular and one looks at the Thom-Gysin sequence of the normal sphere bundle of V in W :

$$\rightarrow H_{p+1}(W) \xrightarrow{I} H_{p-2q+1}(V) \xrightarrow{\tau} H_p(W - V) \rightarrow H_p(W) \rightarrow$$

where τ is tubes over cycles and I is transverse intersection.

In §2 we study V in the case when V has normal crossing, i.e., transverse intersection of nonsingular hypersurfaces. We construct $H_p(V)_\Delta$, a quotient group of a subgroup of the p -cycles of V , such that

COROLLARY 2.14. *The following is exact*

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$$\cdots \rightarrow H_{p+1}(W) \xrightarrow{I} H_{p-1}(V)_\Delta \xrightarrow{\tau} H_p(W - V) \rightarrow H_p(W) \rightarrow \cdots$$

where τ will be induced from the nonsingular hypersurfaces.

The proof is geometric. In §3 it is shown that if one can resolve singularities via monoidal transforms with nonsingular centers, then it suffices to have normal crossings to study $R_p(V)$.

In §4 we consider the general case of V , a subvariety of codimension q in W . There a tubular neighborhood $\tau(V) \subset W - V$ of V in W is constructed such that

COROLLARY 4.17. *The following is exact*

$$\cdots \rightarrow H_{p+1}(W) \rightarrow H_{p-2q+1}(V)_\Delta \xrightarrow{\tau} H_p(W - V) \rightarrow H_p(W) \rightarrow \cdots$$

where τ is the tubes over cycles via this tubular neighborhood and $H_{p-2q+1}(V)_\Delta$ is a quotient group of a subgroup of the $(p - 2q + 1)$ -cycles of V .

The proof is geometric and uses Whitney work on stratifications to construct the tubular neighborhood. At the end of the section, the spectral sequence induced from the inclusion map $W - V \rightarrow W$ is studied and one has

$$E_2^{p,q} \Rightarrow H^{p+q}(W - V)$$

with

$$E_2^{p,0} \Rightarrow \text{image} \{H^p(W) \rightarrow H^p(W - V)\}$$

and

$$E_2^{-q,q} \Rightarrow \text{cokernel} \{H^p(W) \rightarrow H^p(W - V)\}$$

(i.e., $R_p(V)$) for $q > 1$. The d_2 -term of the spectral sequence is essentially the Gysin map and the geometric results of this section are an investigation of this map.

We note here that Corollary 4.17 is true if V satisfies Whitney condition A and W is a differentiable manifold. In particular if V has a regular neighborhood, e.g. V is a polyhedron.

If we take coefficients in \mathbb{C} , rather than \mathbb{Z} , and let $q = 1$, then by dualizing Corollary 4.17 we get

$$H^p(W - V) \xrightarrow{R} H^{p-1}(V)_\Delta = \text{Hom}(H_{p-1}(V)_\Delta, \mathbb{C})$$

where R is the Poincaré residue operator if V is nonsingular. We show that

THEOREM 5.2. *If V has normal crossings, then $\alpha \in H^p(W - V)$ can be represented by a differential form with a pole of order one on V .*

By the de Rham theorem $H^p(W - V; \mathbb{C})$ can be represented by closed C^∞ p -forms on $W - V$ modulo exact ones. Furthermore, if V has normal crossings, ω will be of the type

$$\omega = \theta \wedge \left(\sum_U e_U \frac{df_U}{f_U} \right) + \eta$$

where f_U is a local defining equation of V in U , a suitably chosen coordinate chart in W and e_U a partition of unity subordinate to U . Also η is smooth in W and closed near V . Then $R(\omega) = \theta|_V$, a closed form on V , but not necessarily smooth, i.e., $\theta|_V$ can have a simple pole on the singular locus of V .

It is conjectured that for any hypersurface, ω will have this form, in fact, if V is a subvariety of codimension q , then it is conjectured that $\omega = \theta \wedge \lambda + \eta$ where λ is a $(q, q - 1)$ -form with $d\lambda$ the representative of the Poincaré dual of V in W .

Finally unless otherwise specified, in §§2-4, we always have $H_*(X) = H_*(X; Z)$, i.e., coefficients in Z , and in §5, $H_*(X) = H_*(X; \mathbb{C})$ and $H^*(X) = H^*(X; \mathbb{C})$.

2. Residues of normal crossings. Let W be an n -complex dim manifold and V a hypersurface. If V is nonsingular, then we have a map $\tau: H_p(V) \rightarrow H_{p+1}(W - V)$ called *tubes over cycles* which is adjoint to integration over the fibre $R: H^{p+1}(W - V) \rightarrow H^p(V)$, i.e. if we represent cohomology of $W - V$ by closed C^∞ differential forms on $W - V$ (modulo exact ones), so that if ω is a closed $(p + 1)$ -form on $W - V$ and $\alpha \in H_p(V)$, then $\int_{\tau(\alpha)} \omega = \int_\alpha R(\omega)$. Essentially, if γ_p is a p -cycle which represents α , i.e., $\gamma_p \in \alpha$, and s_α is a local defining equation of V in U_α (where we choose coordinates of W in U_α so that s_α is one of them, which is possible if V is nonsingular), then $\tau(\gamma_p) \cap U_\alpha = (|s_\alpha| = 1) \times (\gamma_p \cap U_\alpha)$. (Note that τ is the map which appears in the Thom-Gysin sequence of the normal bundle of V in W .) For a precise definition and discussion of τ , see Leray [14], where τ is called δ^* = coboundary map.

If V' is another nonsingular hypersurface of W , then we have the maps

$$H_{p-1}(V \cap V') \rightarrow H_p(V - (V \cap V')) \rightarrow H_{p+1}(W - (V \cup V'))$$

and the composition is called *reiterated tubes over cycles*.

We also have a map on the chain level, $T: C_p(V) \rightarrow C_{p+z}(W)$ where if c_p is a p -chain of V and s_α and U_α as above, then $T(c_p) \cap U_\alpha = (|s_\alpha| \leq 1) \times (c_p \cap U_\alpha)$, so that $\partial \circ T = \tau + T \circ \partial$ where ∂ is the boundary map (recall $\partial(a \times b) = \partial a \times b + (-1)^{\dim a} a \times \partial b$).

Note that even if V has a nonempty singular locus S , if c_p is a chain in $V - S$, then we can still speak of the $(p + 1)$ -chain $\tau(c_p) \subset W - V$ and the $(p + 2)$ -chain $T(c_p) \subset W - S$.

2.1 DEFINITION. We say V has *normal crossings* if $V = \bigcup_{i=1}^m V_i$ where each V_i is a nonsingular hypersurface and if $P \in V_1 \cap \dots \cap V_k$ and z_j is a local defining equation for V_j , $1 \leq j \leq k$, then (z_1, \dots, z_k) can be extended to a local coordinate system of W about P .

Let $i: W - V \rightarrow W$ be the inclusion map and so we get $i_*: H_p(W - V) \rightarrow H_p(W)$ and $\text{Ker } i_* = p\text{-residues of } V = R_p(V)$. Also, let $Z_p(V, U)$ denote the relative p -cycles of $V \bmod U$ for $U \subset V$.

2.2 THEOREM. Let W be an n -dim complex manifold and V a hypersurface with normal crossings. Let S be the singular locus of V . Then we can choose S' an arbitrary small neighborhood of S in V such that every class of $R_p(V)$ has a representative of the form $\tau(\gamma_{p-1})$ for some $\gamma_{p-1} \in Z_{p-1}(V, S')$.

PROOF. Let $\alpha \in R_p(V)$ and let γ_p be a representative of α . Then $\gamma_p = \partial c_{p+1}$, $c_{p+1} \subset W$.

Let $M_i = i$ -fold points of V , i.e.,

$$M_i = \bigcup_{\substack{\{j_1, \dots, j_i\} \\ \text{distinct}}} \left\{ V_{j_1} \cap \dots \cap V_{j_i} - \bigcup_{\substack{j_{i+1} \neq j_k \\ 1 \leq k \leq i}} V_{j_1} \cap \dots \cap V_{j_i} \cap V_{j_{i+1}} \right\}.$$

Therefore, $P \in M_i$ implies we can choose coordinates so that V is given by $\prod_{k=1}^i z_k = 0$ in a neighborhood of P and $\dim_{\mathbb{C}} M_i = n - i$ with M_i a nonsingular locally closed submanifold and $\partial M_i = \bigcup_{j>1} M_j$.

Since each M_i is a manifold, it makes sense to speak of the transverse intersection of chains with M_i . Then we let $c_{p+1} \cap M_i = \gamma_i \in Z_{p+1-2i}(M_i)$. γ_i is a cycle because $(\partial c_{p+1}) \cap M_i = \emptyset = (\partial M_i) \cap \gamma_i$. Then we let i_0 be the largest integer such that $0 \neq [\gamma_{i_0}] \in H_{p+1-2i_0}(M_{i_0})$, i.e., if $c_{p+1} \cap M_i = \emptyset$ for $i > i'$ and $\gamma_{i'} \sim 0$ in $M_{i'}$, then we can deform c_{p+1} , keeping $\gamma_p \subset W - V$ fixed so that $c_{p+1} \cap M_i = \emptyset$. We denote by i_0 the largest integer such that this process terminates, and $i_0 \geq 1$, since otherwise $\gamma_p = \partial c'_{p+1}$, $c'_{p+1} \subset W - V$, but then $0 = \alpha \in R_p(V)$. (Recall $R_p(V)$ is a subgroup of $H_p(W - V)$.)

If $P \in c_{p+1} \cap M_{i_0}$ and about P , V is given by $\prod_{k=1}^{i_0} z_k = 0$, then locally about P we get that $c_{p+1} = C_1 \times \dots \times C_{i_0} \times \gamma_{i_0}$ where $C_j = z_j$ -plane. This is because c_{p+1} hits P transversely and $M_{i_0} = (0, \dots, 0, C_{i_0+1}, \dots, C_n)$.

Since V has normal crossings, if A is a set in M_k which misses ∂M_k , then we can speak of the tubular neighborhood of A in \bar{M}_{k-1} , where we put $M_0 = W - V$. That is, if we have

$$A \subset V_1 \cap \dots \cap V_k - \bigcup_{j>k} V_1 \cap \dots \cap V_k \cap V_j$$

then we can take the tubular neighborhood of A in $V_1 \cap \dots \cap \hat{V}_i \cap \dots \cap V_k$, since $V_1 \cap \dots \cap V_k$ is a nonsingular hypersurface in $V_1 \cap \dots \cap \hat{V}_i \cap \dots \cap V_k$. We denote it by $T_{k,i}(A) = \text{solid tube}$, i.e., all fibres of length $\leq \epsilon$.

Let

$$(1) \quad T_k(A) = \sum_{i=1}^k T_{k,i}(A).$$

Furthermore, by putting appropriate Riemannian metrics in each normal bundle, we can assume that, if $\tau_{k,j}(A)$ denotes the boundary of the tube over A , i.e., $T_{k,j}(A) = (\text{all vectors} \leq \epsilon) \times A$ and $\tau_{k,j}(A) = (\text{all vectors} = \epsilon) \times A$, then we have that as sets $\tau_{k-1,i} \tau_{k,j}(A) = \tau_{k-1,j} \tau_{k,i}(A)$.

However, if A is an oriented chain, we have

$$2.3 \text{ LEMMA. } \tau_{k-1,i} \tau_{k,j}(A) = -\tau_{k-1,j} \tau_{k,i}(A).$$

PROOF. See Leray [14, p. 132] for proof. Essentially this is true because in a fibre bundle, if c_1 and c_2 are oriented cycles, then $c_1 \times c_2 = -c_2 \times c_1$. This completes the proof of the lemma.

Let

$$(2) \quad \tau_k(A) = \sum_{i=1}^k \tau_{k,i}(A),$$

2.4 COROLLARY. $\tau_{k-1} \tau_k(A) = 0$, if A is an oriented chain.

PROOF. $\tau_{k-1} \tau_k(A)$ is a sum of pairs of the form $\tau_{k-1,i} \tau_{k,j}(A) + \tau_{k-1,j} \tau_{k,i}(A)$ which is zero by the lemma. Q.E.D. for Corollary 2.4.

If $A \subset V_1 \cap \dots \cap V_k$ let

$$\tilde{T}_i \cdots \tilde{T}_k(A) = \sum_{\substack{i_1 < \dots < i_{k-i+1} \\ i \leq i_j \leq k}} T_{i,i_1} T_{i+1,i_2} \cdots T_{k,i_{k-i+1}}(A).$$

Note that $\tilde{T}_k(A) = T_k(A)$.

Let $\tilde{\gamma}_{i_0} = \gamma_{i_0}$ and by induction define for $i < i_0$

$$\tilde{\gamma}_i = \gamma_i - [\tilde{T}_{i+1} \tilde{\gamma}_{i+1} + \tilde{T}_{i+1} \tilde{T}_{i+2} \tilde{\gamma}_{i+2} + \cdots + \tilde{T}_{i+1} \cdots \tilde{T}_{i_0} \tilde{\gamma}_{i_0}]$$

after deforming C_{p+1} (leaving γ_p fixed) such that $\tilde{T}_{i+1} \tilde{\gamma}_{i+1} + \cdots + \tilde{T}_{i+1} \cdots \tilde{T}_{i_0} \tilde{\gamma}_{i_0}$ is a $(p+1-2i)$ -dim chain contained in γ_i , which is possible by transversality.

2.5 LEMMA. $\partial \tilde{\gamma}_i = -\tau_{i+1} \tilde{\gamma}_{i+1}$ for $i < i_0$.

PROOF. For $i = i_0 - 1$,

$$\partial \tilde{\gamma}_{i_0-1} = \partial \gamma_{i_0-1} - T_{i_0-1} \partial \gamma_{i_0} - \tau_{i_0} \gamma_{i_0} = -\tau_{i_0} \gamma_{i_0}$$

since the γ_j are cycles.

For $i < i_0 - 1$, since $\tau_{i,j} T_{i+1,k} = T_{i,k} \tau_{i+1,j}$ (because the $T_{i,j}$ are locally products of 2-dim chains), we have that

$$\begin{aligned} \sum_{\substack{i_1 < \dots < i_k \\ i < i_j < m}} \sum_{j=1}^k T_{i,i_1} \cdots T_{i+j-1,i_j} \tau_{i+j,i_{j+1}} T_{i+j+1,i_{j+2}} \cdots T_{i+k-1,i_k} \quad (A) \\ = \sum_{\substack{i_1 < \dots < i_{k-1} \\ i < i_j < m}} T_{i,i_1} \cdots T_{i+k-2,i_{k-1}} \tau_{i+k-1} \quad (A) \end{aligned}$$

where τ_{i+k+1} is defined in (2). This is a straightforward combinatorial argument.

Hence by induction on i and the above equality, we get

$$\begin{aligned} \partial(\tilde{T}_{i+1} \cdots \tilde{T}_{i+j} \tilde{\gamma}_{i+j}) &= \tilde{T}_{i+1} \cdots \tilde{T}_{i+j-1} \tau_{i+j} \tilde{\gamma}_{i+j} \\ &\quad - \tilde{T}_{i+1} \cdots \tilde{T}_{i+j} \tau_{i+j+1} \tilde{\gamma}_{i+j+1}. \end{aligned}$$

Then a straightforward computation and the induction hypothesis yield $\partial \tilde{\gamma}_i = -\tau_{i+1} \tilde{\gamma}_{i+1}$. Q.E.D. for Lemma 2.5.

Let $T' = T_1 \tilde{\gamma}_1$ which is $(p + 1)$ -dim chain in W ; deform c_{p+1} (leaving γ_p fixed) so that $T' \subset c_{p+1}$. Then $\partial(c_{p+1} - T') = \gamma_p - \tau_1 \tilde{\gamma}_1 - T_1(\partial \tilde{\gamma}_1)$. Now, $-T_1 \partial \tilde{\gamma}_1 = T_1 \tau_2 \tilde{\gamma}_2$ by Lemma 2.5, and $\partial(T_1(\partial \tilde{\gamma}_1)) = \tau_1 \tau_2 \tilde{\gamma}_2 = 0$ by Corollary 2.4; therefore $T_1(\partial \tilde{\gamma}_1)$ is an absolute cycle which implies $\tau_1(\tilde{\gamma}_1)$ is also an absolute cycle.

2.6 LEMMA. $T_1(\partial \tilde{\gamma}_1) = -T_1 \tau_2 \tilde{\gamma}_2 = \partial T''$ where

$$T'' = \tilde{T}_1 \tilde{T}_2 \tilde{\gamma}_2 + \tilde{T}_1 \tilde{T}_2 \tilde{T}_3 \tilde{\gamma}_3 + \cdots + \tilde{T}_1 \cdots \tilde{T}_{i_0} \tilde{\gamma}_{i_0}.$$

PROOF. We have

$$\tilde{T}_1 \cdots \tilde{T}_k \tilde{\gamma}_k = \tilde{T}_1 \cdots \tilde{T}_{k-1} \tau_k \tilde{\gamma}_k - \tilde{T}_1 \cdots \tilde{T}_k \tau_{k+1} \tilde{\gamma}_{k+1}$$

by Lemma 2.5. Then computing T'' with the above equality yields a telescoping series whose first term is $-T_1 \tau_2 \tilde{\gamma}_2$ and whose last term is $\tilde{T}_1 \cdots \tilde{T}_{i_0} \partial \tilde{\gamma}_{i_0} = 0$. Q.E.D. for Lemma 2.6.

Therefore, let $T = T' + T'' \subset c_{p+1}$ and $c_{p+1} \cap V \subset T$. Then we have

$c_{p+1} - T \subset W - V$ and

$$\partial(c_{p+1} - T) = \gamma_p - \tau_1(\tilde{\gamma}_{p-1}) + T_1(\partial\tilde{\gamma}_1) - T_1(\partial\tilde{\gamma}_1) = \gamma_p - \tau_1(\tilde{\gamma}_{p-1}).$$

Since we have that $\tilde{\gamma}_{p-1,1} \in Z_{p-1}(V, S')$ where S' is an arbitrary small neighborhood in V of $\bigcup_{j>2} M_j = S$ = singular locus of V by choosing the tubular neighborhoods sufficiently small. Q.E.D. for Theorem 2.2.

The τ we have constructed is not well defined on homology, i.e., γ_{p-1} can bound in V but $\tau_1(\tilde{\gamma}_1)$ might not bound in $W - V$. However, if γ_{p-1} does bound in V , then it is a tube over a lower-dimensional cycle in V . Before proving this we will need some more notation.

Again, let $V = \bigcup_{i=1}^m V_i$ have normal crossings. Let M_i be the i -tuple points, i.e.,

$$M_i = \bigcup_{\text{distinct } i\text{-tuples}} M(j_1, \dots, j_i)$$

where

$$M(j_1, \dots, j_i) = V_{j_1} \cap \dots \cap V_{j_i} - \bigcup_{k \neq j_l; 1 \leq l \leq i} V_k \cap V_{j_1} \cap \dots \cap V_{j_i}.$$

Let $\bar{M}(j_1, \dots, j_i)$ = closure of $M(j_1, \dots, j_i) = V_{j_1} \cap \dots \cap V_{j_i}$. For notational completeness, let $M_0 = W - V$. Also, $\bar{M}_i = \bigcup \bar{M}(j_1, \dots, j_i)$.

At each component of M_{i+j} there are $\binom{i+j}{i} = \binom{i+j}{j}$ -components of M_i nearby, i.e., if locally $M_{i+j} = \{z_1 \dots z_{i+j} = 0\}$, then $\{z_{k_1} \dots z_{k_i} = 0\}$ for all distinct subsets $\{k_1, \dots, k_i\} \subset \{1, \dots, i+j\}$ defines the $\binom{i+j}{i}$ -components. Now $\binom{i+j}{i} \geq j+1$ and we say that:

2.7 DEFINITION. $(j+1)$ of the components, say $\{k_{1s}, \dots, k_{is}\}, s = 1, \dots, j+1$, are *tubular* if $\bigcap_{s=1}^{j+1} \{k_{1s}, \dots, k_{is}\} = \{h_1, \dots, h_{i-1}\}$, i.e., $\bigcap_{s=1}^{j+1} M(k_{1s}, \dots, k_{is})$ is a component of \bar{M}_{i-1} .

Note that $|\bigcap_{s=1}^{j+1} \{k_{1s}, \dots, k_{is}\}| \leq i-1$ where the k_{is} are arbitrary. Another way to see this is: if say $M(1, \dots, i+j) \subset M(i+j)$ and take any M_{i-1} nearby, say $M(1, \dots, i-1)$. Then there are exactly $(j+1)$ -components of M_i between $M(1, \dots, i-1)$ and $M(1, \dots, i+j)$, namely $M(1, \dots, i-1, k), k = i, \dots, i+j$, and these $M(1, \dots, i-1, k)$ are tubular.

We let $H_p(\bar{M}_i)_\Delta$ (the *tubular cycles*) be those classes of $H_p(\bar{M}_i)$ which have a representative such that each component γ_p has the following properties:

- (a) If $\gamma_p \cap M(k_1, \dots, k_{i+j}) \neq \emptyset$, then $\dim_R M(k_1, \dots, k_{i+j}) \cap \gamma_p = p - 2j$, i.e., they intersect transversely.
- (b) Of the $\binom{i+j}{i}$ -components of M_i at $M(k_1, \dots, k_{i+j})$, γ_p meets exactly

$(j + 1)$ of them in a nonempty transverse intersection.

(c) These $(j + 1)$ -components form a tubular set.

This means, for example, if $\dim_{\mathbb{C}} W = 2$ and $V = V_1 \cup V_2$ with V_i being nonsingular curves which intersect in double points, then if γ_1 is a line in V_1 from one double point to a different one and a line in V_2 connecting the same double points, then (a) implies that $[\gamma_1] \notin H_1(\bar{M}_1)_{\Delta}$. (b) implies $[V_1] \notin H_2(\bar{M}_1)_{\Delta}$. Also note that in Theorem 2.2 $H_{p-1}(\bar{M}_1)_{\Delta}$ is precisely those classes for which we constructed tubes in $W - V$. We did not have to worry about tubular then, since for $i = 1, ({}^1_1^{+j}) = j + 1$ and they are always tubular.

Let $\tau_i: H_p(\bar{M}_i)_{\Delta} \rightarrow H_{p+1}(M_{i-1})$ be the map defined in Theorem 2.2. There we had it for $i = 1$, but if one checks one easily sees that tubular is precisely what one needs for $i > 1$. In fact, if $M(k_{1s}, \dots, k_{is}), s = 1, \dots, j + 1$, are the tubular components, then they are tubular to some M_{i-1} , say $M(h_1, \dots, h_{i-1})$. Then $M(h_1, \dots, h_{i-1})$ plays the role of $M_0 = W - V$ in Theorem 2.2. Let $\tilde{\tau}_i: H_p(M_i) \rightarrow H_{p+1}(M_{i-1})$ be the map which puts a tube over a cycle in just one normal direction, i.e., $\tilde{\tau}_i = \tau_{i,j}$ for some j .

