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ON COMPOSITE ABSTRACT HOMOGENEOUS POLYNOMIALS

BY

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ABSTRACT. We study the null-sets of composite abstract homogeneous polynomials obtained from a pair of abstract homogeneous polynomials defined on a vector space over an algebraically closed field of characteristic zero. First such study for ordinary polynomials in the complex plane was made by Szegö, Cohn, and Egerváry and Szegö's theorem was later generalized to fields and vector spaces, respectively, by Zervos and Marden. Our main theorem in this paper further generalizes their results and, in the complex plane, improves upon Szegö's theorem and some other classical results. The method of proof is purely algebraic and utilizes the author's vector space analogue [Trans. Amer. Math. Soc. **218** (1976), 115–131] of Grace's theorem on apolar polynomials. We also show that our results cannot be further generalized in certain directions.

1. **Introduction.** Let E be a vector space over a field K of characteristic zero. A mapping P from E to K is called [9] (see also [6, pp. 760–763], [8, p. 55], [10], [13], [14, pp. 52–54]) an *abstract homogeneous polynomial* (a.h.p.) of degree n if for every $x, y \in E$,

$$P(sx + ty) = \sum_{k=0}^{n} A_k(x,y)s^k t^{n-k} \qquad \forall s, t \in K,$$

where $C(n, k) = (n!)/(k!)(n-k)!$ and the coefficients $A_k(x,y)$ are elements of K which are independent of s and t and which depend only on x and y. $\mathbf{P}_n$ shall denote the class of all nth-degree a.h.p.'s. The nth-*polar* of P is the mapping (see [7, Lemma 1] for its existence and uniqueness) $P(x_1, x_2, \ldots, x_n)$ from $E \times E \times \cdots \times E$ to K which is a symmetric n-linear form such that $P(x, x, \ldots, x) = P(x)$ for every $x \in E$. We specify the kth-*polar* of P by

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([1]) The results in this paper are partly contained in the author's Doctoral dissertation (1971) at the University of Wisconsin-Milwaukee, under the supervision of the University of Wisconsin-Milwaukee Distinguished Professor Morris Marden. The author wishes to thank him for his useful suggestions and advice all along.

$$P(x_1, \ldots, x_k, x) = P(x_1, \ldots, x_k, x, \ldots, x).$$

If $P \in \mathbf{P}_n$ the *null-set* $Z_P(x, y)$ of P (relative to given elements $x, y \in E$) is defined by

$$Z_P(x, y) = \{sx + ty \neq 0 \mid s, t \in K;\ P(sx + ty) = 0\}.$$

From now on we shall assume that K is an algebraically closed field of characteristic zero. Then ([1, pp. 38–40], [12, pp. 248–255], [14, p.11]) K has a maximal ordered subfield K_0 such that $K = K_0(i) = \{a + ib \mid a, b \in K_0\}$, where $-i^2 = 1$. For $z = a + ib$, we define $\bar{z} = a - ib$, $\mathrm{Re}(z) = (z + \bar{z})/2$, $\mathrm{Im}(z) = (z - \bar{z})/2i$, and $|z| = (a^2 + b^2)^{\frac{1}{2}}$. A subset A of K is said to be *K_0-convex* if $\sum_{i=1}^n \mu_i a_i \in A$ for every $a_i \in A$ and $\mu_i \in K_{0+}$ (the set of all nonnegative elements of K_0) such that $\sum_{i=1}^n \mu_i = 1$. Adjoin to K an element ω (called infinity) and furnish $K \cup \{\omega\}$ (denoted K_ω) with the following structure: (1) The subset K of K_ω preserves its initial field structure; and (2) $a + \omega = \omega + a = \omega$ for every $a \in K$, $a \cdot \omega = \omega \cdot a = \omega$ for every $a \in K - \{0\}$, and $\omega^{-1} = 0$, $0^{-1} = \omega$. Given an element $\zeta \in K$, we define $\varphi_\zeta(z) = (z - \zeta)^{-1}$ for every $z \in K_\omega$. A subset A of K_ω is called ([14, pp. 25–26], [15, pp. 353, 373]) a *generalized circular region* of K_ω if either A is one of the sets $\varnothing$, K, K_ω, or A satisfies the following two conditions:

(1) $\varphi_\zeta(A)$ is K_0-convex for every $\zeta \in K - A$;

(2) $\omega \in A$ if A is not K_0-convex.

$D(K_\omega)$ denotes the class of all generalized circular regions of K_ω. The characterization (due to Zervos [15, pp. 372–387]) of this class, when K is the field $\mathbf{C}$ of complex numbers, leads to the following [15, p. 352]: *The nontrivial generalized circular regions of* $\mathbf{C}_\omega$ *are the open interior* (*or exterior*) *of circles or the open half-planes*, *adjoined with a connected subset* (*possibly empty*) *of their boundary*. The generalized circular regions of $\mathbf{C}_\omega$, with all or no boundary points included, will be termed as (*classical*) circular regions of $\mathbf{C}_\omega$.

In what follows, we give some definitions and useful properties related to circular cones (see author [13] or [14, pp. 36–40]) in a vector space E over K. Let E^2 denote $E \times E$ and $\mathfrak{L}[x, y]$ the subspace of E generated by the elements $x, y \in E$. Given elements (x_1, y_1) and (x_2, y_2) in E^2, we say that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $\mathfrak{L}[x_1, y_1] = \mathfrak{L}[x_2, y_2]$. Then "$\sim$" is an equivalence relation on E^2 and the equivalence class $[(x, y)]$, containing the element (x, y) of E^2, is called *nontrivial* if x, y are linearly independent (called *trivial*, otherwise). By axiom of choice, we choose a unique element from each nontrivial equivalence class and call the set N ($\subseteq E^2$) of all elements thus chosen as a *nucleus* of E^2. We remark that N is nonempty if the dimension of E is at least two. A *circular cone* in E, relative to a given nucleus N and a given mapping G from N into $D(K_\omega)$ (called a *circular mapping* [13]), is defined to be the subset $E_0(N, G)$ of E given by

$$E_0(N, G) = \cup T_G(x, y), \tag{1.1}$$

where

$$T_G(x, y) = \{sx + ty \neq 0 \mid s, t \in K;\ s/t \in G(x, y)\}. \tag{1.2}$$

The union in (1.1) is taken over all elements $(x, y) \in N$.

