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ON SUBCATEGORIES OF TOP

BY

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ABSTRACT. A categorical characterization of a subcategory S of TOP (or T_2) is one which enables the identification of S in TOP (or T_2) without requiring the reconstruction of the topological structure of its objects. In this paper we so characterize various familiar subcategories of TOP (Hausdorff spaces, normal spaces, compact Hausdorff spaces, paracompact Hausdorff spaces, metrizable spaces, first countable spaces) in terms of the global behavior of the (objects and) morphisms of the subcategory.

0. Introduction. A categorical characterization of a subcategory S of TOP (or of T_2) is one which enables the identification of S in TOP (or T_2) without requiring the reconstruction of the topological structure of its objects. The proof of the triviality of the automorphism class group of TOP [**Fe**, p. 32] assures us of the existence of such characterizations. As will be shown in §1, this proof establishes that all the topological structure of an object of TOP can be recovered from the behavior of the arrows with that object as source and/or target. In §2 we establish a number of lemmas (mostly known), and we use these lemmas in §3 to characterize many important subcategories of TOP and/or T_2, such as the metrizable spaces, the paracompact Hausdorff spaces, the normal spaces, etc.

We can distinguish at least three classes of "categorical" characterizations of subcategories of TOP, which are listed below:

Class 3. Straightforward translations of the familiar definitions. These can always be obtained because of the results in §1. Example: Characterize the concepts "open", "cover", "refinement", "locally finite" categorically and then the standard definition of paracompact spaces is categorical.

Class 2. Characterizations using standard categorical constructions (product, coproduct, epimorphic images, etc.) but in terms of individual objects, and/or constants outside the subcategory. Example: The category of paracompact Hausdorff spaces contains precisely those Hausdorff spaces X such that

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$X \times \beta X$ is normal (presuming we can give Class 2 definitions of normality and of βX).

Class 1. Characterizations in terms of the global behavior of the (objects and) morphisms of the subcategory, preferably without the use of constants, and certainly without the use of constants outside the subcategory. Example: The paracompact Hausdorff spaces form the largest left-fitting subcategory of T_2 preserved by shrinks of T_2 extremal monos.

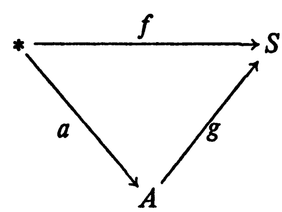
Our aim in this paper is to characterize categorically various familiar subcategories of TOP and T_2, preferably with Class 1 characterizations.

1. **The arrows of** TOP **determine the spaces and functions.** Assume that we are given a collection of dots and arrows and are told that it is the category TOP. We can reconstruct the topological spaces and the functions as follows. See [**Fe**, p. 32].

1.1. The *singleton space*, $*$, is the terminal object of TOP. We can now recover the underlying set of an object A of TOP; it is precisely TOP $(*,A)$. Then, given an arrow $f\colon A \to B$ in TOP, and an element $a \in A$ (i.e., an arrow $a\colon * \to A$), $f(a)$ is the composition $f \circ a\colon * \to B$.

1.2. The *Sierpiński two point space* S, i.e., the two element space with only one isolated point, is the only object, up to isomorphism, with precisely three self-maps. Any space with three or more points has at least four self-maps: each constant and the identity map; $*$ has one self-map. The two point discrete and indiscrete spaces each have four self-maps; hence, only S has three self-maps.

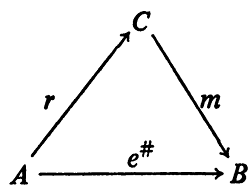
1.3. There are two arrows $* \to S$. We wish to characterize the *open map*, $u\colon * \to S$, and the *closed map*, $c\colon * \to S$. Let $f\colon * \to S$ be one of these arrows, and let A be an object of TOP. For each $g\colon A \to S$ let A_g be the set of all arrows $a\colon * \to A$ such that the following triangle commutes.