2.8 COROLLARY. $R_p(V)$ is generated by $\tau_1 H_{p-1}(\bar{M}_1)_{\Delta}, \tilde{\tau}_1 \tau_2 H_{p-2}(\bar{M}_2)_{\Delta}, \dots, \tilde{\tau}_1 \cdots \tilde{\tau}_{p-1} \tau_p H_0(\bar{M}_p)_{\Delta}$.

2.9 DEFINITION. $\tilde{\tau}_1 \cdots \tilde{\tau}_{q-1} \tau_q H_{p-q}(\bar{M}_q)_{\Delta}$ are called the *geometric $(p - q)$ -residues* of $R_p(V)$.

Note. The notion of geometric q -residue follows Hodge-Atiyah [11], whose definition follows Zariski [28, p. 150]. Also $\tilde{\tau}_i = \tau_{i,j}$ does not depend on which j we choose because $\tilde{\tau}_1 \cdots \tilde{\tau}_{p-q-1} \tau_{p-q}(\gamma_{p-q})$ will only differ by a sign if we choose the j 's in different order. This corollary is true for all $p \leq 2n$, since the only difficulty would be for $p = 2n$, but $R_{2n}(V) = 0$.

PROOF. We prove this by induction on $m =$ number of components of V . For $m = 1$, this is clear. Therefore assume done for less than m .

Let $\alpha \in R_p(V)$ and γ_p be a representative of α so that $\gamma_p = \partial c_{p+1}$, $c_{p+1} \cap V = \gamma_{p-1}$, $[\gamma_{p-1}] \in H_{p-1}(V)_{\Delta}$ and $\gamma_p \sim \tau(\gamma_{p-1})$ by Theorem 2.2.

If $\gamma_{p-1} \neq 0$ in V , then we have $\gamma_p \in \tau_1 H_{p-1}(\bar{M}_1)_{\Delta}$. Suppose $\gamma_{p-1} = \partial c_p$ for $c_p \subset V$. Then choose the largest i_0 such that $\gamma_{p-1} \cap M_{i_0} \neq \emptyset$ and $\dim \gamma_{p-1} \cap M_{i_0} = p + 1 - 2i_0$.

If $i_0 = 1$, then $\gamma_{p-1} \subset M_1$ and we can apply the induction hypothesis; namely, let $W' = V_i, V' = S \cap V_i$ where $S =$ singular locus of $V = \bar{M}_2$ and $\gamma_{p-1} \cap V_i = \gamma_{p-1,i} \in H_{p-1}(W' - V')$. Then since V' has $(m - 1)$ -components, we get $\gamma_{p-1,i}$ is generated by $\tau'_1 H_{p-2}(\bar{M}'_1)_{\Delta}, \dots, \tau'_1 \cdots \tilde{\tau}'_{p-2} \tau'_{p-1} H_0(\bar{M}'_{p-1})_{\Delta}$. But $\bar{M}'_1 \subset \bar{M}_{i+1}$ and $\tau'_i = \tau_{i+1}, \tilde{\tau}'_1 = \tilde{\tau}_{i+1}$ so that we

have $\gamma_{p-1,i}$ is generated by $\tau_2 H_{p-2}(\bar{M}_2)_\Delta, \dots, \tilde{\tau}_2 \cdots \tilde{\tau}_{p-1} \tau_p H_0(\bar{M}_p)_\Delta$ which implies $\tau_1(\gamma_{p-1}) = \sum_i \tau_{1,i}(\gamma_{p-1,i})$ is generated by $\tilde{\tau}_1 \tau_2 H_{p-2}(\bar{M}_2)_\Delta, \dots, \tilde{\tau}_1 \cdots \tilde{\tau}_{p-1} \tau_p H_0(M_p)_\Delta$.

Assume $i_0 > 1$. Let $\gamma_{i_0} = \gamma_{p-1} \cap M_{i_0} \in H_{p+2-2i_0}(M_{i_0})$. Now, $\dim_R c_p \cap \bar{M}_{i_0} \leq p + 2 - 2i_0$. Suppose $\dim_R c_p \cap \bar{M}_{i_0} = p + 2 - 2i_0$ and let $c_p \cap \bar{M}_{i_0} = c_{i_0}$. If $c_{i_0} \subset M_{i_0}$, then we can construct the tube over c_p such that $\tau(c_p) \subset W - V$ and $\partial\tau(c_p) = -\tau(\gamma_{p-1}) = -\gamma_p$, i.e., $\gamma_p \sim 0$ in $W - V$. The construction is as follows: Let $c_p \cap M_i = c_i, 1 \leq i \leq i_0$. Then put $\tilde{c}_{i_0} = c_{i_0}$ and by induction define for $j < i_0$

$$\tilde{c}_j = c_j - [\tilde{T}_{j+1} \tilde{c}_{j+1} + \cdots + \tilde{T}_{j+1} \cdots \tilde{T}_{i_0} \tilde{c}_{i_0}]$$

after deforming as in Theorem 2.2. This is possible because of tubularness.

But $\partial\tilde{c}_j = -\tau_{j+1} \tilde{c}_{j+1}, j < i_0$, and $\partial\tilde{c}_1 = \tilde{\gamma}_1$. So we form $\tau_1(\tilde{c}_1) \subset W - V$ and $\partial\tau_1 \tilde{c}_1 = -\tau_1(\partial\tilde{c}_1) = -\tau_1 \tilde{\gamma}_1 = -\gamma_p$.

Next, suppose $c_p \cap M_{i_0+1} \neq \emptyset$ which implies $\dim_R c_p \cap M_{i_0+1} = p - 2i_0$. However, we cannot, in general, construct the tube over c_p in $W - V$ because we might have M_{i_0+1} contained in some component of \bar{M}_{i_0} with c_p missing this component, e.g., let $p = 4, i_0 = 2$ and locally we have coordinates (x, y, z) with $V = \{xyz = 0\}$.

Let

$$\gamma_3 = [\{x = 0\} \times \mathbf{C}_y \times \{|z| = 1\}] \cup [\mathbf{C}_x \times \{y = 0\} \times \{|z| = 1\}].$$

Then

$$c_4 = [\{x = 0\} \times \mathbf{C}_y \times \{|z| \leq 1\}] \cup [\mathbf{C}_x \times \{y = 0\} \times \{|z| \leq 1\}]$$

and we cannot construct $\tau(c_4) \subset W - V$ because at $(0, 0, 0) \in M_3, \dim c_4 \cdot M_3 = 0$ but $M_3 \subset \bar{M}(y, z)$ and $c_4 \cdot M(y, z) = \emptyset$. We notice, however, that

$$\begin{aligned} \tau(\gamma_3) &= [(|x| = 1) \times \mathbf{C}_y - (|y| \leq 1) \times (|z| = 1)] \\ &\cup [\mathbf{C}_x - |x| \leq 1 \times (|y| = 1) \times (|z| = 1)]. \end{aligned}$$

So if instead of coming down to V so that we get γ_3 , we first come down to V via $|z| = 1$, we get

$$\begin{aligned} \gamma'_3 &= [(|x| = 1) \times \mathbf{C}_y - (|y| \leq 1) \times \{z = 0\}] \\ &\cup [\mathbf{C}_x - (|x| \leq 1) \times (|y| = 1) \times \{z = 0\}] \end{aligned}$$

and $\gamma'_3 = \tau(\gamma'_2), \gamma'_2 \in H_2(\bar{M}_2)_\Delta$ where

$$\gamma'_2 = [\{x = 0\} \times C_y \times \{z = 0\}] \cup [C_x \times \{y = 0\} \times \{z = 0\}].$$

We can also do this in the general case: $\gamma_{i_0} \in H_{p+1-2i_0}(M_{i_0})$ and assume γ_{i_0} is connected (if not, do each component one at a time). This implies $\gamma_{i_0} \subset M(1, \dots, i_0)$. Then if $W' = \bar{M}(1, \dots, i_0)$ and $V' = \bigcup_{j>i_0} \bar{M}(1, \dots, i_0, j)$ then γ_{i_0} is a residue of V' in W' and W' has less than m components which implies by induction that we have

$$\gamma_{i_0} = \tilde{\tau}_{i_0+1} \cdots \tilde{\tau}_{i_0+q-1} \tau_{i_0+q}(\gamma_{i_0+q}), \quad \gamma_{i_0+q} \in H_{p+1-2i_0-q}(\bar{M}_{i_0+q})_\Delta.$$

Thus locally we have for $P \in \gamma_{i_0}$ and U a coordinate chart about P with local coordinates (z_1, \dots, z_n) and $V = \{z_1 \cdots z_{i_0} = 0\}$ that

$$\gamma_{p-1} \cap U = c_{2i_0-2} \times (|z_{i_0+1}| = 1) \times \cdots \times (|z_{q-1}| = 1) \times \{\tau_q(\gamma_{i_0+q}) \cap U\}$$

where

$$c_{2i_0-2} = U \cap [\{z_1 = 0\} \times C_{z_2} \times \cdots \times C_{z_{i_0}} \cup C_{z_1} \times \{z_2 = 0\} \\ \times C_{z_3} \times \cdots \times C_{z_{i_0}} \cup \cdots \cup C_{z_1} \times \cdots \times C_{z_{i_0-1}} \times \{z_{i_0} = 0\}].$$

Claim. $\gamma_p \in \tilde{\tau}_1 \cdots \tilde{\tau}_q \tau_{q+1} H_{p-q-1}(\bar{M}_{q+1})_\Delta$.

We can assume $q = 1$, else rather than dropping down to V by the tube which yields γ_{p-1} , we drop down via $\tilde{\tau}_{i_0+1}$ and this yields a $\gamma'_{p-1} \subset M_1$ and we can proceed by the induction hypothesis.

Then $\gamma_{i_0} = \tau_{i_0+1}(\gamma_{p-1, i_0+1})$, $\gamma_{i_0+1} \in H_{p-2i_0}(\bar{M}_{i_0+1})_\Delta$. Thus

$$\tau_1(c_{2i_0-2} \times \tau_{i_0+1}(\gamma_{i_0+1})) \sim \tilde{\tau}_1 \tau_2(c_{2i_0-2} \times \gamma_{i_0+1}) \in \tilde{\tau}_1 \tau_2 H_{p-2}(\bar{M}_2)_\Delta.$$

To see this, consider the following example: $p = 6, i_0 = 2$ and locally

$$\gamma_{p-1} = [C_{z_1} \times \{z_2 = 0\} \cup \{z_1 = 0\} \times C_{z_2}] \\ \times [C_{z_3} - (|z_3| \leq 1) \times (|z_4| = 1) \cup (|z_3| = 1) \times C_{z_4} - (|z_4| \leq 1)]$$

and

$$\{C_{z_3} - (|z_3| \leq 1) \times |z_4| = 1\} \cup \{|z_3| = 1 \times C_{z_4} - (|z_4| \leq 1\} = \tau_2(\gamma_3)$$

where

$$\begin{aligned} \gamma_3 &= \{z_1 = 0\} \times \{z_2 = 0\} \times C_{z_3} \times \{z_4 = 0\} \\ &\cup \{z_1 = 0\} \times \{z_2 = 0\} \times \{z_3 = 0\} \times C_{z_4}. \end{aligned}$$

So before we constructed tubes, we were considering

$$\begin{aligned} &C_{z_1} \times \{z_2 = 0\} \times C_{z_3} \times \{z_4 = 0\} \cup C_{z_1} \times \{z_2 = 0\} \\ &\quad \times \{z_3 = 0\} \times C_{z_4} \cup \{z_1 = 0\} \times C_{z_2} \times C_{z_3} \\ &\quad \times \{z_4 = 0\} \cup \{z_1 = 0\} \times C_{z_2} \times \{z_3 = 0\} \times \{z_4 = 0\} \\ &= \bar{M}(1, 3) \cup \bar{M}(1, 4) \cup \bar{M}(2, 3) \cup \bar{M}(2, 4). \end{aligned}$$

But $\bar{M}(1, 3) + \bar{M}(2, 3) \sim \bar{M}(3, 4)$ because $\partial\bar{M}(3) = \bar{M}(1, 3) + \bar{M}(2, 3) + \bar{M}(3, 4)$. Therefore, what we want to construct the tube over is really $\bar{M}(1, 4) + \bar{M}(2, 4) + \bar{M}(3, 4) \in H_4(\bar{M}_2)_\Delta$, i.e., $\gamma_p = \tau_1\tau_2(\gamma_2)$, $\gamma_2 \in H_{p-1}(\bar{M}_2)_\Delta$. Thus what we are saying in this example is that if we set

$$\begin{aligned} c_1 &= C_{z_1} - (|z_1| \leq 1) \times (|z_2| = 1), & c_2 &= (|z_1| = 1) \times C_{z_2} - (|z_2| \leq 1), \\ c_3 &= C_{z_3} - (|z_3| \leq 1) \times (|z_4| = 1), & c_4 &= (|z_3| = 1) \times C_{z_4} - (|z_4| \leq 1) \end{aligned}$$

and

$$c_5 = C_{z_1} - (|z_1| \leq 1) \times C_{z_2} - (|z_2| \leq 1) \times (|z_3| = 1) \times (|z_4| = 1),$$

then we have that $\gamma_p = c_1 \times c_3 + c_2 \times c_3 + c_1 \times c_4 + c_2 \times c_4$ is homologous in $W - V$ to $\gamma'_p = c_1 \times c_3 + c_2 \times c_3 + c_5$ with $c_5 \sim c_1 \times c_4 + c_2 \times c_4$ corresponding to $\bar{M}(1, 3) + \bar{M}(2, 3) \sim \bar{M}(3, 4)$.

The general case, locally, is the same as the above example, i.e., given U open with V given in U by $\prod_{i=1}^k z_i = 0$, then we can transform γ_{p-1} in U so that

$$\gamma_{p-1} = \sum_{j=1}^{i_0} C_1 \times \cdots \times C_{j-1} \times \{0\} \times C_{j+1} \times \cdots \times C_{i_0} \times \tau_{i_0+1}(\gamma_{i_0+1}).$$

Then

$$\begin{aligned} \gamma_{i_0+1} &= \sum_{j=1}^l C_{i_0+1} \times \cdots \times C_{i_0+j-1} \times \{0\} \\ &\quad \times C_{i_0+j+1} \times \cdots \times C_{i_0+l} \times \gamma_{i_0+m+1}, \end{aligned}$$

where

$$\gamma_{i_0+m+1} = \begin{cases} P_{i_0+m+1} \times \cdots \times P_n & \text{if } p - 2i_0 = 2l, \\ \gamma_1 \times P_{i_0+m+2} \times \cdots \times P_n & \text{if } p - 2i_0 = 2l + 1, \end{cases}$$

and $P_i \in C_i - \{0\}$, γ_1 is a line in $C_{i_0+m+1} - \{0\}$. (Recall, $\dim_{\mathbb{R}} \gamma_{i_0+1} = p - 2i_0$.) We let $m = \min(k, i_0 + 1)$ and $\pi_m: \mathbb{C}^n \rightarrow \mathbb{C}^m$ be the projection on the first m -coordinates. Then, locally, before we construct τ_{i_0+1} , we have the $(p - 2)$ -dim chain

$$\gamma_{p-2} = \sum_{i=1}^{i_0} \sum_{j=i_0+1}^m \pi_m(M(i, j)) \times \gamma$$

where γ is a $(p - 2m)$ -dim chain in $0 \times \cdots \times 0 \times C_{m+1} \times \cdots \times C_n$ and $M(i, j)$ means $\{z_i = 0\} \cap \{z_j = 0\}$. By keeping γ fixed, we can work in $\pi_m(\mathbb{C}^n) = \pi_m(U)$ and we have

2.10 LEMMA. $\sum_{i=1}^{i_0} \sum_{j=i_0+1}^m \pi_m(M(i, j)) \sim \sum_{i < m} \pi_m(M(i, m))$ in $\pi_m(U)$.

PROOF. To fix some consistent orientation in $M(i)$, we say $M(i, j) = M(j, i)$ and

$$\partial M(i) = \sum_{j \neq i} \epsilon(j, i) M(j, i) \quad \text{where } \epsilon(j, i) = \begin{cases} +1, & j < i, \\ -1, & j > i. \end{cases}$$

Then

$$\partial \pi_m(M(i)) = \sum_{j < m; j \neq i} \epsilon(j, i) \pi_m(M(j, i)).$$

Hence

$$\sum_{i < i_0+k} \pi_m(M(i, i_0+k)) \sim \sum_{i > i_0+k} \pi_k(M(i, i_0+k)) \quad \text{in } \pi_m(U) \text{ for all } k.$$

Thus, for $k = 1$,

$$\sum_{i=1}^{i_0} \sum_{j=i_0+1}^m \pi_m(M(i, j)) \sim \sum_{i=1}^{i_0} \sum_{j=i_0+2}^m \pi_m(M(i, j)) + \sum_{i > i_0+1} \pi_m(M(i, i_0+1));$$

for $k = 2$,

$$\sim \sum_{i=1}^{i_0} \sum_{j=i_0+3}^m \pi_m(M(i, j)) + \sum_{i > i_0+2} \pi_m(M(i, i_0+1)) + \pi_m(M(i, i_0+2));$$

for $k = m - 1$,

$$\sim \sum_{i=1}^{i_0} \pi_m M(i, m) + \sum_{k=1}^{m-i_0-1} \pi_m M(m, i_0 + m) = \sum_{i < m} \pi_m (M(i, m)),$$

Q.E.D. for Lemma 2.10.

Note. This lemma is true for any m chosen, $1 \leq m \leq n = \dim_{\mathbb{C}} W$.

Then let $\gamma'_{p-2} = \sum_{i < m} \pi_m (M(i, m)) \times \gamma$ and $\gamma'_{p-2} \sim \gamma_{p-2}$ in $U \pmod{\partial U}$ and $\tau_2(\gamma'_{p-2}) \subset M_1$ by choice of m with $\tau_2(\gamma'_{p-2}) \sim \gamma_{p-1} \pmod{\partial U}$.

Thus, locally there is no problem. The difficulty is global: Can we make these local changes in a consistent manner, since we change γ_{p-2} at ∂U .

To do the global case, we will construct a polygon Γ of $\dim_{\mathbb{R}} = n_0 - 1$ where

$$n_0 = \max_i \left\{ \exists \text{ local coordinates so that } \prod_{j=1}^i z_j = 0 \text{ defines } V \right\},$$

i.e., $n_0 =$ maximum number of components of V which intersect at a point. Then $n_0 \leq \min(n, m)$, $n = \dim_{\mathbb{C}} W$, $m =$ number of components of V . In general, $n_0 = n$.

Before constructing Γ , we note that we can assume the $M(i_1, \dots, i_k) = V_{i_1} \cap \dots \cap V_{i_k}$ are connected for all $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$. This is because, if not, then perform monoidal transforms with centers at all but one of the components of $M(i_1, \dots, i_k)$ and this will create new $V_i, i > m$, but with only one component for $M(i_1, \dots, i_k)$. Then Proposition 3.1 shows that $R_p(V)$ is invariant under such transformations.

Note. The connectedness of the $M(i_1, \dots, i_k)$ simplifies the geometric arguments which follow but is not necessary.

Now, to construct Γ : to each $M(i) = V_i \subset V$, associate an $(n_0 - 1)$ -dim face, call it S_i . If $\emptyset \neq M(i_1, i_2)$, then the two faces S_{i_1} and S_{i_2} have an $(n_0 - 2)$ -dim face in common, call it $S_{i_1 i_2}$. If $M(i_1, i_2, i_3) \neq \emptyset$, then we have an $(n_0 - 3)$ -dim edge $S_{i_1 i_2 i_3}$ corresponding to where the three $(n_0 - 2)$ -dim edges $S_{i_1 i_2}, S_{i_1 i_3}$ and $S_{i_2 i_3}$ intersect, etc. Finally, each vertex $S_{i_1, \dots, i_{n_0}}$ will correspond to $(M(i_1, \dots, i_{n_0})) \neq \emptyset$.

To see why we need Γ , let us again consider the example on pp. 136, 137 with $i_0 = 2, p = 6$. Suppose for simplicity, $n_0 = 4$ and Γ is a simple closed regular 3-dim tetrahedron (i.e., a cone on a 2-dim pyramid) with 5 3-dim faces $S_i, i = 1, \dots, 5$, 10 2-dim faces $S_{ij}, i, j = 1, \dots, 5, i \neq j$, 10 edges S_{ijk} and 5 vertices S_{\dots} .

Suppose the situation described on pp. 136, 137 in the example is at vertex S_{1234} with S_i corresponding to $\{z_i = 0\} = M(i)$.

Then $\gamma_{i_0+1} = \gamma_3$ is, near S_{1234} , given by $S_{124} + S_{123}$ and before constructing the tube τ_3 , we are considering, near S_{1234} , the 2-cycle (in Γ) $S_{13} + S_{14} + S_{23} + S_{24} \approx \gamma_4 \cap U$. (Note. \approx will mean correspond to.) But γ_3 is a 2-cycle in \bar{M}_3 and $\bar{M}_3 \approx \Gamma_1$, the 1-skeleton of Γ . Hence we must have a 1-cycle in Γ_1 , i.e., S_{123} and S_{124} have S_{1235} and S_{1245} (respectively) as vertices. Suppose that $\gamma_3 \approx S_{123} + S_{124} + S_{125} \in Z_1(\Gamma_1) \subset Z_1(\Gamma)$. Then for $\gamma_5 = \gamma_{p-1}$ to be a cycle, we have that, before constructing tubes, that γ_4 must also have $S_{15} + S_{25}$, i.e., $\gamma_4 \approx S_{13} + S_{14} + S_{23} + S_{24} + S_{15} + S_{25} \in Z_2(\Gamma)$. That is, we start with $\gamma_{p-1} = C_{2i_0-2} \times \tau_{i_0+1}(\gamma_{i_0+1})$, but if we did not construct τ_{i_0+1} , we have a $(p - 2)$ -dim cycle in V , which in U is of the form $\sum_{i=1}^2 \sum_{j=3}^4 \pi_4(M(i, j))$. But as we approach S_5 , if we assume $\gamma_2 \approx S_{123} + S_{124} + S_{125}$, then for γ_4 to remain a cycle, we must have $M(1, 5) + M(2, 5) \approx S_{15} + S_{25}$. Now, $\partial M(3) = M(1, 3) + M(2, 3) - M(3, 4) - M(3, 5)$, but $M(5) \cap U = \emptyset$, so that $\partial M(3) \cap U$ does not have $M(3, 5)$ as was in the example. In Γ , this corresponds to $\partial S_3 = S_{13} + S_{23} - S_{34} - S_{35}$. Also $\partial S_5 = S_{15} + S_{25} + S_{35} + S_{45}$ so that $\gamma_4 \approx S_{14} + S_{24} + S_{34} - S_{45} (= \partial S_4)$. But this is precisely a tubular element, i.e., in V , we have $M(1, 4) + M(2, 4) + M(4, 3) - M(4, 5)$ which is tubular, as at each 4-tuple point, we have exactly 3 4-dim edges intersecting. In Γ , at each vertex S_{1234} , S_{1245} , S_{1345} , and S_{2345} of γ_4 we have exactly 3 edges, i.e., tubular 5-dim cycles of V will correspond to simple closed 2-dim polygons of Γ where a simple 2-dim polygon is one where at each vertex there are exactly 3 edges.

