TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 215, 1976

INVOLUTIONS ON HOMOTOPY SPHERES AND THEIR GLUING DIFFEOMORPHISMS

BY

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ABSTRACT. Let $hS(P^{2n+1})$ denote the set of equivalence classes of smooth fixed-point free involutions on $(2n + 1)$-dimensional homotopy spheres. Browder and Livesay defined an invariant $\sigma(\Sigma^{2n+1}, T)$ for each $(\Sigma^{2n+1}, T) \in hS(P^{2n+1})$, where $\sigma \in Z$ if n is odd, $\sigma \in Z_2$ if n is even. They showed that for $n \geq 3$, $\sigma(\Sigma^{2n+1}, T) = 0$ if and only if (Σ^{2n+1}, T) admits a codim 1 invariant sphere. For any (Σ^{2n+1}, T), there exists an A-equivariant diffeomorphism f of $S^n \times S^n$ such that $(\Sigma^{2n+1}, T) = (S^n \times D^{n+1}, A) \cup_f (D^{n+1} \times S^n, A)$, where A denotes the antipodal map. Let $\beta(f) = \sigma(\Sigma^{2n+1}, T)$. In the case n is odd, we can show that the Browder-Livesay invariant is additive: "$\beta(fg) = \beta(f) + \beta(g)$". But if n is even, then there exists f and g such that $\beta(gf) = \beta(g) + \beta(f) \neq \beta(fg)$. Let $D_0(S^n \times S^n, A)$ be the group of concordance classes of A-equivariant diffeomorphisms which are homotopic to the identity map of $S^n \times S^n$. We can prove that "For $n \equiv 0, 1, 2 \mod 4$, $hS(P^{2n+1})$ is in 1-1 correspondence with a subgroup of $D_0(S^n \times S^n, A)$". As an application of these theorems, we demonstrated that "Let Σ_0^{8k+3} denote the generator of bP_{8k+4}. Then the number of (Σ_0^{8k+3}, T)'s with $\sigma(\Sigma_0^{8k+3}, T) = 0$ is either 0 or equal to the number of (S^{8k+3}, T)'s with $\sigma(S^{8k+3}, T) = 0$, where S^{8k+3} denotes the standard sphere".

0. Introduction. In [7], [8], Browder and Livesay studied differentiable fixed-point free involutions on homotopy spheres. They defined the Browder-Livesay desuspension invariant σ for each free involution (Σ^m, T): $\sigma(\Sigma^m, T) = 0$, for m even; $\sigma(\Sigma^m, T) \in Z$, for $m = 4k + 3$; $\sigma(\Sigma^m, T) \in Z_2$, for $m = 4k + 1$. For $m \geq 6$, they proved that $\sigma(\Sigma^m, T) = 0$ if and only if (Σ^m, T) admits a codim 1 invariant subsphere $(S^{m-1}, T|S^{m-1})$ embedded in it. It was shown by several people that all these desuspension invariants can be realized, [2], [6], [21], [33], and [34] etc.

Livesay and Thomas, [20], showed that any (Σ^{2n+1}, T) can be obtained by gluing $(S^n \times D^{n+1}, A)$ and $(D^{n+1} \times S^n, A)$ together by an A-equivariant diffeomorphism f of their boundaries, where A is the antipodal map. We shall denote this (Σ^{2n+1}, T) by (Σ_f, T_f). The purpose of this paper is to investigate the rela-

Received by the editors August 7, 1974.

AMS (MOS) subject classifications (1970). Primary 57E25, 57E30, 57D65, 55E05.

Key wrods and phrases. Free differentiable involutions, Browder-Livesay invariant, equivariant diffeomorphism, concordance group of diffeomorphisms, curious involutions.

tion between the free involutions on odd dimensional homotopy spheres and their gluing diffeomorphisms.

Let A denote the antipodal map on $S^n \times S^n$, defined by $A(x, y) = (-x, -y)$ for $(x, y) \in S^n \times S^n$. An A-equivariant map $f: S^n \times S^n \to S^n \times S^n$ is a map such that $fA = Af$, we shall call f an A-map. An A-map f induces a map f': $S^n \times S^n/A \to S^n \times S^n/A$, where $S^n \times S^n/A$ denotes the orbit space of $S^n \times S^n$ under the action of A. If f' is a diffeomorphism (or a homotopy equivalence), we will call f an A-diffeomorphism (or an A-homotopy equivalence). Considering the action of $A \times$ identity on $S^n \times S^n \times [0, 1]$, we have the notion of A-homotopy and A-concordance etc. (see §2 below).

Let $D(S^n \times S^n, A)$ denote the group of A-concordance classes of A-diffeomorphisms of $S^n \times S^n$. Define $J_{2n+1} = \{f \in D(S^n \times S^n, A) | f$ is homotopic to identity$\}$, (but f might not be A-homotopic to identity). J_{2n+1} is a subgroup of $D(S^n \times S^n, A)$. We will show that the gluing diffeomorphism can always be chosen from J_{2n+1}, (§4).

Two involutions (Σ, T) and (Σ', T') are called equivalent, $(\Sigma, T) = (\Sigma', T')$, if there exists an orientation-preserving diffeomorphism $f: \Sigma \to \Sigma'$ such that $f \circ T = T' \circ f$. Let $hS(P^{2n+1})$ denote the set of homotopy smoothings of P^{2n+1}, [34], which is also the set of equivalence classes of differentiable free involutions on $(2n + 1)$-homotopy spheres, [21]. In §5 below, we will prove that for $n \equiv 0, 1, 2$ mod 4, $hS(P^{2n+1})$ is in 1-1 correspondence with a subgroup G_{2n+1} of J_{2n+1}. Thus, in these cases, $hS(P^{2n+1})$ forms a group by carrying over the composition law of diffeomorphisms in G_{2n+1}.

Also, we will show in Theorem 6.15 below that the Browder-Livesay index invariant is additive. For $m = 4k + 3$, f and $g \in J_m$, we have $\sigma(\Sigma_f, T_f) + \sigma(\Sigma_g, T_g) = \sigma(\Sigma_{fg}, T_{fg})$.

From Theorems 5.2 and 6.15, we can deduce the following theorem concerning the curious involutions in the sense of [13]. Let Σ_0^n denote the generator of bP_{n+1}, [17]. S^n denotes the standard sphere.

THEOREM 8.2. *For $n = 8k + 3$, the number of curious involutions (Σ_0^n, T) with $\sigma(\Sigma_0^n, T) = 0$ is either 0 or equal to the number of involutions (S^n, T) with $\sigma(S^n, T) = 0$.*

Everything considered here is assumed to be in the smooth category.

This paper is essentially the author's doctoral thesis written at Cornell University. I wish to thank my thesis advisor, Professor G. R. Livesay, for his constant help and encouragement. I also wish to thank Professors I. Berstein, P. J. Kahn, and H. C. Wang for many useful discussions.

1. **Livesay-Thomas decomposition theorem.** We have the following theorem from [20].

THEOREM 1.1. *For any free involution on a homotopy sphere* (Σ^{2n+1}, T), $n \geqslant 3$, *there exists an A-diffeomorphism g of* $S^n \times S^n$ *such that* $(\Sigma^{2n+1}, T) = (S^n \times D^{n+1}, A) \cup_g (D^{n+1} \times S^n, A)$, *denoted by* (Σ_g, T_g).

Note. $M \cup_g N$ denotes a manifold obtained by gluing two manifolds M and N together by a diffeomorphism $g\colon M_0 \to N_0$, where $M_0 \subseteq \partial M$ and $N_0 \subseteq \partial N$.

We shall prove the following proposition in §4 below.

PROPOSITION 1.2. *For any* (Σ^{2n+1}, T), *where* $n \neq 3, 7$, *there exists an A-diffeomorphism g of* $S^n \times S^n$ *such that g is homotopic to the identity and* $(\Sigma^{2n+1}, T) = (\Sigma_g, T_g)$.

Notation. Let "$\sim$" denote homotopic, and "$\sim^A$" denote A-homotopic.

LEMMA 1.3. *If g is an A-diffeomorphism of* $S^n \times S^n$ *such that* $g \sim \mathrm{Id}$ (*the identity*), *then there exists a pair of A-homotopy equivalences* f_1, f_2 *of* $S^n \times S^n$ *such that* $f_1 \sim f_2 \sim \mathrm{Id}$, *where* f_1 (f_2) *extends to an A-homotopy equivalence* h_1 (h_2) *of* $S^n \times D^{n+1}$ ($D^{n+1} \times S^n$); *and* $g \sim^A f_2 f_1$.

PROOF. Let p_j denote the projection of $S^n \times S^n$ onto the jth factor S_j^n, $j = 1$ or 2. p_j is A-equivariant: $p_j(Ax, Ay) = A(p_j(x, y))$. Let $g_j = p_j \circ g$. We define f_1 by $f_1(x, y) = (x, g_2(x, y))$. g^{-1} is also an A-diffeomorphism. Let $k_j = p_j \circ g^{-1}$. We define f_3 by $f_3(x, y) = (k_1(x, y), y)$.

$$\begin{aligned} f_3 \circ g(x, y) &= f_3(g_1(x, y), g_2(x, y)) = (k_1(g_1(x, y), g_2(x, y)), g_2(x, y)) \\ &= (x, g_2(x, y)) = f_1(x, y). \end{aligned}$$

f_1 and f_3 are obviously A-maps.

Since $g \sim \mathrm{Id}$, $(x_0, y) \to (x_0, g_2(x_0, y))$ is a degree 1 map of $x_0 \times S^n$ to itself for each $x_0 \in S^n$. Hence $f_1 | x_0 \times S^n \to x_0 \times S^n$ is a homotopy equivalence for each $x_0 \in S^n$. We have a locally trivial fibre bundle $S_2^n \to S^n \times S^n/A \to S_1^n/A$, with base space S_1^n/A and fibre S_2^n. The map $f_1'\colon S^n \times S^n/A \to S^n \times S^n/A$, induced by f_1, is fibre preserving. The restriction of f_1' to each fibre is a homotopy equivalence. Hence f_1' is a fibre homotopy equivalence by a theorem of Dold [10], and so f_1 is an A-homotopy equivalence. Similarly, we can show that f_3 induces a fibre homotopy equivalence f_3' of the bundle $S_1^n \to S^n \times S^n/A \to S_2^n/A$. Let f_2' be the fibre homotopy inverse of f_3', and write f_2 for the double cover of f_2' such that f_2 is the A-homotopy inverse for f_3. Now since $f_3 \circ g = f_1$, it follows that $g \sim^A f_2 f_1$.

Since the A-map $g_2\colon S^n \times S^n \to S_2^n$ extends to an A-map $\bar{g}_2\colon S^n \times D^{n+1} \to D^{n+1}$ by radial extension, we define an A-homotopy equivalence h_1 of $S^n \times D^{n+1}$ by $h_1(x, y) = (x, \bar{g}_2(x, y))$. An A-homotopy equivalence h_2 of $D^{n+1} \times S^n$ can be defined similarly. $f_1 \sim f_2 \sim \mathrm{Id}$ follows from [19, 2.5]. Q.E.D.

LEMMA 1.4. *Suppose* $(\Sigma^{2n+1}, T) = (\Sigma_g, T_g) = S^n \times D^{n+1} \cup_{\mathrm{Id}} S^n \times S^n \times I \cup_g D^{n+1} \times S^n$ *for an A-diffeomorphism g as in* (1.2). *Then there exists an equivariant homotopy equivalence* F: $(\Sigma^{2n+1}, T) \to (S^{2n+1}, A) = S^n \times D^{n+1} \cup_{\mathrm{Id}} S^n \times S^n \times I \cup_{\mathrm{Id}} D^{n+1} \times S^n$ *such that each summand is mapped into the corresponding one by an A-homotopy equivalence.*

PROOF. For such an A-diffeomorphism g, there exists f_1, f_2, h_1, h_2 as in (1.3). Write f_2^{-1}, h_2^{-1} for A-homotopy inverses of f_2, h_2. Let H be an A-homotopy between $f_2^{-1} \circ g$ and f_1 with $H(x, 0) = f_1(x)$. We then define $F = h, H, h_2^{-1}$ on each summand as follows:

$$\begin{array}{ccccccc}
(\Sigma^{2n+1}, T) = & S^n \times D^{n+1} & \cup_{\mathrm{Id}} & S^n \times S^n \times I & \cup_g & D^{n+1} \times S^n & \\
\big\downarrow F & \big\downarrow h_1 & & \big\downarrow H & & \big\downarrow h_2^{-1} & \\
(S^{2n+1}, A) = & S^n \times D^{n+1} & \cup_{\mathrm{Id}} & S^n \times S^n \times I & \cup_{\mathrm{Id}} & D^{n+1} \times S^n & \text{Q.E.D.}
\end{array}$$

An invariant m-manifold for (Σ^k, T) is an embedded m-manifold $M^m \subseteq \Sigma^k$ which is invariant under T. An invariant M^m for (Σ^k, T) is called characteristic if there is an equivariant map F: $(\Sigma^k, T) \to (S^N, A)$, $N \geqslant k$, such that F is transverse to $S^{N+m-k} \subseteq S^N$ and $F^{-1}(S^{N+m-k}) = M^m$.

PROPOSITION 1.5. *Let* $(\Sigma^{2n+1}, T) = (\Sigma_g, T_g)$ *for an A-diffeomorphism g as in* (1.2). *If one of* f_1, f_2 *corresponding to g in* (1.3) *is A-homotopic to* Id, *then* (Σ^{2n+1}, T) *admits* S^m, *where* $m = 1, \ldots, n$, *as characteristic spheres, such that* $(S^m, T|S^m)$ *is conjugate to* (S^m, A).

PROOF. Suppose $f_1 \sim^A \mathrm{Id}$. We take H in (1.4) to be an A-homotopy between $f_2^{-1} \circ g$ and Id, and $h_1 = \mathrm{Id}$. Let $S^m = S^m \times 0 \subseteq S^n \times D^{n+1}$. From (1.4), we see that $F^{-1}(S^m) = h_1^{-1}(S^m) = S^m$. Since $h_1 = \mathrm{Id}$, $(S^m, T|S^m)$ is equivalent to (S^m, A). Q.E.D.