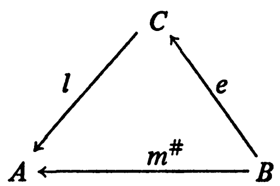
We say A is a *test space* for f if for all $a\colon * \to A$, there is a morphism $g\colon A \to S$ such that $A_g = \{a\}$. (Notice that if $f = c$, the test spaces are the T_1 spaces; and if $f = u$ the test spaces are the discrete spaces.) Then $f = u$ if and only if for every test space A and each $\mathcal{Q} \subset$ TOP $(*,A)$, there exists a $g\colon A \to S$ such that $\mathcal{Q} = A_g$. (This says nothing more than that if points are open then every subset is open, but there are T_1 spaces with nonclosed subsets.) Let 0 denote $u(*)$, and 1 denote $c(*)$.

We now wish to determine the closed and the open subsets of an object A of TOP. We need two standard definitions.

1.4. Definition. A morphism $e^{\#}: A \to B$ in a category $\mathcal{C}$ is an *extremal epimorphism* if $e^{\#}$ is an epimorphism such that whenever $e^{\#} = mr$



for any morphism $r: A \to C$ and any monomorphism $m: C \to B$, then m is an isomorphism. Dually, a morphism $m^{\#}: B \to A$ in a category $\mathcal{C}$ is an *extremal monomorphism* if $m^{\#}$ is a monomorphism such that whenever $m^{\#} = le$



for any morphism $l: C \to A$ and any epimorphism $e: B \to C$, then e is an isomorphism.

1.5. *Notation.* $\rightarrow$ will denote an arbitrary morphism,

$\twoheadrightarrow$ will denote an epimorphism,

$\twoheadrightarrow\!\!\!\!\rightarrow$ will denote an extremal epi,

$\rightarrowtail$ will denote a monomorphism,

$\rightarrowtail\!\!\!\!\!\rightarrowtail$ will denote an extremal mono.

The following three results are in the literature [**H**, pp. 115,116] and [**Fr3**, pp. 22–24].

1.6. *A subspace* $F \overset{m^{\#}}{\rightarrowtail} A$ *of an object* A *of* TOP *is an object* F *of* TOP *together with a* TOP *extremal monomorphism*; *or, the extremal monomorphisms in* TOP *are precisely the embeddings.*

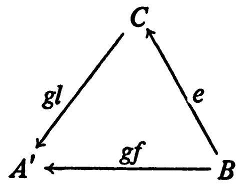
1.7. *The extremal monomorphisms in* T_2 *are precisely the closed embeddings.*

1.8. *In* TOP *and* T_2 *the extremal epimorphisms are the quotient maps.*

We will need the following purely categorical lemma.

1.9. Lemma. *If the composition* $B \xrightarrow{f} A \xrightarrow{g} A'$ *is an extremal monomorphism, then* f *is an extremal monomorphism.*

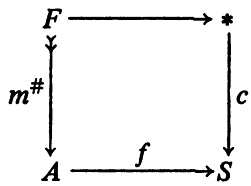
Proof. It is well known that if gf is a monomorphism, then f is a monomorphism. Suppose $f = le$ with e an epimorphism; then the following diagram commutes:



Hence e is an isomorphism.

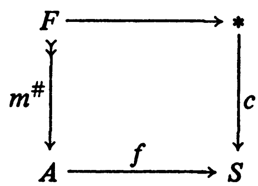
We are now in a position to recover the topology of an object of TOP.

1.10. LEMMA. *A subspace* $F \overset{m^{\#}}{\rightarrowtail} A$ *of an object* A *in* TOP *is closed if and only if there is some* $f\colon A \to S$ *such that*



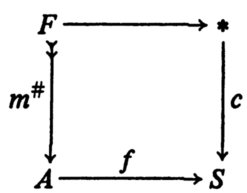
is a pullback diagram.

PROOF. Suppose F is a closed subspace of A and let $f\colon A \to S$ be the (continuous) characteristic function of F (1 is the closed point of S). Then the diagram



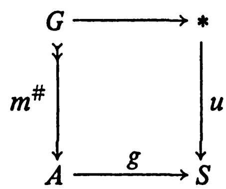
commutes, and is the desired pullback.