Thus, to show that the tubular cycles generate $R_6(V)$ is equivalent to showing that each 2-cycle of Γ is homologous to a simple closed 2-dim polygon. But it is clear that Γ is generated by such, i.e., $H_3(\Gamma) \approx Z$ with a generator $\sum_{i=1}^5 S_i$ and the boundary of each S_i is a simple closed 2-dim polygon.

This is precisely what happens in the global case:

Suppose we are given the representative $\gamma_{p-1} = \partial c_p, c_p \subset V$ and we let $k_0 = \max_k \{c_p \cap M_k \neq \emptyset\}$. We have, of course, $k_0 > i_0$ by the above remarks.

Case I. $n_0 = k_0$.

Then locally $\gamma_{p-1} = C_{2i_0-2} \times \tau_{i_0+1}(\gamma_{i_0+1})$ corresponds to a $\gamma_{p-2} \subset \bar{M}_2$ (before we construct τ_{i_0+1}) which we saw before Lemma 2.10 is essentially $\sum_{i=1}^{i_0} \sum_{j=i_0+1}^m \pi_m(M(i, j))$. The γ is unimportant to finding a tubular representative as we perform the homologies in $\pi_m(U)$.

γ_{p-2} will then correspond to α , an $(n_0 - 2)$ -dim cycle of Γ and γ_{i_0+1} to an $(n_0 - i_0 - 1)$ -dim cycle of $\Gamma_{n_0-i_0-1}$, the $(n_0 - i_0 - 1)$ -skeleton of Γ .

Namely, since $k_0 = n_0$ and π_m in Lemma 2.10 is independent of m , we can assume π_m is never onto an $M(i)$, i.e., $\pi_m: \mathbb{C}^n \rightarrow \mathbb{C}^n$ and we can assume coordinates have been chosen so that for $i > m$, $\{z_i = 0\} \not\subset V$. Then each $\pi_m(M(i, j))$ corresponds to an $(n_0 - 2)$ -dim chain of Γ and if γ_{p-2} is a chain over which we can form a tube to get γ_{p-1} , then these $(n_0 - 2)$ -dim chains must fit together to form an $(n_0 - 2)$ -dim cycle of Γ . That is, if α is not a cycle, so that, say, at the vertex $(1, \dots, n_0)$ we have $M(1, 2) \subset \alpha$ but $M(1, 3) \not\subset \alpha$ and $\partial M(1, 2) \supset M(1, 2, 3)$, then in $V \cap U$ where U is a neighborhood and V is given by $\prod_{i=1}^{n_0} z_i = 0$ in U , we cannot construct τ_{i_0+1} on γ_{i_0+1} .

Hence the proof of Lemma 2.10 is just a local manipulation in Γ and to find a tubular cycle $\gamma'_{p-2} \sim \gamma_{p-2}$ with $\gamma'_{p-1} = \tau_2(\gamma'_{p-2}) \sim \gamma_{p-1}$ in V with $\tau_1 \gamma'_{p-1} \sim \tau_1 \gamma_{p-1}$ in $W - V$ is equivalent to finding a simple closed $(n_0 - 2)$ -dim polygon $\alpha' \sim \alpha$ in Γ , where α' is a simple $(n_0 - 2)$ -dim polygon if at each vertex there are

$$(n_0 - 1) \text{ edges, } \binom{n_0 - 1}{2} \text{ -dim faces, } \dots, (n_0 - 1)(n_0 - 2) \text{-dim faces.}$$

This is because $C_{n_0-1}(\Gamma)$ is generated by the S_i which correspond to $M(i)$ so that $\alpha' \sim \alpha$ in Γ will correspond to $\gamma'_{p-2} \sim \gamma_{p-2}$ in V as shown in Lemma 2.10. Furthermore the condition that α' is simple is precisely the condition of γ'_{p-2} being tubular, i.e., at each vertex α' will have exactly $(n_0 - 1)$ edges, but if a vertex is given by S_{1, \dots, n_0} , then the $(n_0 - 1)$ edges are given by, say, $S_{1, \dots, \hat{i}, \dots, n_0}$ for $i = 1, \dots, n_0 - 1$ (i.e., one of the indices must appear in each edge). This implies that the $(n_0 - 1)(n_0 - 2)$ -dim faces of α' at S_{1, \dots, n_0} are given by S_{in_0} , $i = 1, \dots, n_0 - 1$. This corresponds to having γ'_{p-2} represented by $\sum_{i < n_0} \pi_m(M(i, n_0)) \times \gamma$ at $M(1, \dots, n_0)$ which is a tubular element, so that $\tau_{2, n_0}(\sum_{i < n_0} \pi_m(M(i, n_0)) \times \gamma) \subset M_1$.

However, it is clear that $H_{n_0-1}(\Gamma)$ is a free abelian group with say N generators β_i , $i = 1, \dots, N$, with each β_i having as a representative a simple closed $(n_0 - 1)$ -dim polygon $\sum_j S_j \beta_i$. Also, $H_{n_0-2}(\Gamma) = 0$ and ∂S_i is a simple closed $(n_0 - 2)$ -dim polygon. So that given $\alpha \in Z_{n_0-2}(\Gamma)$, it is easy to see that $\exists \alpha'$ with $\alpha' \sim \alpha$ and α' a simple closed $(n_0 - 2)$ -dim polygon. (One can give a precise proof of this by induction on the number of faces S_i of Γ .)

Case II: $k_0 < n_0$.

Again recall we have $\gamma_{p-1} = \partial c_p$, $c_p \subset V$. Let

$$I(k) = \{(i_1, \dots, i_k) \mid M(i_1, \dots, i_k) \cap c_p \neq \emptyset\}.$$

Then $k_0 = \max_k \{I(k) \neq \emptyset\}$, and let $\bigcup_k I(k) = I$. Form Γ_I , a $(k_0 - 1)$ -dim polyhedron corresponding to I : if $i \in I(i)$ then we take an $(k_0 - 1)$ -dim face, called S_i . If $(i, j) \in I(2)$, then S_i and S_j have a $(k_0 - 2)$ -dim face S_{ij} in common, etc.

Then as in the case $k_0 = n_0$, we can form α , a $(k_0 - 2)$ -dim cycle in Γ_I which correspond to γ_{p-2} . We then have by induction on k_0 , that $\alpha \sim \alpha'$, a simple closed $(k_0 - 2)$ -dim polygon.

Now, if $\prod_{i=1}^{k_0} z_i$ defines V in U and γ_{p-1} is given in U so that

$$\gamma_{p-2} = \sum_{i=1}^{i_0} \sum_{j=i_0+1}^{k_0} \pi_{k_0}(M(i, j)) \times \gamma$$

then $\partial \pi_{k_0}(M(i)) = \sum_{j < k_0} \pm \pi_{k_0}(M(i, j))$. That is, the changes we make in Γ_I correspond to homologous changes in V to form γ'_{p-2} , i.e., π_{k_0} is a local change in U but the changes in Γ_I are global changes in V which give a consistent local change in each U . In Γ_I , we have $\partial S_i = \sum_j \pm S_{ij}$ where S_j corresponds to $\pi_{k_0}(M(i))$, and S_{ij} to $\pi_{k_0}(M(i, j))$. Thus, we have the same situation as on Γ when $n_0 = k_0$.

Note that what we are doing can be seen in another context as follows:

We can embed $\Gamma_I \subset \Gamma$ as a subcomplex of the first barycentric subdivision of Γ :

If $(i_1, \dots, i_{k_0})_{\Gamma_I}$ is a vertex of Γ_I , then at the $(n_0 - k_0)$ -dim face $(i_1, \dots, i_{k_0}) \subset \Gamma$ (i.e., $S_{i_1, \dots, i_{k_0}}$), we take the barycenter to correspond to the vertex. If $(i_1, \dots, \hat{i}_j, \dots, i_{k_0})_{\Gamma_I}$ is an edge of Γ_I joining $(i_1, \dots, i_{k_0})_{\Gamma_I}$ to, say, $(i_1, \dots, i'_j, \dots, i_{k_0})_{\Gamma_I}$, then in the $(n_0 - k_0 + 1)$ -dim face $(i_1, \dots, \hat{i}_j, \dots, i_{k_0})_{\Gamma} \subset \Gamma$, we join the barycenter of $(i_1, \dots, i_{k_0})_{\Gamma}$ and $(i_1, \dots, i'_j, \dots, i_{k_0})_{\Gamma}$ to form an edge. We continue in this fashion.

Then under the embedding, $\alpha \rightsquigarrow \tilde{\alpha}$, a $(k_0 - 2)$ -dim cycle of the first barycentric subdivision of Γ .

Also, note that $\Gamma_I \subset \Gamma - \Gamma_{n_0 - k_0 - 1}$, i.e., Γ_I misses the $(n_0 - k_0 - 1)$ -dim skeleton of Γ and Γ_{k_0} is a deformation retract of $\Gamma - \Gamma_{n_0 - k_0 - 1}$ (as $\dim_{\mathbb{R}} \Gamma = n_0 - 1$ and $n_0 - 1 - (n_0 - k_0 - 1) - 1 = k_0 - 1$). We can deform Γ_I into $\Gamma_{k_0 - 1}$ as a subpolyhedron which is PL-homeomorphic to the original Γ_I . Then we work in $\Gamma_{k_0 - 1}$ to find α' and then push everything back into $\Gamma - \Gamma_{n_0 - k_0 - 1}$. Q.E.D. for Corollary 2.8.

2.11 DEFINITION. We say $V \subset W$ has *local normal crossings* if V is locally given by $\prod_{i=1}^k z_i^{a_i} = 0$ where (z_1, \dots, z_n) are local coordinates.

The proof of Theorem 2.2 is still valid in the local normal crossings case. In fact, a more general result is proven in §4 for arbitrary V . Corollary 2.8 is also valid but it takes some work to prove it. The induction should be changed to an induction on $\dim V$ and n_0 , where $n_0 = \max_k \{\prod_{i=1}^k z_i^{a_i} = 0 \text{ defines } V\}$. The sole difficulty in proving the result is the global construction of Γ (i.e., Lemma 2.10 is a local result) and here we can form Γ which is no longer a polygon but a simplicial complex. The S_i , the $(n_0 - 1)$ -dim simplex of Γ would correspond to the global components of V , e.g. we can have a component intersect itself many times at a point. Also, when S_i intersects itself, we would form S_{ij_k} where j could equal i , e.g., S_{11_1} , S_{11_2} , etc, and the S_{ij_k} would correspond to the global components of \bar{M}_2 , a subvariety of V . Again, if $M(i)$ and $M(i, j_k)$ are the components of V and \bar{M}_2 respectively, then homology in V via $\partial M(i) = \sum_j M(i, j_k)$ would correspond to homology in Γ via $\partial S(i)$. Then the difficult part will be to give a precise proof that simple closed $(n_0 - 2)$ -dim simplices are in each equivalence class of homology. However, this can be done by, e.g. a double induction on the number of components and n_0 . (For $n_0 = 2$, first step in the induction, we have a graph and results are clear.) The details are left to the reader.

To get the relations among the various geometric q -residues, let us say that two geometric q -residues γ_p, γ'_p are equivalent ($\gamma_p \sim \gamma'_p$) if $\gamma_p - \gamma'_p = \partial c_{p+1}$. $c_{p+1} \subset V$ such that $\tau_1(\gamma_p) - \tau_1(\gamma'_p) = -\partial(\tau_1(c_{p+1}))$. This is clearly an equivalence relation.

2.12 DEFINITION. Let $H_{p-1}(V)_\Delta$ denote

$$H_{p-1}(\bar{M}_1)_\Delta \oplus \tau_2 H_{p-2}(\bar{M}_2)_\Delta \oplus \cdots \oplus \tilde{\tau}_2 \cdots \tilde{\tau}_{p-1} \tau_p H_0(\bar{M}_p)_\Delta$$

modulo the equivalence relation.

Then $\tau_1 = \tau: H_{p-1}(V)_\Delta \rightarrow H_p(W - V)$.

2.13 PROPOSITION. The sequence

$$H_{p+1}(W) \xrightarrow{I} H_{p-1}(V)_\Delta \xrightarrow{\tau} H_p(W - V)$$

is exact where I is intersection with V , i.e., essentially cap product with the dual cohomology class of V .

PROOF. Let $\gamma_{p+1} \in H_{p+1}(W)$. Then $I(\gamma_{p+1}) = \gamma_{p+1} \cap V = \gamma_{p-1}$. But by the construction in Corollary 2.8, $\tau(\gamma_{p-1}) = \partial(\gamma_{p+1} - T\gamma_{p-1})$ where T is the solid tube over γ_{p-1} , i.e., all fibres of length less than or equal to ϵ . Therefore $\tau \cdot I = 0$.

Conversely, suppose $\gamma_{p-1} \in H_{p-1}(V)_\Delta$ and $\tau(\gamma_{p-1}) = \partial c_{p+1}$ for $c_{p+1} \subset$

$W - V$. Then set $\gamma_{p+1} = T\gamma_{p-1} - c_{p+1}$. But $\partial\gamma_{p+1} = \tau(\gamma_{p-1}) - \tau(\gamma_{p-1}) = 0$, so that $\gamma_{p+1} \in H_{p+1}(W)$ and clearly $I(\gamma_{p+1}) = \gamma_{p-1}$. Q.E.D. for Proposition 2.13.

Therefore the relations are given by $I(H_{p+1}(W))$. To say $I(\gamma_{p+1}) = 0$ means that if

$$I(\gamma_{p+1}) = (\gamma_{p-1}, \tau_2(\gamma_{p-2}), \dots, \tilde{\tau}_2 \cdots \tilde{\tau}_{p-1} \tau_p \gamma_0)$$

for $\gamma_{p-j} \in H_{p-j}(\bar{M}_j)_\Delta$ then each γ_{p-j} is homologous to zero in $H_{p-j}(\bar{M}_j)$, i.e., $\gamma_{p-j} = \partial c_{p-j+1}$ and $c_{p-j+1} \subset \bar{M}_j$. In fact $\tilde{\tau}_1 \cdots \tilde{\tau}_{j-1} c_{p-j+1}$ is a $(p+1)$ -chain in $W - V$ whose boundary is $\tilde{\tau}_1 \cdots \tilde{\tau}_{j-1} \tau_j \gamma_{p-j}$ by the definition of homologous in $H_{p-1}(V_1)_\Delta$.

2.14 COROLLARY. *The following is exact:*

$$\cdots \rightarrow H_{p+1}(W) \xrightarrow{I} H_{p-1}(V)_\Delta \xrightarrow{\tau} H_p(W - V) \xrightarrow{i} H_p(W) \rightarrow \cdots$$

PROOF. Exactness at $H_*(V)_\Delta$ is Proposition 2.11, while exactness at $H_*(W - V)$ is essentially Corollary 2.8. By the above discussion we have exactness at $H_*(W)$. Q.E.D.

We would like to note here that Theorem 2.2 and Corollary 2.8 are really a geometric investigation of a special sequence. Namely if we take the inclusion map of $W - V \subset W$ and look at the induced Leray spectral sequence, then

$$E_2^{p,q} \simeq \sum_{\{i_1, \dots, i_q\}} H^p(\bar{M}(i_1, \dots, i_q))$$

and

$$E_2^{p,q} \Rightarrow H^{p+q}(W - V).$$

(See Hodge-Atiyah [11] for a detailed investigation of this sequence. However, they use coefficients in \mathbb{C} .) Then $d_2^{p,q}$ is just the Gysin map and $\text{Ker } d_2^{p,q} = H^p(\bar{M}_q, \mathbb{C})_\Delta$. Also see Gordon [5] for a further discussion of this spectral sequence. We also note that Deligne [4] has generalized these results to the case of schemes over \mathbb{C} . For a detailed investigation of V in the general case see the end of §4.

3. **More remarks about the normal crossings case.** In the previous section we investigated the problem of the residues of V when V has normal crossings. We will now show that this is quite sufficient in most cases and also give a further topological reduction if further hypotheses are satisfied.

3.1 PROPOSITION. *If W' is obtained from W by a monoidal transform π whose center $X \subset V$ is nonsingular and if $\pi^{-1}(V) = V'$, then $\pi_*: R_p(V') \simeq R_p(V)$.*

PROOF. From the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R_p(V') & \longrightarrow & H_p(W' - V') & \longrightarrow & H_p(W') \\
 & & & & \downarrow \pi_* & & \downarrow \pi_* \\
 0 & \longrightarrow & R_p(V) & \longrightarrow & H_p(W - V) & \longrightarrow & H_p(W)
 \end{array}$$

we have that $\pi_*R_p(V') \subseteq R_p(V)$.

Suppose $\alpha \in R_p(V)$. Then α has a representative γ_p such that $\gamma_p = \partial C_{p+1}$, $C_{p+1} \subset W$ with C_{p+1} intersecting X transversely, i.e., if $\text{codim}_{\mathbb{C}} X = m$, then $C_{p+1} \cap X = \gamma_{p+1-2m}$, and if $N(X)$ is a normal disk bundle of X in W , then, locally, $C_{p+1} \cap N(X) = \gamma_{p+1-2m} \times D_{2m}$ where D_{2m} is a $2m$ -dim disk.

The difficulty with having $(\pi_*)^{-1}\alpha \in R_p(V')$ is that if we let $\gamma'_p = \pi^{-1}(\gamma_p)$, $C'_{p+1} = \pi^{-1}(C_{p+1} \cap (W - V))$ and C' be the closure of C'_{p+1} in W' , then $C' \cap V$ may not be a $(p - 1)$ -dim chain, i.e., it may be too large.

Set theoretically, we have $W' = (W - X) \cup_X PN(X)$, where $PN(X)$ is the projective-normal bundle of X in W with fibre CP_{m-1} . Then let U be a neighborhood of W with local coordinates (x_1, \dots, x_n) and $\{x_1 = \dots = x_m = 0\} = X \cap U$ and let $CP_{m-1} = [X_1, \dots, X_m]$ in homogeneous coordinates. Then $W' \cap \pi^{-1}(U) = \bar{\Gamma} \subset U \cap CP_{m-1}$ where

$$\Gamma = \{(x_1, \dots, x_n) \times [X_1, \dots, X_m] \mid x_i X_j = x_j X_i, i \leq i, j \leq m\}.$$

Hence, locally, $C_{p+1} \cap U = C_1 \times \dots \times C_m \times (\gamma_{p+1-2m} \cap U)$ and $C' \cap \pi^{-1}(U) \cap V' = CP_{m-1} \times \gamma'_{p+1-2m}$ where γ'_{p+1-2m} is a $(p + 1 - 2m)$ -dim chain in $\pi^{-1}(U)$ which is isomorphic to $\gamma_{p+1-2m} \cap U$. Thus $C' \cap V'$ is a $(p - 1)$ -dim chain, i.e., C' is a $(p + 1)$ -dim chain with $\partial C' = \gamma'_p$. Hence $\pi_*^{-1}\alpha \in R_p(V')$. Q.E.D. for Proposition 3.1.

Hironaka [9] has shown that if W is algebraic or a Stein manifold (in fact, any closed submanifold of a (Stein variety) \times (algebraic variety)), then, if V is a hypersurface of W , after a finite number of monoidal transforms with nonsingular centers contained in V , we can obtain $\pi: W' \rightarrow W$, also algebraic if W is, such that $V' = \pi^{-1}(V)$, the total transform, has normal crossings in the sense of Definition 2.1.

Then recently Lejeune-Teissier [13] have shown that if W is a reduced complex analytic space and if $V \subset W$ is a hypersurface, one can (by a finite number of monoidal transforms centered on nonsingular parts of V) obtain W' with V' having normal crossings..

Hence, in the above cases, the results of §2 are valid.

Next we want to make some further topological investigations of $R_p(V)$

when V has normal crossings. Recall that in Corollary 2.8. we showed that $R_p(V)$ is generated by cycles of the form $\tilde{\tau}_1 \cdots \tilde{\tau}_{q-1} \tau_q(\gamma_{p-q})$ for $\gamma_{p-q} \in H_{p-q}(M_q)_\Delta$. If $\gamma_{p-q} \in H_{p-q}(M_q)_\Delta$ and $\gamma_{p-q} = \Sigma \gamma_{p-q, i_1, \dots, i_q}$ then we first want to consider $\tau_{p-q, i_j}(\gamma_{p-q, i_1, \dots, i_q})$ in $\bar{M}(i_1, \dots, \hat{i}_j, \dots, i_q) - \bar{M}(i_1, \dots, i_q)$. For simplicity assume $q = 1$.

3.2 PROPOSITION. Let $\gamma_{p-1} = \Sigma_{i \in I} \gamma_{p-1, i} \in H_{p-1}(\bar{M}_1)_\Delta$.

(a) If $\gamma_{p-1, i_0} \sim 0$ in V_{i_0} , then $\tau_1(\gamma_{p-1}) \sim \tau_1(\gamma'_{p-1})$ in $W - V$, where $\gamma'_{p-1} = \Sigma_{i \in I; i \neq i_0} \gamma'_{p-1, i}$ and $\gamma'_{p-1, i} \sim \gamma_{p-1, i}$ in V_i .

(b) If $\tau_{1, i_0}(\gamma_{p-1, i_0}) \sim 0$ in $W - V_{i_0}$, then $\tau_1(\gamma_{p-1}) \sim \tau_1(\gamma'_{p-1})$ where $\gamma'_{p-1} = \Sigma_{i \neq i_0} \gamma'_{p-1, i}$.

Furthermore, if $\tau_{1, i_0}(\gamma_{p-1, i_0}) \sim 0$ in $W - (V_{i_0} \cup \bigcup_{j \in I} V_j)$, then $\tau_1(\gamma_{p-1}) \sim \tau_1(\gamma'_{p-1})$ where $\gamma_{p-1} = \Sigma_{i \neq i_0; i \in I} \gamma_{p-1, i}$. Also, if $\tau(\gamma_{p-1, i_0}) \sim 0$ in $W - V_{i_1}$ (or $W - (V_{i_1} \cup \bigcup_{j \in I} V_j)$) then $\tau_1(\gamma'_{p-1, i_1}) \sim 0$ in $W - V_{i_1}$ (or $W - (V_{i_1} \cup \bigcup_{j \in I} V_j)$).