REMARK. (1.1). If E is a two-dimensional vector space, every circular cone $E_0(N, G)$ is of the form

$$E_0(N, G) = \{sx_0 + ty_0 \neq 0 \mid s, t \in K;\ s/t \in A\}$$

for some $A \in D(K_\omega)$, where x_0, y_0 are any two linearly independent elements of E and where $N = \{(x_0, y_0)\}$ and $G(x_0, y_0) = A$.

It is now the right place to discuss the sets employed by Marden [9] toward generalizing Grace's theorem ([5], [8, Theorem (15, 3)]) and Szegö's theorem ([11, §2, Theorem 2], [8, Theorem (16,1)]). The sets of this class, which we term as *hermitian cones* [13], are subsets E_1 of E of the form $E_1 = \{x \in E \mid x \neq 0;\ H(x, x) \geqslant 0\}$ (and the ones obtained by replacing in this expression the inequality "$\geqslant$" by "$>$", "$\leqslant$", or "$<$"), where $H(x, y)$ is a hermitian symmetric form [9] defined from E^2 to K. The relationship between the class of circular cones and the class of hermitian cones is exhibited in the following propositions due to the author [13, Proposition (2.2) and Theorem (2.4)]:

PROPOSITION (1.2). *Let E_1 be a hermitian cone in E defined by a hermitian symmetric form H and let N be a nucleus of E^2. Then there exists a circular mapping G from N into $D(K_\omega)$ such that the circular cone $E_0(N, G) = E_1$ and $E_1 \cap \mathfrak{L}[x, y] = T_G(x, y)$ for every $(x, y) \in N$, where T_G is as defined by* (1.2).

PROPOSITION (1.3). *The class of all circular cones in E properly contains the class of all hermitian cones in E.*

2. A generalization of Szegö's theorem. Before coming to our main theorem, we give the notions of apolarity ([14, p. 101], [13, §4]) and of composite a.h.p.'s. Let $P, Q \in \mathbf{P}_n$ and

$$\begin{aligned} P(s\xi + t\eta) &= \prod_{j=1}^{n} (\beta_j(\xi, \eta)\cdot s - \alpha_j(\xi, \eta)\cdot t) \\ &= \sum_{k=0}^{n} C(n, k) A_k(\xi, \eta) s^k t^{n-k}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} Q(s\xi + t\eta) &= \prod_{j=1}^{n} (\delta_j(\xi, \eta)\cdot s - \gamma_j(\xi, \eta)\cdot t) \\ &= \sum_{k=0}^{n} C(n, k) B_k(\xi, \eta) s^k t^{n-k}, \end{aligned} \tag{2.2}$$

where the coefficients $A_k(\xi, \eta)$, $B_k(\xi, \eta)$, $\alpha_j(\xi, \eta)$, $\delta_j(\xi, \eta)$, etc. are elements of K (depending only on ξ, η and independent of s and t). We say that P and Q

are *apolar* with respect to linearly independent elements $\xi, \eta \in E$ (written briefly, $P \perp Q[\xi, \eta]$) if

$$\sum_{k=0}^{n} (-1)^k C(n, k) A_k(\xi, \eta) \cdot B_{n-k}(\xi, \eta) = 0. \tag{2.3}$$

A composite a.h.p. obtained from P and Q is an a.h.p. $R \in \mathbf{P}_{2n}$ given by

$$R(s\xi + t\eta) = \sum_{k=0}^{n} C(n, k) A_k(\xi, \eta) \cdot B_k(\xi, \eta) s^{2k} t^{2(n-k)} \tag{2.4}$$

for every $s, t \in K$.

The theorems of Marden [9, Theorem (3.1)], Grace [8, Theorem (15,3)], and Zervos [15, p. 363] on the null-sets of apolar polynomials have previously been generalized in the following theorem due to the author ([13, Theorem (4.2)] or [14, Theorem (14.3)]): *Let $E_0(N, G)$ be a circular cone in E and $P, Q \in \mathbf{P}_n$ such that $Z_P(x,y) \subseteq T_G(x,y)$ for some $(x,y) \in N$. If $P \perp Q[\xi, \eta]$ for some $\xi, \eta \in \mathfrak{L}[x,y]$, then $Z_Q(x,y) \cap E_0(N, G) \neq \varnothing$. More precisely $Z_Q(x,y) \cap T_G(x,y) \neq \varnothing$.* This result finds an important application in establishing the following main theorem, which generalizes the theorems of Marden [9, Theorem (3.2)], Zervos [15, p. 363], and Szegö ([11, §2, Theorem 2] or [8, Theorem (16,1)]) on composite polynomials.

THEOREM (2.1). *Let $E_0(N, G)$ be a circular cone in E and $P, Q \in \mathbf{P}_n$ and $R \in \mathbf{P}_{2n}$ be given by* (2.1), (2.2), *and* (2.4). *Suppose that $Z_P(x,y) \subseteq T_G(x,y)$ for some $(x,y) \in N$. If $\mu x + \nu y \in Z_R(x,y)$, then there exist elements $\alpha x + \beta y \in T_G(x,y)$ and $\gamma x + \delta y \in Z_Q(x,y)$ such that $\sigma^2 = -\rho \cdot \Delta$, where $\sigma = \mu/\nu$, $\rho = \gamma/\delta$, and $\Delta = \alpha/\beta$.*

REMARK. When σ^2 is of the form $0 \cdot \omega$ (resp. $\omega \cdot 0$), the equation $\sigma^2 = -\rho \cdot \Delta$ is to be interpreted as $-\sigma^2/\rho = \Delta = \omega$ (resp. $-\sigma^2/\Delta = \rho = \omega$) or $-\sigma^2/\Delta = \rho = 0$ (resp. $-\sigma^2/\rho = \Delta = 0$) according as $\sigma \neq 0$ or $\sigma = 0$.

