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GROWTH PROBLEMS FOR SUBHARMONIC FUNCTIONS OF FINITE ORDER IN SPACE

BY

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ABSTRACT. For a function $u(x)$ subharmonic (or C^2) in $\mathbf{R}^m$, we compare the "harmonics" (defined in §1) of u with those of a related subharmonic function whose total Riesz mass in $|x| \leqslant r$ is the same as that of u, but whose L^2 norm on $|x| = r$ is maximal, for all $0 < r < \infty$. We deduce estimates on the growth of the Riesz mass of u in $|x| \leqslant r$, as $r \to \infty$.

Introduction. Following Hayman [7], [8], we study the growth and distribution of the Riesz mass of subharmonic functions in $\mathbf{R}^m$ ($m \geqslant 2$) from the point of view of classical value distribution theory. Thus, if $u(x)$ is subharmonic we define the *characteristic*

$$T(r,u) = \sigma_m^{-1} \int_{|\omega|=1} u(r\omega)^+ \, d\omega \tag{1}$$

of $u(x)$ and its *order*

$$\lambda = \limsup_{r\to\infty} \frac{\log T(r,u)}{\log r}; \tag{2}$$

$d\omega$ denotes $(m-1)$-dimensional surface area on $\Sigma = \Sigma_m = \{|x| = 1\}$ and $\sigma_m = \int_\Sigma d\omega$. We always suppose u^+ is unbounded: $T(r,u) \to \infty$ when $r \to \infty$, and u is harmonic near 0 with $u(0) = 0$. We compare the growth of $T(r,u)$ with that of

$$N(r) = N_u(r) = \sigma_m^{-1} \int_\Sigma u(r\omega)\, d\omega \tag{3}$$

which, by Jensen's theorem [1, p. 133], is a weighted average of the Riesz mass of u in the ball $|x| \leqslant r$:

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(4) $$n(r) = (\sigma_m d_m)^{-1} \int_{|x| \leqslant r} d(\Delta u(x)), \qquad N(r) = d_m \int_0^r n(t) t^{1-m}\, dt.$$

Here Δ denotes the Laplacian, Δu exists as a distribution and $\mu = (\sigma_m d_m)^{-1} \Delta u$ is a positive measure when u is subharmonic [1, p. 127]; and $d_m = m - 2$ for $m > 2$, $d_2 = 1$. (For definitions and a discussion of basic results, see §1.)

When $f(z)$ is an entire function of one complex variable and $u(x,y) = \log|f(x+iy)|$, $n(r)$ counts the number of zeros of $f(z)$ in $|z| \leqslant r$, and it is a classical problem to find good lower bounds for

(5) $$k(u) = \limsup_{r \to \infty} \frac{N(r)}{T(r,u)}$$

in terms of λ. For example, it is known in this case that

(6) $$k(u) \geqslant \begin{cases} 1 & (0 \leqslant \lambda \leqslant \frac{1}{2}), \\ \sin \pi\lambda & (\frac{1}{2} < \lambda \leqslant 1) \end{cases}$$

(Edrei and Fuchs [3]), where equality holds for $f(z) =$ polynomial ($\lambda = 0$), $= e^z$ ($\lambda = 1$) and

(7) $$f(z) = \prod_{n=1}^{\infty} (1 - z/n^{1/\lambda}) \qquad (0 < \lambda < 1).$$

Hayman has extended (6) to arbitrary subharmonic u in the plane and found the sharp analogue for functions of orders $\lambda < 1$ in $\mathbf{R}^m$, $m \geqslant 3$ ([7], [8]).

For $\lambda > 1$, precise results are not in general available even for entire functions. A recent result in this direction is

(8) $$k(u) \geqslant (0.9) \frac{|\sin \pi\lambda|}{\lambda + 1} \qquad (1 < \lambda < \infty)$$

(Miles and Shea [10]), and well-known examples [2] show that (8) would fail for large λ if the 0.9 factor were replaced by any constant greater than unity. Inequality (8) is an easy corollary of the main result of [10],

THEOREM A. *Let $f(z)$ be an entire function of finite order λ in the plane, and put $u(z) = \log|f(z)|$,*

(9) $$m_2(r,u) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^2\, d\theta \right\}^{1/2}.$$

Then

(10) $$\limsup_{r \to \infty} \frac{N(r)}{m_2(r,u)} \geqslant \frac{|\sin \pi\lambda|}{\pi\lambda} \left\{ \frac{2}{1 + (\sin 2\pi\lambda)/2\pi\lambda} \right\}^{1/2}.$$

Equality is possible in (10) *for each* $\lambda \geqslant 0$.