Conversely, suppose there is an $f\colon A \to S$ such that



is a pullback. Since $A \overset{i}{\leftarrowtail} f^{\leftarrow}(1) \to *$ is also a pullback for $A \xrightarrow{f} S \xleftarrow{c} *$, there must be an isomorphism $g\colon f^{\leftarrow}(1) \to F$ such that $m^{\#} g = i$.

1.11. LEMMA. *A subspace* $G \overset{m^{\#}}{\rightarrowtail} A$ *of an object* A *of* TOP *is open if and only if there is some* $g\colon A \to S$ *such that*



is a pullback diagram.

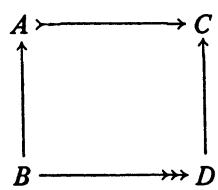
The proof is similar to the proof of Lemma 1.10 with $g = \chi_{A\backslash G}$.

These last two lemmas have the disadvantage that they will have no meaning in categories which do not contain a Sierpiński two point space, e.g., T_2. This same disadvantage is shared by the characterizations of closed maps and open maps in TOP (Lemmas 2.2 and 2.3).

2. **More definitions and lemmas.** We have shown in §1 that the topology of an object of TOP can be recovered from the behavior of the arrows with that object as source and/or target. In this section we establish a number of lemmas which will be used in the characterizations in §3. The impatient reader may skip this section if he is willing to assume that we can establish categorical meanings for the topological terms used in the third section.

The following lemma appears in [**H**, p. 65] and in [**Fr3**, p. 33].

2.1. LEMMA. *In any category with the extremal epi-mono factorization property,*
(i) *factorization is unique and extremal epimorphisms compose if and only if*
(ii) *for each commutative diagram*

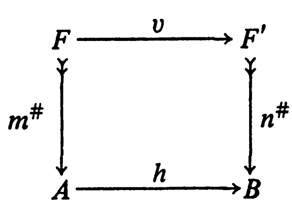


there exists a unique d: $D \to A$ *making everything commute.*

In particular, TOP has these properties and their duals.

2.2. LEMMA. *A* TOP-*morphism* $h: A \to B$ *is closed if and only if for each closed* $m^{\#}: F \rightarrowtail A$, $hm^{\#}$ *can be factored* $hm^{\#} = n^{\#} v$ *where* $n^{\#}: F' \rightarrowtail B$ *is closed and* $v: F \to F'$ *is epi.*

PROOF. Consider the following diagram.



Using 2.1 it is clear that F' is, in fact, $h(F)$ and that the lemma is merely a categorical restatement of the definition of a closed map.

2.3. LEMMA. *A TOP-morphism h: $A \to B$ is open if and only if for each open $m^{\#}: G \rightarrowtail A$, $hm^{\#}$ can be factored $hm^{\#} = n^{\#} v$ where $n^{\#}: F' \rightarrowtail B$ is open and v: $F \to F'$ is epi.*

We can now characterize perfect maps in terms of closed maps. Notice that for us a perfect map need not be onto.

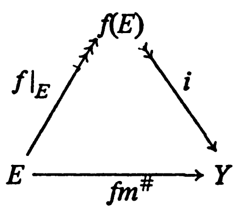
2.4. LEMMA. *A TOP-morphism f: $A \to B$ is perfect if and only if $f \times 1_Z: A \times Z \to B \times Z$ is closed for every identity $1_Z: Z \to Z$.*

PROOF. [**Bo**, p. 117].

If we restrict our attention to T_2 we can describe closed maps and perfect maps without relying on constants.

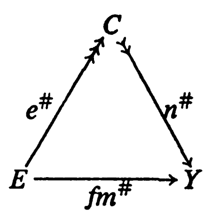
2.5. LEMMA. *In T_2 a morphism f: $X \to Y$ is a closed map if and only if for every extremal mono $m^{\#}: E \rightarrowtail X$, $fm^{\#}$ has an external epi-extremal mono factorization.*

PROOF. Let f: $X \to Y$ be closed and $m^{\#}: E \to X$ be an extremal mono. Then $f(E)$ is a closed subset of Y and $f|_E$: $E \to f(E)$ is a closed map. Thus



is the desired factorization.