PROOF. (a) Suppose $i_0 = 1$. Then if $\gamma_{p-1, 1} \sim 0$ in V_1 , then $\gamma_{p-1, 1} = \partial c_{p, 1}$ for $c_{p, 1} \subset V_1$. Let $c_{p-2, 1i} = c_{p, 1} \cap V_i$ where $c_{p-2, 1i}$ is a $(p-2)$ -chain in $V_1 \cap V_i$, since V_i intersects V_1 transversely. Then we have

$$\gamma_{p-3, 1i} = \gamma_{p-1, 1} \cap V_i = \gamma_{p-1, i} \cap V_1 = \partial(c_{p-2, 1i}).$$

Put

$$\gamma'_{p-1, i} = \gamma_{p-1, i} - T_{2, 1}(\gamma_{p-3, 1i}) - T_{2, 1}(c_{p-2, 1i})$$

for $i \neq 1$. Note that

$$\partial T_{2, 1} c_{p-2, 1i} = \tau_{2, 1} c_{p-2, 1i} + T_{2, 1} \gamma_{p-3, 1i}.$$

If we set $\tilde{c}_{p, 1} = c_{p, 1} - \Sigma_{i \neq 1} T_{1, i}(c_{p-2, 1i})$ then $\tau_{1, 1}(\tilde{c}_{p, 1}) \subset W - V$ and

$$\begin{aligned} \partial \tau_{1, 1}(\tilde{c}_{p, 1}) &= -\tau_{1, 1}(\partial \tilde{c}_{p, 1}) \\ &= \sum_{i > 1} \{\tau_{1, 1} \tau_{2, 1} c_{p-2, 1i} + \tau_{1, 1} \tau_{2, 1} \gamma_{p-3, 1i}\} - \tau_{1, 1} \tilde{\gamma}_{p-1, 1}. \end{aligned}$$

This implies

$$\sum_{i > 1} \tau_{1, i}(\tilde{\gamma}'_{p-1, i}) = \sum_{i \geq 1} \tau_{1, i}(\tilde{\gamma}_{p-1, i}) + \partial \tau_{1, 1}(\tilde{c}_{p, 1}).$$

Thus $\Sigma_{i > 1} \tau(\gamma'_{p-1, i}) \sim \Sigma_{i \geq 1} \tau(\gamma_{p-1, i})$ in $W - V$ and $\gamma'_{p-1, i} \sim \gamma_{p-1, i}$ in V_i for $i > 1$.

(b) Again suppose $i_0 = 1$ and $\tau_{1,1}(\gamma_{p-1,1}) \sim 0$ in $W - V_1$.

Then $\tau_{1,1}(\gamma_{p-1,1}) = \partial c_{p+1}$ for $c_{p+1,1} \subset W - V_1$. Let $c_{p+1,1} \cap V_i = \Gamma_{p-1,i}^1 \subset V_i - V_i \cap V_1$ for $i > 1$.

$$\begin{aligned} \partial \Gamma_{p-1,i}^1 &= (\partial c_{p+1}) \cap V_i = [\tau_{1,1}(\gamma_{p-1,1})] \cap V_i \\ &= \tau_{2,1}(\gamma_{p-1,1} \cap V_i) = \tau_{2,1}(\gamma_{p-3,1i}) \end{aligned}$$

where $\gamma_{p-3,1i} = \gamma_{p-1,1} \cap V_i = \gamma_{p-1,i} \cap V_1$. The second from last equality follows because we can choose the normal bundle of V_1 with respect to V_i so that

$$\tau_{1,1}|_{V_1 \cap V_i} = \tau_{2,1}: V_1 \cap V_i \rightarrow V_i - V_1 \cap V_i$$

by definition of $\tau_{2,1}$.

Then set $\gamma'_{p-1,i} = \tilde{\gamma}_{p-1,i} + \Gamma_{p-1,i}^1$ for $i > 1$ where $\tilde{\gamma}_{p-1,i} = \gamma_{p-1,i} - T_{2,1}(\gamma_{p-3,1i})$. $\partial \gamma'_{p-1,i} - \tau_{2,1}\gamma_{p-3,1i} + \tau_{2,1}\gamma_{p-3,1i} = 0$ by Lemma 2.5. Now,

$$\begin{aligned} \partial \left(\sum_{i>1} T_{1,i} \Gamma_{p-1,i}^1 \right) &= \sum_{i>1} \tau_{1,i} \Gamma_{p-1,i}^1 + \sum_{i>1} T_{1,i} \tau_{2,1} \gamma_{p-3,1i} \\ &= \sum_{i>1} \tau_{1,i} \Gamma_{p-1,i}^1 + \sum_{i>1} \tau_{1,i} T_{2,i} \gamma_{p-3,1i} \end{aligned}$$

since $T_{1,i} \tau_{2,1} = \tau_{1,1} T_{2,i}$ by their definition (i.e., $\partial(D^2 \times S^1) = \partial(S^1 \times D^2)$ so that as oriented chains, $D^2 \times S^1 = S^1 \times D^2$). Also $\sum_{i>1} T_{2,i} \gamma_{p-3,1i} = \gamma_{p-1,1} - \tilde{\gamma}_{p-1,1}$, so that

$$\partial \left(\sum_{i>1} T_{1,i} \Gamma_{p-1,i}^1 \right) = \sum_{i>1} \tau_{1,i} \Gamma_{p-1,i}^1 + \tau_{1,1}(\gamma_{p-1,1} - \tilde{\gamma}_{p-1,1}).$$

Thus

$$\partial \left(c_{p+1} - \sum_{i>1} T_{1,i} \Gamma_{p-1,i}^1 \right) = \tau_{1,1}(\tilde{\gamma}_{p-1,1}) - \sum_{i>1} \tau_{1,i} \Gamma_{p-1,i}^1.$$

Hence

$$\sum_{i>1} \tau_{1,i}(\gamma'_{p-1,i}) = \sum_{i>1} \tau_{1,i}(\gamma_{p-1,i}) - \partial \left(c_{p+1} - \sum_{i>1} T_{1,i} \Gamma_{p-1,i}^1 \right),$$

and $c_{p+1} - \sum_{i>1} T_{1,i} \Gamma_{p-1,i}^1$ is a $(p+1)$ -chain in $W - V_1$.

Furthermore, note that if $c_{p+1} \cap V_j = \emptyset$, then $\gamma'_{p-1,j} = \gamma_{p-1,j}$, so that this proves the second part of (b).

Finally, suppose that $\tau_{1,i_1}(\gamma_{p-1,i_1}) \sim 0$ in $W - V_{i_1}$ (or $W - (V_{i_1} \cup \bigcup_{j \in I} V_j)$). For simplicity suppose $i_1 = 2$. Then if $\gamma_{p-3,12} = \gamma_{p-1,1} \cap V_2 = \gamma_{p-1,2} \cap V_1$, we have that

$$\begin{aligned}
T_{1,2}\tau_{2,1}\gamma_{p-3,12} &= -T_{1,1}\tau_{2,2}\gamma_{p-3,12} + \partial T_{1,1}T_{2,2}\gamma_{p-3,12} \\
&= -\tau_{1,2}T_{2,1}\gamma_{p-3,12} + \partial T_{1,1}T_{2,2}\gamma_{p-3,12} \\
&= -\tau_{1,2}(\gamma_{p-1,2} - \tilde{\gamma}_{p-1,2}) + \partial T_{1,1}T_{2,2}\gamma_{p-3,12},
\end{aligned}$$

and

$$\begin{aligned}
\tau_{1,1}\gamma_{p-1,1} &= \tau_{1,1}(\tilde{\gamma}_{p-1,1}) + \tau_{1,1}(\gamma_{p-1,1} - \tilde{\gamma}_{p-1,1}) \\
&= \partial T_{1,1}\tilde{\gamma}_{p-1,1} - T_{1,1}(\partial\tilde{\gamma}_{p-1,1}) + \tau_{1,1}(\gamma_{p-1,1} - \tilde{\gamma}_{p-1,1}) \\
&= \partial T_{1,1}\tilde{\gamma}_{p-1,1} + T_{1,1}\tau_{2,2}\gamma_{p-3,12} \\
&\quad - T_{1,1}\tau_{2,2}\gamma_{p-3,12} + \partial T_{1,1}T_{2,2}\gamma_{p-3,12} \\
&= \partial T_{1,1}\tilde{\gamma}_{p-1,1} + \partial T_{1,1}T_{2,2}\gamma_{p-3,12}
\end{aligned}$$

where the third equality follows from Lemma 2.5 and the previous sequence of equalities.

Hence, using the above two sets of equalities we have

$$\begin{aligned}
\partial(c_{p+1,1} - T_{1,2}\Gamma_{p-1,2}^1) &= \tau_{1,1}\gamma_{p-1,1} - \tau_{1,2}\Gamma_{p-1,2}^1 - T_{1,2}\tau_{2,1}\gamma_{p-3,12} \\
&= \partial T_{1,1}\tilde{\gamma}_{p-1,1} - \tau_{1,2}\Gamma_{p-1,2}^1 + \tau_{1,2}(\gamma_{p-1,2} - \tilde{\gamma}_{p-1,2}).
\end{aligned}$$

So, if we have $\tau_{1,2}\gamma_{p-1,2} = \partial c_{p+1,2}$ for $c_{p+1,2} \subset W - V_2$ (or $W - (V_2 \cup \bigcup_{j \in I} V_j)$), then

$$\begin{aligned}
\tau_{1,2}(\gamma'_{p-1,2}) &= \tau_{1,2}(\tilde{\gamma}_{p-1,2}) + \tau_{1,2}\Gamma_{p-1,2} \\
&= \partial(c_{p+1,2}) - \tau_{1,2}(\gamma_{p-1,2} - \tilde{\gamma}_{p-1,2}) + \tau_{1,2}\Gamma_{p-1,2}^1 \\
&= \partial(c_{p+1,2}) + \partial T_{1,1}\tilde{\gamma}_{p-1,1} - \partial(c_{p+1,1} - T_{1,2}\Gamma_{p-1,2}^1).
\end{aligned}$$

Also

$$c_{p+1,2} + T_{1,1}\tilde{\gamma}_{p-1,1} - (c_{p+1,1} - T_{1,2}\Gamma_{p-1,2}^1) \subset W - V_2$$

(or $W - (V_2 \cup \bigcup_{j \in I} V_j)$) if $c_{p+1,1}$ and $c_{p+1,2}$ also are contained in $W - (V_2 \cup \bigcup_{j \in I} V_j)$. Q.E.D.

Note. We cannot say just because for all $i \in I$, $\tau_{1,i}(\gamma_{p-1,i}) \sim 0$ in $W - (V_i \cup \bigcup_{j \in I} V_j)$ that $\tau_1(\gamma_{p-1}) \sim 0$ in $W - V$ because

$$\begin{aligned} (c_{p+1,2} + T_{1,1}\tilde{\gamma}_{p-1,1} - c_{p+1,1} + T_{1,2}\Gamma_{p-1,2}^1) \cap V_1 \\ = \Gamma_{p-1,1}^2 + \tilde{\gamma}_{p-1,1} = \gamma'_{p-1,1} \end{aligned}$$

where $\Gamma_{p-1,1}^2 = c_{p+1,2} \cap V_1$. However, we do have

3.3 PROPOSITION. Let $\gamma_{p-1} = \sum_{i \in I} \gamma_{p-1,i} \in H_{p-1}(\bar{M}_1)_\Delta$. Then $\tau(\gamma_{p-1}) \sim 0$ in $W - V$ if and only if

- (i) $\tau_{1,i}(\gamma_{p-1,i}) \sim 0$ in $W - (V_i \cup \bigcup_{j \notin I} V_j)$;
- (ii) if $\Gamma_{p+1,i} \subset W - \bigcup_{j \notin I} V_j$ with $\Gamma_{p+1,i} \cap V_i = \gamma_{p-1,i}$, then $\Gamma_{p+1,i} \cap V_k = \gamma_{p-1,k}$ for all $k \in I$.

PROOF. \Rightarrow Suppose $\tau_1(\gamma_{p-1}) = \sum_{i \in I} \tau_{1,i}(\tilde{\gamma}_{p-1,i}) = \partial c_{p+1} \subset W - V$. Then for $i_0 \in I$, say $i_0 = 1$,

$$\begin{aligned} \partial \left(c_{p+1} - \sum_{i>1} T_{1,i}\tilde{\gamma}_{p-1,i} \right) &= \sum_{i>1} \tau_{1,i}\tilde{\gamma}_{p-1,1} - \sum_{i>1} \tau_{1,i}\tilde{\gamma}_{p-1,i} \\ &\quad + \sum_{i>1} T_{1,i}\tau_{2,1}\gamma_{p-3,1i} \\ &= \tau_{1,1}\tilde{\gamma}_{p-1,1} + \sum_{i>1} \tau_{1,1}T_{2,i}\gamma_{p-3,1i} \\ &= \tau_{1,1}\gamma_{p-1,1} \end{aligned}$$

and $(c_{p+1} - \sum_{i>1} T_{1,i}\tilde{\gamma}_{p-1,i}) \cap V_j = \emptyset$ for $j \notin I$ since $\gamma_{p-1,i} \cap V_j = \emptyset$ implies $T_{1,i}$ can be chosen so that $T_{1,i}\gamma_{p-1,i} \cap V_j = \emptyset$.

Also, notice that by the construction

$$\Gamma_{p+1,i} = -c_{p+1} + \sum_{i>1} T_{1,i}\tilde{\gamma}_{p-1,i} + T_{1,1}\gamma_{p-1,1}$$

so that

$$\Gamma_{p+1,1} \cap V_i = +\tilde{\gamma}_{p-1,i} + T_{2,1}\gamma_{p-3,1i} = \gamma_{p-1,i}$$

\Leftarrow Conversely, if $\Gamma_{p+1,1} \cap V_i = \gamma_{p-1,i}$ for all $i \in I$, then $\gamma_{p-1} = \Gamma_{p+1,1} \cap V$ so that $\tau(\gamma_{p-1}) \sim 0$ in $W - V$. Q.E.D.

If we are given $\tilde{\tau}_1 \cdots \tilde{\tau}_{q-1} \tau_q \gamma_{p-q} \in R_p(V)$ with $\gamma_{p-q} \in H_{p-q}(\bar{M}_q)_\Delta$, then when can we guarantee $\gamma_{p-q} \in H_{p-q}(M_q)$, i.e., $\gamma_{p-q} \cap \bar{M}_{q+1} = \emptyset$? For if we had $\gamma_{p-q} \cap \bar{M}_{q+1} = \emptyset$, for all q , then this would considerably simplify many other problems involving residues, e.g., local invariant cycle problem (Griffiths [7, p. 249]).

There are facts which indicate this might be true i.e., V ample in W implies $R_p(V)$ is generated by $\gamma_{p-q} \in H_{p-q}(M_q) \subset H_{p-q}(\bar{M}_q)_\Delta$.

For example, if V is ample and nonsingular, we have the commutative diagram:

$$\begin{array}{ccccc}
 H_{p+1}(W) & \xrightarrow{I} & H_{p-1}(V) & \xrightarrow{\tau} & H_p(W - V) \\
 & \searrow L_* & \downarrow i_* & & \\
 & & H_{p-1}(W) & &
 \end{array}$$

where L_* is dual to the Hodge map, i.e., $L_*(\gamma) = \gamma \cap [V]^*$ with $[V]^*$ the dual two-dimensional cohomology class of $[V] \in H_{2n-2}(W)$ and “ \cap ” being cap product. If V is ample, L_* is surjective for $p \leq n$ (see Weil [26]), and by the Lefschetz theorem i_* is an isomorphism for $p < n$ and surjective for $p = n$.

Thus for $p < n$, I is surjective, hence $R_p(V) = 0$. For $p = n$, since $H_{n-1}(W) \simeq H_{n+1}(W)$ by Poincaré duality (recall we have coefficients in \mathbb{C}), L_* is an isomorphism, hence $\tau(\gamma_{n-1}) \neq 0$ iff $\gamma_{n-1} \notin \text{image } I$ iff $i_*\gamma_{n-1} = 0$.

3.4 PROPOSITION. $i_*\gamma_{n-1} = 0$ implies

$$\gamma_{n-1} \in \text{image } \{H_{n-1}(V - V \cap V') \rightarrow H_{n-1}(V)\}$$

for any V' which is a nonsingular hypersurface of W such that $V' \cap V$ is nonsingular in V .

PROOF. If $\gamma_{n-1} \notin \text{image } \{H_{n-1}(V - V \cap V') \rightarrow H_{n-1}(V)\}$, then $0 \neq \gamma_{n-1} \cap V' = \gamma_{n-3} \in H_{n-3}(V \cap V')$ since $V \cap V'$ is nonsingular. But V ample in $W \Rightarrow V \cap V'$ obeys the Lefschetz theorem in V' , i.e., $\gamma_{n-3} \neq 0$ in V' since $n - 3 < \dim_{\mathbb{C}} V \cap V'$. But $i_*\gamma_{n-1} = 0$ in W , hence $(i_*\gamma_{n-1}) \cap V' = 0$ in V' , but γ_{n-3} is a representative of their intersection, contradiction. Q.E.D.

3.5 COROLLARY. $i_*\gamma_{n-1} = 0$ implies

$$\gamma_{n-1} \in \text{Im } \left\{ H_{p-1} \left(V - \bigcup_i (V_i \cap V) \right) \rightarrow H_{p-1}(V) \right\}$$

for V_i nonsingular in W and $\{V_i \cap V\}_i$ having normal crossings in V .

PROOF. If γ_{n-1} misses V_i and V_j , then γ_{n-1} misses $V_i \cap V_j$ and hence $V_i \cup V_j$. Q.E.D.

So if V is ample in W , let (W', V', π) be a resolution to normal crossings with $V' = \bigcup_{i=0}^m V_i$. Let V_0 be the proper transformation, i.e., $\pi(V_0) = V$ and $\pi(V_i) \subset S$, singular locus of V , for $i > 0$.

So Corollary 3.5 gives indications it might be true if say V_0 were ample. However, V_0 is in general not ample. If one has enough conditions on the intersection of $V_i \cap V_j$, e.g., $V_i - V_i \cap V_0$ is Stein for all i , or if the injection of $V_0 \cap V_i$ into V_i induces an injection on homology for dimensions up to $n - 2$, then one can prove the result. The difficulty is that one cannot guarantee these extra conditions in general for arbitrary resolutions.

Also, from purely topological techniques, we can show if V has isolated

singularities without any other conditions on V , then $R_p(V)$ is generated by $\gamma_{p-q} \in H_{p-q}(M_q)$ (see Corollary 4.15 below). There are also strong indications that if singular locus of V is a curve, then one can prove the result for ample V . Moreover, Corollary 5.12 (below) says that if ω is a $C^\infty(p, 0)$ form with a pole of order one on V , i.e., if f is a local defining equation of V , then $f\omega$ is smooth, then the dual cycle to ω is of the desired type. As we have mentioned above, $W - V$ is generated by meromorphic p -forms with poles of order k on V . But in general, we need $k > 1$, and when one reduces ω , a pole of order k , to a form with pole of order one (i.e., $\omega = \sum_{q=1}^k \omega_q$ where ω_q is a $(p - q - 1, q - 1)$ -form with a pole of order one (see beginning of §5 for definitions), then the technique of proof breaks down in the general case. So we have

3.6 QUESTION. For V ample in W and V' a resolution to normal crossings, is $R_p(V')$ generated by $\gamma_{p-q} \in H_{p-q}(M_q)$?

4. Residues in the general case. In the previous section we have classified the residues in case the hypersurface has normal crossings, which is the case when one resolves singularities. However, one cannot in general resolve singularities for arbitrary complex manifolds, and even if one could, for technical reasons it might not be the best approach to the problem at hand. As a result, we now will look at the residues of arbitrary hypersurfaces without resolving singularities. The reason we did §2 is that, first of all, it will be necessary to consider the normal crossings in great detail in [5], and secondly, it provides a good example to have in mind in this section.

In this section W will be an arbitrary n -complex dim manifold and V an arbitrary subvariety.

4.1 DEFINITION. A stratification of V is the expression of V as the disjoint union of a locally finite set of complex manifolds, call the *strata*, such that the boundary of each stratum is the union of a set of lower-dimensional strata. (See Whitney [27, esp. pp. 227–230], for a discussion of stratifications.)

From the definition it easily follows that the closure of each stratum is an analytic variety of the same dimension of the stratum and the boundary of a stratum is an analytic variety of lower dimension.

4.2 DEFINITION. The stratification $\{M'_i\}$ of V is a refinement of the stratification $\{M_j\}$ of V if for each stratum M'_i , either M'_i is a stratum of $\{M_j\}$ or $M'_i \subset M_j$ such that $\partial M'_i \cap \bar{M}_j \subseteq \partial M'_i$ as subsets of the stratification.

Note. $\partial M'_i \cap \bar{M}_j$ makes sense as a subset of $\{M'_i\}$ since each stratum of $\partial M'_i$ is either a stratum of $\{M_j\}$ or is contained in some stratum, which must either be a stratum of the boundary of M_j or M_j itself.

4.3 LEMMA. Any two stratifications of V have a common refinement.

PROOF. Let $\{M_i\}$ and $\{M_j'\}$ be two stratifications of V . Then $M_i \cap M_j'$ is a complex subvariety of V , since each M_i is locally given by the zeros of analytic functions. Thus we can stratify $M_i \cap M_j'$ by taking successive nonsingular points, i.e., for any variety V with singular locus S , $\dim_{\mathbb{C}} S < \dim_{\mathbb{C}} V$, so that this process terminates in stratification. Then let $\{M_i''\}$ be the stratification given by the stratification of $M_i \cap M_j'$.

It is clear that $\partial(M_i \cap M_j') \cap \bar{M}_i \subseteq \partial(M_i \cap M_j')$. Q.E.D.

For each stratum M_i , say $\dim_{\mathbb{C}} M_i = d$, let $T_0(M_i, P)$, for $P \in M_i$, denote the $(2d - 1)$ direction of its tangent space, i.e., $T_0(M_i, P) \in G(2n - 1, 2d - 1)$, where $G(2n - 1, 2d - 1)$ is the Grassmannian of $(2d - 1)$ -planes in $(2n - 1)$ -space. Then take all pairs $(P, T_0(M_i, P)) \in W \times G(2n - 1, 2d - 1)$ and let $\tau(M_i)$ be the closure of this set in $W \times G(2n - 1, 2d - 1)$. Then for $P \in \bar{M}_i$, we define $\tau(M_i, P) = \{\alpha \mid (P, \alpha) \in \tau(M_i)\}$. For $P \in M_i$, $\tau(M_i, P) = T_0(M_i, P)$.