Our first purpose in this note is to find the appropriate extension of Theorem A to subharmonic functions. The proof in [10] rests on some simple properties of the Fourier coefficients

$$c_k(r;f) = \frac{1}{2\pi}\int^{2\pi} \log|f(re^{i\theta})|e^{-ik\theta}\,d\theta \qquad (k = 0, \pm 1, \pm 2, \ldots),$$

in particular on the inequality

$$(11) \qquad |c_k(r;f)| \leqslant |c_k(r;f^*)| \qquad (r > 0, k = 0, \pm 1, \pm 2, \ldots)$$

where f^* is a suitable entire function whose zeros have the same moduli as those of f but are projected onto the positive real axis. Thus, if $u^* = \log|f^*|$, then $N_u(r) \equiv N_{u^*}(r)$ and

$$(12) \qquad m_2(r,u) \leqslant m_2(r,u^*) \qquad (0 < r < \infty)$$

by Parseval's theorem, and to prove (10) it suffices to consider just the f^*.

In §2, we study the spherical harmonics of subharmonic functions in $\mathbf{R}^m$ and prove an analogue of (11) for all $m \geqslant 2$ (Theorem 2.1). From this we deduce

THEOREM 1. *Let* $u(x)$ *be subharmonic and of finite order* λ *in* $\mathbf{R}^m$, *and put*

$$m_2(r,u) = \left\{\sigma_m^{-1}\int_\Sigma |u(r\omega)|^2\,d\omega\right\}^{1/2}.$$

Then

$$(13) \qquad \limsup_{r\to\infty} \frac{N(r)}{m_2(r,u)} \geqslant C(\lambda,m) \qquad (0 \leqslant \lambda < \infty, m \geqslant 2),$$

where

(14)

$$C(\lambda,m) = \left\{1 + \frac{\lambda^2(\lambda+m-2)^2}{(m-2)!}\sum_{k=1}^{\infty}\frac{(k+m-3)!\,(2k+m-2)}{k!\,(k-\lambda)^2(k+\lambda+m-2)^2}\right\}^{-1/2}.$$

When $m = 2$, the bound in (13) is the same as that in (10), and when $m = 3$ or 4 inequality (13) remains sharp for all λ, with

$$C(\lambda,3) = \frac{|\sin \pi\lambda|\sqrt{2\lambda+1}}{\pi\lambda(\lambda+1)}\left\{1 - \frac{2}{\pi^2}(\sin^2 \pi\lambda)\sum_{k=1}^{\infty}\frac{1}{(k+\lambda)^2}\right\}^{-1/2},$$

$$C(\lambda,4) = \frac{|\sin \pi\lambda|}{\pi\lambda(\frac{1}{2}\lambda+1)}\left\{1 - \frac{\sin 2\pi\lambda}{2\pi(\lambda+1)}\right\}^{-1/2}.$$

When $m \geqslant 5$ the series in (14) diverges and $C(\lambda, m) \equiv 0$, which just reflects the fact that for these m the extremal functions for this problem (studied in §4) fail to be square-integrable on spheres $|x| = r$, $0 < r < \infty$.

By Schwarz's inequality and Jensen's theorem, $m_2(r,u) \geqslant 2T(r,u) - N(r)$, and we deduce easily a bound for $k(u)$ defined in (5):

COROLLARY 1. *If $u(x)$ is subharmonic*

$$k(u) \geqslant \frac{|\sin \pi\lambda|}{\pi\lambda(\lambda+1)^{\frac{1}{2}m-1}} \qquad (0 \leqslant \lambda < \infty; m = 2,3,4). \tag{15}$$

In §4 we consider a class of examples which, we conjecture, minimize $k(u)$ for any given order λ and dimension m; in particular we show that there exist subharmonic functions $u_{\lambda,m}(x)$ of order λ in $\mathbf{R}^m$ with

$$k(u_{\lambda,m}) \leqslant C_m \frac{|\sin \pi\lambda|}{(\lambda+1)^{\frac{1}{2}m}} \qquad (1 < \lambda < \infty). \tag{16}$$

Thus the bound in (15) has the right order of magnitude for large λ.