REMARK 1.6. In [6], Browder showed that there exists a smooth involution (Σ_0^{4k+1}, T_0) which admits no m-dimensional homotopy sphere, $m \neq 4l + 1$, as characteristic manifold. Hence, any A-diffeomorphism g of $S^n \times S^n$ such that $(\Sigma_g, T_g) = (\Sigma_0^{4k+1}, T_0)$ is not A-homotopic to the identity by (1.5).

2. **Nonuniqueness of the decomposition.** The decomposition for (Σ^{2n+1}, T) in (1.1) is not unique: we may have different A-diffeomorphisms f and g such that $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f) = (\Sigma_g, T_g)$. But we have the following

PROPOSITION 2.1 [20]. *For* $n \geqslant 3$, $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f) = (\Sigma_g, T_g)$ *iff there exist A-diffeomorphisms* H: $S^n \times D^{n+1} \to S^n \times D^{n+1}$ *and* K: $D^{n+1} \times S^n \to D^{n+1} \times S^n$ *such that, when we restrict our attention to the boundary,* $g = KfH$.

Two diffeomorphisms f and g of a manifold M are called concordant, if there exists a diffeomorphism H: $M \times [0, 1] \to M \times [0, 1]$ such that $H(x, 0) =$

$(f(x), 0), H(x, 1) = (g(x), 1)$. Similarly, we have the notion of A-concordance between two A-diffeomorphisms. If f and g are A-concordant diffeomorphisms of $S^n \times S^n$, then $(\Sigma_f, T_f) = (\Sigma_g, T_g)$, which can be seen by constructing an equivariant diffeomorphism F between them as follows:

$$\begin{array}{ccccccc}
(\Sigma_f, T_f) = & S^n \times D^{n+1} & \cup_f & S^n \times S^n \times I & \cup_{\mathrm{Id}} & D^{n+1} \times S^n \\
\downarrow F & \downarrow \mathrm{Id} & & \downarrow H & & \downarrow \mathrm{Id} \\
(\Sigma_g, T_g) = & S^n \times D^{n+1} & \cup_{\mathrm{Id}} & S^n \times S^n \times I & \cup_g & D^{n+1} \times S^n
\end{array}$$

where H^{-1} is an A-concordance between f and g. Q.E.D.

Now, we are going to determine the A-diffeomorphisms H and K in (2.1) within A-concordance classes.

DEFINITION. A bundle map f_a for $S^n \times S^n$ over the first factor is a map of the form $f_a(x, y) = (x, a(x) \cdot y)$, where the homotopy class $\{a\} \in \pi_n(SO_{n+1})$ and a is a smooth map of S^n to SO_{n+1}.

A bundle map f_a is a diffeomorphism. If $\{a\}, \{b\} \in \pi_n(SO_{n+1})$ are homotopic, then f_a and f_b are concordant. Conversely, if f_a and f_b are concordant, then it was shown in [19, 5.2] that a and b are homotopic. Actually, we have

LEMMA 2.2. [19]. *The concordance classes of orientation-preserving diffeomorphisms of $S^n \times S^n$, which can be extended to orientation-preserving diffeomorphisms of $S^n \times D^{n+1}$, are in* 1-1 *correspondence with $\pi_n(SO_{n+1})$.*

We write $S^n = \{x = (x_0, \ldots, x_n) \in R^{n+1} | \Sigma_{i=0}^n x_i^2 = 1\}$. A acts on S^n as an $(n + 1)$-square matrix with -1 on its diagonal and 0 elsewhere. For $(x, y) \in S^n \times S^n, A(x, y) = (-x, -y)$.

LEMMA 2.3. *Let $b \in \pi_n(SO_{n+1})$. Then the bundle map f_b defined above is an A-equivariant bundle map if and only if b factors through P^n by the double covering map π: $S^n \to P^n$.*

PROOF. $f_bA = Af_b \Longleftrightarrow f_bA(x, y) = Af_b(x, y) \Longleftrightarrow (-x, b(-x) \cdot (-y)) = (-x, -b(x) \cdot y) \Longleftrightarrow b(-x) \cdot A \cdot y = A \cdot b(x) \cdot y$. Since A lies in the center of SO_{n+1}, we have $b(x) \cdot y = b(-x) \cdot y$ for all $y \in S^n$. Hence $f_bA = Af_b$ iff $b(x) = b(-x)$ for all $x \in S^n$ iff b factors through π: $S^n \to P^n$. Q.E.D.

LEMMA 2.4. *Every A-diffeomorphism of $S^n \times S^n$, $n \geqslant 3$, which can be extended to an orientation-preserving A-diffeomorphism of $S^n \times D^{n+1}$ is A-concordant to an A-equivariant bundle map over the first factor.*

PROOF. Let f be such an A-diffeomorphism, and h be its A-equivariant extension to $S^n \times D^{n+1}$. $f'(h')$ denotes the map induced by $f(h)$ on the orbit space $S^n \times S^n/A(S^n \times D^{n+1}/A)$. Let i': $S^n \times 0/A \to S^n \times D^{n+1}/A$ be the inclusion. $h'|S^n \times 0/A$ and i' are homotopic by [24]. Hence, they are isotopic by a theorem of Haefliger, [12]. By the equivariant isotopy extension theorem, [28], there exists an A-equivariant diffeomorphism H of $S^n \times D^{n+1}$ such that H is equivariantly isotopic to identity, $H|S^n \times S^n =$ identity, and $H \circ h|S^n \times 0/A = i$, the inclusion. Let B^{n+1} be a small disk in D^{n+1}, with B^{n+1} and D^{n+1} concentric. Both $S^n \times B^{n+1}$ and $H \circ h(S^n \times B^{n+1})$ are equivariant tubular neighborhoods of $S^n \times 0$ in $S^n \times D^{n+1}$. Then by the uniqueness of the equivariant tubular neighborhoods, [4, p. 310], there exists an A-equivariant diffeomorphism G of $S^n \times D^{n+1}$ such that G is A-equivariantly isotopic to identity, $G|S^n \times S^n = \mathrm{Id}$, and $G \circ H \circ h|S^n \times B^{n+1}$ is an A-equivariant bundle map covering the identity on S^n. The restriction of $G \circ H \circ h$ to $S^n \times D^{n+1}$-interior $S^n \times B^{n+1}$ gives us an A-concordance between f and an A-equivariant bundle map. Q.E.D.

Similarly, every A-diffeomorphism of $S^n \times S^n$ which extends equivariantly to $D^{n+1} \times S^n$ is A-concordant to an A-bundle over the second factor.

PROPOSITION 2.5. *The A-concordance classes of orientation-preserving A-diffeomorphism H (or K) in* (2.1) *are in* 1-1 *correspondence with the* Image π^* *of* $[P^n, SO_{n+1}]$ *in* $[S^n, SO_{n+1}]$.

PROOF. Let h be an A-diffeomorphism of $S^n \times S^n$, which can be extended equivariantly to $S^n \times D^{n+1}$. h is A-concordant to an A-equivariant bundle map f_b by (2.4), where $b \in \pi_n(SO_{n+1})$. From (2.3), we know that b factors through $\pi\colon S^n \to P^n$, i.e. $b \in$ Image π^*.

The above correspondence $H \to b$ is well-defined. If H' is A-concordant to H, and $b' \in$ Image π^* corresponds to H', then f_b and $f_{b'}$ are A-concordant, hence concordant. b is homotopic to b' by (2.2). This correspondence is 1-1 and onto, since the mapping given by $b \to f_b$ for $b \in$ Image π^* is its inverse. Q.E.D.

3. The image of π^*: $[P^n, SO_{n+1}] \to [S^n, SO_{n+1}]$. In this section, we will compute $[P^n, SO_{n+1}]$ and its image under π^*: $[P^n, SO_{n+1}] \to [S^n, SO_{n+1}]$. Let us first recall some facts about $\pi_n(SO_{n+1})$, which, for instance, can be found in [16] or [18].

Let $s_{m*}\colon \pi_k(SO_m) \to \pi_k(SO_{m+1})$ denote the homomorphism induced by the natural embedding $s\colon SO_n \to SO_{n+1}$. Consider the following exact sequence,

$$\cdots \xrightarrow{\partial_m} \pi_k(SO_m) \xrightarrow{s_{m*}} \pi_k(SO_{m+1}) \xrightarrow{q_{m*}} \pi_k(S^m) \xrightarrow{\partial_m} \pi_{k-1}(SO_m) \to \cdots.$$

Let ι_m denote the generator of $\pi_m(S^m)$. Write $\tau_m = \partial_{m+1}\iota_{m+1} \in \pi_m(SO_{m+1})$. Putting $m = n, n+1$ in the above exact sequence, we have the following proposition from [18]:

PROPOSITION 3.1. *For n odd,* $\neq 1, 3, 7$, $\pi_n(SO_{n+1})$ *is the direct sum of two cyclic subgroups* image ∂_{n+1} *and* image s_{n*}. *Moreover,* s_{n+1*}: image $s_{n*} \subseteq$

$\pi_n(SO_{n+1}) \to \pi_n(SO_{n+2})$ is an isomorphism. For n even, $\pi_n(SO_{n+1})$ is the direct sum of image ∂_{m+1} *and a certain cyclic subgroup G such that s_{n+1*}: $G \subseteq \pi_n(SO_{n+1}) \to \pi_n(SO_{n+2})$ is an isomorphism.*

From now on, we will write τ for $\tau_n = \partial_{n+1}\iota_{n+1}$, if no confusion will arise. Let σ denote the generator of the other cyclic summand of $\pi_n(SO_{n+1})$. Here we list the values of $\pi_n(SO_{n+1})$ and $\pi_n(SO_{n+2}) = \pi_n(SO)$ for $n > 3$ and $n \neq 7$, from [16].

n(mod 8)	0	1	2	3	4	5	6	7
$\pi_n(SO_{n+1})$	$Z_2 + Z_2$	$Z + Z_2$	Z_2	$Z + Z$	Z_2	Z	Z_2	$Z + Z$
	τ, σ	τ, σ	τ	τ, σ	τ	τ	τ	τ, σ
$\pi_n(SO)$	Z_2	Z_2	0	Z	0	0	0	Z

Consider the maps π: $S^n \to P^n$, i: $P^n \to P^{n+1}$, j: $P^n \to S^n$; which are the double covering, the inclusion, and the map pinching the complement of an open ball to a point. j generates $[P^n, S^n] = H^n(P^n) = Z$ or Z_2, n odd or even; by Hopf's theorem, [23].

The element $\tau \in \pi_n(SO_{n+1})$ is also the characteristic map for the tangent bundle of S^{n+1}, [14] or [30]. We can choose a representative for τ such that $\tau(x) = \tau(-x)$ for $x \in S^n$, [14], which is defined as follows: let α: $S^n \to SO_{n+1}$ be the map defined by the requirement that $\alpha(x)$ be a reflection through the hyperplane in R^{n+1} orthogonal to x, and let e denote the north pole of S^n, then we have $\tau(x) = \alpha(x)\alpha(e)$, [14, p. 89]. Hence this τ factors through π: $S^n \to P^n$, and $\tau_p \in [P^n, SO_{n+1}]$ is defined by $\tau = \tau_p\pi$.

Let q: $SO_{n+1} \to S^n$ denote the projection in the fibration $SO_n \to SO_{n+1} \to S^n$.

PROPOSITION 3.2. *The following diagram is homotopically commutative.*

$$\begin{tikzcd} S^n \arrow[r, "\pi"] \arrow[d, "\tau"'] & P^n \arrow[d, "j"] \arrow[dl, "\tau_p"'] \\ SO_{n+1} \arrow[r, "q"'] & S^n \end{tikzcd}$$

PROOF. We have to show that $q\tau_p$ is homotopic to j. From the above description of τ, $\tau(x) = \alpha(x)\alpha(e)$; we see that $q\tau(x) = \alpha(x)\alpha(e) = \alpha(x)(-e)$, which is the point on S^n obtained by moving e toward x along the great circle passing through e and x by an angle twice the angle between e and x. We note that $q\tau$ maps the interior of the northern hemisphere D^n_+ of S^n onto $S^n - \{e\}$ as a homeomorphism, and maps the equator S^{n-1} to $-e$. Since π maps interior D^n_+ homeomorphically onto $P^n - P^{n-1}$, and $q\tau = q\tau_p\pi$, we see that $q\tau_p$ is just the map pinching the complement of an open ball to a point, which is j. Q.E.D.

LEMMA 3.3. *τj is null-homotopic in* $[P^{2k}, SO_{2k+1}]$.

PROOF. We have $\tau j = \tau_p \pi j$: $P^{2k} \to S^{2k} \to SO_{2k+1}$. Consider $f = \pi j$: $P^{2k} \to P^{2k}$. Since f factors through S^{2k}, $f_*(\pi_1(P^{2k})) = 0$. Hence f is a non-orientation-true map in the language of [25]. By Theorem 1.2 of [25], (see also [25, 1.3(d)]), f is null-homotopic. Q.E.D.

Let $KO^{-k}(-)$ denote the reduced real K-theory. We have $KO^{-k}(X) = [\Sigma^k X, B_{SO}]$ for any finite CW complex X, [14]. $KO^{-1}(P^n) = [\Sigma P^n, B_{SO}] = [P^n, \Omega B_{SO}] = [P^n, SO]$, the latter one is equal to $[P^n, SO_{n+2}]$ because (SO, SO_{n+2}) is $(n + 1)$-connected.

For any fibration $F \to E \to B$, and any finite CW complex X, there is a fibre mapping sequence $\cdots \to [X, \Omega E] \to [X, \Omega B] \to [X, F] \to [X, E] \to [X, B]$, [23], which is exact.

