TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 224, Number 2, 1976

THE CARTESIAN PRODUCT STRUCTURE AND C^∞ EQUIVALENCES OF SINGULARITIES

BY

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ABSTRACT. In this paper the cartesian product structure of complex analytic singularities is studied. A singularity is called indecomposable if it cannot be written as the cartesian product of two singularities of lower dimension. It is shown that there is an essentially unique way to write any reduced irreducible singularity as a cartesian product of indecomposable singularities. This result is applied to give an explicit description of the set of reduced irreducible complex singularities having a given underlying real analytic structure.

1. **Introduction.** Let V be an irreducible germ of a complex analytic set. In this paper I will classify all germs W of complex analytic sets which are diffeomorphic (of class C^∞) to V (Theorem 4.6). It is not surprising that this classification should rest on a structure theorem for such V. What may be surprising is that the relevant structure theorem (Theorem 3.4) concerns nothing more complicated than the decomposition of V as a cartesian product of germs of analytic sets of lower dimension. It will be proven that if V is decomposed into factors to the point where no factor can be further decomposed, then the factors which appear are uniquely determined (up to complex analytic isomorphism) by V.

This type of question has some interesting history. Mumford [7] has shown that a normal two-dimensional germ which is homeomorphic to a manifold germ must actually be nonsingular. If V is a germ of a hypersurface which is homeomorphic as an embedded pair to a manifold embedded as a hypersurface, then W is actually nonsingular. This is a consequence of work of A'Campo [1], but it was noticed, I believe, by Le Dung Trang. Finally, if W is diffeomorphic (of class C^1) to a manifold, then it is nonsingular ([2] or [6]).

The above results all concern singularities homeomorphic (with side con-

Received by the editors October 1, 1975.

AMS (MOS) subject classifications (1970). Primary 32B10, 32C05; Secondary 32K15.

Key words and phrases. Reduced singularity, irreducible singularity, cartesian product, underlying real analytic structure, complexification, complex conjugate, real analytic isomorphism, C^∞ isomorphism.

(1) This work was supported by the Research Foundation of C. U. N. Y., grant No. 10614, and N. S. F. grant No. GP-44011.

ditions) to a nonsingular germ V. The side condition is generally a smoothness condition on the map, and the more general the desired result, the more smoothness is needed. This paper generalizes the above work to the case where V may be any reduced irreducible singularity, and naturally, to have a theorem of this generality, a great deal of smoothness is needed. In [4], I treated by more direct methods the special case where V is a germ of a hypersurface.

2. **Preliminaries.** Let $V \subset C^n$ be a germ at the origin of a complex analytic set. Let O_n be the ring of germs of functions holomorphic in a neighborhood of the origin. Then $I(V)_n$ will denote the ideal of all $f \in O_n$ vanishing on V.

Cartesian products. Suppose $V \subset C^n$ and $W \subset C^m$ are two such germs. Then $V \times W \subset C^{n+m}$ is also a germ at the origin (in C^{n+m}) of a complex analytic set. If $(z_1, \ldots, z_n)$ are coordinates in C^n and $(w_1, \ldots, w_m)$ are coordinates in C^m, then the two projections $(z, w) \longmapsto z$ and $(z, w) \longmapsto w$ define inclusions $O_n \subset O_{n+m}$ and $O_m \subset O_{n+m}$, and we just have $V \times W = V(I(V)_n O_{n+m} + I(W)_m O_{n+m})$. (If $f \in O_n$ then $V(f)$ denotes the germ at the origin of $\{z \in C^n \mid f(z) = 0\}$, and if $J \subset O_n$, then $V(J) = \bigcap_{f \in J} V(f)$ [5].)

In fact, we have

LEMMA 2.1. *Let V and W be as above and suppose V and W are irreducible. Then*

$$I(V \times W)_{n+m} = I(V)_n O_{n+m} + I(W)_m O_{n+m}$$

and $V \times W$ is irreducible.

PROOF. Since V and W are irreducible, $I(V)_n$ and $I(W)_m$ are prime ideals. By [8, 47.5] $I(V)_n O_{n+m} + I(W)_m O_{n+m}$ is prime. Since $I(V)_n O_{n+m} + I(W)_m O_{n+m} \subset I(V \times W)_{n+m}$ and these two ideals both define the same analytic germ $V \times W$, then $I(V)_n O_{n+m} + I(W)_m O_{n+m} = I(V \times W)_{n+m}$ just follows by the Nullstellensatz. $I(V \times W)_{m+n}$ must then be prime, and $V \times W$ is irreducible. Q.E.D.

REMARK 2.2. If $V \times W$ is irreducible, then V and W are certainly both irreducible also.

PROOF. If V is reducible, then $V = V_1 \cup V_2$ where V_1 and V_2 are proper analytic subsets of V, and $V \times W = (V_1 \times W) \cup (V_2 \times W)$ where $V_1 \times W$ and $V_2 \times W$ are proper analytic subsets of $V \times W$, so that $V \times W$ is reducible. Similarly, if W is reducible, $V \times W$ is reducible.