Using other methods, we obtain

THEOREM 2. *If $u(x)$ is subharmonic and of order λ in $\mathbf{R}^m$, then*

$$k(u) \geqslant A_m \frac{|\sin \pi\lambda|}{(\lambda+1)^{\frac{1}{2}(m+1)}} \qquad (0 < \lambda < \infty; m \geqslant 5) \tag{17}$$

where A_m depends only on m.

Hayman [8] has obtained $k(u) \geqslant (q + 1 - \lambda)(\lambda - q)/\lambda(q+1)4^{m+q}$, with $q = [\lambda]$, as a consequence of an inequality between $N(r)$ and $M(r,u) = \sup_{|x|=r} u(x)$. Using the Poisson formula to estimate $M(r,u)$ in terms of $T(\sigma r, u)$, $\sigma > 1$, we can easily adapt the proof of (17) to find that

$$\limsup_{r\to\infty} \frac{N(r)}{M(r,u)} \geqslant B_m \frac{|\sin \pi\lambda|}{(\lambda+1)^{\frac{3}{2}m}} \qquad (0 < \lambda < \infty, m \geqslant 2).$$

The conjectured extremal functions $u_{\lambda,m}$ mentioned above are harmonic in $\mathbf{R}^m$ except on the positive x_1-axis, along which the Riesz mass is distributed regularly: $N_{u_{\lambda,m}}(r) \equiv r^{\lambda}$, and $u_{\lambda,m}(x) = |x|^{\lambda} I(\cos\theta; \lambda, m)$ where θ denotes the angle between the vector x and the positive x_1-axis, and I is defined in §4. If we put

$$K(\lambda, m) \stackrel{\text{def}}{=} k(u_{\lambda,m}) = T(1, u_{\lambda,m})^{-1} \qquad (m \geqslant 2, 0 \leqslant \lambda < \infty)$$

then Hayman's sharp result noted earlier, for $\lambda < 1$ and $m \geqslant 2$, is: $k(u) \geqslant K(\lambda, m)$, and our approximations (15) and (17) for $\lambda > 1$ have been compared with $K(\lambda, m)$ via (16). Complementary to these lower bounds for $k(u)$, when u is an arbitrary subharmonic function, is

THEOREM 3. *Let $u(x)$ be subharmonic in $\mathbf{R}^m$ of finite nonintegral order λ with all its Riesz mass distributed along a ray through* 0. *Then*

$$\liminf_{r\to\infty} \frac{N(r)}{T(r, u)} \leqslant K(\lambda, m) \tag{18}$$

where besides (16) $K(\lambda, m)$ *satisfies*

$$K(\lambda, m) < 1 \qquad (m \geqslant 3, 0 < \lambda < \infty) \tag{19}$$

and

$$\begin{aligned} K(\lambda, 2) &= \frac{|\sin \pi\lambda|}{q + |\sin \pi\lambda|} \qquad (q \leqslant \lambda < q + \tfrac{1}{2}) \\ &= \frac{|\sin \pi\lambda|}{q + 1} \qquad (q + \tfrac{1}{2} \leqslant \lambda < 1) \end{aligned} \tag{20}$$

for $q = 0, 1, 2, \ldots.$

Inequality (18) remains valid for integral orders λ, but then requires different methods; cf. [15].

For entire functions in the plane this is due to Ostrovskiĭ [12]. There exist other related studies of this type, e.g. by Edrei and Fuchs [2], also [4], [5], [9].