PROOF. For convenience, we shall write $A_k \equiv A_k(x,y)$, $B_k \equiv B_k(x,y)$, $\gamma_j \equiv \gamma_j(x,y)$, $\delta_j \equiv \delta_j(x,y)$, $Z_P \equiv Z_P(x,y)$, $Z_Q \equiv Z_Q(x,y)$, and $Z_R \equiv Z_R(x,y)$. We split the proof into three cases.

Case I. $\mu, \nu \neq 0$. We know that

$$R(\mu x + \nu y) = \sum_{k=0}^{n} C(n, k) A_k B_k \mu^{2k} \nu^{2(n-k)} = 0. \tag{2.5}$$

Now we define an a.h.p. $Q^* \in \mathbf{P}_n$ by

$$\begin{aligned} Q^*(sx + ty) &= \prod_{j=1}^{n} (\delta_j^* \cdot s - \gamma_j^* \cdot t) \\ &= \sum_{k=0}^{n} C(n, k) B_k^*(x,y) \cdot s^k t^{n-k} \quad \text{(say)}, \end{aligned}$$

where $\delta_j^* = -\nu^2 \cdot \gamma_j$ and $\gamma_j^* = \mu^2 \cdot \delta_j$. Then

$$\begin{aligned} Q^*(sx+ty) &= \prod_{j=1}^{n}\left(-\nu^2 s\gamma_j - \mu^2 t\delta_j\right) = Q\left(-\mu^2 tx + \nu^2 sy\right) \\ &= \sum_{k=0}^{n} C(n,k)\cdot B_k \cdot \left(-\mu^2 t\right)^k \cdot \left(\nu^2 s\right)^{n-k} \qquad (2.6) \\ &= \sum_{k=0}^{n} C(n,k)\cdot(-1)^{n-k} B_{n-k}\cdot \nu^{2k}\mu^{2(n-k)}\cdot s^k t^{n-k}. \end{aligned}$$

Therefore, $B_k^* = B_k^*(x,y) = (-1)^{n-k}B_{n-k}\nu^{2k}\mu^{2(n-k)}$ and, in view of (2.5), we have

$$\sum_{k=0}^{n}(-1)^k C(n,k)A_k B_{n-k}^* = 0.$$

That is, $Q^* \perp P[x,y]$. Since $Z_P(x,y) \subseteq T_G(x,y)$, our generalized version [13, Theorem (4.2)] (stated above) of Marden's theorem implies the existence of an element $\alpha x + \beta y \in T_G(x,y)$ such that

$$Q^*(\alpha x + \beta y) = Q\left(-\mu^2\beta x + \nu^2\alpha y\right) = 0 \quad \text{(due to (2.6))}.$$

This implies that there exists an element $\gamma x + \delta y \in Z_Q$ such that $-\mu^2\beta x + \nu^2\alpha y = \lambda(\gamma x + \delta y)$ for some nonzero element $\lambda \in K$. That is, $(-\mu^2\beta - \lambda\gamma)x + (\nu^2\alpha - \lambda\delta)y = 0$. Consequently, $\mu^2\beta = -\lambda\gamma$ and $\nu^2\alpha = \lambda\delta$. If $\gamma, \delta \neq 0$, then $\alpha, \beta \neq 0$ and $(\mu/\nu)^2 = -(\gamma/\delta)(\alpha/\beta)$, i.e. $\sigma^2 = -\rho\Delta$. If $\gamma = 0$ and $\delta \neq 0$ (resp. $\gamma \neq 0, \delta = 0$), then $\beta = 0$ and $\alpha \neq 0$ (resp. $\beta \neq 0, \alpha = 0$) and the equation $\sigma^2 = -\rho\Delta$ will be interpreted as $-\sigma^2/\rho = \Delta = \omega$ (resp. $-\sigma^2/\Delta = \rho = \omega$). Hence, the equation $\sigma^2 = -\rho\Delta$ holds, together with the interpretations made in the remark above.

Case II. $\mu \neq 0, \nu = 0$. In this case

$$R(\mu x + \nu y) = R(\mu x) = \mu^{2n}\cdot P(x)\cdot Q(x) = 0 \quad \text{(by (2.4))}.$$

That is, $x \in Z_P$ or $x \in Z_Q$. If $x \in Z_P$, we can choose an element $\alpha x + \beta y \in T_G(x,y)$, with $\alpha \neq 0$ and $\beta = 0$ (so that $\sigma = \omega$ and $\Delta = \omega$). Next, we choose an element $\gamma x + \delta y \in Z_Q$ such that $\rho = \gamma/\delta \neq 0$ (in case $Q(y) \neq 0$), so that $\sigma^2 = -\rho\Delta$. In case $Q(y) = 0$, we may then choose $\gamma x + \delta y \in Z_Q$ with $\gamma = 0$ and $\delta \neq 0$ (i.e. $\rho = 0$), and interpret the equation $\sigma^2 = -\rho\Delta$ as $-\sigma^2/\rho = \Delta = \omega$. Similarly, in case when $x \in Z_Q$, we can verify the truth of the equation $\sigma^2 = -\rho\Delta$ (with the interpretation that $-\sigma^2/\Delta = \rho = \omega$, whenever σ^2 is of the form $\omega \cdot 0$).