Complex conjugate germs. Let $V \subset C^n$ be as above. Define $\overline{V}$, the complex conjugate of V, by $\overline{V} = \{z \in C^n \mid \overline{z} \in V\}$.

REMARK 2.3. $\overline{V}$ is the germ of a complex analytic set.

PROOF. For $f \in O_n$ define $\tilde{f} \in O_n$ by the formula $\tilde{f}(z) = \overline{f(\bar{z})}$. Clearly $z \in \bar{V}$ if and only if $\tilde{f}(z) = 0$ for all $f \in \mathrm{Id}(V)_n$.

REMARK 2.4. $\mathrm{Id}(\bar{V})_n = \{\tilde{f} \in O_n \mid f \in \mathrm{Id}(V)_n\}$.

REMARK 2.5. $\mathrm{Id}(\bar{V})_n$ is prime $\Leftrightarrow \mathrm{Id}(V)_n$ is prime.

REMARK 2.6. The complex conjugate of $\bar{V}$ is just V.

REMARK 2.7. If $V \cong W$ then $\bar{V} \cong \bar{W}$. ("$\cong$" means complex analytically equivalent in all that follows.)

PROOF. Suppose $V \subset C^n$ and $W \subset C^m$ and let $(\phi_1, \ldots, \phi_m)$: $C^n \longrightarrow C^m$ be a map which induces an isomorphism from V to W. Then $(\tilde{\phi}_1, \ldots, \tilde{\phi}_m)$: $C^n \longrightarrow C^m$ induces an isomorphism of $\bar{V}$ to $\bar{W}$.

Complexification of a complex germ. Let $A \subset R^n$ be a germ of a real analytic set, and let $f_1(x), \ldots, f_k(x)$ generate $\mathrm{Id}(A)_n$ the ideal of real analytic functions vanishing on A. Now allow the variables to take on complex as well as real values. We get an embedding $R^n \subset C^n$ with coordinates $(z_1, \ldots, z_n)$, where $z = x + iy$ is the decomposition of the coordinates into real and complex parts, and $R^n = \{z \in C^n \mid y_1 = \cdots = y_n = 0\}$. Since $f_1(x), \ldots, f_k(x)$ can be represented by convergent power series, the same series can be used to define holomorphic functions $f_1(z), \ldots, f_k(z)$. A^*, the complexification of A, is just $A^* = \{z \in C^n \mid f_1(z) = \cdots = f_k(z) = 0\}$. A^* is a germ of a complex analytic variety. Properties of the complexification can be found in [3] or [9], but let me take note of the following obvious

REMARK 2.8. If $A \subset R^n$ and $B \subset R^m$ are real analytically isomorphic, then A^* and B^* are complex analytically isomorphic. In fact, if $(\phi_1, \ldots, \phi_m)$: $R^n \longrightarrow R^m$ is a real analytic map defining an isomorphism of A to B, then by complexifying the variable in the power series expansion for the ϕ's, one obtains a holomorphic map $C^n \longrightarrow C^m$ which defines an isomorphism from A^* to B^*.

Now let $V \subset C^n$ be a complex analytic germ, and think of it as a real analytic germ $V \subset R^{2n}$. It has a complexification V^*.

PROPOSITION 2.9. *If $V \subset C^n$ as above is irreducible, then $V^* \cong V \times \bar{V}$.*

PROOF. Let $f_1(z), \ldots, f_k(z)$ generate $I(V)_n$. Since V is irreducible, the real analytic ideal of $V \subset R^{2n}$ is generated by $(u_1(x,y), v_1(x,y), \ldots, u_k(x,y), v_k(x,y))$ where $u_j(x,y)$ and $v_j(x,y)$ are, respectively, the real and the complex parts of $f_j(z), j = 1, \ldots, k$, and $z_j = x_j + iy_j, j = 1, \ldots, n$, is the decomposition of the coordinates z into real and complex parts [4].

Now let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be extended to be complex coordinates of C^{2n}. Then V^* is defined by $u_1(x,y), v_1(x,y), \ldots, v_k(x,y)$. Equivalently, V^* is defined by $u_1(x,y) + iv_1(x,y), \ldots, u_k(x,y) + iv_k(x,y)$, $u_1(x,y) - iv_1(x,y), \ldots, u_k(x,y) - iv_k(x,y)$. Defining a holomorphic change of coordinates in C^{2n} by $z_j = x_j + iy_j$ and $w_j = x_j - iy_j, j = 1, \ldots, n$, we have $u_j(x, y) + iv_j(x, y) = f_j(z)$ and $u_j(x, y) - iv_j(x, y) = \tilde{f}_j(w), j = 1, \ldots, k$. Thus V^* is defined in C^{2n} by $f_1(z), \ldots, f_k(z), \tilde{f}_1(w), \ldots, \tilde{f}_k(w)$. Thus $V^* \cong V \times \bar{V}$. Q.E.D.