All the results mentioned above for entire functions have extensions to meromorphic functions, provided the definitions of $N(r)$ and $T(r, u)$ are generalized in a natural way. If f is meromorphic in the plane and $u(z) = \log|f(z)| = \log|g(z)| - \log|h(z)|$ where g, h are entire functions having no common zeros, we define $\mu = \Delta u = \Delta \log|g| - \Delta \log|h| = \mu^+ - \mu^-$ where μ^+ and μ^- are positive measures,

$$n(r, u) = \frac{1}{2\pi}\int_{|z|\leqslant r} d\mu^-(z), \qquad n(r, -u) = \frac{1}{2\pi}\int_{|z|\leqslant r} d\mu^+(z),$$

$$N(r, u) = \int_0^r n(t, u)t^{-1}\,dt, \qquad N(r) = N_u(r) = N(r, u) + N(r, -u),$$

with

$$T(r,u) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})^+ \, d\theta + N(r,u), \qquad k(u) = \limsup_{r\to\infty} \frac{N(r)}{T(r,u)}.$$

Thus $k(u)$ gives a measure of the "total deviation from harmonicity" of $u = \log|f|$. The Edrei-Fuchs inequality (6) remains valid in this more general setting [3], as does Theorem A [10].

We prove Theorems 1 and 2 for $u(x)$ in the class $\mathfrak{D}_m$ of functions "delta-subharmonic" in $\mathbf{R}^m$.

DEFINITION 1. A function u defined (a.e.) in $\mathbf{R}^m$ is in $\mathfrak{D}_m$ if there exist subharmonic functions u_1, u_2 in $\mathbf{R}^m$ with $u = u_1 - u_2$.

A more intrinsic definition is: $u \in \mathfrak{D}_m$ if for every compact set F, $u \in L^1(F)$ and

$$\left|\int u(x)\Delta\varphi(x)\,dx\right| \leqslant C(F)\|\varphi\|_\infty \tag{21}$$

for some constant $C(F)$ and every $\varphi \in C^\infty(\mathbf{R}^m)$ vanishing outside of F.

It is immediate from the second definition that any $u \in C^2(\mathbf{R}^m)$ is delta-subharmonic. The equivalence of the two definitions and other basic facts needed here are discussed further in §1.

If f: $\mathbf{C}^M \to \mathbf{C}$ is an entire function of order λ, then Theorem 2 applies to $u = \log|f|$ and yields

$$\limsup_{r\to\infty} \frac{N(r,0;f)}{T(r,f)} \geqslant A(M)\frac{|\sin \pi\lambda|}{\lambda^c + 1} \qquad (0 < \lambda < \infty) \tag{22}$$

with $c = M + \frac{1}{2}$ and $N(r,0;f) \equiv N_u(r)$. Our examples $u_{\lambda,2M}(x)$ show that $c \geqslant M$ for subharmonic functions in $\mathbf{R}^{2M}$ generally, but it remains an interesting question whether (22) with $c \approx M$ is a good estimate for entire functions when $M \geqslant 2$.

1. **Definitions and auxiliary results.** A function u: $\mathbf{R}^m \to [-\infty,\infty)$ is subharmonic, $u \in \mathfrak{S}_m$, if u is upper semicontinuous, $\not\equiv -\infty$ and

$$u(x) \leqslant \sigma_m^{-1}\int_\Sigma u(x+\delta\omega)\,d\omega$$

for all $x \in \mathbf{R}^m$ and $\delta > 0$. It is well known [1, p. 128], [8], [14] that

$$u \in L^1(F) \text{ for every compact } F, \tag{1.1}$$

$$\begin{aligned}&\Delta u \text{ exists as a distribution and } \mu = (\sigma_m d_m)^{-1}\Delta u\\ &\text{is a positive Borel measure, finite for compact sets.}\end{aligned} \tag{1.2}$$

Further, Riesz's theorem holds: If

$$(1.3) \qquad K(x) = \log|x| \quad (m = 2), \qquad = -|x|^{2-m} \quad (m \geqslant 3),$$

then for any compact F,

$$(1.4) \qquad u(x) = \int_F K(x-y)\,d\mu(y) + h(x)$$

where μ is the measure in (1.2) and h is harmonic in the interior of F. Conversely, given a positive locally finite measure μ on $\mathbf{R}^m$, any u having the representation (1.4) for compact F and h harmonic in the interior of F is subharmonic in $\mathbf{R}^m$ with $\Delta u = \sigma_m d_m \mu$.