Conversely, suppose f has the factorization property and E is a closed subset of X. Let $m^{\#}$ be the inclusion map $E \rightarrowtail X$. Then there is a factorization



It follows that $f(E) = fm^{\#}(E) = n^{\#}(C)$ which is a closed subset of Y.

Thus, using Lemma 2.4, we have a characterization of the perfect maps in T_2 which does not use constants. In what follows we will have occasion to use the fact that a perfect epi is necessarily onto.

Now that we have characterized perfect-onto maps categorically, the following definition is also categorical.

2.6. Definition. A subcategory $\mathcal{C}$ of TOP (or T_2) is called *right-fitting* if perfect-onto images of objects in $\mathcal{C}$ are again in $\mathcal{C}$. $\mathcal{C}$ is called *left-fitting* if perfect-onto preimages of objects of $\mathcal{C}$ are again in $\mathcal{C}$. $\mathcal{C}$ is called *fitting* if it is both left-fitting and right-fitting.

We close this section with a categorical characterization of biquotient maps in the style of 2.4.

2.7. Lemma. *A morphism $f\colon A \to B$ in T_2 is biquotient if and only if $f \times 1_C\colon A \times C \to B \times C$ is an extremal epimorphism for every $1_C\colon C \to C$ in T_2.*

Proof. [M_1, p. 288].

3. **Characterizations.** Now that we have shown what is meant categorically by such terms as "subspace", "closed map", etc., we will freely use such common terms as "hereditary", "preserved by closed maps", etc. The word "subcategory" will always denote a full and replete subcategory. For later theorems it will be convenient to have categorical characterizations of T_2 and T_4; for completeness we include categorical characterizations of the major separation axioms.

3.1. Proposition. *T_0 is the epireflective hull of the Sierpiński two point space.*

Proof. This proposition is a restatement of Alexandroff's result that every T_0 space is a subspace of a product of Sierpiński spaces [A] using the fact that epireflective hulls of TOP are produced by taking subspaces of products [H], [Fr3].

3.2. Proposition. *T_1 is the largest epireflective subcategory of* TOP *which is properly contained in T_0.*

Proof. Since T_1 is hereditary and productive, it is epireflective in TOP [H], [Fr3]. Suppose $\mathcal{C}$ is an epireflective subcategory of TOP, with $T_1 \subset \mathcal{C} \subset T_0$, and suppose that there is a space $X \in \operatorname{Ob}\mathcal{C}\setminus\operatorname{Ob}T_1$. Then S is a subspace of X and, since $\mathcal{C}$ is epireflective, $T_0 \subset \mathcal{C}$.

3.3. Proposition. *T_2 is the largest subcategory of* TOP *such that equalizers are precisely the closed* TOP *extremal monos.*

Proof. It is well known that if

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in T_2, then the equalizer of f and g, $\{x \in X \mid f(x) = g(x)\}$, is a closed subspace of X. Conversely, if F is a closed subspace of $X \in \operatorname{Ob}T_2$, then F is the equalizer of the maps

$$\begin{array}{ccccc} & \nearrow & X \times \{0\} = X_1 & \searrow & \\ X & & & & X_1 \cup_F X_2 \\ & \searrow & X \times \{1\} = X_2 & \nearrow & \end{array}$$

and, since F is closed, $X_1 \cup_F X_2$ is indeed a Hausdorff space.

On the other hand, suppose $\mathcal{C}$ is any subcategory of TOP properly containing T_2 and that $X \in \mathrm{Ob}\,\mathcal{C}\backslash\mathrm{Ob}\,T_2$. Then there is a net $S: D \to X$ which converges to two distinct points, say x_1 and x_2, of X. Without loss of generality we may assume that D has no largest element. Then the net space $D^* = D \cup \{\infty\}$ (the tails of D determine a neighborhood basis at ∞) and its discrete subspace D are Hausdorff, and hence belong to $\mathcal{C}$. For $i = 1, 2$ the maps $S_i: D^* \to X$ defined by

$$S_i(d) = \begin{cases} x_i & \text{if } d = \infty, \\ S(d) & \text{otherwise,} \end{cases}$$

are continuous, since S converges to both x_1 and x_2. Their equalizer D is not closed in D^*.