3. Structure of Cartesian products.

DEFINITION 3.1. Let V be a germ of a positive dimensional complex analytic set. By a decomposition of V of length k, I mean a k-tuple $(V_1, \ldots, V_k)$ of complex analytic germs of positive dimension such that $V \cong V_1 \times \cdots \times V_k$.

DEFINITION 3.2. V as above will be called indecomposable if there does not exist a decomposition of V of length > 1.

PROPOSITION 3.3. *Let V be a positive-dimensional germ of an analytic set. Then V has a decomposition of length $k \geqslant 1$ $(V_1, \ldots, V_k)$ for which each V_j, $1 \leqslant j \leqslant k$, is indecomposable.*

PROOF. Since $\dim_C(V_1 \times \cdots \times V_k) = \Sigma_{i=1}^k \dim_C V_i$ it is clear that if $\dim_C V = 1$, then V is indecomposable. The proof will follow by induction on the dimension of V. Suppose the proposition is established for all germs of dimension less than $\dim_C V$. Either V is indecomposable, and we are done, or $V \cong V' \times V''$ with $\dim_C V > \max(\dim_C V', \dim_C V'')$. In that case just apply the induction hypothesis to V' and V'' and put the two decompositions together. Q.E.D.

The purpose of this section is to prove

THEOREM 3.4. *Let V be an irreducible germ of a positive dimensional complex analytic set. Then the decomposition of V into indecomposables (which must exist by Proposition 3.3) is unique, i.e., let $(V_1, \ldots, V_k)$ and $(W_1, \ldots, W_l)$ be two decompositions of V into indecomposables. Then $k = l$ and, after a permutation of $(W_1, \ldots, W_k)$ we have $V_j \cong W_j, j = 1, \ldots, k$.*

Before giving a proof of this theorem several preliminaries are necessary. Let $V \subset C^n$ be a germ at the origin of a complex analytic set. Recall that a holomorphic vector field on V can be thought of as a holomorphic vector field on an ambient neighborhood of V which preserves $I(V)_n$. This definition is easily seen to be independent of the choice of embedding $V \subset C^n$ [10]. Following Whitney [11]

DEFINITION 3.5. $C_1(V, 0) = \{v \in T_0 V \mid$ there is a holomorphic vector field X on V for which $v = X(0)\}$ ($T_0 V$ denotes the Zariski tangent space to V at $0 \in C^n$).

Clearly $C_1(V, 0)$ is a complex linear subspace of $T_0 V$. Rossi [10] showed that if $\dim_C C_1(V, 0) = k$ then $V \cong C^k \times V'$. On the other hand, if $V \cong$

$C^k \times V'$, then clearly $\dim_C C_1(V, 0) \geqslant k$ (in fact $\partial/\partial z_i$, $1 \leqslant i \leqslant k$, are linearly independent vectors in $C_1(V, 0)$). Therefore we have

REMARK 3.6. $\dim_C C_1(V, 0)$ is precisely the largest integer k for which one can write $V \cong C^k \times V'$. We will need

PROPOSITION 3.7. *If* $V \cong V' \times V''$ *then* $C_1(V, 0) = C_1(V', 0) \times C_1(V'', 0)$.

PROOF. Choose embeddings $V' \subset C^n$, $V'' \subset C^m$ as in the previous section. Let $v \in C_1(V', 0)$ and choose $X(z) = \Sigma_{i=1}^n a_i(z)\,\partial/\partial z_i$ a holomorphic vector field on C^n preserving $I(V')_n$ for which $X(0) = v$. Extend $X(z)$ to a vector field on C^{n+m} by $X(z, w) = \Sigma_{i=1}^n a(z)\,\partial/\partial z_i$. Clearly, for any $f(w) \in O_m \subset O_{n+m}$, $X(z, w)f(w) = 0$. Since $I(V)_{n+m} = I(V')_n O_{n+m} + I(V'')_m O_{n+m}$, it follows (by the product rule for derivatives) that $X(z, w)$ preserves $I(V)_{n+m}$. Thus $(v, 0) = X(0, 0) \in C_1(V, 0)$ so that $C_1(V', 0) \times \{0\} \subset C_1(V, 0)$. Similarly, $\{0\} \times C_1(V'', 0) \subset C_1(V, 0)$, and since $C_1(V, 0)$ is a vector space, we have $C_1(V', 0) \times C_1(V'', 0) \subset C_1(V, 0)$.