Case III. $\mu = 0, \nu \neq 0$. Here

$$R(\mu x + \nu y) = R(\nu y) = \nu^{2n}\cdot P(y)\cdot Q(y) = 0 \quad \text{(by (2.4))}.$$

If $y \in Z_P$, we choose an element $\alpha x + \beta y \in Z_P \subseteq T_G(x,y)$ with $\alpha = 0$ and $\beta \neq 0$ (so that $\Delta = 0$ and $\sigma = 0$). Next, we choose $\gamma x + \delta y \in Z_Q$ such that

$\rho = \gamma/\delta \neq \omega$ (in case $Q(x) \neq 0$), so that $\sigma^2 = -\rho\Delta$. In case $Q(x) = 0$, we may then choose $\gamma x + \delta y \in Z_Q$ with $\gamma \neq 0$ and $\delta = 0$ (i.e. $\rho = \omega$) and interpret the equation $\sigma^2 = -\rho\Delta$ as $-\sigma^2/\rho = \Delta = 0$. Similarly, in case when $y \in Z_Q$, the equation $\sigma^2 = -\rho\Delta$ holds along with the interpretation that $-\sigma^2/\Delta = \rho = 0$, whenever σ^2 is of the form $0 \cdot \omega$.

This completes the proof of Theorem (2.1).

Our first specialization of this theorem to hermitian cones leads to the following corollary, a result due to Marden [9, Theorem (3.2)].

COROLLARY (2.2.). *Let H be a hermitian symmetric form on E and E_1 be the hermitian cone given by*

$$E_1 = \{x \in E | x \neq 0; H(x, x) < 0\}.$$

Let $P, Q \in \mathbf{P}_n$ and $R \in \mathbf{P}_{2n}$ be given by (2.1), (2.2), *and* (2.4). *Suppose that $Z_P(x, y) \subseteq E_1$ for some linearly independent elements $x, y \in E$. If $\mu x + \nu y \in Z_R(x, y)$ then there exist elements $\alpha x + \beta y \in E_1$ and $\gamma x + \delta y \in Z_Q(x, y)$ such that $\sigma^2 = -\rho\Delta$, where $\sigma = \mu/\nu$, $\rho = \gamma/\delta$, and $\Delta = \alpha/\beta$.*

REMARKS. (I) Whenever σ^2 is of the form $0 \cdot \omega$ or $\omega \cdot 0$, the same interpretations will be made as in the remark following Theorem (2.1). (II) The conclusion "$\mu^2 x + \nu^2 y = -\alpha\gamma x + \beta\delta y$" in Marden's theorem is equivalent to "$\sigma^2 = -\rho\Delta$" in our corollary.

PROOF. Take a nucleus N of E^2 such that $(x, y) \in N$ (this is always possible). By Proposition (1.2), there exists a circular cone $E_0(N, G)$ such that $E_0(N, G) = E_1$ and $E_1 \cap \mathfrak{L}[x, y] = T_G(x, y)$. Obviously, the hypothesis on $Z_P(x, y)$ then implies that $Z_P(x, y) \subseteq T_G(x, y)$. Therefore, by Theorem (2.1), if $\mu x + \nu y \in Z_R(x, y)$, there exist elements $\alpha x + \beta y \in T_G(x, y)$ and $\gamma x + \delta Y \in Z_Q(x, y)$ such that $\sigma^2 = -\rho\Delta$, where $\sigma = \mu/\nu$, $\rho = \gamma/\delta$ and $\Delta = \alpha/\beta$. Finally, we notice that $\alpha x + \beta y \in E_1$. This completes the proof of our corollary.

Our second application of Theorem (2.1) leads to an improved version of a theorem in the complex plane due to Szegö (see [11, §2, Theorem 2] or [8, Theorem (16,1)]), namely: *Let $\mathbf{C}$ denote the field of complex numbers. Given nth-degree polynomials* (*from $\mathbf{C}$ to $\mathbf{C}$*)

$$f(z) = \sum_{k=0}^{n} C(n, k) A_k z^k, \qquad g(z) = \sum_{k=0}^{n} C(n, k) B_k z^k, \tag{2.7}$$

let us form the composite polynomial

$$h(z) = \sum_{k=0}^{n} C(n, k) A_k B_k z^k. \tag{2.8}$$

If all the zeros of $f(z)$ lie in a (*classical*) *circular region C of $\mathbf{C}$, then every zero σ of $h(z)$ has the form $\sigma = -\rho\Delta$, where $\Delta \in C$ and ρ is a zero of $g(z)$.* Our improved version of this theorem is the following.

COROLLARY (2.3). *Let $f(z)$, $g(z)$, and $h(z)$ be the polynomials given by* (2.7) *and* (2.8). *If all the zeros of $f(z)$ lie in a generalized circular region C of $\mathbf{C}_\omega$, then every zero σ of $h(z)$ is of the form $\sigma = -\rho\Delta$, where $\Delta \in C$ and ρ is a zero of $g(z)$.*

REMARK. Whenever σ is of the form $0 \cdot \omega$ or $\omega \cdot 0$, the equation $\sigma = -\rho\Delta$ has interpretations similar to that of the equation $\sigma^2 = -\rho\Delta$ in Theorem (2.1).