It would be particularly pleasing to find a characterization of T_2 which avoids identifying the closed subspaces, since T_2 is such an important subcategory of TOP.

3.4. PROPOSITION. *T_3 is the smallest nontrivial, fitting, hereditary subcategory of T_2.*

It is clear that T_3 is hereditary and right-fitting in T_2. To see that T_3 is left-fitting let $p: X \to Y$ be a perfect map with $Y \in \mathrm{Ob}\,T_3$ and $X \in \mathrm{Ob}\,T_2$. Let F be a closed set in X and let x be a point of X not in F. If $p(x) \notin p(F)$ there is no problem separating x from F, so suppose $p(x) \in p(F)$. This means that $p^{\leftarrow}[p(x)] \cap F \neq \varnothing$. Since X is Hausdorff the compact set $p^{\leftarrow}[p(x)] \cap F$ and the point x can be separated by disjoint open sets U_1 and V_1, say $x \in U_1$ and $p^{\leftarrow}[p(x)] \cap F \subset V_1$. Thus $p^{\leftarrow}[p(x)] \cap (F\backslash V_1) = \varnothing$ and $p(x) \notin p(F\backslash V_1)$. Hence x and $F\backslash V_1$ can be separated by disjoint open sets U_2 and V_2, say $x \in U_2$, $F\backslash V_1 \subset V_2$. Then $U_1 \cap U_2$ and $V_1 \cup V_2$ are open sets separating x from F.

Let $\mathcal{C}$ be a subcategory of T_2 satisfying the above hypotheses. Since $\mathcal{C}$ is nontrivial and hereditary it contains $*$; since $\mathcal{C}$ is left-fitting it contains all compact Hausdorff spaces. The hypothesis that $\mathcal{C}$ is hereditary now puts all Tychonoff spaces in $\mathcal{C}$; and since $\mathcal{C}$ is right-fitting it contains T_3, since every regular space is the perfect image of its projective cover which is Tychonoff (see [**Ba**]).

The standard result that $T_{3\,1/2}$ is the hereditary hull of the category of compact Hausdorff spaces will be meaningful in this context once a categorical characterization has been given for CH; see Propositions 3.8, 3.9, and 3.10.

The following definition is needed for the characterization of T_4.

3.5. Definition. A morphism $f\colon X \to Y$ is a *shrink* of $m\colon F \to X$ if there is a $g\colon * \to Y$ such that

$$\begin{array}{ccc} F & \xrightarrow{\ m\ } & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ * & \xrightarrow{\ g\ } & Y \end{array}$$

is a pushout.

3.6. Proposition. *T_4 is the largest subcategory of T_2 preserved by shrinks, in* TOP, *of T_2 extremal monos.*

3.7. Lemma. *In* TOP, *a shrink of an extremal mono $m^{\#}: F \rightarrowtail X$ is the quotient map which collapses F to a point.*

Proof. The quotient map $q\colon X \to X/F$ has the pushout property, and pushouts are unique.

Proof of Proposition 3.6. Since a shrink of a T_2 extremal mono is a closed map, T_4 is preserved by them. Suppose that $\mathcal{C}$ is a subcategory of T_2 preserved by shrinks of T_2 extremal monos, and let F_1 and F_2 be disjoint closed subsets of a space X in $\mathcal{C}$. First shrink F_1 to a point, and then shrink F_2 to a point. Since the resulting space must be Hausdorff, X itself must be normal. Thus any such $\mathcal{C}$ can contain only normal spaces, and T_4 is the largest such subcategory.

We give three characterizations of the subcategory CH of compact Hausdorff spaces.

3.8. Theorem (Herrlich and Strecker). *CH is the only nontrivial, epireflective subcategory of T_2 which is varietal.*

This theorem is proved in [**H–S**].