Now suppose $(v_1, v_2) \in C_1(V, 0)$ and extend it to

$$X(z, w) = \sum_{i=1}^{n} a_i(z, w)\,\frac{\partial}{\partial z_i} + \sum_{j=1}^{m} b_i(z, w)\,\frac{\partial}{\partial w_j},$$

a vector field on C^{n+m} preserving $I(V)_{n+m}$. Let $f(z) \in I(V')_n \subset I(V)_{n+m}$. Then

$$(Xf)(z, w) \in I(V)_{n+m} = I(V')_n O_{n+m} + I(V'')_m O_{n+m}$$

so that $(Xf)(z, 0) \in I(V')_n$. But

$$(Xf)(z, w) = \sum_{i=1}^{n} a_i(z, w)\,\frac{\partial f(z)}{\partial z_i}$$

and

$$(Xf)(z, 0) = \sum_{i=1}^{n} a_i(z, 0)\,\frac{\partial f(z)}{\partial z_i} = \left(\sum_{i=1}^{n} a_i(z, 0)\,\frac{\partial}{\partial z_i}\right) f(z).$$

Thus $\Sigma_{i=1}^n a_i(z, 0)\,\partial/\partial z_i$ is a vector field on C^n preserving $I(V')_n$ and $v_1 = \Sigma_{i=1}^n a_i(0, 0)\,\partial/\partial z_i \in C_1(V', 0)$. Similarly $v_2 \in C_1(V'', 0)$ so that $(v_1, v_2) \in C_1(V', 0) \times C_1(V'', 0)$, and $C_1(V, 0) \subset C_1(V', 0) \times C_1(V'', 0)$. Q.E.D.

COROLLARY 3.8. *If* $V \cong V' \times V''$ *then* $\dim C_1(V, 0) = \dim C_1(V', 0) + \dim C_1(V'', 0)$.

PROPOSITION 3.9. *Suppose* $C \times V \cong C \times W$ *is irreducible; then* $V \cong W$.

PROOF. By Corollary 3.8, $\dim_C C_1(V, 0) = \dim_C C_1(W, 0) = r$. Applying Remark 3.6, $V = C^r \times V'$ and $W = C^r \times W'$ where $\dim_C C_1(V', 0) =$

$\dim_C C_1(W', 0) = 0$. The proposition will follow if it can be shown that $V' \cong W'$. In any event, $C^{r+1} \times V' \cong C^{r+1} \times W'$. Choose embeddings $V' \subset C^n$ and $W' \subset C^n$ and let $\{f_1, \ldots, f_k\}$ be a minimal set of generators of $\mathrm{Id}(V')_n$ and $\{g_1, \ldots, g_l\}$ be a set of generators for $\mathrm{Id}(W')_n$. Then $\{f_1, \ldots, f_k\}$ generate $\mathrm{Id}(C^{r+1} \times V')_{n+r+1}$ and are minimal and $\{g_1, \ldots, g_l\}$ generate $\mathrm{Id}(C^{r+1} \times W')_{n+r+1}$. Now choose an isomorphism $\Phi = (\phi_1, \ldots, \phi_{n+r+1})$: $C^{r+1+n} \longrightarrow C^{r+1+n}$, $\Phi(0) = 0$, which induces an isomorphism Φ: $C^{r+1} \times V' \longrightarrow C^{r+1} \times W'$ and let $\Psi = (\psi_1, \ldots, \psi_{n+r+1})$: $C^{r+1+n} \longrightarrow C^{r+1+n}$ be the inverse of Φ.

Define $\Omega = (\omega_1, \ldots, \omega_n)$: $C^n \longrightarrow C^n$ by

$$\omega_i(z_1, \ldots, z_n) = \phi_{r+1+i}(0, \ldots, 0, z_1, \ldots, z_n), \quad i = 1, \ldots, n.$$

(The coordinates in C^{n+r+1} are $(w_1, \ldots, w_{r+1}, z_1, \ldots, z_n)$.) Also define $\Lambda = (\gamma_1, \ldots, \gamma_n)$: $C^n \longrightarrow C^n$ by

$$\gamma_i(z_1, \ldots, z_n) = \psi_{r+1+i}(0, \ldots, 0, z_1, \ldots, z_n), \quad i = 1, \ldots, n.$$

I will show that Ω is an isomorphism which induces an isomorphism Ω: $V' \longrightarrow W'$. The proof will be based on the theorem [4, 3.2].

THEOREM. *Suppose* Ω: $C^n \longrightarrow C^n$, $\Omega(0) = 0$. *Let* $f_1, \ldots, f_k \in O_n$ *be germs of holomorphic functions vanishing at the origin and suppose* $f \circ \Omega = Af$ *where* f *is the column vector with components* $f_1, \ldots, f_k$, *and* $A \in \mathrm{Gl}(k, O_n)$. *Then either* Ω *is an isomorphism, or there exists a nowhere vanishing vector field near* $0 \in C^n$ *which preserves the ideal generated by the* $f_1, \ldots, f_k$.

PROOF. For a proof see [4, 3.2]. The proof there was given for $k = 1$, but the same proof works for general k with no modifications being necessary except for trivial changes in terminology; e.g. replace "function" by "column vector of functions" and "unit in O_n" by "element of $\mathrm{Gl}(k, O_n)$".