LEMMA 3.4. *For n odd, $Z \to [P^n, SO_{n+1}] \to [P^n, SO_{n+2}] \to 0$ is exact. For n even, $Z_2 \to [P^n, SO_{n+1}] \to [P^n, SO_{n+2}] \to 0$ is exact.*

PROOF. Substitute $SO_{n+1} \to SO_{n+2} \to S^{n+1}$ and P^n into the above fibre mapping sequence. $[P^n, S^{n+1}] = 0$. $[P^n, \Omega S^{n+1}] = [\Sigma P^n, S^{n+1}] = H^{n+1}(\Sigma P^n) = H^n(P^n) = Z$, n odd; $= Z_2$, n even. Q.E.D.

We also need the following from [6] or [11].

PROPOSITION 3.5. *$KO^{-1}(P^m) = Z_2$, for $m \not\equiv 3 \bmod 4$; $KO^{-1}(P^m) = Z + Z_2$, for $m \equiv 3 \bmod 4$, where the Z summand is the image of $KO^{-1}(S^m)$ under the degree 1 map j: $P^m \to S^m$. The inclusion $P^k \subseteq P^m$ induces $KO^{-1}(P^m) \to KO^{-1}(P^k)$, which is an isomorphism on Z_2 and annihilates the Z factor.*

Replacing P^k by the mapping cylinder M_π of π: $S^k \to P^k$, we can change π into a cofibration π': $S^k \to M_\pi$, and we may consider i: $P^k \to P^{k+1}$ as the cofibre. For a simple space X, we have the following Puppe exact sequence, [27],

$$\cdots \longrightarrow [\Sigma P^k, X] \xrightarrow{\Sigma\pi^*} \pi_{k+1}(X) \xrightarrow{j^*} [P^{k+1}, X] \xrightarrow{i^*} [P^k, X] \xrightarrow{\pi^*} \pi_k(X).$$

Putting $k = n - 1$ and $X = SO_{n+1}$ in the above Puppe sequence, we have the following exact sequence:

$$\cdots \longrightarrow \pi_n(SO_{n+1}) \xrightarrow{j^*} [P^n, SO_{n+1}] \xrightarrow{i^*} [P^{n-1}, SO_{n+1}] \xrightarrow{\pi^*} \pi_{n-1}(SO_{n+1}). \tag{3.6}$$

For n odd, $j\pi$: $S^n \to P^n \to S^n$ is of degree 2. Hence $\pi^* j^*$: $\pi_n(SO_{n+1}) \to [P^n, SO_{n+1}] \to \pi_n(SO_{n+1})$ is just the multiplication by 2.

If $n \equiv 3 \mod 4$, then $\pi_n(SO_{n+1}) = Z + Z$, generated by τ and σ. $\pi^* j^*$ is 1-1, hence j^* is 1-1. Thus $[P^n, SO_{n+1}]$ contains $Z + Z$. One of the generators is τ_p, since $\pi^*\tau_p = \tau_p \pi = \tau$. Therefore $j^*\tau = 2\tau_p$. For $m \equiv 2 \mod 4$, $\pi_m(SO_{m+2}) = 0$ and $[P^m, SO_{m+2}] = KO^{-1}(P^m) = Z_2$ by (3.5). Hence, the exact sequence (3.6) becomes $0 \to Z + Z \xrightarrow{j^*} [P^n, SO_{n+1}] \to Z_2 \to 0$. Since $j^*\tau = 2\tau_p$, we see that $[P^n, SO_{n+1}] = Z + Z$, generated by τ_p and $b = j^*\sigma$. $\pi^* b = \pi^* j^* \sigma = 2\sigma$.

If $n \equiv 5 \mod 8$, then $\pi_n(SO_{n+1}) = Z$, generated by τ. The argument in the preceding paragraph shows that $[P^n, SO_{n+1}] = Z$, generated by τ_p, and $j^*\tau = 2\tau_p$.

Now consider the following commutative diagram, where the rows are Puppe sequences and the columns are fibre mapping sequences.

$$
\text{(3.7)}\qquad
\begin{tikzcd}
\pi_{n+1}(S^{n+1}) \arrow[r, "j_1"] \arrow[d, "\partial"] & {[P^n, \Omega S^{n+1}]} \arrow[r] \arrow[d, "\partial_1"] & {[P^{n+1}, \Omega S^{n+1}] = 0} \arrow[d] \\
\pi_n(SO_{n+1}) \arrow[r, "j^*"] \arrow[d, "s_*"] & {[P^n, SO_{n+1}]} \arrow[r, "i^*"] \arrow[d, "s'_*"] & {[P^{n-1}, SO_{n+1}]} \arrow[d] \\
\pi_n(SO_{n+2}) \arrow[r, "j_2"] & {[P^n, SO_{n+2}]} \arrow[r] & {[P^{n-1}, SO_{n+2}]}
\end{tikzcd}
$$

If $n \equiv 1 \mod 8$, then $\pi_n(SO_{n+1}) = Z + Z_2$, generated by τ and σ respectively. Since j^* is 1-1 on the Z summand, and $\pi^*\tau_p = \tau$, $[P^n, SO_{n+1}]$ contains a Z subgroup which is generated by τ_p. Let ι_{n+1} denote the generator of $\pi_{n+1}(S^{n+1})$. In the diagram (3.7), $\partial\iota_{n+1} = \tau$ and $[P^n, \Omega S^{n+1}] = [\Sigma P^n, S^{n+1}] = H^n(P^n) = Z$ is generated by $j_1 \iota_{n+1}$. $[P^n, SO_{n+2}] = KO^{-1}(P^n) = Z_2$ by (3.5). The middle column of (3.7) reads $Z \xrightarrow{\partial_1} [P^n, SO_{n+1}] \xrightarrow{s'_*} Z_2$. If $\partial_1(aj_1\iota_{n+1}) = \tau_p$ for some integer a, then $j^*(a\tau) = aj^*\partial\iota_{n+1} = \partial_1(aj_1\iota_{n+1}) = \tau_p$, a contradiction. Hence $\tau_p \in$ image ∂_1, $s'_*\tau_p \neq 0$. $[P^n, SO_{n+1}] = Z$, generated by τ_p. Also $j^*\tau = 2\tau_p$, $j^*\sigma = 0$.

We now consider the case n is even.

Lemma 3.8. *For n even, $[P^n, SO_{n+1}] = Z_2$, which is generated by τ_p.*

Proof. In the diagram (3.7), j_1 is onto. Let ι_{n+1} generate $\pi_{n+1}(S^{n+1})$. $j_1\iota_{n+1}$ generates $[P^n, \Omega S^{n+1}] = H^{n+1}(\Sigma P^n) = Z_2$. $\partial_1(j_1\iota_{n+1}) = j^*\partial\iota_{n+1} = j^*\tau = \tau j = 0$ by (3.3). The map s'_* in the middle column in (3.7) is onto by (3.4). Hence s'_*: $[P^n, SO_{n+1}] \to [P^n, SO_{n+2}] = KO^{-1}(P^n) = Z_2$ is an isomorphism. Thus $[P^n, SO_{n+1}] = Z_2$, generated by τ_p. Q.E.D.

Summing up, we have the following:

THEOREM 3.9. *Assume* $n \neq 1, 2, 3, 7$. *For n even,* $[P^n, SO_{n+1}] = Z_2$, *generated by* τ_p, *and* $\pi^*\tau_p = \tau \in \pi_n(SO_{n+1})$. *For* $n \equiv 1 \mod 4$, $[P^n, SO_{n+1}] = Z$, *generated by* τ_p, *and* $\pi^*\tau_p = \tau$. *For* $n \equiv 3 \mod 4$, $[P^n, SO_{n+1}] = Z + Z$, *generated by* τ_p *and* b, *where* $b = j^*\sigma$, *and* $\pi^*\tau_p = \tau$, $\pi^*b = 2\sigma$.

Let Im π^* denote the image of $[P^n, SO_{n+1}]$ in $\pi_n(SO_{n+1})$ under π^*.

n(mod 8)	0	1	2	3	4	5	6	7
$\pi_n(SO_{n+1})$	$Z_2 + Z_2$	$Z + Z_2$	Z_2	$Z + Z$	Z_2	Z	Z_2	$Z + Z$
	τ, σ	τ, σ	τ	τ, σ	τ	τ	τ	τ, σ
Im π^*	τ	τ	τ	$\tau, 2\sigma$	τ	τ	τ	$\tau, 2\sigma$

COROLLARY 3.10. $n > 3$ *and* $\neq 7$. *Let* s'_*: $[P^n, SO_{n+1}] \to [P^n, SO_{n+2}] = KO^{-1}(P^n)$ *be induced by the inclusion. Then* $s'_*(\tau_p)$ *generates the* Z_2 *summand of* $KO^{-1}(P^n)$; *and for* $n \equiv 3 \mod 4$, $s'_*(b)$ *generates the Z summand of* $KO^{-1}(P^n)$.

PROOF. We have proved the corollary for n even in (3.8). For n odd, s'_*: $[P^n, SO_{n+1}] \to [P^n, SO_{n+2}]$ is onto, (3.4). In (3.7), let ι_{n+1} generate $\pi_{n+1}(S^{n+1})$, $j_1\iota_{n+1}$ generates $[P^n, \Omega S^{n+1}]$. $\partial_1(j_1\iota_{n+1}) = j^*\partial\iota_{n+1} = j^*\tau = 2\tau_p$. Hence s'_* maps τ_p to the generator of the Z_2 summand of $KO^{-1}(P^n)$, and maps b to the generator of the Z summand for $n \equiv 3 \mod 4$ by the exactness of the middle column in (3.7). Q.E.D.

4. Proof of Proposition 1.2. We are going to prove Proposition 1.2 in this section. Let $D(S^n \times S^n)(D(S^n \times S^n, A))$ denote the group of concordance (A-concordance) classes of diffeomorphisms (A-diffeomorphisms) of $S^n \times S^n$. $\varnothing$: $D(S^n \times S^n, A) \to D(S^n \times S^n)$ is the homomorphism forgetting the action. Define $D_0(S^n \times S^n)$ = the subgroup of $D(S^n \times S^n)$ consisting of those elements which are homotopic to Id. Let $J_{2n+1} = \varnothing^{-1}(D_0(S^n \times S^n))$. Given a diffeomorphism (an A-diffeomorphism) f of $S^n \times S^n$, we will write $\{f\}$ ($\{f\}_A$) for its concordance class in $D(S^n \times S^n)(D(S^n \times S^n, A))$.

If f is a diffeomorphism of $S^n \times S^n$, then f_* induces an automorphism of $H_n(S^n \times S^n)$. We can associate to f_* its matrix representative M_f with respect to the natural basis $\{S^n \times 0, 0 \times S^n\}$ of $H_n(S^n \times S^n)$. M_f is an element of $GL(2, Z)$ the group of 2×2-unimodular matrices. Let ψ: $D(S^n \times S^n) \to GL(2, Z)$ be the homomorphism defined by $f \to M_f$. We have $fg \to M_g \cdot M_f$.

From [19], we have the following:

PROPOSITION 4.1 [19]. *If n is even, then image* ψ *consists of eight matrices:*

$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$. *If* $n = 1, 3, 7$, *then image* $\psi = GL(2, Z)$. *If n is odd, but* $n \neq 1, 3, 7$, *then image* ψ *is the subgroup of* $GL(2, Z)$ *consisting of matrices* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *where* $ab \equiv cd \equiv 0 \bmod 2$.

LEMMA 4.2. *If f is a diffeomorphism of* $S^n \times S^n$ *such that* $S^n \times D^{n+1} \cup_f D^{n+1} \times S^n$ *is a homotopy sphere, then* $M_f = \begin{pmatrix} a & b \\ c & \pm 1 \end{pmatrix}$.

PROOF. Write $V_1 = S^n \times D^{n+1}$, $V_2 = D^{n+1} \times S^n$, $\partial V_1 = \partial V_2 = S^n \times S^n$. Let $i_k\colon \partial V_k \to V_k$, $k = 1$ or 2, be the inclusion. Let $\{x, y\}$ denote the natural basis $\{S^n \times 0, 0 \times S^n\}$ of $H_n(S^n \times S^n)$, and u_1, u_2 the generator of $H_n(S^n \times D^{n+1})$, $H_n(D^{n+1} \times S^n)$ respectively. From the Mayer-Vietoris sequence, we have

$$0 \longrightarrow H_n(\partial V_1) \xrightarrow{(j_1, j_2)} H_n(V_1) \oplus H_n(V_2) \longrightarrow 0$$

where $j_1 = i_{1*}$ and $j_2 = i_{2*}f_*$. $i_{1*}x = u_1$, $i_{1*}y = 0$, $i_{2*}x = 0$, $i_{2*}y = u_2$. Let $M_f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $f_*x = ax + by$ and $f_*y = cx + dy$. Hence $j_1 x = u_1$, $j_1 y = 0$, $j_2 x = i_{2*}f_*x = bu_2$, $j_2 y = i_{2*}f_*y = du_2$. Thus the matrix for (j_1, j_2) with respect to the basis $\{x, y\}$ and $\{u_1, u_2\}$ is $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$, which is unimodular, $d = \pm 1$. Q.E.D.

LEMMA 4.3. $n > 3$ *and* $\neq 7$. *If* $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$ *for some A-diffeomorphism f of* $S^n \times S^n$, *then there exists another A-diffeomorphism g such that* M_g *is the identity matrix, and* $(\Sigma^{2n+1}, T) = (\Sigma_g, T_g)$.

PROOF. If n is even, then it follows from (4.1) and (4.2) that $M_f = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. Consider the A-diffeomorphisms h_1 and h_2 of $S^n \times S^n$ defined by $h_1(x, y) = (x, -y)$ and $h_2(x, y) = (-x, y)$. h_1 (or h_2) extends equivariantly to $(S^n \times D^{n+1}, A)$ (or $(D^{n+1} \times S^n, A)$). One of the A-diffeomorphisms f, fh_1, $h_2 fh_1$, or $h_2 f$ has the corresponding matrix = identity matrix. Take g to be this map. Also, $(\Sigma^{2n+1}, T) = (\Sigma_g, T_g)$ by (2.1).