4.4 DEFINITION. We say $\{M_i\}$ is a *Whitney stratification* A of V if $P \in M_i \subset \bar{M}_j$ and $T(M_i, P)$ is the tangent plane of P in M_i , then $T(M_i, P) \subset T$ for all $T \in T(M_j, P)$, i.e., if $v \in T(M_i, P)$, then $\exists P_m \in M_j$ and $v_m \in T(M_j, P_m)$ such that $v_m \rightarrow v$.

Whitney [27] has shown

4.5 THEOREM. *Any stratification has a refinement to a Whitney stratification A .*

As a corollary we have

4.6 COROLLARY. *$V \subset W$ may be stratified so that the following is true: If $P_0 \in M$, a stratum of dimension d , and if M_0 is a neighborhood of P_0 in M such that through each point $P \in M_0$ there is a $2(n - d)$ -dimensional plane $T(P)$ transverse to M at P with the $T(P)$ continuous in P and disjoint, then there exists a neighborhood U of P_0 in W such that for all $P \in U \cap M_0$, $T(P)$ is transverse to $M' \cap U$ for all strata M' .*

PROOF. Since this is a local result, we consider M_0 contained in some coordinate chart in W and then a plane means a plane in \mathbb{C}^n .

Suppose there does not exist such a neighborhood U . Then there exists $P_m \in M_0$ such that $T(P_m)$ meets some stratum M' nontransversely at Q_m and $Q_m \rightarrow P_0$ as $m \rightarrow \infty$.

Since the stratification is locally finite, we can assume all the Q_m belong to the same strata M' . Then we have at Q_m , $\dim(T(P_m) + T(M', Q_m)) < 2n$. But as $m \rightarrow \infty$, $T(P_m) \rightarrow T(P_0)$ since $T(P_m)$ are continuous in P_m ; also $Q_m \rightarrow P_0$ which implies by passing to a subsequence if necessary, that we have $T(M', Q_m) \rightarrow T \in T(M', P_0)$. By the choice of the stratification of Theorem 4.5, $T(M, P_0)$

$\subset T$. However, by continuity at P_0 , $\dim(T(P_0) + T) < 2n$, contradiction of the fact that $T(P_0)$ meets M transversely at P_0 . Q.E.D.

Let $M \subset \bar{M}'$ where M and M' are strata. By using the corollary we construct the tubular neighborhood of M in M' :

Let M be a stratum of complex dimension d . Then let $T(M)$ be a tubular neighborhood of M in W and $\pi: T(M) \rightarrow M$ be the projection. For $P \in M$, $\pi^{-1}(P)$ is a $2(n - d)$ -dim ball transverse to M at P and $\pi^{-1}(P)$ depend continuously on P and they are disjoint. Therefore there exists U , a neighborhood of P in W , such that $\forall Q \in U \cap M$, $\pi^{-1}(Q)$ meets $U \cap M'$ transversely. Then shrink the fibres of $T(M)$ such that $\forall Q \in U \cap M$, $\pi^{-1}(Q) \subset U$ and $\pi^{-1}(Q)$ is a $2(n - d)$ -dim disk.

Cover M by a locally finite cover $\{U_\alpha\}$ of such sets. Then patch together the new fibres of $T(M)$ via partition of unity to get a new tubular neighborhood of M in W such that, if $\pi: T(M) \rightarrow M$ is the projection, then for $Q \in M$, $\pi^{-1}(Q)$ is a $2(n - d)$ -dim ball which intersects M' transversely, for all M' .

We pick the fibres of each stratum such that they respect the fibres of the previous strata in the following sense:

Suppose M_d is one of the lowest-dimensional strata and its complex dimension is d . We form $\pi: T(M_d) \rightarrow M_d$ with all $P \in M_d$, $\pi^{-1}(P)$ is a $2(n - d)$ -dim ball and if $M_d \subset M_{d+j}$ ($\dim_{\mathbb{C}} M_{d+j} = d + j$), then $\pi^{-1}(P) \cap M_{d+j}$ is a $2j$ -dim chain. Furthermore $\partial(\pi^{-1}(P)) \cap M_{d+j}$ is a transverse intersection of a $2(d + j)$ -dimension manifold with an arbitrary small $2(n - d) - 1$ -dimension sphere, say $\tau(P)$, in a $2n$ -dimensional space, i.e., $2d + 2j + 2n - 2d - 1 - 2n = 2j - 1$ -dim cycle in M_{d+j} . As P varies in a C^∞ fashion in M_d , the $2n - 2d - 1$ -dimensional sphere varies C^∞ with each sphere disjoint.

In fact $\partial(\pi^{-1}(P)) \cap M_{d+j}$ is a $(2j - 1)$ -dimensional C^∞ manifold, since the intersection is transverse. Thus, $\pi^{-1}(P) \cap M_{d+j}$ is a cone over a $(2j - 1)$ -dimensional manifold. So, for $P \in M_d$, a neighborhood U of P in M_d can be chosen so that $\pi^{-1}(U) \cap M_{d+j}$ is the product of $U \times$ (cone over a manifold) and each of the fibres will be homeomorphic. We have a $2(n - d - j)$ normal direction in $\tau(P)$ to $M_{d+j} \cap \tau(P)$ and we pick this direction for the normal bundle of M_{d+j} in W at $M_{d+j} \cap \tau(P)$. Since $\tau(P)$ varies in a C^∞ way with P , we can pick these directions in a C^∞ way. Then extend this to all of $\tilde{M}_{d+j} = M_{d+j} - \text{int } T(M_d)$, where $\text{int } T(M_d) = \text{interior in } W \text{ of } T(M_d)$. Thus, we have constructed $\pi: T(\tilde{M}_{d+j}) \rightarrow \tilde{M}_{d+j}$.

For all $P \in \tilde{M}_{d+j} \cap \tau(M_d)$, where $\tau(M_d) = \partial(T(M_d))$, we have that $\pi^{-1}(P) \subset \tau(M_d)$ and since $T(M_d)$ meets all the strata transversely, we can make sure that $T(\tilde{M}_{d+j})$ also has this property.

Now, suppose all $\pi: T(\tilde{M}_k) \rightarrow \tilde{M}_k$ have been constructed for all dimensions

less than d . Then for $P \in \tilde{M}_{d-k} \subset M_d$ we have $\pi^{-1}(P) = 2(n - d - k)$ -dimensional ball which meets M_d transversely and $[\tau(P) = (\partial\pi^{-1}(P))] \cap M_d$ is the transverse intersection of a $2(n - d + k) - 1$ sphere with a $2d$ -dim manifold. We have chosen the smallest k possible. There is a $2(n - d)$ normal direction in $\tau(P)$ to this intersection which varies in a C^∞ fashion with P until P meets $\partial\tilde{M}_{d-k}$ and then we have $P \in \tau(\tilde{M}_{d-k-j}) \cap M_{d-k}$. We then move to $\pi^{-1}(Q) = 2(n - d + k + j)$ -dim ball for $Q \in \tilde{M}_{d-k-j}$. But since $P \in \tau(\tilde{M}_{d-k-j})$, we choose the fibre of $T(\tilde{M}_{d-k})$ to lie in $\tau(Q) = \partial\pi^{-1}(Q)$, a $2(n - d + k + j) - 1$ -dim sphere; we can continue in a C^∞ fashion, i.e., $\tau(Q)$ intersects M_d in a $(2k + 2j - 1)$ -dim submanifold and hence the intersection has a $2(n - d)$ -dim normal direction in $\tau(Q)$ and the point $P \in \partial\tilde{M}_{d-k}$ also lies in $\tau(Q)$ and we already chose a $2(n - d)$ -dim normal direction near P , so we just continue along $M_d \cap \tau(Q)$.

We then extend this to all of $\tilde{M}_d = M_d - \bigcup_{k \geq 1} \text{int } T(\tilde{M}_{d-k})$ in a C^∞ fashion yielding a normal bundle $\pi: T(\tilde{M}_d) \rightarrow \tilde{M}_d$ such that each fibre meets each stratum transversely.

Thus we have constructed a normal bundle of each \tilde{M}_d such that the fibres respect those of $\tilde{M}_{d-j} \subset \tilde{M}_d$.

Let us introduce some notation: If $T(M_j)$ is the normal disk bundle of M_j in W , let $\tau(M_j) = (\partial T)(M_j)$ denote the induced normal sphere bundle, i.e., fibres will be $(2n - 2j - 1)$ -dimensional spheres. Then denote by $T(\tilde{M}_d)$ the "bundle" we have just constructed over \tilde{M}_d and let $\tau(\tilde{M}_d) = \partial(T\tilde{M}_d)$.

Then we can perform the above construction in such a way that $\tau(\tilde{M}_d)$ is a smooth $(2n - 1)$ -dim submanifold of W and also that $\partial\tilde{M}_d$ is a smooth $(2d - 1)$ -dim submanifold of M_d .

First of all, $\tau(\tilde{M}_d)$ is a PL-manifold (i.e., underlying space is a topological manifold, but the induced differential structure has corners). To see this note that $T(\tilde{M}_d)$ has as boundary $\tau(\tilde{M}_d) + T(\partial\tilde{M}_d)$ and $T(\partial\tilde{M}_d) \subset T(\tilde{M}_{d-j})$ as a submanifold. What we have done is glued $T(\tilde{M}_d)$ to $T(\tilde{M}_{d-j})$ along $T(\partial\tilde{M}_d)$ (this is essentially what the topologist refer to as plumbing; see Figure 1 where the shaded region is $T(\partial\tilde{M}_d)$). This will have a corner along $\tau(\partial\tilde{M}_d)$. However by standard arguments involving plumbing, see e.g. Milnor [16], we can smooth these corners so that this is a differentiable manifold.

To see this smoothing intuitively, suppose we have $P \in M_0$, i.e., a zero-dim strata and then $S_p^{2n-1} \cap M_d$ is a $(2d - 1)$ -dimensional submanifold. Since this is purely a local problem (i.e., if we can smooth in a small neighborhood, then we can patch together via a partition of unity to get a global smoothing), choose a small neighborhood R^{2d-1} of $S_p^{2n-1} \cap M_d$. Then we have that $\partial R_+^{2d} = R^{2d-1}$ where $R_+^{2d} (\subset M_d)$ is the half-plane of R^{2d} , i.e., $S_p^{2n-1} \cap M_d$ is the boundary of \tilde{M}_d , submanifold of M_d . Let D^{2n-2d} be the normal plane to M_d (i.e., $R_+^{2d} \times$

D^{2n-2d} is the normal bundle) such that $R^{2d-1} \times D^{2n-2d} \subset S_p^{2n-1}$. Then

$$\partial(R_+^{2d} \times D^{2n-2d}) = R^{2d-1} \times D^{2n-2d} + R_+^{2d} \times S^{2n-2d-1}$$

and

$$\partial(R^{2d-1} \times D^{2n-2d}) = -R^{2d-1} \times S^{2n-2d-1}.$$

So on S_p^{2n-1} we cut out $R^{2n-1} \times D^{2n-2d}$ and attach $R_+^{2d} \times D^{2n-2d}$ along its boundary. This has a corner at $R^{2d-1} \times S^{2n-2d-1}$, i.e., if $Q \in R^{2d-1} \times S^{2n-2d-1}$, then a neighborhood of Q in this $(2n - 1)$ -dimensional PL-manifold is of the form $(R^{2d-1} \times S^{2n-2d-1}) \times (R_+ \cup R_+)$ where $R_+ \cup R_+ \subset R \times R =$ normal plane of $R^{2d-1} \times S^{2n-2d-1}$ in W . Then we just smooth out this corner in the normal plane.

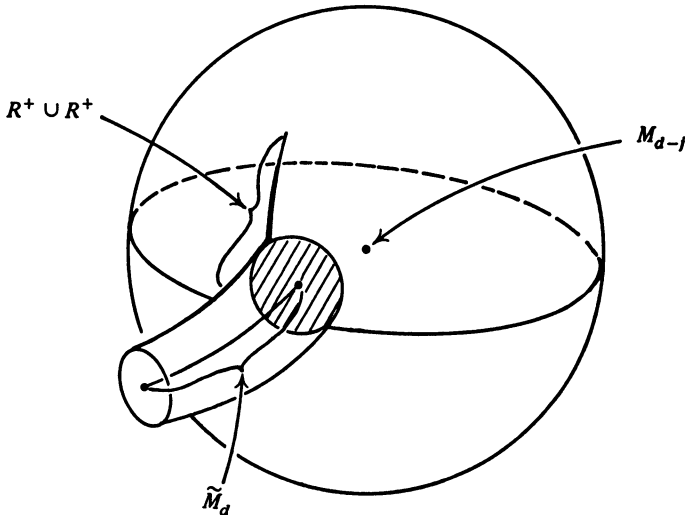


FIGURE 1

4.7 LEMMA. $T(\bar{M}_d)$ is the cone over $\tau(\bar{M}_d)$ with vertex \bar{M}_d , i.e., \bar{M}_d is a strong deformation retract of $T(\bar{M}_d)$ and if $\{f_t\}$ gives the deformation (with f_0 the identity), then $f_t|_{\tau(\bar{M}_d)}$ is a diffeomorphism for $t < 1$ and $f_t(\tau(\bar{M}_d)) \cap f_{t'}(\tau(\bar{M}_d)) = \emptyset$ for $t \neq t'$. Furthermore for any stratum M_{d+j} with $M_d \subset \bar{M}_{d+j}$, $f_t|M_{d+j} \cap T(\bar{M}_d) \subset M_{d+j}$ for all t . For $Q \in M_j$, $f_1^{-1}(Q)$ is a $2(n - 2j)$ -dimensional chain.

PROOF. The deformation is given by shrinking along the fibres of the normal bundles of \tilde{M}_{d+k} . The plumbing causes no difficulty because we are glueing together along normal bundles.

To see that $f_1^{-1}(Q)$ is a $(2n - 2j)$ -dim chain $Q \in M_j$, let M_k be the lowest-dimensional stratum of \bar{M}_d . Then it is clear that for $Q \in M_k$ that $f_1^{-1}(Q)$ is a $2(n - k)$ -dim ball since $T(\bar{M}_d)|_{M_k} = T(M_k)$, M_k being a complex k -dim manifold without boundary.

Suppose $Q \in M_{k+j} \cap T(M_k)$. Choose a Riemann metric in $T(M_d)$ so that the fibres are balls of radius one. Then suppose $Q \in \pi^{-1}(P)$, $P \in M_k$, and Q is of distance t , $0 < t \leq 1$, from P . Choose a collared neighborhood $\partial\tilde{M}_{k+j} \times I$ of $\partial\tilde{M}_{k+j}$ in \tilde{M}_{k+j} . Let $Q' \in \tau(\bar{M}_k) \cap M_{k+j} \subset \partial\tilde{M}_{k+j}$ be that unique point which passes through the given Q when we shrink $T(\bar{M}_k)$ along its fibres via the Riemannian metric. Then the collapsing of $T(\tilde{M}_{k+j})|(\partial\tilde{M}_{k+j} \times I)$ will send the fibre over $(Q', 2t) \in \partial\tilde{M}_{k+j} \times I$ onto Q and will send the fibre over (Q', t) for $\frac{1}{2} \leq t \leq 1$ onto $(Q', 2t - 1)$. The collapsing of $T(\tilde{M}_{k+j})|(\tilde{M}_{k+j} - (\partial\tilde{M}_{k+j} \times I))$ will be along the fibres via a Riemannian metric in $T(M_{k+j})$. Q.E.D.

Of course the construction of $T(\bar{M}_d)$ and the deformation retract depend on the choice of various normal bundles of the M_j . However, we have

4.8 LEMMA. $T(\bar{M}_d)$ is isotopic to $T'(\bar{M}_d)$ for any other choice of normal bundles $T'(\tilde{M}_j)$. Furthermore, if $\{g_t\}$ is the isotopy, then $g_1|f_1^{-1}(Q) = (f'_1)^{-1}(Q)$ where f_1, f'_1 are from Lemma 4.7 and $g_1(\bar{M}_d) = \bar{M}_d$.

Note. Recall two submanifolds $M_1, M_2 \subset W$ are isotopic if there is a smooth map $G: M_1 \times I \rightarrow W$ such that $g_t = G(\cdot, t)$ is a diffeomorphism for all t and g_0 is the identity map, $g_1(M_1) = M_2$.

PROOF. The isotopy follows from the uniqueness (up to isotopy) of normal bundles, e.g. if M_k is the smallest dimensional stratum in M_d , then $T(M_k)$ is isotopic via $\{g_t\}$ to $T'(M_k)$. Let M_{k+j} be the smallest stratum M_d which has M_k as part of its boundary and set

$$\tilde{M}_{k+j} = M_{k+j} - T(M_k), \quad \tilde{M}'_{k+j} = M_{k+j} - T'(M_k).$$

If we take $T(\partial\tilde{M}_{k+j})$ in $\tau'(M_k)$, then $g_1|T(\partial\tilde{M}_{k+j})$ is a normal bundle to $\partial\tilde{M}'_{k+j}$ in $\tau'(M_k)$, hence is isotopic via $\{h_t\}$ to $T'(\partial\tilde{M}'_{k+j})$ in $\tau'(M_k)$.

But $\{g_t\}$ extends to an isotopy between \tilde{M}_{k+j} and \tilde{M}'_{k+j} , because we have an isotopy between two boundaries of two manifolds which are diffeomorphic, so by taking a collared neighborhood $(\partial\tilde{M}_{k+j}) \times I$ of the boundaries we can extend the isotopies such that $g_t|\partial\tilde{M}_{k+j} \times \{1\} = g_0|\partial\tilde{M}_{k+j} \times \{1\} = \partial\tilde{M}'_{k+j} \times \{1\}$ and extend to the rest of the manifold by the diffeomorphism g_0 . Hence we can extend $\{g_t\}$ to an isotopy of $T(\tilde{M}_{k+j})$ such that $g_1|T(\tilde{M}_{k+j})$ is a normal bundle to \tilde{M}'_{k+j} . Then $g_1|T(\tilde{M}_{k+j})$ is isotopic via $\{h_t\}$ to $T'(M_{k+j})$ in such a way that it extends the above $h_t|g_1T(\partial\tilde{M}_{k+j})$, because if two normal bundles

have the same fibre, then the isotopy can be chosen so that on this fibre it is the identity map at each t .

Then we continue in this fashion to get the isotopy between $T(M_d)$ and $T'(M_d)$. But $g_t|_{\bar{M}_d}$ is not the identity, only a diffeomorphism for all t . Actually, it is more: $g_t|_{\bar{M}_{k+j}}$ is an isotopy on each stratum. Furthermore, from the above construction it is clear that $g_1|_{f_1^{-1}(Q)} = (f'_1)^{-1}(Q)$ for all Q . Q.E.D.

Let V be any variety in a complex manifold W , and $\{M_i\}$ a Whitney stratification A of V . Then we can construct the tubular neighborhoods of M_i in W and denote it by $T(V; M_i)$, which we call the *tubular neighborhood of V in W relative to $\{M_i\}$* . If $\pi: T(V; M_i) \rightarrow V$ is the retraction of Lemma 4.7, then $\pi^{-1}(Q)$ is the *fibre over Q* .

4.9 THEOREM. *Let V be a subvariety of W and $\{M_i\}$ and $\{M'_i\}$ Whitney stratifications A of V . Then $T(V; M_i)$ is isotopic via $\{g_t\}$ to $T(V; M'_i)$ and $g_t(V) = V$ for all t .*

PROOF. It suffices to prove 4.9 for $\{M'_i\}$ a refinement of $\{M_i\}$ because of Lemma 4.3 and Theorem 4.5. But once we have a refinement, the result will follow from the tubular neighborhood theorem for manifolds with boundary: if N is a submanifold of W such that $\partial N \cap M \subset \partial M$ with N transverse to ∂M or if $\partial N \subset M - \partial M$, then N has a tubular neighborhood in M which is unique up to isotopy (see Milnor [15, p. 57]). So the same construction of the isotopy in Lemma 4.8 will go through with the obvious modifications, because any stratum $M'_i \subset M_i$, then either $\partial M'_i \subset M_i$ or if $\partial M'_i$ contains a stratum M'_j which is not a stratum of ∂M_i , then \bar{M}'_i meets ∂M_i transversely at M'_j . Q.E.D.

Let S be any subvariety of a variety V in some manifold W . Then let $\{M_i\}$ be a Whitney stratification A of V in W relative to S , i.e., $M_i \cap S \neq \emptyset$ implies $M_i \subset S$. Such a stratification clearly exists and will also be a Whitney stratification A of S in W . Then we can form $T(S; M_i)$ and put $T(S; M_i) \cap V = T(S, V; M_i)$ which will be called the *tubular neighborhood of S in V relative to M_i* . Thus we have shown

4.10 COROLLARY. *Let $\{M_i\}$ and $\{M'_i\}$ be Whitney stratifications A of V in W relative to S . Then $T(S, V; M_i)$ is homotopic to $T(S, V; M'_i)$ via a family of homeomorphisms which will come from the restrictions of diffeomorphisms. If both $\tau(S, V; M_i)$ are contained in the singular part of V , then they will be isotopic.*

4.11 DEFINITION. Let S be a subvariety of a variety $V \subset W$, a manifold. Then $T(S, V)$ will be called the *tubular neighborhood of S in V (relative to W)*, where $T(S, V)$ is constructed for any stratification of V relative to S . If $\gamma_p \in H_p(\tau(S, V))$, where $\tau(S, V) = \partial T(S, V) = (\partial T(S)) \cap V = \tau(S) \cap V$, then γ_p will be called a *tube over a cycle in S* .

4.12 THEOREM. *Let W be an n -complex dimensional manifold and V a subvariety of codimension q , i.e. $\dim_{\mathbb{C}} V = n - q$. Then $\ker \{H_p(W - V) \rightarrow H_p(W)\} =$ geometric p -residues of $V = R_p(V)$ is generated by tubes over cycles in V .*

PROOF. Let $\gamma_p \in H_p(W - V)$ and $\gamma_p = \partial c_{p+1}$. Choose a Whitney stratification A of V in W and construct $T(V, W)$. Then since M_i are manifolds and we are working in a manifold W , we can choose c_{p+1} such that it intersects each stratum of V transversely. That is c_{p+1} respects the fibres of V , i.e., if $\gamma_{p-2q+1} = c_{p+1} \cap V$ (i.e. V is a codimension q , hence has a dual cohomology class of dimension $2q$ and the cap product of c_{p+1} with V will then define a $p - 2q + 1$ -dim chain on V , but $\partial c_{p+1} \cap V = \emptyset$, so that this $p - 2q + 1$ -dim chain, γ_{p-2q+1} , is actually a cycle on V), then $T\gamma_{p-2q+1} \subset c_{p+1}$, where $T\gamma_{p-2q+1} = c_{p+1} \cap T(V)$.