Returning to the present situation, since Ψ: $C^{r+1} \times W' \longrightarrow C^{r+1} \times V'$, $f_j \circ \Psi \in \mathrm{Id}(C^{r+1} \times W')_{n+r+1}$ for any $j = 1, \ldots, k$. Remembering that the f_j's were chosen independent of the first $r + 1$ variables, we get

(3.10) $f \circ \Psi = f(\psi_{r+2}, \ldots, \psi_{r+n+1}) = Ag$ where
f is the k-column vector with components $f_1, \ldots, f_k$,
g is the l-column vector with component $g_1, \ldots, g_l$, and
A is a holomorphic $k \times l$ matrix.

Similarly, we get

(3.11) $g \circ \Phi = g(\phi_{r+2}, \ldots, \phi_{r+n+1}) = Bf$
where B is a holomorphic $l \times k$ matrix.

Setting $w_1 = \cdots = w_{r+1} = 0$ in equations (3.10) and (3.11), we get

(3.12) $f \circ \Lambda = A'g$ where $A'(z_1, \ldots, z_n) = A(0, \ldots, 0, z_1, \ldots, z_n)$

and

(3.13) $g \circ \Omega = B'f$ where $B'(z_1, \ldots, z_n) = B(0, \ldots, 0, z_1, \ldots, z_n)$.

Composing (3.10) with Φ, substituting in (3.11), and using the fact that $\Psi = \Phi^{-1}$, we get

$$f = f \circ \Psi \circ \Phi = (A \circ \Phi)Bf. \tag{3.14A}$$

Let $C = (A \circ \Phi)B$ be a $k \times k$ holomorphic matrix. Since $f = Cf$ and $f_1, \ldots, f_k$ were chosen to be a *minimal* set of generators for $\mathrm{Id}(C^{r+1} \times V')_{n+r+1}$, we must have

$$C(0) = A(\Phi(0))B(0) = A(0)B(0) = I, \quad \text{the } k \times k \text{ identity matrix.}$$

Now compose (3.12) with Ω and substitute in (3.13),

$$f \circ (\Lambda \circ \Omega) = (A' \circ \Omega)B'f. \tag{3.14B}$$

Letting $C' = (A' \circ \Omega)B'$ we have $C'(0) = A'(\Omega(0))B'(0) = A'(0)B'(0)$. By definition of A' and B', $A'(0) = A(0)$ and $B'(0) = B(0)$. Therefore $C'(0) = A(0)B(0) = C(0) = I$, the $k \times k$ identity matrix, and $C \in \mathrm{Gl}(k, O_n)$. Equation (3.14B) becomes

$$f \circ (\Lambda \circ \Omega) = Cf, \qquad C \in \mathrm{Gl}(k, O_n). \tag{3.14C}$$

Since $f_1, \ldots, f_k$ generate $\mathrm{Id}(V')_n$ and $\dim_C C_1(V', 0) = 0$, every holomorphic vector field on a neighborhood of $0 \in C^n$ which preserves $\mathrm{Id}(V')_n$ must vanish at $0 \in C^n$. Apply the just quoted theorem to (3.14C), and it follows that $\Lambda \circ \Omega: C^n \longrightarrow C^n$ is an isomorphism. Hence $\Lambda: C^n \longrightarrow C^n$ is also an isomorphism, and by (3.12) so is $\Lambda: V' \longrightarrow W'$. Q.E.D.

We now have everything needed for a proof of Theorem 3.4, but before giving the proof it is convenient to make a few simple observations.

OBSERVATION 3.15. If $V \cong W$ then $\mathrm{Sg}(V) \cong \mathrm{Sg}(W)$ ($\mathrm{Sg}(V)$ denotes the singular locus of V), and every irreducible component of $\mathrm{Sg}(V)$ is isomorphic to an irreducible component of $\mathrm{Sg}(W)$.

OBSERVATION 3.16. Let V be irreducible and $(V_1, \ldots, V_k)$ be a decomposition of V into indecomposables. Then every irreducible component Z of $\mathrm{Sg}(V)$ is of the form

$$Z = V_1 \times \cdots \times V_{i-1} \times Z' \times V_{i+1} \times \cdots \times V_k \tag{3.17}$$

where Z' is an irreducible component of $\mathrm{Sg}(V_i)$, $1 \leq i \leq k$. Conversely, any such Z' gives rise (through (3.17)) to an irreducible component Z of $\mathrm{Sg}(V)$.

OBSERVATION 3.18. Let $V, (V_1, \ldots, V_k), Z$ and Z' be as in 3.16. Suppose $(Z_1, \ldots, Z_r)$ is a decomposition of Z' into indecomposables. Then

$$(3.19) \qquad (V_1, \ldots, V_{i-1}, Z_1, \ldots, Z_r, V_{i+1}, \ldots, V_k)$$

is a decomposition of Z into indecomposables.

OBSERVATION 3.20. Let V, Z be as in 3.16 and 3.18. Let $(W_1, \ldots, W_l)$ be another decomposition of V into indecomposables. Applying Observations 3.15, 3.16 and 3.18 we have $Z \cong W_1 \times \cdots \times W_{j-1} \times Y' \times W_{j+1} \times \cdots \times W_l$ where Y' is an irreducible component of $\mathrm{Sg}(W_j)$ for some j, $1 \leq j \leq l$. If $(Y_1, \ldots, Y_s)$ is a decomposition of Y' into indecomposables, then

$$(3.21) \qquad (W_1, \ldots, W_{j-1}, Y_1, \ldots, Y_s, W_{j+1}, \ldots, W_l)$$

is another decomposition of Z into indecomposables.