If n is odd and $\neq 1, 3, 7$, then $M_f = \begin{pmatrix} a & b \\ c & \pm 1 \end{pmatrix}$ by (4.2). We can compose f with h_1 or h_2 if necessary, to make $M_f = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ and $\det M_f = +1$. From (4.1), $b \equiv c \equiv 0 \bmod 2$. Since $a - bc = 1$, we see that $\begin{pmatrix} a & b \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.

We choose a representative for τ as in §3, satisfying $\tau(x) = \tau(-x)$. Consider the maps f_τ and g_τ, defined by $f_\tau(x, y) = (x, \tau(x) \cdot y)$ and $g_\tau(x, y) = (\tau(y) \cdot x, y)$. f_τ and g_τ are A-diffeomorphisms. Theorem 7.10.1 of [14] showed that the map $q \circ \tau$ in (3.2) is of degree 2. Hence $M_{f_\tau} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $M_{g_\tau} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Define $g = g_\tau^{(-c/2)} f f_\tau^{(-b/2)}$. Since $M_{fg} = M_g M_f$, we see that M_g is the identity matrix. $(\Sigma_f, T_f) = (\Sigma_g, T_g)$ by (2.1). Q.E.D.

LEMMA 4.4. *A diffeomorphism f of* $S^n \times S^n$ *is homotopic to* Id *if and only if* M_f *is the identity matrix.*

PROOF. If f and g are diffeomorphisms of $S^n \times S^n$ such that $\{f\} = \{g\}$ in $D(S^n \times S^n)$, then $M_f = M_g$. Let $G = \{\{f\} \in D(S^n \times S^n) | M_f$ is the identity matrix$\}$. Theorem II of [28] showed that $0 \to H_1 + \Gamma^{2n+1} \to G \to H_2 \to 0$ is exact, where H_i, $i = 1$ or 2, is isomorphic to image$\{s_*: \pi_n(SO_n) \to \pi_n(SO_{n+1})\}$. Let a be a smooth map representing $\{a\} \in$ image $s_* \subseteq \pi_n(SO_{n+1})$. We define two diffeomorphisms f_a and g_a of $S^n \times S^n$ by $f_a(x, y) = (x, a(x) \cdot y)$ and $g_a(x, y) = (a(y) \cdot x, y)$. The maps $\{a\} \to \{f_a\}$ and $\{a\} \to \{g_a\}$ are isomorphisms of image s_* to H_1 and H_2 respectively, (compare 2.2). Γ^{2n+1} is the Kervaire-Milnor group [17] and acts by leaving the complement of a $2n$-disk in $S^n \times S^n$ fixed.

Let a be a smooth map representing a homotopy class $\{a\} \in \pi_n(SO_{n+1})$ such that $\{a\} = s_*\beta$ for some $\beta \in \pi_n(SO_n)$. We can take $a|D^n_- = \mathrm{Id}$, where D^n_- (D^n_+) denotes the lower (upper) hemisphere of S^n. Hence $f_a|D^n_- \times S^n = \mathrm{Id}$. Let e denote the north pole of S^n. Since $\{a\} = s_*\beta$, we can take $f_a|S^n \times \{e\} = \mathrm{Id}$. By the homotopy extension theorem, f_a is homotopic to a map h such that h is the identity on a neighborhood N of $S^n \times \{e\}$ and $f_a = h$ on $D^n_- \times S^n$. h keeps the complement of a disk $S^n \times S^n)|M_f$ is the identity matrix$\}$.

Since every element $\{h\}$ of G has a representative h such that h leaves the complement of a $2n$-disk fixed, we can apply the Alexander trick to see that h is homotopic to Id. Thus $G = D_0(S^n \times S^n)$. Q.E.D.

Combining (4.3) and (4.4) together, we have proved Proposition (1.2). From now on, when $n > 3$ and $\neq 7$, we will assume the A-diffeomorphism f in the decomposition $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$, be homotopic to Id.

5. **Group structure on $hS(P^{2n+1})$.** Given an involution (Σ^{2n+1}, T), $n > 3$ and $\neq 7$, there exists an A-diffeomorphism f of $S^n \times S^n$ such that $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$, and $(\Sigma_f, T_f) = (\Sigma_g, T_g)$ if $g \in \{f\}_A \in J_{2n+1} = \emptyset^{-1}(D_0(S^n \times S^n))$. But $\{f\}_A \in J_{2n+1}$ is not uniquely determined by (Σ^{2n+1}, T). Suppose we can find a subgroup G_{2n+1} of J_{2n+1} such that G_{2n+1} is in 1-1 correspondence with $hS(P^{2n+1})$ under the mapping $\{f\}_A \to (\Sigma_f, T_f)$. Then $(hS(P^{2n+1}), *)$ forms a group by carrying over the composition law in G_{2n+1}: $(\Sigma_f, T_f)^*(\Sigma_g, T_g) = (\Sigma_{fg}, T_{fg})$. In this section, we will show that such a subgroup G_{2n+1} exists for $n \equiv 0, 1, 2 \bmod 4$.

THEOREM 5.1. *For n even, and > 2, such a subgroup G_{2n+1} of J_{2n+1} exists, hence $(hS(P^{2n+1}), *)$ is a subgroup.*

PROOF. From [19] or (4.4) above, we know that $D_0(S^n \times S^n)$ is the semi-direct product of $H_1 + \Gamma^{2n+1}$ and H_2, where H_1 and H_2 are isomorphic to image$\{s_*: \pi_n(SO_n) \to \pi_n(SO_{n+1})\}$, which is equal to $\pi_n(SO_{n+1})$, [18]. $\pi_n(SO_{n+1}) = Z_2$, generated by τ for $n \not\equiv 0 \bmod 8$; $\pi_n(SO_{n+1}) = Z_2 + Z_2$, generated by τ and σ for $n \equiv 0 \bmod 8$. Every element of $D_0(S^n \times S^n)$ can be uniquely expressed in the form h_2yh_1, where $h_i \in H_i$ and $y \in \Gamma^{2n+1}$, [17]. We define a subgroup F_{2n+1} of $D_0(S^n \times S^n)$ as follows: $F_{2n+1} = \Gamma^{2n+1}$ for $n \not\equiv 0$ mod 8; if $n \equiv 0$ mod 8, then F_{2n+1} is the semidirect product of $(\sigma_1) + \Gamma^{2n+1}$ and (σ_2), where (σ_i) denotes the cyclic group of order 2 generated by σ_i: $\sigma_1(x, y) = (x, \sigma(x) \cdot y)$, $\sigma_2(x, y) = (\sigma(y) \cdot x, y)$. Let $G_{2n+1} = \emptyset^{-1}(F_{2n+1})$.

We choose a smooth representative for $\tau \in \pi_n(SO_{n+1})$ such that $\tau(x) = \tau(-x)$. Define τ_1 and τ_2 by $\tau_1(x, y) = (x, \tau(x) \cdot y)$ and $\tau_2(x, y) = (\tau(y) \cdot x, y)$. τ_1 and τ_2 are A-diffeomorphisms of $S^n \times S^n$. Since τ commutes with σ in $\pi_n(SO_{n+1})$, [19], we see that any element of $D_0(S^n \times S^n)$ can be uniquely expressed in the form bya, where $b \in (\tau_2)$, $a \in (\tau_1)$, and $y \in F_{2n+1}$.

If $h \in \{h\}_A \in J_{2n+1}$, then $\emptyset(\{h\}_A) = \{\tau_2^d\}\{f\}\{\tau_1^c\}$, where $\{f\} \in F_{2n+1}$, and $c, d = 0$ or 1 uniquely determined by $\{h\}_A$. We define $g = \tau_2^d h \tau_1^c$, which is an A-diffeomorphism of $S^n \times S^n$. Since $\{\tau_i\}$ is of order two in $D_0(S^n \times S^n)$, $\emptyset(\{g\}_A) \in F_{2n+1}$. $(\Sigma_g, T_g) = (\Sigma_h, T_h)$ by (2.1). On the other hand, if $\{f\}_A$, $\{g\}_A \in G_{2n+1}$ and $(\Sigma_f, T_f) = (\Sigma_g, T_g)$, then g is A-concordant to $\tau_2^d f \tau_1^c$, where $c, d = 0$ or 1, by (2.1) and (3.9). But $\emptyset(\{f\}_A)$, $\emptyset(\{g\}_A) \in F_{2n+1}$. Hence $c = d = 0$, $\{f\}_A = \{g\}_A$. Q.E.D.

THEOREM 5.2. *For $n \equiv 1$ mod 4, $hS(P^{2n+1})$ is in 1-1 correspondence with J_{2n+1}; hence $(hS(P^{2n+1}), *)$ forms a group.*

PROOF. As in (5.1), we know that $D_0(S^n \times S^n)$ is the semidirect product of $H_1 + \Gamma^{2n+1}$ and H_2, where H_1 and H_2 are isomorphic to image$\{s_*: \pi_n(SO_n) \to \pi_n(SO_{n+1})\}$, which is 0 for $n \equiv 5$ mod 8; and Z_2, generated by σ for $n \equiv 1$ mod 8. $\sigma \notin$ image$\{\pi^*: [P^n, SO_{n+1}] \to [S^n, SO_{n+1}]\}$ by (3.9). Hence no element in $D_0(S^n \times S^n)$ is concordant to an A-bundle map by (2.5). Thus $hS(P^{2n+1})$ is in 1-1 correspondence with $J_{2n+1} = \emptyset^{-1}(D_0(S^n \times S^n))$ by (1.2) and (2.1). Q.E.D.

REMARK 5.3. For $n > 7$ and $n \equiv 3$ mod 4, we have the exact sequence: $0 \to Z + \Gamma^{2n+1} \to D_0(S^n \times S^n) \to Z \to 0$, where each Z is isomorphic to image$\{s_*: \pi_n(SO_n) \to \pi_n(SO_{n+1})\}$, generated by σ. We know that $2m\sigma \in$ image π^* but $(2m + 1)\sigma \notin$ image π^* by (3.9). Let σ_1 and σ_2 be defined by $\sigma_1(x, y) = (x, \sigma(x) \cdot y)$, $\sigma_2(x, y) = (\sigma(y) \cdot x, y)$. If none of the four diffeomorphisms σ_1, σ_2, $\sigma_2\sigma_1$, and $\sigma_1\sigma_2$ is concordant to an A-diffeomorphism, i.e. $\{\sigma_1\}$, $\{\sigma_2\}$, $\{\sigma_2\sigma_1\}$, $\{\sigma_1\sigma_2\} \notin$ image $\emptyset$, then we can take $G_{2n+1} = \emptyset^{-1}(\Gamma^{2n+1})$ as in (5.1) and (5.2).

REMARK 5.4. *The case $n = 3$, $hS(P^7)$.* Viewing S^3 as the unit sphere in the quaternionic space, we define $\{r\}, \{t\} \in \pi_3(SO_4)$ by $r(x) \cdot y = xyx^{-1}$, $t(x) \cdot y = xy$. $\pi_3(SO_4) = Z + Z$ is generated by $\{r\}$ and $\{t\}$, [14, p. 94]. As in (3.9), we can show that image$\{\pi^*: [P^3, SO_4] \to \pi_3(SO_4)\}$ is generated by $\{r\}$ and $2\{t\}$.

Let t_1 and t_2 be defined in the same way as σ_1 and σ_2 in (5.3). If $\{t_1\}, \{t_2\}, \{t_1t_2\}, \{t_2t_1\} \notin$ image $\emptyset$, then (1.2) is also true in this case and $(hS(P^7), *)$ forms a group.

6. Additivity of Browder-Livesay index invariant. In [7], [8], Browder and Livesay defined a desuspension invariant σ for any free involution (Σ^{2n+1}, T) as follows: construct an $(n-1)$-connected characteristic submanifold N^{2n} for (Σ^{2n+1}, T), i.e. $\Sigma^{2n+1} = A \cup B$, $A \cap B = N$, $TA = B$, $TN = N$, and A, B are $(2n+1)$-submanifolds of Σ^{2n+1}. Let $K_n = \mathrm{Ker}(H_n(N) \to H_n(A))$. If n is odd, they define a unimodular even symmetric bilinear form B on K_n (modulo torsion) by $B(x, y) = x \cdot T_* y$. Let $\sigma(\Sigma^{2n+1}, T) = (1/8)$ index $B \in Z$. If n is even, they use Z_2 as coefficients and define the unimodular bilinear form B_2 on K_n (with Z_2 coefficients) by $B_2(x, y) = x \cdot T_* y$. They also defined a quadratic form $\psi\colon K_n \to Z_2$ associated to B_2, $B_2(x, y) = \psi(x+y) + \psi(x) + \psi(y)$, such that, if $x \in K_n$ is represented by an immersed sphere d, then $\psi(x) = 1$ iff $d \cap Td$ in general position consists of an odd number of pairs of points. Write $\sigma(\Sigma^{2n+1}, T) = c(\psi) \in Z_2$, the Art invariant of ψ. They also showed that for $n \geqslant 3$, $\sigma(\Sigma^{2n+1}, T) = 0$ iff (Σ^{2n+1}, T) admits a codimension 1 invariant sphere (for details, see [8]).

From now on, we assume $n \geqslant 3$. Suppose $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$ for an A-diffeomorphism f of $S^n \times S^n$, let $j\colon S^n \times S^n/A \to S^n \times S^{m-1}/A \to P^{n+m} = (S^n \times D^m/A) \cup_{\mathrm{Id}} (D^{n+1} \times S^{m-1}/A)$, with m large, be the natural inclusion. Both $j \circ (f/A)$ and j are classifying maps for the same Z_2-bundle. Hence they are homotopic, by a map $F\colon (S^n \times S^n \times I)/A \times \mathrm{Id} \to P^{n+m}$. We may suppose that F is smooth and transverse regular on $P^{n+m-1} = (S^n \times D^{m-1}/A) \cup_{\mathrm{Id}} (D^{n+1} \times S^{m-2}/A)$, relative boundary. The double cover M of $F^{-1}(P^{n+m-1})$ is a characteristic submanifold of $(S^n \times S^n \times I, A \times \mathrm{Id})$, with $\partial M = M_1 - M_0$, where $M_1 = f^{-1}(S^n \times S^{n-1}) \times 1$, $M_0 = (S^n \times S^{n-1}) \times 0$. We recall that a codim 1 characteristic submanifold M of a free involution (W, T) is a codim 1 submanifold of W such that $W = A \cup B$, where A and B are codim 0 submanifolds of W, $A \cap B = M$, and $TA = B$, [8].