The measure μ in (1.2) is termed the *Riesz measure* of u.

Let $u \in \mathfrak{D}_m$, so that $u = u_1 - u_2$ where $u_j \in \mathfrak{S}_m$. Then it is clear that (21) holds with $C(F) = \mu_1(F) + \mu_2(F)$ if $\mu_j = \Delta u_j$ for $j = 1, 2$. Conversely, suppose $u \in L^1_{\text{loc}}$ satisfies (21). Then Δu is a (locally finite, signed) Borel measure $\sigma_m d_m \mu$ [1, p. 93]. Let $|\mu|$ be the total variation of μ, and let $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, $\mu^- = \frac{1}{2}(|\mu| - \mu)$. Then as in Weierstrass's classical theorem we can construct [8, Chapter 4] functions $u^+, u^- \in \mathfrak{S}_m$ with $\Delta u^\pm = \sigma_m d_m \mu^\pm$ and $u = u^+ - u^- + h$ where h is harmonic in $\mathbf{R}^m$; thus $u \in \mathfrak{D}_m$ according to Definition 1.

For convenience, we shall continue to refer to the measure defined in (1.2) as the Riesz measure of u, for any $u \in \mathfrak{D}_m$, and to the mass of the total variation measure $|\mu| = \mu^+ + \mu^-$ as the Riesz mass of u.

If $u \in \mathfrak{D}_m$,

$$(1.5) \qquad \mu = (\sigma_m d_m)^{-1} \Delta u = \mu^+ - \mu^-$$

and we assume throughout §§1–3 that μ^+, μ^- have no mass in a neighborhood of 0, that

$$(1.6) \qquad u(0) = 0,$$

and that

$$(1.7) \qquad T(r, u) \to \infty \qquad (r \to \infty).$$

This involves no restriction for the kind of asymptotic problems studied here.

Generalizing definitions (1), (4) we put

$$(1.8) \qquad \begin{aligned} n(r,u) &= \mu^-(\{|x| \leqslant r\}), \quad n(r,-u) = \mu^+(\{|x| \leqslant r\}), \\ N(r,u) &= d_m \int_0^r n(t,u) t^{1-m}\,dt, \\ N(r) &= N_u(r) = N(r,u) + N(r,-u), \\ T(r,u) &= \sigma_m^{-1} \int_\Sigma u^+(r\omega)\,d\omega + N(r,u), \end{aligned}$$

and (2), (5) remain unchanged.

Applying Green's formula to u_2, we have

$$(1.9)\qquad u_2(0) = \sigma_m^{-1}\int_\Sigma u_2(r\omega)\,d\omega + \int_{|y|\leqslant r}[K(y) - K(re)]\,d\mu^-(y)$$

where $e = (1, 0, \ldots, 0)$, and integration by parts converts the last integral in (1.9) to $N(r, u)$. Thus

$$\begin{aligned} T(r,u) &= \sigma_m^{-1}\int_\Sigma [(u_1 - u_2)^+ + u_2](r\omega)\,d\omega - u_2(0) \\ &= \sigma_m^{-1}\int_\Sigma v(r\omega)\,d\omega - u_2(0) \end{aligned}$$

where $v = \max(u_1, u_2) \in \mathbb{S}_m$, so that by (3) and (4), $T(r,u)$ is a continuous, increasing function convex in $\log r$ $(m = 2)$, r^{2-m} $(m \geqslant 3)$.