3.9. Theorem (de Groot). *CH is the only nontrivial, productive subcategory $\mathcal{C}$ of* TOP *which is preserved by closed epis and satisfies*

> *if $X \in \mathrm{Ob}\,\mathcal{C}$ and $m^{\#}: F \rightarrowtail X$, then*
> *$F \in \mathrm{Ob}\,\mathcal{C}$ if and only if $m^{\#}$ is a closed map.*

Proof. In [**W**, p. 51] there is a topological version of this result, due to de

Groot, which needs only to be translated into categorical language to yield 3.9 [F–T].

3.10. PROPOSITION. *CH is the only nontrivial, productive, left-fitting subcategory of T_2 preserved by shrinks, in* TOP, *of T_2 extremal monos.*

PROOF. If $\mathcal{C}$ is a subcategory of T_2 satisfying these conditions, then $\mathcal{C}$ contains some nonempty space X. Since $\mathcal{C}$ is left-fitting, $* \in \mathrm{Ob}\,\mathcal{C}$ and, hence, CH $\subset \mathcal{C}$. On the other hand $\mathcal{C} \subset T_4$ since T_4 is the largest subcategory preserved by shrinks of T_2 extremal monos. Thus for any $X \in \mathrm{Ob}\,\mathcal{C}$, $X^\kappa \in \mathrm{Ob}\,\mathcal{C} \subset \mathrm{Ob}\,T_4$ for any cardinal κ, and it follows from Noble's theorem [N] that X is a compact Hausdorff space.

3.11. PROPOSITION. *The paracompact spaces form the largest left-fitting subcategory of T_2 which is preserved by shrinks, in* TOP, *of T_2 extremal monos.*

PROOF. Since shrinks of T_2 extremal monos are closed maps, the category of paracompact spaces clearly satisfies the conditions of the theorem.

Let $\mathcal{C}$ be a subcategory of T_2 satisfying these conditions; then CH $\subset \mathcal{C} \subset T_4$ by previous arguments (Propositions 3.6 and 3.10). Furthermore, since $\mathcal{C}$ is left-fitting, for each $X \in \mathrm{Ob}\,\mathcal{C}$, $X \times \beta X \in \mathrm{Ob}\,\mathcal{C}$ (see **[Fr2]**), and thus by the theorem of Tamano and Morita [**T**], [**Mo**], X is paracompact.

3.12. PROPOSITION. *The locally compact Hausdorff spaces form the smallest nontrivial, coproductive, left-fitting subcategory of T_2 preserved by biquotient maps (in T_2).*

PROOF. Let $\mathcal{C}$ be a category satisfying the above conditions. Since $\mathcal{C}$ is left-fitting and nontrivial, CH $\subset \mathcal{C}$. Let X be any locally compact Hausdorff space and let $\{K_\alpha\}_{\alpha\in A}$ be all the compact subsets of X; then the disjoint union of the K_α's, $\bigcup_\alpha K_\alpha$ is an object of $\mathcal{C}$. The natural quotient map $\bigcup_\alpha K_\alpha \to X$ is, in fact, biquotient, and hence $X \in \mathrm{Ob}\,\mathcal{C}$. Since the locally compact Hausdorff spaces satisfy the condition of the proposition, they form the smallest such category.

3.13. PROPOSITION. *The separable Hausdorff spaces form the smallest nontrivial, countably coproductive subcategory of T_2 preserved by epis (in T_2).*

PROOF. Every separable Hausdorff space is the codomain of some epimorphism (in T_2) with domain N, which is the countable coproduct of singleton spaces.

It is interesting to note the (superficially) minor differences in the next three characterizations.

3.14. PROPOSITION. *The metrizable spaces form the smallest nontrivial subcategory of T_2 which is*

(1) *countably productive,*

(2) *hereditary,*
(3) *coproductive,*
(4) *right-fitting (in T_2).*

PROOF. Franklin has shown [Fr1] that any metric space is the perfect image of a subspace of a countable product of discrete spaces.