In the rest of this section, we will write $(W, T) = (S^n \times S^n \times I, A \times \mathrm{Id})$, $W_1 = S^n \times S^n \times 1$, $W_0 = S^n \times S^n \times 0$; also, let $W = V \cup TV$, $V \cap TV = M$, $V_i = V \cap W_i$ for $i = 0, 1$, where $V_0 = S^n \times D^n$, $V_1 = f^{-1}(S^n \times D^n)$.

Since $S^n \times S^n/A$ is the total space of a spherical fibre bundle $S^n \to S^n \times S^n/A \to S^n/A$, (§1). By Gysin sequence, $H_k(S^n \times S^n/A; Z_2) = Z_2$ for $k \neq n$, and $= Z_2 + Z_2$ for $k = n$. Hence $H_k(S^n \times S^n/A; Z_2) = H_k(W/T; Z_2) = H^{2n+1-k}(W/T, \partial W/T; Z_2)$.

We want to make a characteristic submanifold M of (W, T) as highly connected as possible.

LEMMA 6.1. *There exists a connected characteristic submanifold M for (W, T) with $\partial M = M_1 - M_0$.*

PROOF. Let M be a characteristic submanifold constructed above with $\partial M = M_1 - M_0$. Then $(M, \partial M)/T$ carries the unique nonzero element of $H_{2n}(W/T, \partial W/T; Z_2) = Z_2$, dual to the 1-dimensional cohomology class F^*x, where x generates $H^1(P^{n+m}; Z_2)$ and F is the classifying map constructed above, [32]. Hence a component of M/T carries this element. Let M' be the double cover of this component. If $M' \cap M_0 = \emptyset$, then we can take $y \in H^1(W_0/T; Z_2) = H^1(W/T; Z_2) = Z_2$ representing F^*x, hence $H_{2n}(M'/T, \partial M'/T; Z_2) \to H_{2n}(W/T, \partial W/T; Z_2)$ is trivial, a contradiction. Hence $M_0 \cap M' \neq \emptyset$. Since M_0 is a closed connected manifold, $M_0 \subseteq M'$. Similarly $M_1 \subseteq M'$. It is clear that T interchanges the two components of $W - M'$, so that M' is a characteristic submanifold. We will write M for M'. Q.E.D.

LEMMA 6.2. *There exists a simply connected characteristic submanifold M for (W, T) with $\partial M = M_1 - M_0$.*

PROOF. Since $\dim W = 2n + 1 \geqslant 7$ and $\pi_j(W) = 0$ for $j \leqslant 2$, the proof is exactly the same as in Lemma 2.2 of [8] by applying [4]: We apply equivalent handle exchanges in the interior of W to make M 1-connected. Q.E.D.

From now on, we assume the characteristic submanifold M is 1-connected, with $\partial M = M_1 - M_0$. Consider the following diagram of exact sequences:

$$
\begin{array}{ccccccc}
 & H_k(M_1) & \longrightarrow & H_k(V_1) \oplus H_k(TV_1) & \longrightarrow & H_k(W_1) & \longrightarrow H_{k-1}(M_1) \\
 & \downarrow & & \downarrow & & \downarrow & \\
H_{k+1}(W) \to & H_k(M) & \longrightarrow & H_k(V) \oplus H_k(TV) & \longrightarrow & H_k(W) & \longrightarrow H_{k-1}(M) \\
 & \downarrow & & \downarrow & & & \\
0 \longrightarrow & H_k(M, M_1) & \longrightarrow & H_k(V, V_1) \oplus H_k(TV, TV_1) & & & \\
(6.3) & \downarrow & & \downarrow & & & \\
 & H_{k-1}(M_1) & \longrightarrow & H_{k-1}(V_1) \oplus H_{k-1}(TV_1) & & & \\
 & \downarrow & & & & & \\
 & H_{k-1}(M) & & & & &
\end{array}
$$

We can replace M_1, V_1 by M_0, V_0 in (6.3).

LEMMA 6.4. *By performing equivariant surgery (equivariant handle exchanges) in the interior of W, we can transform M into an $(n-2)$-connected $2n$-characteristic submanifold. We also have $\pi_k(M, M_1) = \pi_k(M, M_0) = \pi_k(V, V_1) = \pi_k(V, V_0) = 0$ for $k \leqslant n-2$.*

PROOF. Suppose M is already $(k-1)$-connected, $k - 1 < n - 2$. From (6.3), we have

$$\begin{tikzcd}
0 \arrow[r] & H_k(M) \arrow[r] \arrow[d] & H_k(V) \oplus H_k(TV) \arrow[r] \arrow[d, "\approx"] & 0 \\
0 \arrow[r] & H_k(M, M_1) \arrow[r] & H_k(V, V_1) \oplus H_k(TV, TV_1) \arrow[r] & 0
\end{tikzcd}$$

Using the first exact sequence, we can perform equivariant handle exchanges in the interior of W to kill $H_k(M)$ as in [8], (see [8, 2.3] for details). The other part of the lemma follows from the above diagram, the induction hypothesis, and the Hurewicz theorem. Q.E.D.

From now on, we will assume M to be $(n-2)$-connected. Letting $k = n - 1$ in (6.3), we have

$$\begin{tikzcd}
Z + Z \arrow[r, two heads] \arrow[d, tail] & H_{n-1}(M_1) \arrow[d] & \\
H_n(W) \arrow[r] & H_{n-1}(M) \arrow[r, two heads, "{(\alpha, \alpha')}"] \arrow[d, tail] & H_{n-1}(V) \oplus H_{n-1}(TV) \arrow[d, tail] \\
& H_{n-1}(M, M_1) \arrow[r, tail, two heads, "{(\beta, \beta')}"] & H_{n-1}(V, V_1) \oplus H_{n-1}(TV, TV_1)
\end{tikzcd}$$

where "$\rightarrowtail$" means 1-1, "$\twoheadrightarrow$" onto, and "$\rightarrowtail\!\!\!\!\twoheadrightarrow$" isomorphic; $\alpha, \alpha', \beta, \beta'$ are maps induced by inclusions.

As in [8, 2.3], we see that $H_{n-1}(M, M_1) = \operatorname{Ker} \beta \oplus \operatorname{Ker} \beta'$, $T_*(\operatorname{Ker} \beta) = \operatorname{Ker} \beta'$, both β: $\operatorname{Ker} \beta' \to H_{n-1}(V, V_1)$ and β': $\operatorname{Ker} \beta \to H_{n-1}(TV, TV_1)$ are isomorphisms. Similarly, $H_{n-1}(M) = \{a + b \mid a \in \operatorname{Ker} \alpha, b \in \operatorname{Ker} \alpha'\}$ with

$$\begin{aligned} \operatorname{Ker} \alpha \cap \operatorname{Ker} \alpha' &= \operatorname{Image}(H_n(W) \to H_{n-1}(M)) \\ &= \operatorname{Image}(H_{n-1}(M_1) \to H_{n-1}(M)) = \{a\}, \end{aligned}$$

a cyclic group. Suppose $a = px$, for some integer P, where $\{x\}$ is a direct summand of $H_{n-1}(M)$. Since both $\alpha = \operatorname{Ker} \alpha' \to H_{n-1}(V)$ and α': $\operatorname{Ker} \alpha \to H_{n-1}(TV)$ are onto with kernel $= \{a\}$, if $p \neq 0$ or 1, then x contributes two copies of Z_p in $H_{n-1}(V) \oplus H_{n-1}(TV)$, which is impossible by a simple counting argument. Hence $\operatorname{Ker} \alpha \cap \operatorname{Ker} \alpha' = \{a\}$ is a direct summand of $H_{n-1}(M)$ · $H_{n-1}(M) = \{a\} + H$, and $H \rightarrowtail\!\!\!\!\twoheadrightarrow H_{n-1}(V) \oplus H_{n-1}(TV)$. We can perform equivariant handle exchanges in the interior of W to kill H as in [8, 2.3]. Thus we have the following:

LEMMA 6.6. *By equivariant handle exchanges in the interior of W, we can make M $(n-2)$-connected, with $H_{n-1}(M)$ a cyclic group; and $\pi_k(M, M_1) = \pi_k(M, M_0) = 0$ for $k < n$.*

In the rest of this section, all homology will be taken with rational coefficient Q, except where explicitly stated.

Letting $k = n$ in (6.3), we have

$$
\begin{tikzcd}
Q \arrow[d, equal] & Q+Q \arrow[d, equal] & Q+Q \arrow[d, equal] & Q \arrow[d, equal] \\
H_n(M_1) \arrow[r, tail, "{(\delta, \delta')}"] \arrow[d, "i_1"] & H_n(V_1) \oplus H_n(TV_1) \arrow[r] \arrow[d, "j_1"] & H_n(W_1) \arrow[r, two heads] \arrow[d] & H_{n-1}(M_1) \arrow[d] \\
H_n(M) \arrow[r, tail, "{(\alpha, \alpha')}"] \arrow[d, "i_2"] & H_n(V) \oplus H_n(TV) \arrow[r] \arrow[d, "j_2"] & H_n(W) \arrow[r, two heads] & H_{n-1}(M) \\
H_n(M, M_1) \arrow[r, tail, "{(\beta, \beta')}"] \arrow[d, "i_3"] & H_n(V, V_1) \oplus H_n(TV, TV_1) & & \\
Q = H_{n-1}(M_1) \arrow[d] & & & \\
H_{n-1}(M) & & &
\end{tikzcd}
\tag{6.7}
$$

where M is the characteristic submanifold for (W, T) as in (6.6).

LEMMA 6.8. *In the diagram* (6.7), i_2 *maps* Ker α *into* Ker β *injectively.*

PROOF. i_2 maps Ker α into Ker β by the commutativity of (6.7). $H_n(M_1) \to H_n(V_1)$ is an isomorphism. Since $H_n(V_1) \to H_n(TV)$ is trivial, $H_n(V_1) \to H_n(V)$ is 1-1. But $\alpha i_1 = j_1 \delta$ from (6.7). Hence Ker $\alpha \cap$ Im $i_1 = 0$. Q.E.D.

LEMMA 6.9. *In the diagram* (6.7), *if* $x \in$ Ker $\beta \cap$ Ker i_3, *then there exists* $y \in$ Ker α *such that* $i_2(y) = x$.

PROOF. By exactness, there exists $w \in H_n(M)$ such that $i_2(w) = x$. Let $\alpha(w) = t$, $j_2(t) = j_2\alpha(w) = \beta i_2(w) = 0$. Hence there is a $z \in H_n(V_1)$ such that $j_1(z) = t$. But δ: $H_n(M_1) \to H_n(V_1)$ is an isomorphism. Define $y = w - i_1\, \delta^{-1}(z)$. We have $\alpha(y) = 0$ and $i_2(y) = x$. Q.E.D.

In the diagram (6.7), $H_{n-1}(M) = Q$ or 0 by (6.6).

LEMMA 6.10. *Let* $m =$ rank $H_n(V, V_1)$ *in the diagram* (6.7)

(a) *If* $H_{n-1}(M) = Q$, *then* rank $H_n(M) = 2m + 1$, rank Ker $\alpha = m$.

(b) *If* $H_{n-1}(M) = 0$, *then* rank $H_n(M) = 2m$, rank Ker $\alpha = m - 1$.

PROOF. (a) If $H_{n-1}(M) = Q$, then $H_{n-1}(M_1) \to H_{n-1}(M)$ is an isomorphism. Since rank $H_n(M, M_1) = 2$ rank $\beta = 2m$, rank $H_n(M) = 2m + 1$. Hence rank Ker $\alpha =$ rank Ker $\alpha' \leqslant m$. But $i_3 = 0$. rank Ker $\alpha \geqslant$ rank Ker $\beta = m$ by (6.8) and (6.9). Thus rank Ker $\alpha = m$.

(b) If $H_{n-1}(M) = 0$, then i_3 is onto. Since rank $H_n(M, M_1) = 2m$ as in (a), rank $H_n(M) = 2m$. Hence rank Ker $\alpha \leqslant m - 1$ from (6.8). But rank Ker $\alpha \geqslant m - 1$ by (6.8) and (6.9). Thus rank Ker $\alpha = m - 1$. Q.E.D.

Given such a characteristic submanifold M for (W, T) as in (6.6), we can define a bilinear form C_M^f on $H_n(M)$ by $C_M^f(x, y) = x \cdot T_* y$. Let $B_M^f = C_M^f | \mathrm{Ker}\, \alpha$, where (α, α'): $H_n(M) \to H_n(V) \oplus H_n(TV)$ is the map induced by inclusion as in (6.7).

Now, we assume n odd in the rest of this section, hence C_M^f is symmetric.

LEMMA 6.11. index $C_M^f = 2$ index B_M^f.