PROOF. Any element x of $\mathbf{C}^2$ can be uniquely written as $x = sx_0 + ty_0$ for some $s, t \in \mathbf{C}$, where $x_0 = (1, 0)$ and $y_0 = (0, 1)$. Define a.h.p.'s $P, Q \in \mathbf{P}_n$ and $R \in \mathbf{P}_{2n}$ (from $\mathbf{C}^2$ to $\mathbf{C}$) by

$$(2.9)\qquad \begin{aligned} P(x) &\equiv P(sx_0 + ty_0) = \sum_{k=0}^{n} C(n, k)A_k s^k t^{n-k} = t^n \cdot f(s/t), \\ Q(x) &\equiv Q(sx_0 + ty_0) = \sum_{k=0}^{n} C(n, k)B_k s^k t^{n-k} = t^n \cdot g(s/t), \end{aligned}$$

$$(2.10)\qquad R(x) \equiv R(sx_0 + ty_0) = \sum_{k=0}^{n} C(n, k)A_k B_k s^{2k} t^{2(n-k)} = t^{2n} \cdot h(s^2/t^2),$$

for every $x = (x, t) \in \mathbf{C}^2$. Note that the second equality in each of the expressions for P, Q, and R hold for all nonzero elements $x \in \mathbf{C}^2$. Let us take the circular cone $E_0(N, G)$, where $N = \{(x_0, y_0)\}$ and $G(x_0, y_0) = C$, so that $E_0(N, G) = T_G(x_0, y_0) = \{sx_0 + ty_0 \neq 0 | s, t \in \mathbf{C}; s/t \in C\}$. Obviously, the hypothesis on f implies that $Z_P(x_0, y_0) \subseteq T_G(x_0, y_0)$. If $h(\sigma) = 0$ and if $\sigma'^2 = \sigma$, then $R(\sigma' x_0 + y_0) = h(\sigma'^2) = 0$. By Theorem (2.1), there exist elements $\alpha x_0 + \beta y_0 \in T_G(x_0, y_0)$ and $\gamma x_0 + \delta y_0 \in Z_Q(x_0, y_0)$ such that $\sigma'^2 = -(\gamma/\delta) \cdot (\alpha/\beta)$. Since $\alpha/\beta \in C$ and γ/δ is a zero of $g(z)$, we have $\sigma = -\rho\Delta$ for some element $\Delta \in C$ and for some zero ρ of $g(z)$. This completes our proof.

Our last application of Theorem (2.1) gives the following corollary , a result due to Zervos [15, p. 363].

COROLLARY (2.4). *Let $f(z)$, $g(z)$, and $h(z)$ be polynomials (from K to K) given by the expressions* (2.7) *and* (2.8). *If all the zeros of $f(z)$ lie in a generalized circular region C of K_ω, then every zero σ of $h(z)$ is of the form $\sigma = -\rho\Delta$ for some $\Delta \in C$ and some zero ρ of $g(z)$. (This theorem entails the same remark as the one following Corollary* (2.3).)

PROOF. The proof is exactly the same as that of Corollary (2.3) when $\mathbf{C}$ is replaced by K.

3. **Further applications in the complex plane.** In this section, we give some applications of our Corollary (2.3), leading to improved versions of the

corresponding results of Szegö [11], Cohn [3], Egerváry [4], and De Bruijn [2]. First, we prove the following

THEOREM (3.1). *Let $f(z)$, $g(z)$, and $h(z)$ be complex-valued polynomials given by* (2.7) *and* (2.8). *Let C_1, C_2, and C be the generalized circular regions of $\mathbf{C}_\omega$ given by*

$$C_1 = \{z||z|< r\} \cup \{re^{i\theta}|\alpha_1 \leqslant \theta < \alpha_1 + \beta_1\},$$

$$C_2 = \{z||z|< s\} \cup \{se^{i\theta}|\alpha_2 \leqslant \theta < \alpha_2 + \beta_2\},$$

$$C = \{z||z|< rs\} \cup \{rse^{i\theta}|\pi + \alpha_1 + \alpha_2 \leqslant \theta < \pi + \alpha_1 + \alpha_2 + \beta_1 + \beta_2\},$$

where α_j, β_j ($j = 1, 2$) are constants with $0 \leqslant \beta_j \leqslant 2\pi$. If all the zeros of f lie in C_1 and if all the zeros of g lie in C_2, then all the zeros of h lie in C. Furthermore, the same conclusion holds when the inequality '$<$' is replaced by '$>$' throughout the first parentheses in each of the expressions for C_1, C_2, and C.

PROOF. Let $h(\sigma) = 0$. Since all the zeros of f lie in C_1, Corollary (2.3) implies the existence of an element $\Delta \in C_1$ and a zero ρ of $g(z)$ such that $\sigma = -\rho \cdot \Delta$. Therefore $|\sigma| = |\rho| \cdot |\Delta|$, where $\Delta \in C_1$ and $\rho \in C_2$. We consider the following cases.

Case I. If Δ or ρ is not a boundary point, then (since $|\Delta| < r$ or $|\rho| < s$) $|\sigma| < rs$ and $\sigma \in C$.

Case II. If both Δ and ρ are boundary points, then $\Delta = re^{i\alpha}$ and $\rho = se^{i\beta}$ for some $\alpha \cdot \beta$ for which $\alpha_1 \leqslant \alpha < \alpha_1 + \beta_1$ and $\alpha_2 \leqslant \beta < \alpha_2 + \beta_2$, and so

$$\sigma = -rse^{i(\alpha+\beta)} = rse^{i(\pi+\alpha+\beta)}.$$

Since $\alpha_1 + \alpha_2 \leqslant \alpha + \beta < \alpha_1 + \alpha_2 + \beta_1 + \beta_2$, we see that $\sigma \in C$.

Consequently, Cases I and II imply that every zero of $h(z)$ lies in C. Furthermore, the proof for the second part of our theorem is exactly similar. This completes the proof of Theorem (3.1).

REMARK. If all boundary points are included in C_1 and in C_2 (i.e., if $\beta_1, \beta_2 = 2\pi$), then the above theorem leads to the following result of Szegö ([11, §2, Theorem 3′], or [8, Corollary (16, 1a)]), namely: Let $f(z)$, $g(z)$, and $h(z)$ be polynomials given by (2.7) and (2.8). If all the zeros of $f(z)$ and $g(z)$ lie respectively in $|z| \leqslant r$ (resp. $|z| \geqslant r$) and $|z| \leqslant s$ (resp. $|z| \geqslant s$), then all the zeros of $h(z)$ lie in $|z| \leqslant rs$ (resp. $|z| \geqslant rs$). This result was also proved by Egerváry [4] for $r = s = 1$ and by Cohn [3, Theorem 4, p. 14] as a special case of his more general theorem.