3.15. PROPOSITION. *The separable metrizable spaces form the smallest nontrivial subcategory of T_2 which is*
(1) *countably productive,*
(2) *hereditary,*
(3) *countably coproductive,*
(4) *right-fitting (in T_2).*

PROOF. If $\mathcal{C}$ is any category satisfying the stated conditions, it contains $*$, by (2) and, hence by (3), it contains the two point discrete space. Use of (1) puts the Cantor set in $\mathcal{C}$, and (4) yields the unit interval. Another application of (1) gives us the Hilbert cube, and (2) all the separable metric spaces.

Another proof can be given along the lines of the previous proposition, i.e., we could show more directly that every separable metric space is the perfect image of a subspace of a countable product of countable discrete spaces.

3.16. PROPOSITION. *The first countable spaces form the smallest nontrivial subcategory of* TOP *which is*
(1) *countably productive,*
(2) *hereditary,*
(3) *coproductive,*
(4) *preserved by open* TOP *epis.*

PROOF. As we have seen in previous proofs, any such category must contain all discrete spaces. Ponomarev has shown that every first countable T_1 space is the open image of a subspace of a countable product of discrete spaces [P]. Michael has since shown that any first countable space, T_1 or not, is the open image of a first countable Hausdorff space [M2].

BIBLIOGRAPHY

[A] P. Alexandroff, *Zur Theorie der topologischen Räume*, C. R. (Dokl.) Acad. Sci. USSR **2** (1936), 55–58.

[Ba] B. Banaschewski, *Projective covers in categories of topological spaces and topological algebras*, General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 63–91. MR **44** #1616.

[Bo] N. Bourbaki, *Éléments de mathématique*. Livre III. *Topologie générale*. Chap. 1: *Structures topologiques*, 3ième éd., Actualités Sci. Indust., no. 1142, Hermann, Paris, 1961. MR **25** #4480.

[Fe] P. Freyd, *Abelian categories. An introduction to the theory of functors*, Harper's Ser. in Modern Math., Harper & Row, New York, 1964. MR **29** #3517.

[Fr1] S. P. Franklin, *A new metrization proof*, General Topology and its Relations to Modern

Analysis and Algebra, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, pp. 127–130. MR **44** #987.

[Fr2] ———, *On epi-reflective hulls*, Carnegie-Mellon Univ. Math. Dept. Research Report 70–23, 1970.

[Fr3] ———, *Topics in categorical topology*, Carnegie-Mellon University, Lecture Notes, 1970.

[F-T] S.P. Franklin and B.V.S. Thomas, *A categorical characterization of CH*, Carnegie-Mellon Univ. Math. Dept. Research Report 70–33, 1970.

[H] H. Herrlich, *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Math., no. 78, Springer-Verlag, Berlin and New York, 1968. MR **41** #988.

[H-S] H. Herrlich and G. E. Strecker, *Algebra* $\cap$ *Topology = compactness*, General Topology and Appl. **1** (1971), 283–287. MR **46** #7349.

[I] J. R. Isbell, *Subobjects, adequacy, completeness and categories of algebras*, Rozprawy Mat. **36** (1964). MR **29** #1238.

[M1] E. Michael, *Bi-quotient maps and Cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) **18** (1968), fasc. 2, 287–302 (1969). MR **39** #6277.

[M2] ———, *On representing spaces as images of metrizable and related spaces*, General Topology and Appl. **1** (1971), 329–343. MR **45** #2681.

[Mo] K. Morita, *Paracompactness and product spaces*, Fund. Math. **50** (1961/62), 223–236. MR **24** #A2365.

[N] N. Noble, *Products with closed projections*. II, Trans. Amer. Math. Soc. **160** (1971), 169–183. MR **44** #979.

[P] V. I. Ponomarev, *Axioms of countability and continuous mappings*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **8** (1960), 127–134. (Russian) MR **22** #7109.

[T] H. Tamano, *On paracompactness*, Pacific J. Math. **10** (1960), 1043–1047. MR **23** #A2186.

[W] E. Wattel, *The compactness operator in set theory and topology*, Math. Centre Tracts, 21, Math. Centrum, Amsterdam, 1968. MR **39** #7551.

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