PROOF. There are two cases: (a) $H_{n-1}(M) = Q$. $H_n(M) = \mathrm{Ker}\,\alpha \oplus \mathrm{Ker}\,\alpha' \oplus \mathrm{Im}\, i_1$ by (6.8) and (6.10). For $x \in H_n(M)$, let $\bar{x} \in H^n(M, \partial M)$ denote its Poincaré dual. If $x, y \in \mathrm{Ker}\,\alpha$, then $\bar{x}, \bar{y} \in \mathrm{Image}\{\alpha^*: H^n(V, \partial V) \to H^n(M, \partial M)\}$. Let $\bar{x} = \alpha^* u$, $\bar{y} = \alpha^* V$. $x \cdot y = \langle \alpha^* u \cup \alpha^* v, [M] \rangle = \langle u \cup v, \alpha_* [M] \rangle = \langle u \cup v, 0 \rangle = 0$. Similarly, $x \cdot y = 0$ for $x, y \in \mathrm{Ker}\,\alpha'$. Since $\mathrm{Im}\, i_1 \subseteq \mathrm{Image}\{i: H_n(\partial M) \to H_n(M)\}$, we see that $x \cdot z = 0$ for $x \in H_n(M)$ and $z \in \mathrm{Im}\, i_1$. We have $T_*\, \mathrm{Ker}\,\alpha = \mathrm{Ker}\,\alpha'$, $T_*\,\mathrm{Ker}\,\alpha' = \mathrm{Ker}\,\alpha$, and $T_*\,\mathrm{Im}\, i_1 = \mathrm{Im}\, i_1$. Hence $C_M^f(x, y) = 0$ for $x \in \mathrm{Ker}\,\alpha$, $y \in \mathrm{Ker}\,\alpha'$; or $x \in \mathrm{Im}\, i_1$; $y \in H_n(M)$. Thus $C_M^f = C_M^f|\mathrm{Ker}\,\alpha + C_M^f|\mathrm{Ker}\,\alpha' +$ a 1-dim trivial form. But $C_M^f|\mathrm{Ker}\,\alpha = C_M^f|\mathrm{Ker}\,\alpha' = B_M^f$. Hence index $C_M^f = 2$ index B_M^f.

(b) $H_{n-1}(M) = 0$. Let i: $H_n(\partial M) \to H_n(M)$ be induced by the inclusion. i is injective, because $H_{n+1}(M, \partial M) \cong H^{n-1}(M) \cong H_{n-1}(M) = 0$. $H_n(\partial M) = H_n(M_0) \oplus H_n(M_1) = Q + Q$. The proof of (6.8) shows that

$$\mathrm{Im}\, i \cap (\mathrm{Ker}\,\alpha \oplus \mathrm{Ker}\,\alpha') = 0.$$

Hence $H_n(M) = \mathrm{Ker}\,\alpha \oplus \mathrm{Ker}\,\alpha' \oplus \mathrm{Im}\, i$ by (6.10). As in the case (a), we can show that $C_M^f = C_M^f|\mathrm{Ker}\,\alpha + C_M^f|\mathrm{Ker}\,\alpha' + a$ 2-dim trivial form. Hence index $C_M^f = 2$ index B_M^f. Q.E.D.

Let S be an involution on a manifold X^{2p}. Let B' denote the bilinear form defined on $H_p(X^{2p})$ by $B'(x, y) = x \cdot S_* y$. If B' is symmetric, define $\sigma_s(X) =$ index B'. We need the following theorem from [31].

THEOREM 6 .12. [31, II. 4]. *Let S_1 and S_2 be involutions on X_1 and X_2, with $\partial X_1 =$ disjoint union $Y_1 \cup X_0$ and $\partial X_2 =$ disjoint union $Y_2 \cup X_0$, and $S_1|X_0 = S_2|X_0$. Let (X, S) denote $(X_1 \cup_{X_0} X_2, S_1 \cup S_2)$. Then $\sigma_s(x) = \sigma_{s_1}(X_1) + \sigma_{s_2}(X_2)$.*

Now, we are ready to prove the folowing:

THEOREM 6.13. *Let (Σ^{2n+1}, T) be a free involution on homotopy sphere Σ^{2n+1}, where n is odd and $\geqslant 3$, and $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$ for an A-diffeomorphism f of $S^n \times S^n$. Then* index $B_M^f = 8\sigma(\Sigma^{2n+1}, T)$.

PROOF. From the Mayer-Vietoris sequences, we see that $N = S^n \times D^n \cup_{\mathrm{Id}} M \cup_f D^{n+1} \times S^{n-1}$ is an $(n-1)$-connected characteristic submanifold of

$$(\Sigma_f, T_f) = (S^n \times D^{n+1}, A) \cup_{\mathrm{Id}} (S^n \times S^n \times I, A \times \mathrm{Id}) \cup_f (D^{n+1} \times S^n, A).$$

$\Sigma_f = E \cup T_f E$, $E \cap T_f E = N$, and

$$H_n(N) = \operatorname{Ker}(H_n(N) \to H_n(E)) \oplus \operatorname{Ker}(H_n(N) \to H_n(T_f E)),$$

[8]. Let $K_n = \operatorname{Ker}(H_n(N) \to H_n(E))$, and B = the symmetric unimodular bilinear form defined on K_n by $B(x \cdot y) = x \cdot T_{f*}y$. $\sigma(\Sigma_f, T_f)$ is defined to be (1/8) index B.

Consider the sysmetric bilinear forms C, C_1, C_2, defined on

$$H_n(N), H_n(S^n \times D^n), H_n(D^{n+1} \times S^{n-1})$$

respectively by $C(x, y) = x \cdot T_{f*}y$, $C_1(x, y) = x \cdot A_* y$, $C_2(x, y) = x \cdot A_* y$. Since $H_n(D^{n+1} \times S^{n-1}) = 0$ and $H_n(S^n \times S^{n-1}) \to H_n(S^n \times D^n)$ is onto, index C_2 = index $C_1 = 0$. Hence index C = index C_M^f by (6.12). But index $C_M^f = 2$ index B_M^f by (6.11). Similarly index $C = 2$ index B [8, p. 75]. Hence index B_M^f = index $B = 8\sigma(\Sigma_f, T_f) = 8\sigma(\Sigma^{2n+1}, T)$. Q.E.D.

REMARK 6.14. If the characteristic submanifold M in (6.13) satisfies (6.8)(a), i.e. $H_{n-1}(M) = Q$, then the symmetric bilinear form B_M^f defined on Ker α in (6.7) is actually isomorphic to the unimodular symmetric bilinear form B defined on K_n in (6.14). By the Mayer-Vietoris sequence we can show that if j: Ker $\alpha \to K_n$ is an isomorphism under the map induced by the inclusion, then we show that for $x, y \in \operatorname{Ker} \alpha$, $x \cdot T_* y = jx \cdot jT_* y$, which follows from the fact that some multiples of x and $T_* y$ can be represented by the immersions h_1 and h_2 of manifolds $M^P, N^P \to X^{2P}$, [32], and Theorem V.1.3 of [5]: The geometric intersection number of M and N = the intersection number of the homology classes $h_{1*}[M] \cdot h_{2*}[N]$. Hence B_M^f is isomorphic to B. But given an involution (Σ_f, T_f), we do not know whether we can always find such an M or not.

Given a free involution on a homotopy sphere (Σ^{2n+1}, T), $n \neq 3, 7$, we can always find an A-diffeomorphism f of $S^n \times S^n$, which is homotopic to identity, such that $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$ by (1.2). The next theorem tells us that the Browder-Livesay index invariant is additive: Given two involutions (Σ_f, T_f) and (Σ_g, T_g) with f and g homotopic to identity, we have $\sigma(\Sigma_f, T_f) + \sigma(\Sigma_g, T_g) = \sigma(\Sigma_{gf}, T_{gf})$.

THEOREM 6.15. *If $f, g \in J_{2n+1} = \varnothing^{-1}(D_0(S^n \times S^n))$, n odd, then* $\sigma(\Sigma_f, T_f) + \sigma(\Sigma_g, T_g) = \sigma(\Sigma_{gf}, T_{gf})$.

PROOF. Let M, N' be characteristic submanifolds for $(W, T) = (S^n \times S^n \times I, A \times \mathrm{Id})$ associated to f, g respectively as in (6.6). M and N' are $(n-2)$-connected, $H_{n-1}(M)$ and $H_{n-1}(N')$ are cyclic, $\partial M = f^{-1}(S^n \times S^{n-1}) \times 1 - S^n \times S^{n-1} \times 0$ and $\partial N' = g^{-1}(S^n \times S^{n-1}) \times 1 - S^n \times S^{n-1} \times 0$. Since $f^{-1} \times \mathrm{Id}$ is an equivariant diffeomorphism for (W, T), $N = (f^{-1} \times \mathrm{Id})(N')$ is a characteristic submanifold of (W, T) and $\partial N = f^{-1}g^{-1}(S^n \times S^{n-1}) \times 1 - f^{-1}(S^n \times S^{n-1})$

$\times 0$. Furthermore, the bilinear form $C_{N'}$ defined on $H_n(N')$ by $C_{N'}(x, y) = x \cdot T_* y$ is isomorphic to C_N^g, hence, index $C_{N'}$ = index C_N^g.

Now, we glue two copies W' and W'' of W together along W_1 of W' and W_0 of W'' by the identity. Consider $M \subseteq W'$ and $N \subseteq W''$. $P = M \cup N$ is a characteristic submanifold for $(W' \cup W'', T) = (W, T)$ with $\partial P = f^{-1}g^{-1}(S^n \times S^{n-1}) \times 1 - S^n \times S^{n-1} \times 0$, and P satisfies (6.6). P is $(n-2)$-connected and $H_{n-1}(P)$ is cyclic by the Mayer-Vietoris sequences. Hence C_P^{gf} and B_P^{gf} are defined.

By (6.12) again, we have index C_P^{gf} = index $C_{N'}$ + index C_M^f. Hence index C_P^{gf} = index C_N^g + index C_M^f, which implies index B_P^{gf} = index B_N^g + index B_M^f by (6.11). Thus $\sigma(\Sigma_{gf}, T_{gf}) = \sigma(\Sigma_g, T_g) + \sigma(\Sigma_f, T_f)$ by (6.13). Q.E.D.

REMARK 6.16. Actually, we have showed that given an A-diffeomorphism f of $S^n \times S^n$, n odd, we can associate an index $\beta(f)$ to f which is defined to be the index of the form B_M^f above. By the standard argument as in [8, 3.2], we see that B_M^f is independent of the choice of the characteristic submanifold M. The proof of Theorem 6.15 shows that the induced map $\bar{\beta}$: $D(S^n \times S^n, A) \to Z$ is a homomorphism for $n \geqslant 3$.

7. **The Arf invariant case.** Theorem (6.15) is no longer valid in the case n is even, as shown by the following example.

PROPOSITION 7.1. *If n is even and > 2, then there exist two A-diffeomorphisms f, g of $S^n \times S^n$ such that f and g are homotopic to* Id, *and $\sigma(\Sigma_f, T_f) = \sigma(\Sigma_g, T_g) = \sigma(\Sigma_{gf}, T_{gf}) = 0$ but $\sigma(\Sigma_{fg}, T_{fg}) = 1$.*

PROOF. Let τ be one of the generators of $\pi_n(SO_{n+1})$. We choose a representative for τ such that $\tau(x) = \tau(-x)$, and define two A-diffeomorphisms f and g by $f(x, y) = (x, \tau(x) \cdot y)$, $g(x, y) = (\tau(y) \cdot x, y)$. f and g are homotopic to Id, (4.4). It follows from (2.1) that $(\Sigma_f, T_f) = (\Sigma_g, T_g) = (\Sigma_{gf}, T_{gf}) = (S^{2n+1}, A)$. Hence their Browder-Livesay invariant is 0.

P. Orlik showed that if (Σ^{4k+1}, T) extends to an involution with fixed point on a π-manifold W^{4k+2} whose boundary is Σ^{4k+1}, then $\sigma(\Sigma^{4k+1}, T) = C(W^{4k+2})$, the Arf invariant of W^{4k+2}, [26], [21, p. 69]. We will construct such a W to show that $\sigma(\Sigma_{fg}, T_{fg}) = 1$.

Following [22], we define W to be

$$(D^{n+1} \times D^{n+1})_1 \cup_g (D^{n+1} \times D^{n+1})_2 \cup_f (D^{n+1} \times D^{n+1})_3,$$

where g is the diffeomorphism gluing $(D^{n+1} \times S^n)_1$ and $(D^{n+1} \times S^n)_2$ together, f is the gluing map from $(S^n \times D^{n+1})_2$ to $(S^n \times D^{n+1})_3$. Since f and g are A-equivariant, we define an involution T' on W by gluing the antipodal map A on each summand. The restriction of T' to $\partial W = \Sigma_{fg}$ is T. $C(W^{4k+2}) = 1$ follows from [6, V]. Q.E.D.

As in the index case, given an involution $(\Sigma^{2n+1}, T) = (\Sigma_f, T_f)$ we can find an

$(n-2)$-connected characteristic submanifold M of $(W, T) = (S^n \times S^n \times I, A \times \mathrm{Id})$ with $\partial M = S^n \times S^{n-1} \times 0 \cup f^{-1}(S^n \times S^{n-1}) \times 1$ and $H_{n-1}(M)$ is cyclic.

In the rest of this section, all homology will be taken with Z_2 coefficients, unless stated otherwise explicitly.

As in (6.10), we have two cases: either (a) $H_{n-1}(M) = Z_2$, or (b) $H_{n-1}(M) = 0$. In case (a), since $Z = H_{n-1}(M_i; Z) \to H_{n-1}(M; Z)$ is onto (6.7), we see that $H_{n-1}(M_1) \to H_{n-1}(M)$ is an isomorphism.

Suppose $H_{n-1}(M) = Z_2$. We see that the map i_3 in (6.7) is trivial (taken with Z_2 coefficient). We define a bilinear form B_M^f on Ker α in (6.7) by $B_M^f = x \cdot T_* y$. By using the Mayer-Vietoris sequences, and applying [32] and [5, V. 1.3] as in (6.14), we can show that B_M^f is isomorphic to the bilinear form B_2 defined by Browder and Livesay in [8]. In particular, B_M^f is unimodular.

Following [8], we can define a cohomology operation ψ_M^f on Ker α, (for details, see [8, §4]). For $x, y \in$ Ker α, we have $B_M^f(x, y) = \psi_M^f(x + y) + \psi_M^f(x) + \psi_M^f(y)$, [8, 4.5]. Since B_M^f is unimodular, the Arf invariant for ψ_M^f is well-defined, [5], as follows. Choose a sympletic basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ for Ker α such that $B_M^f(x_i, y_j) = \delta_{ij}, B_M^f(x_i, x_j) = B_M^f(y_i, y_j) = 0$, the Arf invariant $c_1(f, M) = \Sigma_{i=1}^n \psi_M^f(x_i)\psi_M^f(y_i)$.