Applying (1.9) to u, we obtain the analogue for $u \in \mathfrak{D}_m$ of Nevanlinna's first fundamental theorem,

$$(1.10)\qquad T(r,u) = T(r,-u) \qquad (0 < r < \infty).$$

If $x, y \in \mathbf{R}^m$ we write

$$x \vee y = x \cdot y/|x|\,|y| = \cos\theta$$

where θ is the angle between $\overrightarrow{0x}$ and $\overrightarrow{0y}$. Then

$$(1.11)\qquad \begin{aligned} K(x-y) &= -\sum_{k=0}^{\infty} P_k(x \vee y)\frac{|x|^k}{|y|^{k+m-2}} \qquad (|x| < |y|) \\ &= -\sum_{k=0}^{\infty} P_k(x \vee y)\frac{|y|^k}{|x|^{k+m-2}} \qquad (|y| < |x|) \end{aligned}$$

where the P_k are the Gegenbauer polynomials [16, pp. 302, 329]. On the other hand, for fixed y, $K(x-y)$ is real-analytic in x and thus $P_k(x \vee y) \cdot |x|^k/|y|^{k+m-2}$ is the sum of terms of degree k in the Taylor expansion of $K(x-y)$ in a neighborhood of the origin. Thus $P_k(x \vee y)|x|^k$ is a homogeneous harmonic polynomial of degree k in x (except when $m = 2$, $k = 0$), and [1, p. 169]

$$(1.12)\qquad \int_\Sigma P_j(r\omega \vee y)P_k(r\omega \vee z)\,d\omega = 0 \qquad (j \neq k)$$

for all $r = |x| > 0$ and $y, z \in \mathbf{R}^m - \{0\}$.

For any integer $q \geqslant 0$, we define

$$(1.13)\qquad K_q(x,y) = K(x-y) + \sum_{k=0}^{q} P_k(x \vee y)\frac{|x|^k}{|y|^{k+m-2}} \qquad (x, y \in \mathbf{R}^m).$$

Thus

$$(1.14) \qquad K_q(x,y) = -\sum_{k=q+1}^{\infty} P_k(x \vee y)\frac{|x|^k}{|y|^{k+m-2}} \qquad (|x| < |y|).$$

Assume that $u \in \mathfrak{D}_m$ is of finite order λ, so that by (1.8) and (1.10):

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log r} \leqslant \lambda,$$

and let μ be the associated Riesz measure. Then for $\alpha > \lambda$,

$$(1.15) \qquad \begin{aligned} \int_0^\infty \frac{N(r)}{r^{\alpha+1}}\,dr &= \frac{d_m}{\alpha}\int_0^\infty \frac{n(t)}{t^{\alpha+m-1}}\,dt \\ &= \frac{d_m}{\alpha(\alpha+m-2)}\int_{\mathbf{R}^m} \frac{d|\mu|(x)}{|x|^{\alpha+m-2}} \end{aligned}$$

converges. Let $q = q(\mu)$ denote the least integer $\geqslant 0$ for which

$$(1.16) \qquad \int \frac{d|\mu|(x)}{|x|^{q+m-1}} < \infty,$$

and put

$$(1.17) \qquad u_\mu(x) = \int_{\mathbf{R}^m} K_q(x,y)\,d\mu(y).$$

By (1.16), (1.13), (1.11) and (1.4), $u_\mu \in \mathfrak{D}_m$ and

$$(1.18) \qquad u_\mu(r\omega) \in L^1(\Sigma, d\omega) \qquad (0 < r < \infty).$$

For some purposes it is convenient to have explicit estimates of K_q, and we state

LEMMA 1.1. *There exists a constant $C = C(m,q)$ such that, if $|x| = r$,*

$$|K_q(x,y)| \leqslant Cr^{q+1}/|y|^{q+m-1} \qquad (r \leqslant \tfrac{1}{2}|y|),$$

$$K_q(x,y) \leqslant Cr^{q+1}/|y|^{q+m-2}(r+|y|) \qquad (x,y \in \mathbf{R}^m),$$

the latter except when $m = 2$ and $q = 0$, in which case

$$K_0(x,y) = \log|1 - x/y| \leqslant \log(1 + r/|y|).$$

When $m = 2$, Lemma 1.1 is well known [6, p. 26]; analogous estimates yield the result for $m \geqslant 3$, e.g. see [8].