Thus $\partial(c_{p+1} - T\gamma_{p-2q+1}) = \gamma_p - \tau(\gamma_{p-2q+1})$, where $\tau(\gamma_{p-2q+1}) = c_{p+1} \cap \tau(V)$. Q.E.D.

Note that a stratification of V can be chosen so that the M_{n-q} strata ($\dim_{\mathbb{C}} M_i = i$) are the connected components of the nonsingular part of V . Also if $\tilde{\gamma}_i = \tilde{M}_i \cap c_{p+1}$ where $\tilde{M}_i = M_i - \sum_{j>1} T(\tilde{M}_{i-j})$ and $\tilde{\gamma}_i \in H_{p+1+2i-2n}(\tilde{M}_i, \partial\tilde{M}_i)$, then $\gamma_p = \sum_{i=[(2n-p-1)/2]}^{n-q} \tau(\tilde{\gamma}_i)$.

Note. The essential criterion to construct the tubular neighborhood in Theorem 4.12 was Theorem 4.5. Thus, if V is any simplicial complex contained in a real smooth manifold W such that V has a Whitney stratification A , then Theorem 4.12 is also true for V with respect to W . In particular, any locally finite polyhedron in W will have a regular neighborhood which will satisfy the Whitney stratification A . Also any real subanalytic set V will satisfy the Whitney stratification A . We say V is subanalytic at P if there exists an open set $U \ni P$ and proper real analytic maps $f_i: X_i \rightarrow W|U$, $g_i: Y_i \rightarrow W|U$ for X_i, Y_i and W real analytic spaces such that $V \cap U = \bigcup_i \{\text{Im}(f_i) - \text{Im}(g_i)\}$. Any real semianalytic set is subanalytic (Hironaka [10]).

Given Theorem 4.12, the obvious question is: What happens if $\gamma_{p-2q+1} \sim 0$ in V ? I.e., Corollary 2.8 says that if $\gamma_{p-1} \sim 0$, then $0 \neq [\gamma_{p-2q+1}] \in H_{p-1}(\tau(S, V))$ where S is the singular locus of V . So if $\gamma_{p-1} \sim 0$ in V , does this imply $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(\tau(S, V))$? Instead of this, we have

4.13 COROLLARY. $[\gamma_p] \in R_p(V)$ with $\gamma_p = \tau(\gamma_{p-2q+1})$ such that $\gamma_{p-2q+1} \sim 0$ in V , then $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(\tau(S', V))$ for S' some subvariety of S , the singular locus of V .

PROOF. If $\gamma_{p-2q+1} \cap S = \emptyset$, i.e.,

$$\gamma_{p-2q+1} \in \text{image} \{H_{p-2q+1}(V - \tau(S, V)) \rightarrow H_{p-2q+1}(V)\}$$

and $\gamma_{p-2q+1} = \partial c_{p-2q+2}$, then

$$c_{p-2q+2} \cap \tau(S, V) = \gamma'_{p-2q+1} \in H_{p-2q+1}(\tau(S, V))$$

and $\tau(\gamma_{p-2q+1}) \sim \tau(\gamma'_{p-2q+1})$ in $W - V$.

Suppose $\gamma_{p-2q+1} \cap S \neq \emptyset$, i.e., $\gamma_{p-2q+1} \cap M_i = \gamma_{p+2i+1-2n} \in H_{p+2i+1-2n}(M_i)$, then $\gamma_{p+2i+1-2n} \sim 0$ in M_i with homology taken with closed support, i.e.,

$$\gamma_{p+2i+1-2n} = (\partial c_{p-2q+2}) \cap M_i = \partial(c_{p-2q+2} \cap M_i) \pm c_{p-2q+2} \cap \partial M_i.$$

Let M_d be the lowest-dimensional stratum with which γ_{p-2q+1} has nonzero intersection. Then if $c_{p-2q+2} \cap \partial M_d = \emptyset$, we can construct $\tau(c_{p-2q+1}) \subset W - V$ such that $\partial(\tau(c_{p-2q+2})) = -\tau(\gamma_{p-2q+1})$.

Thus suppose $c_{p-2q+1} \cap \partial M_d \neq \emptyset$. Set $S' = \partial M_d$, a subvariety of S , i.e. $S' = \bar{M}_d - M_d$, and $\gamma_{p-2q+1} \cap M_d$ will be a $(p+1+2d-2n)$ -cycle of $\tau(S', M_d)$. If we set $\gamma_{p-2q+1, T} = \gamma_{p-2q+1} \cap T(S)$, then $\gamma_{p-2q+1, T}$ is a $(p-2q+1)$ -dimensional chain of $\tau(S', V)$ because $\gamma_{p-2q+1} \cap M_d \subset \tau(S', M_d)$ and $\gamma_{p-2q+1, T}$ has transverse intersection with each stratum; hence in the construction of $\tau(S', V)$, the fibres can be chosen to respect the transverse intersection of $\gamma_{p-2q+1, T}$.

Let $\gamma_{p-2q} = \gamma_{p-2q+1, T} \cap \tau(S, V)$. Then

$$\gamma_{p-2q} = (\partial c_{p-2q+2}) \cap \tau(S, V) = \partial(c_{p-2q+2} \cap \tau(S, V)).$$

Also

$$c_{p-2q+2} \cap \tau(S, V) = c_{p-2q+1, \tau} \subset \tau(S', V)$$

because $c_{p-2q+2} \cap \bar{M}_d$ is the cone over $\gamma_{p-2q+1} \cap M_d$ in \bar{M}_d relative to ∂M_d . Thus

$$\gamma'_{p-2q+1} = \gamma_{p-2q+1, T} - c_{p-2q+1, \tau} \in H_{p-2q+1}(\tau(S', V)).$$

Also note that $\tau(\gamma'_{p-2q+1}) \subset W - V$, since $\gamma_{p-2q+1, T} = \gamma_{p+1} \cap T(S, V)$ by transverse intersection and $c_{p-2q+1, \tau} \subset V - S$. Furthermore $\tau(\gamma'_{p-2q+1}) \sim \tau(\gamma_{p-2q+1})$ in $W - V$ because

$$\partial(c_{p-2q+2} - c_{p-2q+2} \cap T(S, W)) = \gamma_{p-2q+1} - \gamma_{p-2q+1, T} + c_{p-2q+1, \tau}.$$

Furthermore, $0 \neq \gamma'_{p-2q+1} \in H_{p-2q+1}(\tau(S', V))$. This is because if $\gamma'_{p-2q+1} = \partial c'_{p-2q+2}$ for $c'_{p-2q+2} \subset \tau(S', V)$, then $\tau(c'_{p-2q+2}) \subset W - V$, i.e., the only obstruction to constructing the tube over c'_{p-2q+2} lies in S' , as

$$\gamma'_{p-2q+1} \cap M_d = (\partial c'_{p-2q+2}) \cap M_d = \partial(c'_{p-2q+2} \cap M_d) \pm c'_{p-2q+2} \cap \partial M_d,$$

but $c'_{p-2q+2} \cap \partial M_d = c'_{p-2q+2} \cap S' = \emptyset$. Thus we have $\partial(\tau(c'_{p-2q+2})) = -\tau(\gamma'_{p-2q+1})$ in $W - V$. Q.E.D.

4.14 COROLLARY. *If $\dim S = 0$, then $\gamma_p = \tau(\gamma_{p-2q+1})$ implies either $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(V - \tau(S, V))$ or $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(\tau(S, V))$.*

4.15 COROLLARY. *If the dimension of the singular locus of V is zero and (W', V', π) is a resolution of V to normal crossings with V_0 being the proper transform, then $R_p(V')$ is generated by cycles of the form $\tau_1(\gamma_{p-1,0})$ for $\gamma_{p-1,0} \subset V_0 - \bar{M}_2$.*

PROOF. Let U_i be small neighborhoods in W of the isolated points P_i and set $U = \bigcup_i U_i$. Put $U' = \pi^{-1}(U)$. Then $\pi|_{W' - U'} \rightarrow W - U$ is a biholomorphic homeomorphism because when we resolve singularities, we never change anything outside U .

But $0 \rightarrow R_p(V') \xrightarrow{\pi_*} R_p(V)$ and $R_p(V)$ is generated by $\tau(\gamma_{p-2q+1})$ for $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(V - V \cap U)$ or $[\gamma_{p-2q+1}] \in H_{p-2q+1}(\partial U)$ and ∂U is unchanged. Q.E.D. for Corollary 4.15.

Note. Recall by Proposition 3.1, $R_p(V) \simeq R_p(V')$, so that if $V \subset W$ can be blown down to a hypersurface with isolated singularities, we have the results discussed in §3.

Note. Just because $\gamma_{p-1} \sim 0$ in V does not imply $\gamma_{p-1} \sim 0$ in V_0 , e.g., let $V = \{x^3 + y^3 + z^3 = 0\}$ in $\mathbb{C}P_3$ with homogeneous coordinates $[x, y, z, w]$. V is the cone over the elliptic curve $C \subset \mathbb{C}P_2$. Let ν be the normal bundle of C in $\mathbb{C}P_2$, and ν^* the projectification of ν , i.e., replace each fibre C' by $\mathbb{C}P_1$ by adding a point at infinity. If C_∞ is the curve we obtain at infinity in constructing ν^* , then by collapsing C_∞ to a point P_∞ we get V . If we blow up V at P_∞ , we get $V' = V_0 \cup V_1$ where $V_0 \approx \nu^*$ and $V_1 = \mathbb{C}P_2$ with $V_0 \cap V_1 = C_\infty$. (In V_0 , $C_\infty = C - 3F$, F being the fibre.) Also $\tau(P_\infty, V) \approx \tau(C_\infty, \nu^*)$ and $H_1(\tau(C_\infty, \nu^*)) = Z \oplus Z \oplus Z_3$ (Z_3 because $C_\infty \cdot C_\infty = -3$ in ν^*). But the free generators of $H_1(\tau(C_\infty, \nu^*))$ do not bound in $V_0 = \nu^*$.

4.16 COROLLARY. *$R_{2n-1}(V) = 0$ and if $\gamma_{2n-2} = \tau(\gamma_{2n-2q-1})$ with $\gamma_{2n-2q-1} \sim 0$, then $[\gamma_{2n-2q-1}] \in H_{2n-2q-1}(\tau(S, V))$ where S is the singular locus.*

PROOF. $R_{2n-1}(V)$ can have at most one cycle, $\tau(V) = \partial(W - T(V))$.

If $\gamma_{2n-2} = \tau(\gamma_{2n-2q-1})$ and $0 \neq [\gamma_{2n-2q-1}] \in H_{2n-2q-1}(\tau(S', V))$, but $\dim_R \tau(S', V) = 2n - 2q - 1$, hence if $b_{2n-2q-1}(\tau(S', V)) = 1$, $b_i = i$ th Betti number, then $\gamma_{2n-2q-1} = k\tau(S', V)$ for $k \neq 0$ and $\tau_V(\gamma_{2n-2q-1}) = \partial(k\tau(S') - kT_V(\tau(S', V)))$.

If $b_{2n-2q-1}(\tau(S', V)) > 1$ and S' is a proper subvariety of S , then let V_1, \dots, V_k be the components of V near S' . If $\gamma_{2n-2q-1}$ misses one of the components, say V_1 , then we are in the situation discussed in the proof of Corollary 2.8, i.e., $\gamma_{2n-2q-1} \cap S \subset V - V_1$ will be the cone over part of $S' \subset V_1$, so that instead of taking the tube so that we get $\gamma_{2n-2q+1}$, we first take the tube in the V_1 direction to get a $\gamma_{2n-2q+1} \subset V_1 - S$, i.e., $\gamma'_{2n-2q+1} \in H_{2n-2q+1}(\tau(S, V))$. If $\gamma_{2n-2q-1}$ intersects each component, then $\gamma_{2n-2q-1} = \sum_{i=1}^k k_i \tau(S', V_i)$ and $k_i \neq 0$ for all i ; hence $\tau(\gamma_{2n-2q-1}) \sim 0$. Q.E.D.

Note. If $\dim_{\mathbb{C}} S = 1$, we have that if $\gamma_{p-2q+1} \sim 0$ in V , then $\gamma_{p+1} \cap M_1 = p + 1 + 2 - 2n \geq 0$, i.e., $p \geq 2n - 3$. But for $p = 2n - 3$, we can have $\gamma_{2n-3-2q-1} \in H_{2n-3-2q-1}(\tau(S', V))$ for S' a proper subvariety of S , i.e., $S' = \{M_0\}$. E.g, if $n = 3$, this says we have a $\gamma_3 \in H_3(W - V)$, $\gamma_3 = \tau(\gamma_2)$ with $\gamma_2 \subset \tau(P, V)$ where P is a singular point of S and $\tau(P, V) \cap S$ will be a real one-dimensional submanifold such that $\gamma_2 \cdot (\tau(P, V) \cap S) = m > 0$ in $\tau(P, V)$, a real three-dimensional algebraic variety. By $\gamma_2 \cdot (\tau(P, V) \cap S) = m > 0$, we mean γ_2 intersects these circles transversely in isolated points. Further we can also assume V is locally irreducible at P , i.e., $\tau(P, V)$ has its third Betti number equal to one, since if $V = \bigcup_i V_i$, then $\tau(P, V) = \sum_i \tau(P, V_i) / \sum \tau(P, V_i \cap V_j)$ with $\tau(P, V_i \cap V_j) \subset S$, singular curve. But γ_2 having transverse intersection with $\tau(P, V) \cap S$ implies that $\gamma_2 \subset \tau(P, V_i)$ for some i . I do not know of an example where this occurs. The obvious choices, $z^a = x^b y^c$ with g.c.d. $(a, b, c) = 1$ (iff V is locally irreducible at $(0, 0, 0) = P$) have as singular locus $C_x \cup C_y$ if $b, c > 1$. But in all these cases $H_2(\tau(P, V)) = 0$. One shows this by mapping $f: D^2 \times D^2 \rightarrow V \cap B$ where B is the ball in C^3 of radius one by, if $l = \text{l.c.m.}(a, b, c)$, $f(u, v) = (u^{l/b}, v^{l/c}, (uv)^{l/a})$ which is onto if g.c.d. $(a, b, c) = 1$. Then set $(u, v) \sim (p, q)$ iff $f(u, v) = f(p, q)$ and $\tau(P, V) = V \cap \partial B = \partial(D^2 \times D^2 / \sim)$.

Now let us return to constructing the Gysin sequence for an arbitrary variety of codimension q in W . Locally, $V = \bigcup V_{q',i}$ where $V_{q',i}$ is an irreducible component of codim $q' \geq q$. Then let $V_{q'} = \bigcup_i V_{q',i}$ which is a subvariety of V_q of codim $q - q'$. Then $V = \bigcup_{q'} V_{q'}$. Note that if $\gamma_p \subset W - V$ and $\gamma_p = \partial c_{p+1}$, then $c_{p+1} \cap V_{q'} = \gamma_{p-2q'+1}$. So we let $Z_{p-2q+1}(V)_{\Delta}$ be those $(p - 2q' + 1)$ -cycles of $V_{q'}$ for $q' \geq q$ over which the tube can be constructed in $W - V$, i.e., we take a $(p - 2q' + 1)$ -cycle of $V_{q'}$ and (perhaps) plumb it together with a $(p - 2q + 1)$ -cycle of V_q to get a p -cycle of $W - V$.

If $\gamma, \gamma' \in Z_{p-2q+1}(V)_{\Delta}$, then we shall say that γ is equivalent to γ' , denoted $\gamma \sim \gamma'$, if $\gamma - \gamma' = \partial c$ where $c = \sum_{q'} c_{p-2q'+2}$ and $c_{p-2q'+2}$ is a $(p - 2q' + 2)$ -chain of $V_{q'}$ and we have $\tau(\gamma) - \tau(\gamma') = -\partial\tau(c)$ in $W - V$. Then define

$$H_{p-2q+1}(V)_{\Delta} = (Z_{p-2q+1}(V)_{\Delta}) / \sim.$$

We call $H_{p-2q+1}(V)_{\Delta}$ the *tubular cycles*.

4.17 COROLLARY. *The following sequence is exact*

$$\cdots \rightarrow H_{p+1}(W) \xrightarrow{I} H_{p-2q+1}(V)_\Delta \xrightarrow{\tau} H_p(W - V) \xrightarrow{i_*} H_p(W) \rightarrow \cdots$$

PROOF. The exactness at $H_p(W - V)$ follows from Theorem 4.12 and the others are exact by the same proof in Corollary 2.12. Q.E.D. for Corollary 4.17.

In Corollary 4.17, we do not say that if $I(\gamma_{p+1}) = 0$ in $H_{p-2q+1}(V)$, then γ_{p+1} has a representative disjoint from V , i.e., we might have $I(\gamma_{p+1}) \neq 0$ in $H_{p-2q+1}(V)_\Delta$. E.g., let $W = Q_2$, the nonsingular quadratic in CP_3 , i.e., $W = \{x^2 + y^2 + z^2 + w^2 = 0\} \subset [x, y, z, w] = CP_3$ and W is topologically equivalent to $S^2 \times S^2$ where S^2 is the two sphere. Let H_t be a generic family of hyperplanes and $V_t = W \cap H_t$, i.e., $H_t, t \in CP_1$, is a pencil of hyperplanes in CP_3 such that all but two of the V_t are nonsingular, in fact quadratic Riemann surfaces, i.e., topologically a two-sphere, and two of them (because degree of W is 2), say V_0 and V_1 , have the equator collapsed to a point, i.e., having a quadratic singularity. Such a family always exists, see. e.g., Mořsezon [18]. Let $V = V_0$ be the hypersurface of W . We define a two cycle $\gamma \in H_2(W)$ by taking a real line L in the t -plane from $t = 0$ to $t = 1$ and let $\gamma = \bigcup_{t \in L} c_t$ where c_t is the equator of V_t for $t \neq 0, 1$ and $c_0 = c_1 =$ pinch point (topologically $\gamma \approx S^2$).

Let $S = \bigcup_{t \in L} S_t$ where S_t is the upper hemisphere of $V_t, t \in L$. If we choose a, b as generators of $H_2(W)$ such that $V_0 = a + b$, then $\partial S = \gamma - (a - b)$. We have $(a - b)$ because if we start at V_1 with a , the upper sphere, then as we move towards V_0 along L we twist toward the lower sphere b of V_0 because W is quadratic.

Therefore $\gamma \sim (a - b)$ and $a \cdot a = 0 = b \cdot b$ and $a \cdot b = 1 = b \cdot a$. But $(a - b) \cdot (a + b) = 0$, i.e., $\gamma \cdot V_0 = 0$, and $a - b$ has no representative disjoint from V_0 , because if it did, then it would have a representative disjoint from b , but $(a - b) \cdot b = -1$. Also $(a - b) \cdot (a - b) = -2$, hence $a - b \neq 0$.

In Corollary 4.17 in this case we have

$$0 \rightarrow H_2(W) \xrightarrow{I} H_0(V_0)_\Delta \rightarrow H_1(W - V_0) \rightarrow 0$$

where the injection on the left-hand side follows because a and b generate $H_2(W)$ and neither has a representative disjoint from V_0 . We choose as stratum $M_{1,1} = a - \{\text{pinch point}\}, M_{1,2} = b - \{\text{pinch point}\}, M_0 = \{\text{pinch point}\}$ and since $a, b \sim \frac{1}{2} CP_1$ in W the $M_{1,i}$ are strata and it is clear they satisfy Theorem 3.2.

Then $a \cdot (a + b) = +1$ implies a representative of a can be chosen so that it intersects $M_{1,2}$ in one point with +1 algebraic intersection. Then for $P \in M_{1,2}$, this implies $\tau_1(P) \sim 0$.

Similarly $Q \in M_{1,1}$ implies $\tau_1(Q) \sim 0$. Thus $H_1(W - V_0) = 0$. Then $H_0(V)_\Delta = Z \oplus Z$ where the generators are $P \in M_{1,2}, Q \in M_{1,1}$ because where

we can join P to Q by a line c_1 in V_0 , we cannot construct $\tau(c_1) \subset W - V_0$ such that $\partial\tau(c_1) = \tau(P) - \tau(Q)$. We also have $I(\gamma) = (1, -1) \neq 0$ in $H_0(V_0)_\Delta$, although $I(\gamma) = 0$ in $H_0(V_0)$. (See Proposition 4.18(a) below.)

We also note that even if I is onto $H_{p-1}(V)$, i.e., $H_{p+1}(W) \xrightarrow{I} H_{p-1}(V) \rightarrow 0$, this does not imply that τ is the zero map. E.g., let C be the elliptic curve $\{x^3 + y^3 + z^3 = 0\}$ in CP_2 . Then let V be the cone over C in CP_3 , i.e., $V = \{x + y + z = 0\}$ in $W = CP_3$ with $[x, y, z, w]$ as homogeneous coordinates. Then V is ample in CP_3 , hence $H_1(V) = 0$. Thus $H_3(CP_3) \xrightarrow{I} H_1(V) \rightarrow 0$ but if $\alpha_i, i = 1, 2$, are the generators of $H_1(C)$, then $\tau(\alpha_i) \neq 0, i = 1, 2$ in $H_2(CP_3 - V)$ since $H_3(CP_3) = 0$. In fact, $H_3(CP_3 - V) = Z \oplus Z$.

Finally, just as we noted at the end of §2, these geometric results can be obtained by looking at the Leray spectral sequence induced from the inclusion map $W - V \subset W$.

Again in this spectral sequence we will have $E_2^{p,q} \simeq H^p(W, R^q(V))$ where $R^q(V)$ is the sheaf obtained from the presheaf $U \rightarrow H^q(U - U \cap V)$ for U an open set. Then the stalk of $R^q(V)$ at P , denoted by $R^q(V)_P$, is given by $\text{proj lim } H^q(U - U \cap V)$ where U is a neighborhood of P in W and we take the inverse limit over all such U . We note that a cofinal family of U can be chosen such that $\partial U - \partial U \cap V$ is a strong deformation retract of $U - U \cap V$. This follows from Milnor [17, Lemma 5.9], where he actually shows that $U - U \cap V$ is the cone over $\partial U - \partial U \cap V$.

Then $E_2^{p,q} \simeq H^{p+q}(W - V)$.

If V is a hypersurface, let f be a local defining equation of V in U . Then Milnor [17] proves that $\partial U - \partial U \cap V = E$ is a locally trivial fibre bundle over a circle with the fibre F being diffeomorphic to $f^{-1}(c) \cap U$ for c a sufficiently small complex number. The fibration is given by $z \mapsto f(z)/|f(z)|$. We have the Wang sequence of the fibration

$$\rightarrow H_S(F) \xrightarrow{T-I} H_S(F) \rightarrow H_S(E) \rightarrow H_{S-1}(F) \rightarrow \dots$$

where T is the map induced by the action of $\pi_1(S^1) = Z$. (Note: T is called the Picard-Lefschetz transform by algebraic geometers) and I denotes the map induced from the identity.