OBSERVATION 3.22. Let V, Z be as in 3.16; then $\dim_C Z < \dim_C V$.

PROOF OF THEOREM 3.4. Let V be an irreducible, positive dimensional germ of a complex analytic set, and let $(V_1, \ldots, V_k)$ and $(W_1, \ldots, W_l)$ be two decompositions of V into indecomposables. Theorem 3.4 will be proven by induction on $N = \dim_C V$.

If $\dim_C V = 1$ then V must be indecomposable, so that $l = k = 1$ and $V_1 \cong V \cong W_1$. This establishes Theorem 3.4 in this case.

Now suppose Theorem 3.4 is established for all V'', $1 \leq \dim_C V'' < N = \dim_C V$. Theorem 3.4 must be established for V. I will begin by making two reductions in the problem.

REDUCTION 1. We may as well assume that $k \geq 2$ and $l \geq 2$.

PROOF. If either $k = 1$ or $l = 1$ then V is indecomposable. But then $k = l = 1$ and $V_1 \cong V \cong W_1$, establishing Theorem 3.4 for such a V.

REDUCTION 2. We may as well assume that $\dim_C C_1(V, 0) = 0$.

PROOF. By Reduction 1 we may assume $\min\{k, l\} \geq 2$. Suppose $\dim_C C_1(V, 0) \neq 0$. Then, by Corollary 3.8 and Remark 3.6, at least one of the V_i's and at least one of the W_j's must both be isomorphic to C. By reordering the V_i's and the W_j's we may assume $V_1 \cong W_1 \cong C$. Then $V \cong C \times V_2 \times \cdots \times V_k \cong C \times W_2 \times \cdots \times W_l$. By Proposition 3.9 we have $V_2 \times \cdots \times V_k \cong W_2 \times \cdots \times W_l$. Since $\dim_C V_2 \times \cdots \times V_k = \dim_C V - 1$ we may apply the induction hypothesis to conclude $k - 1 = l - 1$, so that $k = l$, and, we may reorder $(W_2, \ldots, W_l)$ to achieve $V_i \cong W_i$, $2 \leq i \leq l$. But, we already have $V_1 \cong W_1 \cong C$. Theorem 3.4 is thus established in this case.

We are now reduced to establishing Theorem 3.4 for a V for which

$\min\{k, l\} \geq 2$, and $\dim_C C_1(V, 0) = 0$. This last condition says that $\mathrm{Sg}(V_i) \neq \varnothing$, $\mathrm{Sg}(W_j) \neq \varnothing$ for all i, j, $1 \leq i \leq k$ and $1 \leq j \leq l$. Suppose we are in this situation.

For convenience, let us reorder the V_i's and the W_j's so that $\dim_C V_1 \geq \dim_C V_2 \geq \cdots \geq \dim_C V_k$ and $\dim_C W_1 \geq \dim_C W_2 \geq \cdots \geq \dim_C W_l$. The proof of Theorem 3.4 will follow from a careful application of Observations and descriptions 3.15–3.22, particularly (3.19) and (3.21), together with the induction hypothesis.

CLAIM 3.23. $\dim_C V_1 = \dim_C W_1$.

PROOF. Let Z be any irreducible component of $V_1 \times \cdots \times V_{k-1} \times \mathrm{Sg}(V_k)$. Then (3.19) (with $i = k$) and (3.21) give two decompositions of Z into indecomposables. By the induction hypothesis they must be the same (in the sense of Theorem 3.4). Thus, either $V_1 \cong W_h$ for some $h \neq j$, $1 \leq h \leq l$, or $V_1 \cong Y_g$ for some g, $1 \leq g \leq s$. But by the ordering of the W_h's and the definition of the Y_g's, $\dim_C W_1 \geq \dim_C W_h$ and $\dim_C W_1 \geq \dim_C W_j > \dim_C Y_g$. Thus $\dim_C W_1 \geq \dim_C V_1$. The claim follows by symmetry. Let $N_1 = \dim_C V_1 = \dim_C W_1$.

CLAIM 3.24. Let $M = \sup\{i \mid \dim_C V_i = N_1\}$ and $M' = \sup\{j \mid \dim_C W_j = N_1\}$. Then $M = M'$.

PROOF. By symmetry we may as well assume $M \geq M'$. By 3.23, $M' \geq 1$. Let Z be any irreducible component of $\mathrm{Sg}(W_1) \times W_2 \times \cdots \times W_l$. Then (3.19) and (3.21) (with $j = 1$) give two decompositions of Z into indecomposables, and by the induction hypothesis must really be the same. Since $N_1 = \dim_C W_1 > \dim_C Y_i$ for any i, $1 \leq i \leq s$, exactly $M' - 1$ ($\leq M - 1$) of the terms in (3.21) are of dimension N_1. No fewer than $M - 1$ of the terms of (3.19) can be of dimension N_1. Therefore $M' - 1 = M - 1$, and $M' = M$.