Using Lemma 1.1 we find that if

$$\lambda_0 = \limsup_{r\to\infty} \frac{\log N(r)}{\log r}, \tag{1.19}$$

then u_μ has order λ_0, $q \leqslant \lambda_0 \leqslant q+1$. Further, arguments like those used for the classical Hadamard representation theorem (worked out in Hayman's book [8, Chapter IV]), give

LEMMA 1.2. *Let $u \in \mathfrak{D}_m$ have finite order λ, let $q(\mu)$ be determined as in* (1.16) *and put $g = \max(q, [\lambda])$.*
Then

$$u(x) = u_\mu(x) + h(x) \tag{1.20}$$

where h is a harmonic polynomial of degree at most g.

Observe that $g = q(\mu)$ when λ is not a positive integer.

Finally, we collect some facts about spherical harmonics needed for Theorem 1; for proofs see [1, pp. 168–170] and [11, pp. 43, 44]. Let $\mathcal{H}_k$ denote the space of all homogeneous harmonic polynomials of degree k. The restrictions of these to Σ are the spherical harmonics of order k, and they form a finite-dimensional subspace $\mathfrak{L}_k$ of $L^2(\Sigma, d\omega)$. For each $k \geqslant 0$, let $\{\varphi_{k,j}\}_{j=0}^{n(k)}$ be an orthonormal basis of $\mathfrak{L}_k$; then the set $\Phi = \{\varphi_{k,j}\colon k \geqslant 0, 0 \leqslant j \leqslant n(k)\}$ is complete in $L^2(d\omega)$. If $\varphi, \psi \in \Phi$ are of different degrees then $\int_\Sigma \varphi(\omega)\psi(\omega)\,d\omega = 0$; this fact generalizes (1.12).

Let $f \in L^1(d\omega)$, and define the kth harmonic of f to be

$$f_k = \sum_{j=0}^{n(k)} \left\{\int_\Sigma f(\omega)\varphi_{k,j}(\omega)\,d\omega\right\}\varphi_{k,j}. \tag{1.21}$$

We note that

$$\|f_k\|_\infty \leqslant C(k)\|f\|_1 \qquad (k \geqslant 0),$$

that $f = \sum f_k$ holds for all f in the linear span Φ^* of Φ, and that if $f_k \equiv 0$ for each $k \geqslant 0$ then $f \equiv 0$, since Φ^* is dense in $C(\Sigma)$. Further, f_k is the orthogonal projection of f onto $\mathfrak{L}_k$ for all $f \in L^2(d\omega)$, and thus f_k does not depend on the basis chosen.

Finally, we write

$$c_k = c_k(f) = \left\{\int_\Sigma f_k^2(\omega)\,d\omega\right\}^{1/2} = \|f_k\|_2 \tag{1.22}$$

and observe that, if $f \in L^2(d\omega)$,

$$\|f\|_2 = \left\{\sum_{k=0}^\infty c_k^2\right\}^{1/2} \tag{1.23}$$

since Φ is complete.

In the next section we study the harmonics of u_μ defined in (1.17), and for this we must compute the harmonics of K_q. For a given $r > 0$ and $y \in \mathbf{R}^m$, let $\{\varphi_{k,j}\}_{j=0}^{n(k)}$ be as described above with $\varphi_{k,0}(\omega) = \alpha_k P_k(\omega \vee y)$, where the positive number α_k is determined by $\|\varphi_{k,0}\|_2 = 1$. Then it is obvious from (1.12)–(1.14) that the kth harmonic of $f(\omega) = K_q(r\omega, y)$ is

$$f_k(\omega) = Q_k P_k(\omega \vee y) \qquad (\omega \in \Sigma)$$

for a suitable factor $Q_k(r, |y|)$. When $|y| > r$ we compute Q_k using (1.14),

$$Q_k(r, |y|) = -r^k/|y|^{k+m-2} \quad (k > q), \qquad = 0 \quad (k \leqslant q).$$

When $|y| < r$ we use (1.13) in a similar way and, for $|y| = r$, Q_k is defined by continuity since $K_q(r\omega, \sigma y) \to K_q(r\omega, y)$ in $L^1(d\omega)$ when $\sigma \to 1$. Then the values of Q_k can be tabulated as follows:

(1.24)

$Q_k(r,t)$	$t < r$	$t \geqslant r$
$k > q$	$-t^k/r^{k+m-2}$	$-r^k/t^{k+m-2}$
$1 \leqslant k \leqslant q$	$r^k/t^{k+m-2} - t^k/r^{k+m-2}$	0
$k = 0$	$K(r) - K(t)$	0