Thus to compute $R^s(V)_P$, we need only know the action of T . This T has been well studied. In case V is a Brieskorn variety at P , i.e., $f(z_1, \dots, z_n) = z_1^{a_1} + \dots + z_n^{a_n}$ for $a_i > 0$, then Pham [21] has computed the action of T . What he does is let $\omega_i: C^n \rightarrow C^n$ be multiplying the i th coordinate by a primitive a_i th root of unity. Then $\omega_i: F \rightarrow F$ ($F = f^{-1}(1)$). If we let $e = \{(z_1, \dots, z_n) \in F \mid z_i \text{ are real and nonnegative}\}$ (e is homeomorphic to an $(n-1)$ -cell) then $\prod_{i=1}^n (1 - \omega_i)e = h$ is an $(n-1)$ -cycle in F and F is a free group of rank $\prod_{i=1}^n (a_i - 1)$ with a set of generators given by $\{\omega_1^{i_1} \dots \omega_n^{i_n} h \mid$

$0 \leq i_j < a_j$ where the product $\omega_i \omega_j$ means composition of maps. Furthermore T can be chosen as action $\prod_{i=1}^n \omega_i$ where we have the relations $\omega_i^{a_i} = -(1 + \omega_i + \dots + \omega_i^{a_i-1})$.

Another way to compute T is to resolve the singularities of V to normal crossings. $V' = \bigcup V_i$, i.e., when we resolve V we do not change $W - V$ and $E \subset W - V$, so that now V' is given by $\prod_{i=1}^n z_i^{a_i} = 0$ where $a_i \geq 0$, a_i being the multiplicity of V_i in W' and z_i a local defining equation of V_i . Then F is given by $\prod_{i=1}^n z_i^{a_i} = 1$ and $T: F \rightarrow F$ is given by $\prod_{i=1}^n \omega_i$ where $\omega_i: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is multiplying the i th coordinate by a primitive a_i th root of unity. Clemens [2] has computed the minimal polynomial of T , which is $\prod_{i=1}^n (T^{a_i} - 1)$, and given a formula to compute T^a , where $a = \text{l.c.m.}(a_1, \dots, a_n)$, if one knows $H_s(F)$.

Another way to compute E follows from Alexander duality, viz, $E \subset S^{2n-1} = \partial U$, so that

$$\begin{aligned} H^s(E) &\simeq H^s(S^{2n-1} - S^{2n-1} \cap V) \\ &\simeq H_{2n-2-s}(S^{2n-1} \cap V) = H_{2n-2-s}(\tau(P, V)). \end{aligned}$$

So if one knows $\tau(P, V)$ one is done. If P is an isolated singular point of V , then when one resolves singularities to $V' = \bigcup V_i$, then $\tau(P, V)$ is diffeomorphic to $\tau(S, V_0)$ where S is the singular locus of V , and V_0 is the proper transform of V , i.e., $S = \bigcup_{i>0} V_i \cap V_0$. In many instances this is very easy to compute. The difficulty lies in computing the "multiplicity" of $V_i \cap V_0$ in V_i , i.e., the Euler class of the induced normal bundle. In the case when V is a surface, Orlik and Wagreich [20] have computed $\tau(P, V)$ by this method if V admits a \mathbb{C}^* -action. (These include the Brieskorn varieties.) Wagreich [25] has computed $\tau(P, V)$ in case V is what he calls an elliptic singularity.

Let us return to the case of a variety V of codimension q in a complex manifold W of dimension n . Then

$$R^s(V)_P = \begin{cases} Z, & s = 0 \text{ for all } P, \text{ if } n > 2, \\ 0, & s > 0, P \in W - V, \\ Z, & s = 2q - 1, P \in V - S, \\ 0, & 0 < s < 2q - 1, P \in V, \\ 0, & s \geq n + 1, P \in S \text{ and } V \text{ is a hypersurface.} \end{cases}$$

For $n = 2$, $R^0(V)_P$ will equal the free group on the number of components of V at P . By Alexander duality, $H^s(E) \simeq H_{2n-2-s}(\tau(P, V))$ and $\tau(P, V)$ is of dimension $2n - 2q - 1$. If V is a hypersurface, Milnor [17] has shown that $\tau(P, V)$ is $(n - 3)$ -connected. He has also shown that F has the homotopy type of a finite CW-complex of real dimension $n - 1$.

Suppose V has only isolated singularities $\{P_i\}$ and $n > 2$. Then $R^s(V)_P \simeq$

$H^{s-2q+1}(\tau(P_i, V))$ by Alexander and Poincaré duality. Then $R^{2q-1}(V) \simeq Z_V$, i.e., the constant sheaf whose stalk over each point of V is Z and whose stalk off of V is zero, so that $E_2^{p,0} \simeq H^p(W)$, $E_2^{p,2q-1} \simeq H^p(V)$ and $E_2^{p,s} = 0$ for $p > 1$ and $s \neq 0, 2q - 1$.

$$E_2^{0,s} = \bigoplus_i H^0(P_i, R^s(V)_{P_i}) \simeq \bigoplus_i H^{s-2q+1}(\tau(P_i, V))$$

where the last isomorphism follows from Alexander-Poincaré duality.

4.18 PROPOSITION. *If V only has isolated singularities and is of codimension q , then*

- (a) $E_2^{p,0} \Rightarrow \text{image } \{H_p(W - V) \rightarrow H_p(W)\}$;
- (b) $E_2^{p-2q+1,2q-1} \Rightarrow \text{image } \{H_{p-2q+1}(V)_\Delta \rightarrow H_{p-2q+1}(V)\}$, i.e., those γ_{p-2q+1} of $H_{p-2q+1}(V)_\Delta$ which do not bound in V ;
- (c) $E_2^{0,p} \Rightarrow \ker \{H_{p-2q+1}(V)_\Delta \rightarrow H_{p-2q+1}(V)\}$.

PROOF. (a) $\ker d_2^{p,0} = E_2^{p,0} = H^p(W)$. The first nonzero term in the spectral sequence to map onto $E_r^{p,0}$ is

$$E_{2q}^{p-2q,2q-1} = H^{p-2q}(V) \rightarrow H^p(W) = E_{2q}^{p,0}$$

and $d_{2q}^{p-2q,2q-1}$ is adjoint to intersection $H_p(W) \xrightarrow{I} H_{p-2q}(V)$ so that $\text{coker } d_{2q}^{p-2q,2q-1} = \ker I$, i.e., $E_{2q+1}^{p,0} = \{\gamma_p \in H_p(W) \mid \gamma_p \cdot V = 0\}$.

We know from the above examples that a cycle having a zero intersection with V does not guarantee that it has a representative disjoint from V , but this is a first approximation.

The next nonzero term to map onto $E_r^{p,0}$ is

$$d_p^{0,p-1}: E_p^{0,p-1} = \bigoplus_i H^{p-2q}(\tau(P_i, V)) \rightarrow E_{2q+1}^{p,0} \subset H^p(W)$$

and this is also adjoint to the map $H_p(W) \rightarrow \{\bigoplus_i H_{p-2q}(\tau(P_i, V))\} \subset H_{p-2q}(V)$ given by intersection in V . The kernel of this intersecting map are those p -cycles of W which have zero intersection with $\tau(P_i, V)$, which means precisely that if $\gamma_p \cap V = \gamma_{p-2q} = \partial c_{p-2q+1}$ for $c_{p-2q+1} \subset V$, then c_{p-2q+1} can be chosen to miss the P_i ; for if not, then γ_{p-2q} is a cone over P_i , i.e., $0 \neq \gamma_{p-2q} \in H_{p-2q}(\tau(P_i, V))$.

Thus we get $\text{image } d_p^{0,p-1} = \{\gamma_p \in H_p(W) \mid \gamma_p \cdot V = \gamma_{p-2q} = \partial c_{p-2q+1}$ but c_{p-2q+1} cannot be pulled off some $P_i\}$, i.e.,

$$E_\infty^{p,0} = \text{image } \{H_p(W - V) \rightarrow H_p(W)\}.$$

(b) $E_2^{p-2q+1,2q-1} = H^{p-2q+1}(V) \rightarrow E_2^{p-2q+2,2q-2}$ will be the zero map until

$$E_{2q}^{p-2q+1,2q-1} = H^{p-2q+1}(V) \rightarrow H^{p+1}(W) = E_{2q-1}^{p+1,0}$$

which is adjoint to intersection, so

$$\ker d_{2q}^{p-2q+1, 2q-1} = \{\gamma_{p-2q+1} \in H_{p-2q+1}(V) \mid \gamma_{p-2q+1} \neq \gamma_{p+1} \cap V \text{ for } \gamma_{p+1} \in H_{p+1}(W)\}.$$

There will not be terms which map onto $E_r^{p-2q+1, 2q-1}$ until

$$E_{p-2q+1}^{0, p-1} = \bigoplus_i H^{p-2q}(\tau(P_i, V)) \rightarrow E_{p-2q+1}^{p-2q+1, 2q-1} \subset H^{p-2q+1}(V)$$

which is adjoint to the intersection map. Thus

$$\begin{aligned} E_{p-2q+2}^{p-2q+1, 2q-1} &= E_{\infty}^{p-2q+1, 2q-1} \\ &= \{\gamma_{p-2q+1} \in H_{p-2q+1}(V) \mid \gamma_{p-2q+1} \in H_{p-2q+1}(V)_{\Delta}\} \end{aligned}$$

since we can construct the tube over γ_{p-2q+1} if and only if γ_{p-2q+1} misses all the P_i if and only if $\gamma_{p-2q+1} \cap \tau(P_i, V)$ is zero in $\tau(P_i, V)$.

(c) $d_2^{0, p}: E_2^{0, p} \rightarrow E_2^{2, p-1}$ will be the zero map until we reach $E_r^{p-2q+2, 2q-1}$, i.e.,

$$\begin{aligned} E_{p-2q+2}^{0, p} &= E_2^{0, p} \\ &= \bigoplus_i H^{p-2q+1}(\tau(P_i, V)) \xrightarrow{d_{p-2q+2}^{0, p}} E_{p-2q+2}^{p-2q+2, 2q-1} \\ &\simeq H^{p-2q+2}(V) \end{aligned}$$

which is adjoint to the intersection map $I_{0, p}$. Also

$$\bigoplus_i H^{p-2q+1}(\tau(P_i, V)) \supset E_{p-2q+3}^{0, p} = E_{p+1}^{0, p} \xrightarrow{d_{p+1}^{0, p}} H^{p+1}(W)$$

is adjoint to the intersection map with V , so that $d_{p-2q+2}^{0, p}$ and $d_{p+1}^{0, p}$ together give those cycles $\sum_i \gamma_{p-2q+1, i} \in \bigoplus_i H_{p-2q+1}(\tau(P_i, V))$ such that $\sum_i \tau(\gamma_{p-2q+1, i}) \neq 0$ in $W - V$, i.e., $\sum_i \gamma_{p-2q+1, i} \in \text{image } I_{0, p}$ implies $\sum_i \gamma_{p-2q+1, i} = \sum_i \tau(P_i, V) \cap \gamma_{p-2q+2}$ for $\gamma_{p-2q+2} \in H_{p-2q+2}(V)$ so that

$$\sum_i \tau(\gamma_{p-2q+1, i}) = -\partial \tau(\gamma_{p-2q+2} - \sum \tau(P_i, V) \cap \gamma_{p-2q+2}) \sim 0$$

in $W - V$ while $d_p^{0, p+1}(\sum_i \gamma_{p-2q+1, i}) = 0$ if and only if $\sum_i \tau(\gamma_{p-2q+1, i}) = 0$. Therefore

$$E_{p+2}^{0, p} \simeq E_{\infty}^{0, p} = \ker \{H_{p-2q+1}(V)_{\Delta} \rightarrow H_{p-2q+1}(V)\}$$

because $H_{p-2q+1}(\tau(P_i, V)) \rightarrow H_{p-2q+1}(V)$ has zero image. Q.E.D.

In general, if V does not have isolated singularities, we have that $H_{2n-2q-1}(\tau(P_i, V)) \simeq R^{2q-1}(V)_{P_i} \simeq$ free group on the number of components

V has at P_i . Thus $E_2^{p,2q-1} \simeq \bar{H}^p(V)$ where the “ $\bar{}$ ” over H means we count the multiplicity of the number of components of V at each point, i.e. $\bar{H}^p(V) \simeq \sum_i H^p(V_i)$ where V_i are the (local) irreducible components of V . Also $R^s(V) = 0$ for $0 < s < 2q - 1$. Also support $R^s(V) \subseteq S$, singular locus of V for $s \geq 2q - 1$, so that $E_2^{p-s,s} \simeq H^*(\tau(S', V))$ for some subvariety S' of S .

4.19 COROLLARY. *Let V be an arbitrary subvariety of codimension q , then:*

- (a) $E_2^{p,0} \Rightarrow \text{image} \{H_p(W - V) \rightarrow H_p(W)\}$;
- (b) $E_2^{p-2q+1,2q-1} \Rightarrow \text{image} \{H_{p-2q+1}(V)_\Delta \rightarrow H_{p-2q+1}(V)\}$;
- (c) for $s > 2q - 1$,

$$E_2^{p-s,s} \Rightarrow \text{ker} \{H_{p-2q+1}(V)_\Delta \rightarrow H_{p-2q+1}(V)\}.$$

Thus Theorem 4.12 is a study of (b) and (c) while Corollary 4.17 studies (c).

Furthermore, Corollary 4.17 gives us an intersection pairing on V , a variety of dim q , i.e., there is a natural mapping $H^p(V) \rightarrow H_{2q-p}(V)_\Delta$ which is an isomorphism, and also a group $H^p(V)_\Delta$ can be defined which corresponds to a cohomology theory to the homology theory such that $H_p(V) \rightarrow H^{2q-p}(V)_\Delta$ is also an isomorphism. This gives us an intersection pairing on V just as Poincaré duality does for nonsingular V . Also $H_p(V)_\Delta$ can be shown to be an intrinsic property of V , i.e., independent of the embedding of V into W , ambient manifold. In fact, if V is any polyhedra embedded as a subcomplex of some C^∞ -manifold W , then $H_q(V)_\Delta$ (resp. $H^q(V)_\Delta$) can be constructed such that Corollary 4.17 is true and we have the duality theorem. For details, see Gordon [6].

5. **The residue operator.** If V is a hypersurface of a complex manifold W , then we have shown that we have an exact sequence in Corollaries 2.12 or 4.17 which reduces to the Thom-Gysin sequence in case V is nonsingular. We can dualize the sequence, i.e., take Hom, and since our coefficients are over a field \mathbb{C} , we get the long exact sequence in cohomology,

$$(3) \quad \dots \rightarrow H^p(W) \xrightarrow{i^*} H^p(W - V) \xrightarrow{R} H^{p-1}(V)_\Delta \xrightarrow{\cup[V]^*} H^{p+1}(W) \rightarrow \dots$$

where $H^{p-1}(V)_\Delta$ is by definition equal to $\text{Hom}_{\mathbb{C}}(H_{p-1}(V)_\Delta, \mathbb{C}) = H_{p-1}(V)_\Delta^*$. The map $\cup[V]^*$ is essentially cup product with the cohomology class of V , i.e., V defines a homology class $[V] \in H_{2n-2}(W)$ and hence by Poincaré duality, has a dual two-dimensional cohomology class $[V]^*$.

The map R is the Poincaré residue operator in case V is nonsingular. We know that if $\omega \in H^p(W - V)$, then by de Rham's theorem we can represent ω by a closed differential form which is at least C^∞ in $W - V$. Furthermore the representation is unique up to exact such forms.

Leray [14] has shown that if V is nonsingular, then ω has a representation of the form $\theta \wedge \lambda + \eta$ where θ and η are smooth forms of W , i.e., they are C^∞ with no singularities. If $j: V \rightarrow W$ is the inclusion map, then $j^*\theta = \theta|_V$ is a closed form of V and λ has a pole of order one on V , i.e.,

5.1 DEFINITION. A differential form λ has a pole of order k on V if $(f_U)^k \cdot \lambda$ is smooth in U , where U is any coordinate chart of W and f_U is a local defining equation of V in W .

In fact Leray shows that if $\{U\}$ is a locally finite cover of W by coordinate charts and $\{e_U\}$ is a partition of unity subordinate to $\{U\}$, then λ can be chosen to have the form $\sum_U e_U df_U/f_U$. Since the f_U are analytic, this implies λ is a $(1, 0)$ form and $d\lambda$ is essentially the Poincaré dual of $[V]$, i.e., if Ω is a fundamental $(n - 1, n - 1)$ -form of V , then $\int_W \Omega \wedge d\lambda > 0$. Then Leray proves $R(\omega) = j^*\theta/2\pi\sqrt{-1}$.

Hence to analyze R in (3), we will first consider the case of normal crossings.

5.2 THEOREM (ROBIN). If V has normal crossings, then $H^*(W - V)$ can be represented by differential forms having a pole of order one on V . In fact, if $\{U\}$ is a locally finite cover of W by coordinate charts, $\{e_U\}$ a partition of unity subordinate to the cover $\{U\}$ and f_U a local defining equation of V in U , then $\alpha \in H^*(W - V)$ has a representation ω , a closed form, of the type $\omega = (\theta \wedge \sum_U e_U df_U/f_U) + \eta$ where $\theta \wedge e_U df_U$ and $f_U \cdot \eta$ are smooth.

PROOF OF THEOREM 5.2. Robin [23] has shown that each class is represented by a form having a pole of order one, and Leray [14] has shown that if a closed form ω of degree p in $W - V$ has a pole of order one on V , then ω has a representation in U of the type

$$\theta_p + \sum_{q=1}^{p-1} \sum_{\{i_1, \dots, i_q\}} \theta_{i_1 \dots i_q} \wedge \frac{df_{i_1, U}}{f_{i_1, U}} \wedge \dots \wedge \frac{df_{i_q, U}}{f_{i_q, U}}$$

where the θ 's are smooth and $f_U = \prod_{i=1}^k f_{i, U}$. But

$$\sum_{i=1}^k \frac{df_{i, U}}{f_{i, U}} = \frac{df_U}{f_U},$$

hence the representation as stated in the theorem. Q.E.D for Theorem 5.2.

Note. The choice of θ is not unique, e.g., if $f(x, y) = xy$ defines V in $W = C^2 = (x, y)$, then $H^2(W - V)$ can be represented by $dx/x \wedge df/f$ as well as by $\theta_1 \wedge df/f$ where $\theta_1 = (\bar{x} d\bar{x} - \bar{y} d\bar{y})/(x\bar{x} + y\bar{y})$.

Note. Suppose we let

$$\Psi_q = \sum_{i_1 \dots i_q} \theta_{i_1 \dots i_q} \wedge \frac{df_{i_1, U}}{f_{i_1, U}} \wedge \dots \wedge \frac{df_{i_q, U}}{f_{i_q, U}},$$

using the notation of the proof of Theorem 5.2. Let $\gamma_{p-q} \in H_{p-q}(\bar{M}_q)_\Delta$, i.e., we have $\tilde{\tau}_1 \cdots \tilde{\tau}_q \gamma_{p-q} \in H_p(W - V)$. We would like to say that the homology class which has period one on $\tilde{\tau}_1 \cdots \tilde{\tau}_q \gamma_{p-q}$ and zero periods on the other classes can be represented by a form which is locally of the type $\Psi_q + \eta$ where η is smooth, i.e., we do not need Ψ_r for $r \neq q$.

But consider, for example, $V = V_1 \cup V_2$ and $\gamma_{p-2} \in H_{p-2}(V_1 \cap V_2)$. Suppose θ_{12} is a closed form on V_{12} which represents the dual cohomology class of γ_{p-2} via integration. Then let θ'_{12} be a smooth form on W which vanishes off $T_{1,1}T_{2,2}V_{12}$ and equals θ_{12} near V_{12} ; and let f_i be a local defining equation of V_i . Then we would like to say that $\theta'_{12} \wedge df_1/f_1 \wedge df_2/f_2$ should (modulo a smooth form on W) represent the dual cohomology class of $\tau_1 \tau_2 \gamma_{p-2}$.

One can show that $d(\theta'_{12} \wedge df_1/f_1 \wedge df_2/f_2)$ has no periods on W , but it is not necessarily smooth, i.e., $d\theta'_{12}$ is closed near V_{12} , but $d\theta'_{12}|_{V_1}$ is not zero, hence $d\theta'_{12} \wedge df_1/f_1 \wedge df_2/f_2$ need not be smooth on V_1 . Hence, we might need a correction term, $\theta_1 \wedge df_1/f_1$, to make it smooth on V_1 , and another correction term, $\theta_2 \wedge df_2/f_2$, to make it smooth on V_2 , where the θ_i are smooth forms.

However, we would have that $R(\theta_i \wedge df_i/f_i) = 0$. This is because in the spectral sequence of the map $j: W - V \subset W$, we showed that $E_2^{p-1,1} \Rightarrow \tau_1 H_{p-1}(\bar{M}_1)_\Delta$, while $E_2^{p-2,2} \Rightarrow \tau_1 \tau_2 H_{p-2}(\bar{M}_2)_\Delta$. But we are working over a field, hence there are no relations among $E_\infty^{p-1,1}$ and $E_\infty^{p-2,2}$. But if $R(\theta_i \wedge df_i/f_i) \neq 0$ and $\theta_i \wedge df_i/f_i$ is part of the dual cohomology class of $\tau_1 \tau_2 \gamma_{p-2}$, these would give nontrivial relations.

Hence, to analyze R in (3) when V has normal crossings, we have that if $\gamma_{p-q} \in H_{p-q}(\bar{M}_q)_\Delta$ and α is the dual homology class of $\tilde{\tau}_1 \cdots \tilde{\tau}_q \gamma_{p-q}$, then

$$R(\alpha) = (2\pi\sqrt{-1})^q \sum_{i_1 \cdots i_q} [\theta_{i_1 \cdots i_q} | V_{i_1} \cap \cdots \cap V_{i_q}] \in H^{p-q}(\bar{M}_q)_\Delta.$$

Thus R is a mapping on differential forms of degree p to differential forms of degree less than or equal to $p - 1$. If we represent ω by

$$\omega = \theta' \wedge \sum_U e_U \frac{df_U}{f_U} + \eta$$

then $R(\omega) = (2\pi\sqrt{-1})\theta'|_V$ where θ' is a closed $(p - 1)$ -form on V with singularities (in fact logarithm singularities) on the singular locus of V , i.e., R maps closed p -forms with poles on V to closed $(p - 1)$ -forms with (perhaps) poles on the singular locus of V .