Note that

$$MN_1 = \dim_C(V_1 \times \cdots \times V_M) = \dim_C(W_1 \times \cdots \times W_M).$$

CLAIM 3.25. $V_1 \times \cdots \times V_M \cong W_1 \times \cdots \times W_M$.

PROOF. If $MN_1 = N$ then $k = l = M$ and $V_1 \times \cdots \times V_M \cong V \cong W_1 \times \cdots \times W_M$.

If $MN_1 < N$ then $M < k$ and $M < l$. Let Z be any irreducible component of $V_1 \times \cdots \times V_{k-1} \times \mathrm{Sg}(V_k)$. Then (3.19) with $i = k$ and (3.21) give two decompositions Z into indecomposables, and by the induction hypothesis they are the same. Since there are M terms in the decomposition (3.19) which are of dimension N_1, there must be M such terms in the decomposition (3.21). Thus, $j > M$ where j is the integer j from (3.21). Apply the induction hypothesis to the terms of the decomposition (3.21)

$$(W_1, \ldots, W_M, \ldots, W_{j-1}, Y_1, \ldots, Y_s, W_{j+1}, \ldots, W_l)$$

so that after the permutation, the pth term is isomorphic to the pth term of $(V_1, \ldots, V_{k-1}, Z_1, \ldots, Z_r)$ for any p. But, out of $(W_1, \ldots, W_{j-1}, Y_1, \ldots, Y_s, W_{j+1}, \ldots, W_l)$, the only terms of dimension N_1 are the W_p's $(1 \leqslant p \leqslant M)$. The permutation of the terms of (3.21) must therefore restrict to a permutation of $(W_1, \ldots, W_M)$, and after reordering the $(W_1, \ldots, W_M)$ by this permutation we have $V_i = W_i$, $1 \leqslant i \leqslant M$. Thus $V_1 \times \cdots \times V_M \cong W_1 \times \cdots \times W_M$, proving the claim.

Let X be any germ of a complex analytic set, and let $M(X) = \#\{j \mid V_j \cong X\}$ and $M'(X) = \#\{j \mid W_j \cong X\}$. Then to prove Theorem 3.4 it clearly suffices to prove

CLAIM 3.26. $M(X) = M'(X)$ for all X.

PROOF. First suppose that $N = N_1 M$. Then, as already noted, $M = k = l$, $\dim_C V_i = \dim_C W_i = N_1$ for all i, $1 \leqslant i \leqslant n$.

For any X, either $M(X) \geqslant M'(X)$ or $M(X) \leqslant M'(X)$. By symmetry we may as well assume $M(X) \geqslant M'(X)$. We may also assume $M(X) \geqslant 1$, since if $M(X) = 0$ then $M'(X) = 0 = M(X)$ and we would be done for this X.

First I will show that $M'(X) \geqslant 1$. The proof of this is similar in idea to the proof of Claim 3.23. Using this I will show that $M(X) = M'(X)$. The proof is similar in idea to the proof of Claim 3.24.

Since $M(X) \geqslant 1$, I can find an i so that $V_i \cong X$. Choose a $q \neq i$, $1 \leqslant q \leqslant M$ (this can be done since $M = k \geqslant 2$) and consider an irreducible component Z of $V_1 \times \cdots \times V_{q-1} \times \mathrm{Sg}(V_q) \times \cdots \times V_M$. It has $V_i \cong X$ as a factor and, hence, X must be isomorphic to some term of the decomposition into indecomposables (3.21) for Z (using the induction hypothesis). But $\dim_C Y_i < N_1$, $1 \leqslant i \leqslant s$. Thus $X \cong W_j$ for some j and $M'(X) \neq 0$.

Choose a j for which $X \cong W_j$, and now let Z be an irreducible component of $W_1 \times \cdots \times W_{j-1} \times \mathrm{Sg}(W_j) \times W_{j+1} \times \cdots \times W_M$. By the induction hypothesis (3.19) and (3.21) must give the same decomposition of Z into indecomposables. But precisely $M'(X) - 1$ of the terms in (3.21) are isomorphic to X, and there are at least $M(X) - 1$ such terms in (3.19). Combining this with $M(X) \geqslant M'(X)$ gives $M(X) = M'(X)$ and the claim is proven in case $N = MN_1$.

Now suppose $N > MN_1$. Then, as already noted, $M < k$ and $M < l$. By Claim 3.25 $V_1 \times \cdots \times V_M \cong W_1 \times \cdots \times W_M$ and we can use the induction hypothesis to conclude that after reordering the $(W_1, \ldots, W_M)$ we have $V_i \cong W_i$, $1 \leqslant i \leqslant M$. Thus, we clearly have $M(X) = M'(X)$ provided $\dim_C X = N_1$.