As another application of our Corollary (2.3), we prove the following

THEOREM (3.2) *Let f, g, and h be the polynomials and C_1, C_2 the generalized circular regions of Theorem* (3.1) *and let*

$$C = \{z||z|< rs\} \cup \{rse^{i\theta}|\pi + \alpha_1 + \alpha_2 \leqslant \theta < -\pi + \alpha_1 + \alpha_2 + \beta_1 + \beta_2\}.$$

Suppose that S is a subset of the complex plane. If $f(z) \in S$ for every $z \in C_1$ and $g(z) \neq 0$ for every $z \in C_2$, then

(i) *$h(z) \in B_0 S$ for every $z \in C$ if $\beta_1, \beta_2 < 2\pi$, and*

(ii) *$h(z) \in B_0 S$ for every $|z| \leqslant rs$ if β_1 or $\beta_2 = 2\pi$, where $B_0 S = \{B_0 s | s \in S\}$.*

PROOF. (i) Let $\beta_1, \beta_2 < 2\pi$. Putting $F(z) = f(z) - \lambda$, we get

$$F(z) = (A_0 - \lambda) + \sum_{k=1}^{n} C(n, k) A_k z^k.$$

Then the composite polynomial $H(z)$ (obtained from $F(z)$ and $g(z)$) is given by $H(z) = h(z) - \lambda B_0$. Suppose that $\lambda \notin S$. Then all the zeros of $F(z)$ and $g(z)$ lie respectively in the complements of C_1 and C_2 (denoted respectively by C_1' and C_2'). Since C_1' and C_2' are also generalized circular regions of $\mathbf{C}_\omega$, by Corollary (2.3), every zero σ of $H(z)$ can be written as $\sigma = -\rho\Delta$, where

$$\Delta \in C_1' = \{z \,|\, |z| > r\} \cup \{re^{i\theta} | \alpha_1 + \beta_1 \leqslant \theta < 2\pi + \alpha_i\},$$

$$\rho \in C_2' = \{z \,|\, |z| > s\} \cup \{se^{i\theta} | \alpha_2 + \beta_2 \leqslant \theta < 2\pi + \alpha_2\}.$$

In case ρ or Δ does not lie on the boundary, $|\sigma| = |\rho| \cdot |\Delta| > rs$. If both ρ and Δ lie on the boundary, then

$$\sigma = -(re^{i\theta_1}) \cdot (se^{i\theta_2}) = rse^{i(\pi + \theta_1 + \theta_2)}$$

for some θ_j $(j = 1, 2)$ such that $\alpha_j + \beta_j \leqslant \theta_j < 2\pi + \alpha_j$. That is, $\sigma = rse^{i\theta}$ such that

$$\pi + \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leqslant \theta < 5\pi + \alpha_1 + \alpha_2.$$

Since the complement C' of C is given by

$$C' = \{z \,|\, |z| > rs\} \cup \{rse^{i\theta} | -\pi + \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leqslant \theta < 3\pi + \alpha_1 + \alpha_2\},$$

$$= \{z \,|\, |z| > rs\} \cup \{rse^{i\theta} | \pi + \alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leqslant \theta < 5\pi + \alpha_1 + \alpha_2\},$$

we see that $H(z) \neq 0$ for every $z \in C$. So far, we have shown that if $\lambda \notin S$, $h(z) \neq \lambda B_0$ for every $z \in C$. Since $g(z) \neq 0$ for every $z \in C_2$, we see that $B_0 \neq 0$. Given any $z_0 \in C$, let $h(z_0) = c_0 = (c_0/B_0) \cdot B_0 = \lambda_0 B_0$ (say). We claim that $\lambda_0 \in S$. For, if not, then $H(z) = h(z) - \lambda_0 B_0 \neq 0$ for every $z \in C$, contradicting the fact that $H(z)$ does vanish at the point $z_0 \in C$. Consequently, $h(z) \in B_0 S$ for every $z \in C$.

(ii) Let β_1 or $\beta_2 = 2\pi$. Then, either

$$C_1' = \{z \,|\, |z| > r\} \quad \text{or} \quad C_2' = \{z \,|\, |z| > s\}.$$

In either case, we find that $|\sigma| = |\rho| \cdot |\Delta| > rs$. Therefore, $H(z) \neq 0$ for every $|z| \leqslant rs$. Similarly, as in part (i), we can prove that $h(z) \in B_0 S$ for every $|z| \leqslant rs$. This completes the proof of Theorem (3.2)

REMARK. If $r = s = 1$, $\beta_1 = 2\pi$, and $\beta_2 = 0$, then part (ii) of this theorem deduces the following result of De Bruijn ([2], [8, Corollary (16, 1c)]): *Let $f(z)$, $g(z)$, and $h(z)$ be polynomials of Theorem* (3.2) *and S be a subset of the complex plane. If $f(z) \in S$ for every $|z| \leq 1$ and $g(z) \neq 0$ for every $|z| < 1$, then $h(z) \in B_0 S$ for every $|z| \leq 1$.* As an application of Theorem (3.2), we now prove the following

THEOREM (3.3). *Let f, g, and h be the polynomials and C_1, C_2, C the generalized circular regions as in Theorem* (3.2). *If $|f(z)| \leq 1$ for every $z \in C_1$ and $|g(z)| \leq 1$ for every $z \in C_2$, then*

(i) $|h(z)| \leq 1 - |\,|A_0| - |B_0|\,|$ *for every $z \in C$, if $\beta_1 \cdot \beta_2 < 2\pi$, and*

(ii) $|h(z)| \leq 1 - |\,|A_0| - |B_0|\,|$ *for every $|z| \leq rs$, if β_1 or $\beta_2 = 2\pi$.*