LEMMA 7.2. *Suppose $H_{n-1}(M) = Z_2$ in (6.7), and ψ_M^f is defined as in* [8]. *Then any $x \in$ Ker α can be represented by an immersed manifold X^n, and $\psi_M^f(x) = 1$ iff $X \cap TX$ in general position consists of an odd number of pairs of points.*

PROOF. The representability of $x \in$ Ker α by an immersed manifold follows from [32], since the coefficient group is Z_2. The other assertion can be proved by exactly the same argument in [8, 4.6]. Q.E.D.

PROPOSITION 7.3. *For n even, if the manifold M associated to (Σ_f, T_f) as in* (6.7) *with Z_2-coefficient satisfies $H_{n-1}(M) = Z_2$, then $c_1(f, M) = \sigma(\Sigma_f, T_f)$.*

PROOF. We noted before that B_M^f is isomorphic to B_2 in this case. Let ψ be the quadratic form associated to B_2 defined in [8]. From (7.2), [8, 4.6], and [5, V. 1.3] again, we see that ψ_M^f and ψ are isomorphic. Hence their Arf invariants are equal. Q.E.D.

PROPOSITION 7.4. *For n even, and $f, g \in J_{2n+1}$. If there exist M, N associated to $(\Sigma_f, T_f), (\Sigma_g, T_g)$ as in* (6.6) *such that $H_{n-1}(M) = H_{n-1}(N) = Z_2$, then $\sigma(\Sigma_f, T_f) + \sigma(\Sigma_g, T_g) = \sigma(\Sigma_{gf}, T_{gf})$.*

PROOF. As in (6.15), let P denote $M \cup N$, the characteristic submanifold associated to (Σ_{gf}, T_{gf}). We denote the domain on which B_M^f, B_N^g, B_P^{gf} is defined by Ker α, Ker β, Ker γ respectively. By the Mayer-Vietoris sequence, we see that $H_{n-1}(P) = Z_2$, and Ker γ = Ker $\alpha \oplus$ Ker β under the inclusion. Using (7.2) and

[5, V. 1.3] as in (7.3), we see that the quadratic form ψ_P^{gf} is the direct sum of ψ_M^f and ψ_N^g. Hence $\sigma(\Sigma_f, T_f) + \sigma(\Sigma_{gf}, T_{gf})$ by (7.3). Q.E.D.

COROLLARY 7.5. *Let f be the A-diffeomorphism defined in* (7.1). *M is an $(n-2)$-connected characteristic submanifold for $(S^n \times S^n \times I, A \times \mathrm{Id})$ such that $\partial M = f^{-1}(S^n \times S^{n-1}) \times 1 - S^n \times S^{n-1} \times 0$. If $H_{n-1}(M)$ is cyclic, then $H_{n-1}(M; Z_2) = 0$.*

PROOF. Let g be the A-diffeomorphism defined in (7.1) by $g(x, y) = (\tau(y) \cdot x, y)$. Since $g(S^n \times S^{n-1}) = S^n \times S^{n-1}$, we can take $N = S^n \times S^{n-1} \times I$ to be a characteristic manifold associated to (Σ_g, T_g) as in (6.6). $H_{n-1}(N; Z_2) = Z_2$. Assume $H_{n-1}(M; Z_2) = Z_2$. From (7.4), we would have $\sigma(\Sigma_{fg}, T_{fg}) = \sigma(\Sigma_f, T_f) + \sigma(\Sigma_g, T_g) = 0$. This contradicts (7.1). Hence $H_{n-1}(M; Z_2) = 0$. Q.E.D.

8. **Curious involutions.** Let Σ_0^{4k-1} denote the generator of bP^{4k}, a cyclic subgroup of Γ^{4k-1}, consisting of those homotopy spheres which bound parallelizable manifolds, [17]. Let (Σ^{4k-1}, T) be a fixed point free involution such that $\Sigma^{4k-1} \in bP_{4k}$, we can write $\Sigma^{4k-1} = m\, \Sigma_0^{4k-1}$ for some integer m, which is well-defined mod 2. Following [13], we will call an involution (Σ^{4k-1}, T) curious if $m + \sigma(\Sigma^{4k-1}, T)$ mod 2 is equal to 1.

LEMMA 8.1. *The number of curious involutions (Σ_0^{4k-1}, T) with $\sigma(\Sigma_0^{4k-1}, T) = 0$ is finite.*

PROOF. The number of the normal cobordism classes $[P^{4k-1}, G/O]$ is finite, [21]. In each normal cobordism class, there is exactly one p.l. involution with the zero Browder-Livesay's index invariant, [21] or [33]. Since $\pi_j(PL/O)$ is finite, the number of differentiable involutions with zero index invariant in each normal cobordism class is finite by smoothing theory. Q.E.D.

Let S^n denote the standard sphere. As an application of our previous theorems, we have the following:

THEOREM 8.2. *Let $n = 8k + 3$; the number of curious involutions (Σ_0^n, T) with $\sigma(\Sigma_0^n, T) = 0$ is either 0 or equal to the number of involutions (S^n, T) with $\sigma(S^n, T) = 0$.*

PROOF. From (5.2), we know that $hS(P^{8k+3})$ is in 1-1 correspondence with $J_{8k+3} = \varnothing^{-1}(D_0(S^{4k+1} \times S^{4k+1}))$, a subgroup of the group

$$D_0(S^{4k+1} \times S^{4k+1}, A)$$

of concordance classes of A-diffeomorphisms of $S^{4k+1} \times S^{4k+1}$, consisting of those A-diffeomorphisms which are homotopic to identity.

Let $C = \{f \in J_{8k+3} | \Sigma_f = \Sigma_0^{8k+3}, \sigma(\Sigma_f, T_f) = 0\}$, and $C' = \{f \in J_{8k+3} | \Sigma_f = S^{8k+3}, \sigma(\Sigma_f, T_f) = 0\}$. There are two cases:

(a) k is odd. $D_0(S^{4k+1} \times S^{4k+1}) = \Gamma^{8k+3}$ by (5.2). Let $\gamma \in \Gamma^{8k+3}$ be the element which corresponds to Σ_0^{8k+3}. If C is not empty, then $\varnothing(f) = \gamma$ for all $f \in C$, where $\varnothing$: $D_0(S^{4k+1} \times S^{4k+1}, A) \to D_0(S^{4k+1} \times S^{4k+1})$ is the forgetting map in §4. Take $g \in C$, $\varnothing(g^{-1}) = \gamma^{-1}$, hence $\Sigma_{g^{-1}} = -\Sigma_0^{4k+3}$; and $\sigma(\Sigma_{g^{-1}}, T_{g^{-1}}) = -\sigma(\Sigma_g, T_g) = 0$ by (6.15). Using (6.15) again, we see that the mapping $f \to f \circ g^{-1}$ for $f \in C$ maps C into C' because $\Sigma_{fg^{-1}} = S^{4k+3}$. This correspondence is 1-1 and onto, since the inverse is given by $h \to h \circ g$ for $h \in C'$.

(b) k is even. From (5.2), we know that $D_0(S^{4k+1} \times S^{4k+1})$ is the semi-direct product of $(\sigma_1) + \Gamma^{8k+3}$ and (σ_2), where σ_1 and σ_2 are defined by $\sigma_1(x, y) = (x, \sigma(x) \cdot y)$ and $\sigma_2(x, y) = (\sigma(y) \cdot x, y)$, $\sigma_1^2 = \sigma_2^2 = \mathrm{Id}$ in $D_0(S^{4k+1} \times S^{4k+1})$, [19]. If C is not empty, then for $f \in C$, $\varnothing(f) = \gamma, \sigma_2\gamma\sigma_1$, $\gamma\sigma_1$, or $\sigma_2\gamma$, [19], where γ is the element of Γ^{8k+3} corresponding to Σ_0^{8k+3}.

(i) If there exists a $g \in C$ such that $\varnothing(g) = \gamma$. Since γ^{-1} lies in the center of $D_0(S^{4k+1} \times S^{4k+1})$, [19], and $\varnothing(g^{-1}) = \gamma^{-1}$, we see that $\varnothing(gf^{-1}) = \mathrm{Id}$, $\sigma_2\sigma_1$, σ_1, or σ_2. Hence $\Sigma_{fg^{-1}} = S^{8k+3}$. By applying (6.15) as in (a), we see that the mapping $f \to f \circ g^{-1}$ for $f \in C$ gives a 1-1 correspondence between C and C'.

(ii) If $\varnothing(f) \neq \gamma$ for every $f \in C$, but there exists $g \in C$ such that $\varnothing(g) = \gamma\sigma_1$. Then $\varnothing(g^{-1}) = \sigma_1^{-1}\gamma^{-1} = \sigma_1\gamma^{-1}$, and $\sigma(\Sigma_{g^{-1}}, T_{g^{-1}}) = -\sigma(\Sigma_g, T_g) = 0$ by (6.15). In this case, $\varnothing(f) = \gamma\sigma_1$, $\sigma_2\gamma$, or $\sigma_2\gamma\sigma_1$ for $f \in C$. As in (i), we have $\varnothing(fg^{-1}) = \mathrm{Id}$, $\sigma_2\sigma_1$, or σ_2. Hence $\Sigma_{fg^{-1}} = S^{8k+3}$. By (6.15) again, the mapping $f \to f \circ g^{-1}$ for $f \in C$ gives a 1-1 correspondence between C and C'.

(iii) If $\varnothing(f) \neq \gamma, \gamma\sigma_1$, for every $f \in C$, but there exists $g \in C$ such that $\varnothing(g) = \sigma_2\gamma$. Then the mapping $f \to g^{-1}f$ gives a 1-1 correspondence between C and C' as in (i).

(iv) If $\varnothing(f) = \sigma_2\gamma\sigma_1$ for all $f \in C$. Take $g \in C$, $\varnothing(g^{-1}) = \sigma_1^{-1}\gamma^{-1}\sigma_2^{-1}$, and $\sigma(\Sigma_{g^{-1}}, T_{g^{-1}}) = 0$ by (6.15) as before. For $f \in C, f \to f \circ g^{-1}$ gives 1-1 correspondence between C and C' by (6.15) as before. Q.E.D.

9. Decomposition of (Σ^{2n}, T).

In this section, we will prove an analogue of (1.1) for free involutions on even dimensional homotopy spheres.

PROPOSITION 9.1. *For $n > 3$, $(\Sigma^{2n}, T) = (S^n \times D^n, A) \cup_g (D^{n+1} \times S^{n-1}, A)$ for some A-diffeomorphism g of $(S^n \times S^{n-1}, A)$.*

PROOF. Let P^m denote the real projective space. There is a homotopy equivalence f: $P^{2n} \to Q^{2n} = \Sigma^{2n}/T$. Let i: $P^n \to P^{2n}$ be the inclusion. For dimensional reasons, $f|P^n$ is homotopic to an embedding by [12]. By the homotopy extension theorem, we see that f homotopic to a smooth map g such that $g|P^n$ is an embedding.

Let ν_1 denote the normal bundle of P^n in P^{2n}, and ν_2 the normal bundle of gP^n in Q^{2n}. Let τ_1 and τ_2 denote the tangent bundles of P^{2n} and Q^{2n}. By

Theorem 3.6 in [1], $g^*\tau_2$ and τ_1 are J-equivalent. Since the projection $\widetilde{KO}(P^{2n}) \to J(P^{2n})$ is an isomorphism, $g^*\tau_2$ and τ_1 are stably equivalent. Let τ_p denote the tangent bundle of P^n. $g^*\tau_2|P^n$ is stably equivalent to $\tau_1|P^n$. Since $g|P^n$ is an embedding, $(g|P^n)^*(\tau_1|gP^n)$ is stably equivalent to $\tau_1|P^n$. The induced map commutes with the Whitney sum; hence $\tau_p \oplus \nu_1$ is stably equivalent to $\tau_p \oplus g^*\nu_2$. By adding a stable inverse for τ_p, we see that ν_1 is stably equivalent to $g^*\nu_2$. But ν_1, the normal bundle of P^n in P^{2n}, is equivalent to $n\eta = \eta \oplus \eta \oplus \cdots \oplus \eta$, n times, where η is the canonical line bundle over P^n. Hence by Corollary 1.10 in [5], $g^*\nu_2$ and ν_1 are actually equivalent. By lifting this equivalence of normal bundles to the double cover, we see that there is an equivariant embedding $h = (S^n \times D^n, A) \to (\Sigma^{2n}, T)$. The image solid torus is unknotted by [12], the complement is diffeomorphic to $D^{n+1} \times S^{n-1}$ by the h-cobordism theorem.

Consider $(S^n \times D^n, A) \xrightarrow{h} (\Sigma^{2n}, T) \xleftarrow{k'} (D^{n+1} \times S^{n-1}, U)$, where we define an involution on the right-hand torus by $U = k'^{-1}Tk'$. Both h and k' are equivariant embeddings. The A-invariant diagonal sphere in $S^{n-1} \times S^{n-1} \subseteq S^n \times S^{n-1}$ on the left-hand side is mapped by $k'h|S^n \times S^n$ onto a U-invariant sphere S^{n-1}_Δ on the right. On the boundary of $D^{n+1} \times S^{n-1}$, U is equivalent to A. We equivariantly collar $S^n \times S^{n-1}$ in $(D^{n+1} \times S^{n-1}, U)$ by [9, 21.2], and push S^{n-1}_Δ a little way inside the boundary. U is equivalent to A on a tubular neighborhood N of this interior copy of S^{n-1}_Δ, which can be proved by applying Lemma 2 of [20] to show that the normal bundle of $P^{n-1} = S^{n-1}_\Delta/A$ in N is equivalent to the normal bundle of P^{n-1} in P^{2n}. The orbit space $(D^{n-1} \times S^{n-1} - \mathring{N})/U$ is an h-cobordism between two copies of a manifold diffeomorphic to $S^n \times S^{n-1}/A$. Since the Whitehead group $Wh(Z_2) = 0$, this h-corbodism is diffeomorphic to $(S^n \times S^{n-1}/A) \times I$ by the s-cobordism theorem. Therefore $(D^{n+1} \times S^{n-1}, U)$ is equivalent to $(D^{n+1} \times S^{n-1}, A)$. Q.E.D.