Furthermore R commutes with d , i.e. $R(d\omega) = dR(\omega)$, because if $\lambda = \sum_U e_U df_U/f_U$ then $d\lambda$ is smooth, in fact $d\lambda$ is a $(1, 1)$ form which is the Poincaré dual of V in W . We also remark that this description of R is independent of

the representation of ω , i.e., if $\omega = \theta'' \wedge \sum_U e_U df_U/f_U + \eta$, then $\theta'|V = \theta''|V$, and if f'_U is another defining equation, i.e., $f'_U = g_U f_U$ where g_U is nonvanishing in U , then $\omega = \theta' \wedge \sum e_U df'_U/f'_U + \eta$ will have $R(\omega) = \theta'|V$ as well since $df'_U/f'_U = df_U/f_U + dg_U/g_U$ and g_U is nonvanishing.

Let us state this explicitly.

5.3 COROLLARY. *Let V have normal crossings and $\lambda = \sum_U e_U df_U/f_U$ and $\omega \in H^p(W - V)$. Then in (3), $R(\omega) = \theta|V$, a closed form on $V - S$ with (perhaps) a pole of order one on S , and $\bigcup[V]^*$ is essentially the wedge product with $d\lambda$, a smooth form.*

Note. We say essentially because we actually have

$$\begin{array}{ccc} H^{p-q}(\bar{M}_q)_\Delta & \xrightarrow{\bigcup[V]^*} & H^{p+1}(W) \\ j^* \uparrow & \nearrow \wedge d\lambda_q & \\ H^{p-q}(W) & & \end{array}$$

where

$$\lambda_q = \sum_U e_U \frac{df_{1,U} \wedge \dots \wedge df_{q,U}}{f_{1,U} \dots f_{q,U}}$$

where $\bar{M}(1, \dots, q)$ is given by $\bigcap_{i=1}^q (f_{i,U} = 0)$ in U . $d\lambda_q$ will be a $(q, 1)$ -form and j^* is the restriction map, i.e., $j: \bar{M}_q \subset W$.

We would like to say that if V is any hypersurface and $\lambda = \sum_U e_U df_U/f_U$ where the $\{U\}$ are suitably chosen coordinate charts, then $\alpha \in H^*(W - V)$ has a representative of the form $\omega = \theta \wedge \lambda + \eta$ where $f_U \cdot \theta \wedge \lambda$ and $f_U \cdot \eta$ are smooth with $R(\alpha) = [\theta|V]$. Unfortunately, proving it for the normal crossings case does not suffice.

Suppose, then, we have given $V \subset W$ and we take a resolution (W', V', π) where $V' = \pi^{-1}(V)$, $\pi: W' \rightarrow W$ and V' has local normal crossings. Then if U is a local coordinate chart in W with f_U a local defining equation of V and $\pi^{-1}(U) = U'$, then we set $f_U \circ \pi = f_{U'}$ and $f_{U'} = \prod_{i=1}^k (f_{i,U'})^{\alpha_i}$ with $\alpha_i \geq 1$ and $\prod_i f_{i,U'}$ is a local defining equation of V' in U' .

Let $\lambda = \sum_U e_U df_U/f_U$ for $\{U\}$ a suitably chosen locally finite cover of W . Then $\pi^{-1}(U) = U'$ is a locally finite cover of W' and one can choose a refinement $\{U''\}$ of $\{U'\}$ with $f_{U''}|_{U' \cap U''} = f_{U'}|_{U' \cap U''}$ such that U'' are coordinate charts which satisfy the representation of ω in Theorem 5.2. Then $df_{U''}/f_{U''} = \sum_{i=1}^k \alpha_i df_{i,U''}/f_{i,U''}$. But in the proof of Theorem 5.2, one can replace all of the $df_{i,U''}/f_{i,U''}$ by $\alpha_i df_{i,U''}/f_{i,U''}$ and the conclusions will still be valid, i.e., since the α_i are invariant of the coordinate chart representation, one has only multiplied the representative by a nonzero constant. Hence one will have by The-

orem 5.2 that $\alpha \in H^*(W' - V')$ has a representative $\omega' = \theta' \wedge \lambda' + \eta'$ where $\lambda' = \sum_{U''} e_{U''} df_{U''}/f_{U''}$. But if $I_U = \{U'' | U'' \subset \pi^{-1}(U)\}$, then $\bigcup_{I_U} \pi^{-1}(U'') = U'$ and $\sum_{I_U} e_{U''} f_{U''} = e_U f_U = (e_U \circ \pi) (f_U \circ \pi)$ so that $\pi^* \lambda = \lambda'$.

Hence $(\pi^*)^{-1} \omega = (\pi^*)^{-1} \theta' \wedge \lambda' + (\pi^*)^{-1} \eta'$. If S denotes the singular locus of V , and $S' = \pi^{-1}(S)$, then $\pi | W' - S'$ is a bianalytic isomorphism so that $(\pi^*)^{-1}(\theta')$ is a form on $V - S$ which is smooth as $\theta' | W' - S'$ is smooth and θ' is closed near V' implies $(\pi^*)^{-1} \theta'$ is closed near $V - S$.

Hence, we have shown

5.4 PROPOSITION. *If V is a hypersurface in W and $\alpha \in H^*(W - V)$, then α has a representative of the form $\theta \wedge \lambda + \eta$ where $\lambda = \sum_U e_U df_U/f_U$. Moreover, $R(\alpha) = [\theta | V]$, which will be a closed form with (perhaps) singularities on S , the singular locus of V .*

However, we cannot say that $f_U \cdot (\pi^*)^{-1} \eta'$ is smooth, i.e., if $P \in S$, then we can have $P_i, P'_i \in W - S$ with $P_i, P'_i \rightarrow P$ but $\lim \eta'(P_i) \neq \lim \eta'(P'_i)$. Also, we cannot say that $f_U \cdot (\pi^*)^{-1} \theta' \wedge \lambda$ is smooth as $(\pi^*)^{-1} \theta'$ may acquire singularities, even if θ is smooth, as in the example after Corollary 4.15 we have $\gamma_1 \not\sim 0$ in V' hence $\theta' | V'$ is smooth, but $\gamma_1 \sim 0$ in V so that $(\pi^*)^{-1} \theta' | V$ will have a singularity at the singular point of V . But in this example, $f_U (\pi^*)^{-1} \theta' \wedge \lambda$ is smooth as λ has a zero at the singular point of sufficiently high order to make the form smooth.

However, we can say if $\Phi^p(U)$ denotes the closed, smooth p -forms in U , $\Omega(U)$ are smooth forms in U , and $d\Omega(U)$ are exact ones, then

5.5 PROPOSITION. *Let $\theta \in \Phi^p(W - V)$ and (W', V', π) a resolution of (W, V) whose centers are nonsingular. Then if $\pi^* \theta$ is smooth, either*

- (i) $\theta - d\eta \in \Phi^p(W)$ for some $\eta \in \Omega^{p-1}(W - V)$ or
- (ii) $\theta \in d\Omega(W - V)$.

PROOF. By Proposition 3.1, since $\text{Ker } i_* = R_p(V) \simeq R_p(V') = \text{Ker } i'_*$ where $i: V \subset W, i': V' \subset W'$, we have that (as coefficients are in \mathbb{C}) $\pi^*: \text{Coker } i^* \simeq \text{Coker } (i')^*$, i.e., $\pi^*: \text{Ker } R \simeq \text{Ker } R'$ where

$$\begin{array}{ccccc} H^p(W') & \xrightarrow{(i')^*} & H^p(W' - V') & \xrightarrow{R'} & H^{p-1}(V')_{\Delta} \\ \uparrow \pi^* & & \uparrow \pi^* & & \\ H^p(W) & \xrightarrow{i^*} & H^p(W - V) & \xrightarrow{R} & H^{p-1}(V)_{\Delta} \end{array}$$

Hence θ closed implies $[\theta] \in H^p(W - V)$ and $\pi^*[\theta]$ being smooth implies $R' \pi^*[\theta] = 0$ which implies $R[\theta] = 0$, i.e., either $\theta - d\eta \in \Phi(W)$ or $\theta \in d\Omega(W - V)$. Q.E.D. for Proposition 5.5.

5.6 LEMMA. $d(f_U \cdot (\pi^*)^{-1}(\eta'))$ has no periods on W .

PROOF OF LEMMA 5.6. Let $[\gamma_{p+1}] \in H_{p+1}(W)$, then $\gamma_{p+1} \cap V = \gamma_{p-1}$ with $[\gamma_{p-1}] \in H_{p-1}(V)_\Delta$. Hence,

$$\begin{aligned} \int_{\gamma_{p+1}} d(f_U \cdot (\pi^*)^{-1}(\eta')) &= \lim_{\epsilon \rightarrow 0} \int_{\gamma_{p+1} - T_\epsilon \gamma_{p-1}} d(f_U \cdot (\pi^*)^{-1}(\eta')) \\ &= - \lim_{\epsilon \rightarrow 0} \int_{T_\epsilon \gamma_{p-1}} f_U \cdot (\pi^*)^{-1}(\eta') \end{aligned}$$

by Stokes' theorem where $T_\epsilon \gamma_{p-1}$ are all the fibres of length less than or equal to ϵ and $\tau_\epsilon \gamma_{p-1}$ are all the fibres of length equal to ϵ .

By Proposition 3.1, $\pi^* \tau_\epsilon \gamma_{p-1} = \tau_\epsilon \gamma'_{p-1}$ for some $\gamma'_{p-1} \in H_{p-1}(V')_\Delta$, i.e., in the proof of that proposition, it was shown that $\pi^{-1}(\gamma_{p-1}) \in Z_{p-1}(V')_\Delta$ if $\gamma_{p-1} \in H_{p-1}(V)_\Delta$. Hence

$$- \lim_{\epsilon \rightarrow 0} \int_{\tau_\epsilon \gamma_{p-1}} f_U \cdot (\pi^*)^{-1}(\eta') = - \lim_{\epsilon \rightarrow 0} \int_{\tau_\epsilon \gamma'_{p-1}} \pi^*(f_U) \cdot \eta' = 0$$

since $\pi^* f_U = f \cdot f'_U$ and $f'_U \cdot \eta'$ is smooth, while $\lim_{\epsilon \rightarrow 0} \tau_\epsilon \gamma'_{p-1}$ is a $(p - 1)$ -dimensional chain. Q.E.D. for Lemma 5.6.

5.7 PROPOSITION. If $d(f_U \cdot (\pi^*)^{-1}(\eta'))$ is smooth, then $\alpha \in H^*(W - V)$ has a representative of the form $\theta \wedge \lambda + \eta$ where $f_U \cdot \eta$ is smooth.

PROOF OF PROPOSITION 5.7. If $d(f_U \cdot (\pi^*)^{-1}(\eta'))$ is smooth, then by Lemma 5.6 and de Rham's theorem, we have a $\xi \in \Omega(W)$ such that $d(f_U \cdot (\pi^*)^{-1}(\eta')) = d\xi$. Hence $f_U \cdot (\pi^*)^{-1}(\eta') - \xi$ is a closed form on $W - V$ such that $\pi^*(f_U \cdot (\pi^*)^{-1}(\eta') - \xi)$ is smooth, so that by Proposition 5.5 either

$$f_U \cdot (\pi^*)^{-1}(\eta') - \xi - d\rho = \Psi \in \Omega(W)$$

or

$$f_U \cdot (\pi^*)^{-1}(\eta') - \xi = d\rho \quad \text{for } \rho \in \Omega(W - V).$$

Then let $\omega = (\pi^*)^{-1} \omega' - d\rho = (\pi^*)^{-1} \theta' \wedge \lambda + \eta$ where η is now smooth. If we set $\theta = (\pi^*)^{-1} \theta'$, then θ is a smooth form on $W - S$ with $\theta|_V$ being closed. Q.E.D. for Proposition 5.7.

5.8 CONJECTURE. If V is a hypersurface in W , a complex manifold and $\alpha \in H^*(W - V)$, then α has a representative of the form $\omega = \theta \wedge \lambda + \eta$ where $f \cdot \eta$ is smooth if f is a local defining equation of V . Then $R(\alpha) = [\theta|_V]$, a closed form on $V - S$ with perhaps singularities on S , the singular locus of V .

Given any variety V in a complex manifold W , we can associate to it a differential form λ , called its kernel, where λ is locally summable on W , has singular support in V and such that $d\lambda = 1_V + \psi$ where ψ is a smooth form on W and 1_V is the current of integration over V , i.e., $1_V[\omega] = \int_{V-S} \omega$ (cf. Poly [22] for

the existence of λ). For V a hypersurface, $\lambda = \sum_U e_U df_U/f_U$ is the kernel associated to V .

5.9 CONJECTURE. If $\alpha \in H^*(W - V)$ where V is a variety of codim q and λ is the kernel of V , then α has a representative of the form $\omega = \theta \wedge \lambda + \eta$ where η is smooth and $R(\alpha) = \theta|V$ which will be a closed form on $V - S$ with (perhaps) singularities on S , the singular locus of V .

Note. In general, $\text{Res}(\alpha) = R(\alpha)$ is considered as a current with support in V , but what Conjecture 5.11 says is that $\text{Res}(\alpha)$ can be represented by a closed differential form on V but perhaps with singularities on the singular locus of V .

5.10 THEOREM (POLY). A necessary and sufficient condition that $\alpha \in H^p(W - V)$ has a representative of the form $\omega = \theta \wedge \lambda + \eta$ where θ is smooth and $\theta|V$ is closed is that if α^* is the dual homology class of α via integration, then α^* has a representative of the form $\gamma_p = \tau(\gamma_{p-2q+1})$ where $0 \neq [\gamma_{p-2q+1}] \in H_{p-2q+1}(V)$.

PROOF. This is proven in [22] where it is stated slightly differently.

In Gordon [6], it is shown that $\cap[V]: H^{2n-p-1}(V) \simeq H_{p-2q+1}(V)_\Delta$ where $n = \dim_{\mathbb{C}} W$ and $\cap[V]$ is cap product with the homology class carried by V . But if $\beta \cap [V] = [\gamma_{p-2q+1}] \in H_{p-2q+1}(V)_\Delta \cap H_{p-2q+1}(V)$, then $0 \neq \delta_*\beta \in H^{2n-p}(W, V)$, i.e., we have the commutative diagram

$$\begin{array}{ccc} H^{2n-p}(W, V) & \xleftarrow{\delta_*} & H^{2n-p-1}(V) \\ \uparrow & & \uparrow \cap[V] \\ H_p(W - V) & \xleftarrow{\tau} & H_{p-2q+1}(V)_\Delta \end{array}$$

where the left-hand isomorphism is Lefschetz duality. Hence Theorem 5.12 is equivalent to:

$\alpha \in H^p(W - V)$ has a representative of the form $\theta \wedge \lambda + \eta$ where θ, η are smooth and λ is kernel associated to V if and only if we let $\alpha^* \in H_{2n-p}(W, V)$ be the Lefschetz dual to α , then $\delta_*\alpha^* = [V] \cap \beta$ for some $\beta \in H^{p-2q+1}(V)$ and $\delta_*\alpha^* \in H_{2n-p-1}(V)$. This is the form of the theorem that Poly proves. Q.E.D. for Theorem 5.10.

We note that a direct proof of Theorem 5.12 can be proved without use of currents in the theory of Bloom-Herrera and Herrera-Lieberman [12]. The basic idea is that given γ_{p-2q+1} , we can find a $(p - 2q + 1)$ -form θ on W which is closed near V such that θ is dual to V to γ_{p-2q+1} via integration. Then one shows that $d(\theta \wedge \lambda) = d\xi$ for $\xi \in \Omega(W)$ by the fundamental identity

$$\int \tau(\gamma'_{p-2q+1}) \theta \wedge \lambda = K \int \gamma'_{p-2q+1} \theta|V \quad \text{for } \gamma'_{p-2q+1} \in H_{p-2q+1}(V)_\Delta$$

and K is some constant depending only on q . Then $\theta \wedge \lambda - i^* \xi - i^* \rho \in \alpha$ for some $\rho \in \Omega(W)$.

We note that Robin [23] has shown that any $\alpha \in H^*(W - V)$ can be represented by a form with a pole of order k for some k , but has no bounds on k . We can state

5.11 COROLLARY. *Any class $\alpha \in H^*(W - V)$ has a representative ω such that $f^2 \omega$ is continuous, where f is a local defining equation.*

PROOF. Choose a resolution with normal crossings (W', V', π) such that $f \circ \pi = f'$ locally. But we can choose $\omega \in \alpha \in H^*(W - V)$ with $f' \pi^* \omega = \pi^* f \omega$ smooth. Hence for $P \in S$, $\pi^{-1}(P)$ is a compact set since π is proper, so that $\pi^* f \omega|_{\pi^{-1}(P)} < \infty$. Thus, $\lim_{Q \rightarrow P; Q \in W - S} f^2(Q) \omega(Q) = 0$, so that $f^2 \omega$ is continuous. Q.E.D. for Corollary 5.11.

The question of semimeromorphic forms will be discussed in a subsequent paper.

I would like to conclude this paper by returning to Theorem 5.2 and consider the case when ω is a $(p, 0)$ -form. This will tie up with the question asked in §3.

5.12 COROLLARY. *If V has normal crossings and ω is a $(p, 0)$ -form with a pole of order one and $R(\omega) \in H^{p-q}(\bar{M}_q)_\Delta$, then there is a sufficiently small neighborhood U_q of \bar{M}_{q+1} in W such that for all $\gamma_{p-q} \in H_{p-q}(V_{i_1} \cap \dots \cap V_{i_q})$,*

$$\int_{\gamma_{p-q}} R\omega = \int_{\gamma_{p-q} \cap (\bar{M}_q - U_q)} R\omega = \int_{\gamma_{p-q} \cap M_q} R\omega.$$

Note. In general for ω a $(p, 0)$ -form with a pole of order one, $R(\omega) \in \Sigma_{q=1}^p H^{p-q}(\bar{M}_q)_\Delta$, i.e., $R(\omega) = \Sigma_q R_q(\omega)$ and the corollary will be true for each $R_q(\omega)$ and some U_q .

PROOF. It suffices to assume ω is of the type

$$\tilde{\theta} \wedge \sum_U e_U \frac{df_{1,U}}{f_{1,U}} \wedge \dots \wedge \frac{df_{q,U}}{f_{q,U}} + \eta.$$

Since η is smooth and $df_{i,U}$ is a $(1, 0)$ form, this implies $\tilde{\theta}$ is a $(p - q, 0)$ -form. Then let $\theta = R(\omega) = (2\pi\sqrt{-1})^q \tilde{\theta} | \bar{M}_q$.

Let $\gamma_{p-q} \in H_{p-q}(\bar{M}(1, \dots, q))$ and suppose $\gamma_{p-q} \cap \bar{M}_{q+1} \neq \emptyset$ for all representatives of γ_{p-q} . Then we have $\gamma_{p-q} \cap V_{q+1} = \gamma_{p-q-2} \neq \emptyset$ in $\bar{M}(1, \dots, q, q + 1)$.

Also,

$$\int_{\gamma_{p-q}} \theta = \sum_{U \cap M_{q+1} = \emptyset} \int_{\gamma_{p-q} \cap U} e_U \cdot \theta + \sum_{U \cap M_{q+1} \neq \emptyset} \int_{\gamma_{p-q} \cap U} e_U \cdot \theta$$

where $\{U\}$ is a cover of W by coordinate charts.

To prove the corollary it suffices to show if $U \cap M_{q+1} = \emptyset$, then

$\int \gamma_{p-q} \cap U e_U \cdot \theta = 0$. But this is so because in U , we have that

$$\gamma_{p-q} \cap U = \underbrace{(0 \times \cdots \times 0 \times \mathbb{C}_z \times \gamma_{p-q-2})}_{(q-1)\text{-times}} \cap U$$

since γ_{p-q} intersects V_{q+1} transversely. But θ is a $(p - q, 0)$ -form, hence does not have a $d\bar{z}_q$ -term which implies the integral must be zero. Q.E.D.

Note. The corollary does not say that if γ_{p-q} is the dual cycle to $R(\omega)$ in $\bar{M}(1, \dots, q)$ that γ_{p-q} has a representative disjoint from \bar{M}_{q+1} . We can have $\gamma_{p-q} \cap V_{q+1} = \gamma_{p-q-2} \neq 0$ in $\bar{M}(1, \dots, q + 1)$ but $\gamma_{p-q-2} \sim 0$ in $\bar{M}(1, \dots, q)$. What this says is that the restriction of $R\omega$ to $U_q \cap \bar{M}_q$ is exact. However we have

5.13 COROLLARY. *Let V have normal crossings and let ω be a $(p, 0)$ -form with a pole of order one and $R(\omega) \in H^{p-q}(\bar{M}_q)_\Delta$. If in*

$$H_{p-q}(\bar{M}(1, \dots, q)) \xrightarrow{I} H_{p-q-2}(\bar{M}(1, \dots, q + 1)) \xrightarrow{i_*} H_{p-q-2}(\bar{M}(1, \dots, q))$$

where I is intersection and i_* is induced from the inclusion, we have $(\text{image } I) \cap \text{Ker } i_* = 0$, then $R(\omega) \in H^{p-q}(M_q)$, i.e., dual cycle to $R(\omega)$ lies in $H_{p-q}(M_q)$.

PROOF. In the notation of Corollary 5.11, we have $\gamma_{p-q-2} \neq 0$ in $H_{p-q-2}(\bar{M}(1, \dots, q + 1))$. But from the commutative diagram

$$\begin{array}{ccc} H^{p-q-2}(\bar{M}(1, \dots, q + 1)) & \rightarrow & H^{p-q}(\bar{M}(1, \dots, q)) \\ i_* \uparrow & \nearrow \wedge [V_{q+1}]^* & \\ H^{p-q-2}(\bar{M}(1, \dots, q)) & & \end{array}$$

this implies that $\theta = \theta' \wedge [V_{q+1}]^*$ where $[V_{q+1}]^*$ is the Poincaré dual cycle to $V_1 \cap \cdots \cap V_q \cap V_{q+1}$ in $\bar{M}(1, \dots, q)$. But $[V_{q+1}]^*$ is a $(1, 1)$ -form which implies θ cannot be a $(p - q, 0)$ -form contradiction. Q.E.D.

Note. If V comes from a resolution, then the hypothesis that $\text{image } I \cap \text{Ker } i_* = 0$ may always be true.

We know that if V is ample, then $H^p(W - V)$ is generated by meromorphic $(p, 0)$ -forms with poles of order $k \leq n = \dim_{\mathbb{C}} W$. Griffiths [8, §8] has given an algorithm for representing $\omega = \sum_{i=0}^{k-1} \omega_i$ where ω_i has a pole of order one and ω_i is a $(p - i, i)$ -form. Using the fact that ω is meromorphic, one might be able to generalize Corollary 5.12 without too restrictive topological conditions on V which are satisfied if V is a resolution.

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