PROOF. (i) Let $\beta_1, \beta_2 < 2\pi$. Take a scalar λ such that $|\lambda| > 1$. Since $|g(0)| = |B_0| \leq 1$,

$$G(z) = (g(z) - \lambda)/(B_0 - \lambda) \neq 0 \quad \forall z \in C_2.$$

Now the composite polynomial $H(z)$ obtained from $f(z)$ and $G(z)$ is given by

$$H(z) = A_0 + \frac{1}{B_0 - \lambda} \cdot \sum_{k=1}^{n} C(n, k) A_k B_k z^k = \frac{h(z) - \lambda A_0}{B_0 - \lambda}.$$

In Theorem (3.2) if we replace S, $g(z)$ and $h(z)$ respectively by $\{z \mid |z| \leq 1\}$, $G(z)$, and $H(z)$, we conclude that (since $G(0) = 1$) $H(z) \in S$ for every $z \in C$, i.e.,

$$|[h(z) - \lambda A_0]/(B_0 - \lambda)| \leq 1 \quad \forall z \in C,$$

or

$$|h(z)| \leq |\lambda A_0| + |B_0 - \lambda| \quad \forall z \in C.$$

Since this inequality holds for every $|\lambda| > 1$, it holds also in the limit for $|\lambda| = 1$. If $B_0 \neq 0$, we may choose λ equal to $\exp(i \arg B_0)$ and obtain (since $|B_0| \leq 1$)

$$|h(z)| \leq |A_0| + 1 - |B_0| \quad \forall z \in C.$$

In case $B_0 = 0$, this inequality immediately follows on taking any λ such that $|\lambda| = 1$. Now, symmetry of hypotheses on f and g implies that

$$|h(z)| \leq |B_0| + 1 - |A_0| \quad \forall z \in C.$$

Consequently,

$$|h(z)| \leq 1 - |\,|A_0| - |B_0|\,| \quad \forall z \in C,$$

as was to be proved.

(ii) Whenever β_1 or $\beta_2 = 2\pi$, the proof is exactly similar to that of part (i), except we apply Theorem (3.2)(ii) in this case. This completes the proof of Theorem (3.3).

REMARK. If $r = s = 1$ and $\beta_1 = \beta_2 = 2\pi$, Corollary (3.3) deduces the following result of De Bruijn ([2], [8, Corollary (16, 1d)]): *Let f, g, and h be polynomials of Theorem* (3.3). *If* $|f(z)| \leqslant 1$ *and* $|g(z)| \leqslant 1$ *for every* $|z| \leqslant 1$, *then*

$$|h(z)| \leqslant 1 - \big|\, |A_0| - |B_0| \,\big| \quad \forall |z| \leqslant 1.$$

4. **Some useful remarks about the main theorem.** In this section, we discuss the validity of hypotheses and the degree of generality of our Theorem (2.1). Furthermore, we furnish examples to show that our results cannot be further generalized in certain directions.

The following example shows that Theorem (2.1) and its Corollaries (2.2)–(2.4) are best possible in the sense that they cannot be generalized for vector spaces over nonalgebraically closed fields of characteristic zero.

EXAMPLE (4.1). Let K_0 be a maximal ordered field (so that K_0 is a nonalgebraically closed field of characteristic zero [12, pp. 233 – 250]). Take $E = K_0^2$ with a basis $\{x_0, y_0\}$, where $x_0 = (1, 0)$ and $y_0 = (0, 1)$. Let C be the generalized circular region of K_0 given by $C = \{1\}$. (Though we have defined generalized circular regions for algebraically closed fields of characteristic zero, but the definition remains the same for a general field as in [14, p. 26] or [15, pp. 353, 373].) Then (cf. Remark (1.1)) the set

$$E_0(N, G) = T_G(x_0, y_0) = \{sx_0 + ty_0 \neq 0 | s, t \in K_0;\, s/t = 1\}$$

is a circular cone in E, where $N = \{(x_0, y_0)\}$ and $G(x_0, y_0) = C$. Note that $E_0(N, G)$ is easily seen to be a hermitian cone E_1 defined by a 'real' hermitian-symmetric form H from $E \times E$ to K_0 defined by

$$H(sx_0 + ty_0, s'x_0 + t'y_0) = ss' - st' - s't + tt' \quad \forall s, t, s', t' \in K_0.$$

In fact, $E_1 = \{sx_0 + ty_0 \neq 0 | H(sx_0 + ty_0, sx_0 + ty_0) \leqslant 0\} = E_0(N, G)$.

Now, define a.h.p.'s $P, Q \in \mathbf{P}_3$ by

$$P(x) \equiv P(sx_0 + ty_0) = t^3 - st^2 + s^2t - s^3 = (-s + t)\cdot(s^2 + t^2),$$

$$Q(x) \equiv Q(sx_0 + ty_0) = 2t^3 + st^2 + 10s^2t + 5s^3 = (s + 2t)\cdot(5s^2 + t^2)$$

for every $x = (s, t) \in E$. Then (cf. (2.4)) the composite a.h.p. R obtained from P and Q is given by

$$R(x) \equiv R(sx_0 + ty_0) = 2t^6 - (1/3)\cdot s^2t^4 + (10/3)\cdot s^4t^2 - 5s^6$$

$$= -5(s + t)\cdot(s - t)\cdot\left[(s^2 + t^2/6)^2 + (67/180)t^4\right].$$

Since the second factor in each of the expressions for P, Q, and R cannot vanish unless $s = t = 0$ (cf. [1, p. 36]), we notice that $Z_P(x_0, y_0) = T_G(x_0, y_0)$. Therefore, P, Q, R, and $E_0(N, G)$ satisfy the hypotheses of Theorem (2.1), whereas if $\mu x_0 + \nu y_0 \in Z_R(x_0, y_0)$, there do not exist elements $\alpha x_0 + \beta y_0 \in T_G(x_0, y_0)$ and $\gamma x_0 + \delta y_0 \in Z_Q(x_0, y_0)$ for which $(\mu/\nu)^2 = -(\gamma/\delta) \cdot (\alpha/\beta)$, i.e., Theorem (2.1) and Corollary (2.2) cannot be generalized for vector spaces over nonalgebraically closed fields of characteristic zero.