Similar to (2.1), we have the following:

PROPOSITION 9.2. *$n > 3$, $(\Sigma^{2n}, T) = (\Sigma_f, T_f) = (\Sigma_g, T_g)$ for some A-diffeomorphisms f and g of $S^n \times S^{n-1}$ iff there exists A-diffeomorphisms H: $S^n \times D^n \to S^n \times D^n$ and K: $D^{n+1} \times S^{n-1} \to D^{n+1} \times S^{n-1}$ such that $f = KgH$ on $S^n \times S^{n-1}$.*

PROOF. Exactly the same as in [20]. Suppose we have two distinct decompositions, $(S^n \times D^n, A) \xrightarrow{h_i} (\Sigma^{2n}, T) \xleftarrow{k_i} (D^{n+1} \times S^{n-1}, A)$, $i = 1$ or 2. On $S^n \times S^{n-1}$, $f = k_1^{-1}h_1$ and $g = k_2^{-1}h_1$. Note that $h_1, h_2 = S^n \times 0 \to \Sigma^{2n+1}$ are equivariantly homotopic embeddings, since both are lifted classifying maps for P^n in Q^{2n}. Hence $h_1(S^n \times 0)$ and $h_2(S^n \times 0)$ are equivariantly isotopic by a global isotopy by [12] as in the proof of (2.4). By the equivariant tubular neighborhood theorem, [3], there is an equivariant isotopy of Σ^{2n} such that, after composing with the first isotopy, there is an equivariant diffeomorphism r: $\Sigma^{2n} \to$

Σ^{2n} with $H = h_2^{-1} r h_1$, an equivariant bundle map. Let $K = k_1^{-1} r^{-1} k_2 \colon D^{n+1} \times S^{n-1}$; we note that $k_1 = r^{-1} k_2 K^{-1}$. Hence $f = k_1^{-1} h_1 = K k_2^{-1} r h_1 = K k_2^{-1} h_2 H = KgH$. H extends equivariantly to all of $S^n \times D^n$, and H extends equivariantly to all of $D^{n+1} \times S^{n-1}$. Q.E.D.

10. **Equivariant Milnor's pairing.** Milnor defined in [22] a certain pairing Σ on $s_* \pi_n(SO_n) \otimes \pi_n(SO_{n+1})$ to Γ^{2n+1}, the group of homotopy spheres, where $s_* \pi_n(SO_n)$ denotes the image of $s_* \colon \pi_n(SO_n) \to \pi_n(SO_{n+1})$. The pairing is defined as follows. Let $a \in \pi_n(SO_n)$, $a' \in \pi_n(SO_{n+1})$; define two diffeomorphisms f_1 and f_2 on $S^n \times S^n$ by $f_1(x, y) = (x, s_*a(x) \cdot y)$ and $f_2(x, y) = (a'(y) \cdot x, y)$. Let $h = f_1 \circ f_2$, $\Sigma(s_*a, a') = S^n \times D^{n+1} \cup_h D^{n-1} \times S^n$.

If the above s_*a and a' lie in the image of $\pi^* \colon [P^n, SO_{n+1}] \to \pi_n(SO_{n+1})$, then we can take representatives of s_*a and a' such that $s_*a(x) = s_*a(-x)$ and $a'(x) = a'(-x)$. Hence $h = f_1 \circ f_2$ is an A-diffeomorphism, and $\Sigma(s_*a, a')$ admits a free involution, which is (Σ_h, T_h). In §7, we used $a' = s_*a = \tau$ in $\pi_n(SO_{n+1})$, n even, to construct an involution on the Kervaire sphere with non-zero Arf invariant. For n odd, we will see that all the involutions obtained in this way have zero index invariant.

From (3.1) and (3.9), we know that in $\pi_n(SO_{n+1})$, $s_* \pi_n(SO_n) \cap \pi^*[P^n, SO_{n+1}] = \{2\sigma\}$ for $n > 7$, $n \equiv 3 \mod 4$; and $= 0$ for $n \equiv 1 \mod 4$. Hence we only have to consider the case where $n \equiv 3 \mod 4$ and $n \neq 3, 7$.

LEMMA 10.1. *If f is an A-diffeomorphism of $S^n \times S^n$ such that f leaves $S^n \times S^{n-1}$ or $S^{n-1} \times S^n$ invariant, then $\sigma(\Sigma_f, T_f) = 0$.*

PROOF. $S^n \times D^n \cup_f D^{n+1} \times S^{n-1}$ or $S^{n-1} \times D^{n+1} \cup_f D^n \times S^n$ is a codim 1 invariant sphere of (Σ_f, T_f), because $S^n \times S^{n-1}$, $S^n \times D^n$, $D^{n+1} \times S^{n-1}$, etc. are all invariant under A. Q.E.D.

In $\pi_n(SO_{n+1})$ for $n \equiv 3 \mod 4$ and $n \neq 3, 7$, $\pi^*[P^n, SO_{n+1}] = Z + Z$, generated by τ and 2σ; $\pi^*[P^n, SO_{n+1}] \cap s_*\pi_n(SO_n) = Z$, generated by 2σ, (3.9).

We first consider $\Sigma(2\sigma, 2\sigma)$. Let $\sigma = s_*a$, where $a \in \pi_n(SO_n)$, [18]. We can choose a representative for $2a$ such that $2a(x) = 2a(-x)$. Let f represent $a \in \pi_n(SO_n)$ such that $f \mid$ the southern hemisphere $=$ identity. Since n is odd, Af is homotopic to f. The map $g \colon S^n \to SO_n$ defined by $g = f$ on the northern hemisphere and $= Af$ on the southern hemisphere represents $f + Af$, hence $2a$, and $g(x) = g(-x)$. We have $f_1(x, y) = (x, s_*2a(x) \cdot y)$, $f_2(x, y) = (s_*2a(y) \cdot x, y)$, and $h = f_1 \circ f_2$. Since $2a \in \pi_n(SO_n)$, $s_*2a(S^n) \cdot S^{n-1} \subseteq S^{n-1}$, $f_1(S^n \times S^{n-1}) \subseteq S^n \times S^{n-1}$. But $2a(x) \in SO_n$, which has a matrix representation $[2a(x)]$. We define $d \in \pi_n(SO_n)$ by $d(x) = [2a(x)]^{-1}$, the inverse matrix for $[2a(x)]$. The diffeomorphism f_3 defined by $f_3(x, y) = (x, s_*d(x) \cdot y)$ is the inverse for f_1, and $f_3(S^n \times S^{n-1}) \subseteq S^n \times S^{n-1}$. Hence $f_1(S^n \times S^{n-1}) = S^n \times S^{n-1}$. Similarly,

$f_2(S^n \times S^{n-1}) = S^n \times S^{n-1}$. Thus $h = f_1 \circ f_2$ leaves $S^n \times S^{n-1}$ invariant. $\sigma(\Sigma_h, T_h) = 0$ by (10.1).

Now, we consider the A-diffeomorphisms g_1 and g_2 of $S^n \times S^n$ defined by $g_2 = f_2$ in the preceding paragraph and $g_1(x, y) = (\tau(y) \cdot x, y)$, where $\tau(y) = \alpha(y)\alpha(e)$ as in §3. e is the north pole of S^n, $\alpha(y)$ = the reflection through the hyperplane orthogonal to y. This representative of τ satisfies $\tau(x) = \tau(-x)$. $\tau(S^{n-1}) \cdot S^n = \alpha(S^{n-1})\alpha(e)S^n = \alpha(S^{n-1})S^n = S^n$. Hence $g_1(S^n \times S^{n-1}) \subseteq S^n \times S^{n-1}$. But $\alpha(y)\alpha(y)$ = identity. Thus $g_1(S^n \times S^{n-1}) = S^n \times S^{n-1}$. $h = g_2 \circ g_1$ leaves $S^n \times S^{n-1}$ invariant. $\sigma(\Sigma_h, T_h) = 0$ by (10.1).

Suppose β_1 and β_2 are two representatives for $\beta \in \pi_n(SO_{n+1})$ such that $\beta_i(x) = \beta_i(-x)$; we define two A-diffeomorphisms h_1 and h_2 by $h_i(x, y) = (x, \beta_i(x) \cdot y)$. Since $\pi^*[P^n, SO_{n+1}] \to \pi_n(SO_{n+1})$ is 1-1 for $n \equiv 3 \bmod 4$ and $n > 7$, (3.9), we see that h_1 and h_2 are A-concordant by (2.4) and (2.5). Hence the construction of the (Σ_h, T_h) is independent of the choice of representatives for $\beta \in$ Image π^*. Thus we have

PROPOSITION 10.2. *Every involution (Σ^{8n+7}, T) constructed above by using Milnor's pairing: $\Sigma = \Sigma(2m\sigma, 2n\sigma)$ or $\Sigma(2m\sigma, n\tau)$, has zero Browder-Livesay index invariant.*

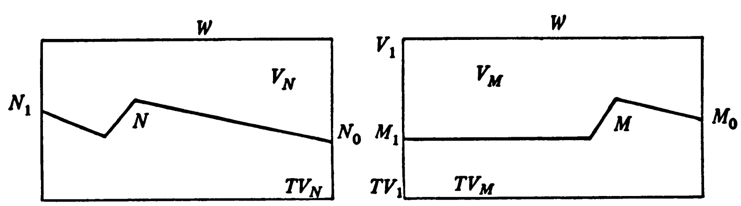
Added in proof. Lemma 6.11 is not true. Since T changes the orientation of the characteristic submanifold M, we have

$$\text{index } C_M^f \mid \operatorname{Ker} \alpha = -\text{ index } C_M^f \mid \operatorname{Ker} \alpha'.$$

Thus index $C_M^f = 0$.

Here we will adapt the proof of [31, Theorem II.4] (instead of applying the Theorem itself, which was stated as Theorem 6.12 above) to verify Theorems 6.13 and 6.15.

Let M, N, and P be the characteristic submanifolds for $(W, T) = (S^n \times S^n \times I, A \times \text{id})$ associated with f, g, and gf respectively as in (6.15)



$$M_0 = S^n \times S^{n-1}, M_1 = f^{-1}(S^n \times S^{n-1}) = N_0, \quad N_1 = f^{-1}g^{-1}(S^n \times S^{n-1}),$$
$$V_1 = f^{-1}(S^n \times D^n), \quad P = M \cup_{M_1} N, \quad V = V_M \cup_{V_1} V_N, \quad \text{etc.}$$

Let

$$(\alpha_1, \alpha_1'): H_nM \longrightarrow H_nV_M \oplus H_nTV_M,$$
$$(\alpha_2, \alpha_2'): H_nN \longrightarrow H_nV_N \oplus H_nTV_N, \quad \text{and}$$
$$(\alpha, \alpha'): \quad H_nP \longrightarrow H_nV \oplus H_nTV$$

be the maps induced by inclusion as in (6.7).

THEOREM.

$$\text{index } B_P^{gf} = \text{index } B_N^g + \text{index } B_M^f.$$

PROOF. (6.10) and (6.11) stated that

$$H_nM = \text{Ker } \alpha_1 \oplus \text{ Ker } \alpha_1' \oplus \text{ Im } i_1,$$

where $i_1: H_n\partial M \longrightarrow H_nM$ is the inclusion. We take Q as coefficient from now on. Let $m = \text{rank } H_n(V, V_1)$. We can classify M into two types:

(a) $H_{n-1}M = Q$, rank $H_nM = 2m + 1$, rank Ker $\alpha_1 = m$, and $Q = H_{n-1}M_1 \longrightarrow H_{n-1}M$ is onto.

(b) $H_{n-1}M = 0$, rank $H_nM = 2m$, rank Ker $\alpha_1 = m - 1$.

Consider the following exact sequence.

$$\begin{tikzcd}
H_n(M_1) \arrow[r] \arrow[d] & H_nM \oplus H_nN \arrow[r] \arrow[d] & H_nP \arrow[r, two heads] \arrow[d] & H_{n-1}M_1 \arrow[r] \arrow[d] & {} \\
Q + Q \arrow[r] & H_nV_M \oplus H_nTV_M \oplus H_nV_N \oplus H_nTV_n \arrow[r] & H_nV \oplus H_nTV \arrow[r] & 0 &
\end{tikzcd}$$

If one of M, N is of type (a), then we can show that Ker α_1 + Ker $\alpha_2 \longrightarrow$ Ker α is an isomorphism by a simple counting argument, and index B_P^{gf} = index B_N^g + index B_M^f follows from the statement in (6.14).

Now we assume that both M and N are of type (b). Let (α^*, α'^*): $H^nV \oplus H^nTV \longrightarrow H^nP$ be the map induced by inclusions. We have the following exact sequence.

$$\begin{tikzcd}
H_nM_1 \arrow[r] \arrow[rr, bend left, "k"] & \text{Ker } \alpha_1 \oplus \text{ Ker } \alpha_2 \arrow[r, "j_1 + j_2"] & \text{Ker } \alpha \arrow[r, two heads] & H_{n-1}M_1 \\
 & & \text{Im } \alpha'^* \arrow[u, "\cong"] \arrow[r, "k^*"] & H^nM_1 \arrow[u, "\cong"]
\end{tikzcd}$$

Let A_B Im k denote the annihilator of Im k under B_P^{gf}. Then we may use the argument in [31. II4] to show that A_B Im k = Im j_1 + Im j_2. But we also have

LEMMA [31, II.3]. *If B is a symmetric bilinear form on a vector space V, and if there is a subspace $C \subseteq V$ with $C \subseteq AC$, then* index B = index $B \mid AC$.

Thus our theorem follows from this Lemma and (6.14) as above. Q.E.D. The proof of Theorem 6.13 is similar.

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