Now choose any X such that $\dim_C X < N_1$. Either $M(X) \geqslant M'(X)$ or $M'(X) \geqslant M(X)$, and by symmetry we may assume $M(X) \geqslant M'(X)$.

Let V'' be any irreducible germ of a complex analytic set of dimension less than N. Let $(V_1'', \ldots, V_t'')$ be a decomposition of V'' into indecomposables. Then, by the induction hypothesis, $M(X, V'') = \#\{j \mid V_j'' \cong X\}$ depends only on X and V'', and not on the choice of decomposition.

Choose an irreducible component Z_1 of $\mathrm{Sg}(V_1 \times \cdots \times V_M)$ in such a way that $M(X, Z_1)$ is as large as possible. $Z = Z_1 \times V_{M+1} \times \cdots \times V_k$ is an irreducible component of $\mathrm{Sg}(V_1 \times \cdots \times V_k)$. By the induction hypothesis, the two decompositions (3.19) and (3.21) of Z must be the same. But in (3.19) we have $1 \leqslant i \leqslant M$ (by the choice of Z). (3.19), and thus also (3.21), must then have exactly $M - 1$ terms of dimension N_1. But then in (3.21) we must have $1 \leqslant j \leqslant M$, so that $Z \cong Z_2 \times W_{M+1} \times \cdots \times W_l$ where Z_2 is an irreducible component of $\mathrm{Sg}(W_1 \times \cdots \times W_M) \cong \mathrm{Sg}(V_1 \times \cdots \times V_M)$. We now clearly have

$$M(X, Z) = M(X, Z_1) + M(X) = M(X, Z_2) + M'(X)$$

since $\dim_C Z < N$. Since $M(X) \geqslant M'(X)$ and $M(X, Z_1) \geqslant M(X, Z_2)$ (by the choice of Z_1), we can conclude that $M(X) = M'(X)$. This completes the proof of 3.26 and hence, of Theorem 3.4.

Theorem 3.4 may be reformulated as

THEOREM 3.27. *Let V be an irreducible germ of a complex analytic set having a decomposition $(V_1, \ldots, V_k)$ into indecomposables. Then, if X is any germ of a complex analytic set, the integer $M(X, V) = \#\{j \mid X \cong V_j\}$ is well defined, i.e., $M(X, V)$ depends only X and V, not on the choice of $(V_1, \ldots, V_k)$.*

PROOF. This is clearly equivalent to Theorem 3.4.

4. Equivalences of singularities.

DEFINITION 4.1. Let V be an irreducible germ of a complex analytic set having a decomposition $(V_1, \ldots, V_k)$ into indecomposables. A germ W of a complex analytic set will be said to be *partially conjugate* to V if and only if $W \cong W_1 \times \cdots \times W_k$ where for each i, $1 < i < k$, either $W_i \cong V_i$ or $W_i \cong \bar{V}_i$.

REMARK 4.2. Let V be as above. Then, up to isomorphism, there are at most finitely many analytically distinct partial conjugates of V. In fact, the number of distinct partial conjugates $\leqslant 2^l$ where

$$l = \min\{\dim_C V - \dim_C C_1(V, 0), \dim_C T_0 V - \dim_C V, k\}.$$

PROPOSITION 4.3. *Let V and W be irreducible germs of complex analytic sets. Then V and W are partially conjugate if and only if*

$$M(X, V) + M(\bar{X}, V) = M(X, W) + M(\bar{X}, W) \tag{4.4}$$

for every germ X of a complex analytic set.

PROOF. This is an obvious consequence of Theorem 3.4 and its reformulation, Theorem 3.27.

REMARK 4.5. If V and W are partially conjugate they are real analytically equivalent.

We can now easily prove

THEOREM 4.6. *Let V be an irreducible germ of a complex analytic set which is C^∞ isomorphic to W, a germ of a complex analytic set. Then V and W are partially conjugate* (*thus, by Remark* 4.2, *there are, up to complex analytic isomorphism, only finitely many possibilities for* W).

PROOF. By [4, 1.1], V and W must actually be real analytically equivalent. Hence, W is also irreducible. Also $V^* \cong W^*$ where V^* (resp. W^*) is the complexification of V (resp. W) viewed as a real analytic set (Remark 2.8).

By Proposition 2.9, $V^* \cong V \times \bar{V}$, and $W^* \cong W \times \bar{W}$. Thus, one clearly has

$$M(X, V^*) = M(X, V) + M(X, V) \tag{4.7}$$

and

$$M(X, W^*) = M(X, W) + M(X, \bar{W}). \tag{4.8}$$

Since $V^* \cong W^*$ we have $M(X, V^*) = M(X, W^*)$, which give, together with (4.7) and (4.8),

$$M(X, V) + M(X, \bar{V}) = M(X, W) + M(X, \bar{W}). \tag{4.9}$$

The theorem now follows from Proposition 4.3 and the obvious fact that $M(\bar{X}, V) = M(X, \bar{V})$ and $M(\bar{X}, W) = M(X, \bar{W})$.

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