If we employ the familiar reduction (cf. equations (2.7)–(2.10) used in the proof of Corollary (2.3)) and express the above facts in terms of ordinary polynomials from K_0 to K_0, we conclude the following: If

$$f(z) = 1 - z + z^2 - z^3 = (1 - z) \cdot (1 + z^2),$$

$$g(z) = 2 + z + 10z^2 + 5z^3 = (z + 2) \cdot (5z^2 + 1),$$

$$h(z) = 2 - (1/3)z + (10/3)z^2 - 5z^3 = 5(1 - z) \cdot \left[(z + 1/6)^2 + 67/180\right],$$

then h is a composite polynomial obtained from f and g, where the zeroes of f lie in the generalized circular region $C = \{1\}$ of K_0. Notice that 1, -2, 1 are, respectively, the only zeros of f, g, and h. Therefore, f, g, h, and C satisfy the hypotheses of Corollaries (2.3) – (2.4), whereas if σ is any zero of h, then we cannot find an element $\Delta \in C$ and a zero ρ of g for which $\sigma = -\rho\Delta$, i.e., Corollaries (2.3)–(2.4) cannot be generalized for nonalgebraically closed fields.

Next, we give another example to show that Theorem (2.1) and its Corollaries (2.2)–(2.4) are best possible in the sense that the generalized circular regions $G(x, y)$ or C (used in those results) cannot be replaced, in general, by generalized circular regions adjoined with arbitrary subsets of the boundary.

Example (4.2). Take $E = \mathbf{C}^2$, $K = \mathbf{C}$, and

$$C = \{z \in \mathbf{C} | \mathrm{Im}(z) < 0\} \cup \{1, 2\}.$$

Then $C \not\in D(\mathbf{C}_\omega)$, but the interior of C does belong to $D(\mathbf{C}_\omega)$. Take $N = \{(x_0, y_0)\}$ and $G(x_0, y_0) = C$, where $x_0 = (1, 0)$ and $y_0 = (0, 1)$. So that

$$E_0(N, G) = T_G(x_0, y_0) = \{sx_0 + ty_0 \neq 0 | s, t \in \mathbf{C}; s/t \in C\}.$$

Now, define a.h.p.'s P, $Q \in \mathbf{P}_2$ by

$$P(x) \equiv P(sx_0 + ty_0) = 2t^2 - 3st + s^2 = (s - t) \cdot (s - 2t),$$

$$Q(x) \equiv Q(sx_0 + ty_0) = 2t^2 + 3st + s^2 = (s + t) \cdot (s + 2t),$$

for every $x = (s, t) \in E$. Then (cf. (2.4)) the composite a.h.p. R obtained from P and Q is given by

$$R(x) \equiv R(sx_0 + ty_0) = 4t^4 - (9/2)s^2t^2 + s^4$$

for every $x = (s, t) \in E$. Obviously, $Z_P(x_0, y_0) \subseteq T_G(x_0, y_0)$ and $(\mu/\nu)^2 = (9 \pm \sqrt{17})/16$ for every $\mu x_0 + \nu y_0 \in Z_R(x_0, y_0)$. Hence, P, Q, R and $E_0(N, G)$ satisfy the hypotheses of Theorem (2.1), whereas if $\mu x_0 + \nu y_0 \in$

$Z_R(x_0, y_0)$, there do not exist elements $\alpha x_0 + \beta y_0 \in T_G(x_0, y_0)$ and $\gamma x_0 + \delta y_0 \in Z_Q(x_0, y_0)$ for which $(\mu/\nu)^2 = -(\gamma/\delta)\cdot(\alpha/\beta)$. For, otherwise (as $\gamma/\delta = -1, -2$), Δ would be equal to $(-9 \pm \sqrt{17})/16$ or $(-9 \pm \sqrt{17})/32$, contradicting the fact that $\Delta \in C$. Therefore, Theorem (2.1) and Corollary (2.2) cannot be further generalized to sets $E_0(N, G)$ by replacing $G(x,y)$ by generalized circular regions adjoined with arbitrary subsets of their boundary.

If we express the above example in terms of ordinary polynomials from **C** to **C**, we conclude the following: If

$$f(z) = z^2 - 3z + 2, \qquad g(z) = z^2 + 3z + 2,$$

then the compositie polynomial $h(z)$ obtained from f and g is given by (cf. (2.8)) $h(z) = z^2 - 9z/2 + 4$. Now, all the zeros of $f(z)$ lie in the set C (defined above), $(9 \pm \sqrt{17})/4$ are the zeros of $h(z)$, and $-1, -2$ are the only zeros of $g(z)$. But no zero σ of $h(z)$ can be written in the form $\sigma = -\rho\Delta$, where $\Delta \in C$ and ρ is a zero of $g(z)$. Hence, Corollaries (2.3)–(2.4) cannot be further generalized by replacing C (in those results) by generalized circular regions adjoined with arbitrary subsets of their boundary.

Lastly, we remark that the author (cf. [14, Examples (12.1)–(12.5) and Remark (12.6)] or [13, Examples (3.7)–(3.9) and the succeeding remark]) has already shown the existence of a.h.p.'s $P \in \mathbf{P}_n$ and circular cones $E_0(N, G)$ (hermitian, or otherwise) such that $Z_P(x,y) \subseteq T_G(x,y)$ for every $(x,y) \in N$. This establishes that the hypotheses in Theorem (2.1) are valid and that, in view of Proposition (1.3), our Theorem (2.1) is a strengthened generalization of Marden's theorem expressed in Corollary (2.2).

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