Finally, we observe from (1.17) and Fubini's theorem that the kth harmonic of $u_\mu(r\omega)$ is

$$\int_{\mathbf{R}^m} Q_k(r, |y|) P_k(\omega \vee y)\, d\mu(y). \tag{1.25}$$

2. An extremal property of spherical symmetrizations of potentials; proof of Theorem 1. Let $u \in \mathfrak{D}_m$ have finite nonintegral order λ, and let $q = [\lambda]$. Let μ be the Riesz measure of u, and denote by $\tilde{\mu}$ the measure obtained by projecting the mass of μ onto the positive x_1-axis according to

$$\tilde{\mu}([a,b]) = \mu(\{a \leqslant |x| \leqslant b\}) \qquad (0 < a < b < \infty)$$

where $[a,b]$ denotes the interval on the x_1-axis with endpoints $(a, 0, \ldots, 0)$, $(b, 0, \ldots, 0)$. We also introduce the total variation measure $\mu^* = |\tilde{\mu}|$ and the associated subharmonic function

$$u_\mu^*(x) = \int_0^\infty K_q(x, te)\, d\mu^*(t). \tag{2.1}$$

We shall compare the harmonics of u_μ and u_μ^*. Recalling (1.22) and (1.25) we define

$$C_k(r, u_\mu) = c_k(u_\mu(r\omega)) = \left\| \int_{\mathbf{R}^m} Q_k(r, |y|) P_k(\omega \vee y)\, d\mu(y) \right\|_2.$$

If $u_\mu(r\omega) \in L^2(d\omega)$, $m_2(r, u_\mu)$ defined in Theorem 1 satisfies

$$m_2(r,u_\mu) = \sigma_m^{-1/2}\|u_\mu\|_2 = \left\{\sigma_m^{-1}\sum_{k=0}^{\infty} C_k(r,u_\mu)^2\right\}^{1/2}, \tag{2.2}$$

see (1.23). In any case, we have

THEOREM 2.1. *Let μ be a measure satisfying* (1.16). *Then*

$$C_k(r,u_\mu) \leqslant C_k(r,u_\mu^*) \qquad (0 < r < \infty; k \geqslant 0). \tag{2.3}$$

Thus

$$m_2(r,u_\mu) \leqslant m_2(r,u_\mu^*) \tag{2.4}$$

for all r such that $m_2(r,u_\mu^*) < \infty$. [This holds everywhere when $m = 2$, and a.e. when $m = 3, 4$; for, by (2.1) it is sufficient to show, a.e.,

$$\psi_r(\omega) = \int_{r/2}^{2r} |r\omega - te|^{2-m}\, d\mu^*(t) \in L^2(d\omega). \tag{2.5}$$

Fix any r such that $\varphi(t) = \mu^*\{[0,t]\}$ has a finite derivative at r, and let δ and K satisfy $|\varphi(t) - \varphi(r)| \leqslant K|t - r|$ when $|t - r| \leqslant 2\delta$. Then

$$\begin{aligned}
\psi_r(\cos\theta) &= \int_{r/2}^{2r} \{t^2 + r^2 - 2tr\cos\theta\}^{-\nu}\, d\varphi(t) \qquad (\nu = \tfrac{1}{2}(m-2)) \\
&\leqslant C\int_{r/2}^{2r} \{|r - t| + \theta\}^{-2\nu}\, d\varphi(t) \\
&\leqslant C\left\{\int_{|r-t|\leqslant\theta} \theta^{-2\nu}\, d\varphi(t) + \sum_{j=0}^{k}\int_{2^j\theta<|r-t|\leqslant 2^{j+1}\theta} |r-t|^{-2\nu}\, d\varphi(t)\right. \\
&\qquad\left. + \int_{\delta<|r-t|\leqslant r} |r-t|^{-2\nu}\, d\varphi(t)\right\}
\end{aligned}$$

where C depends only on r and $k = [\log(\delta/\theta)/\log 2]$. It follows that $\psi_r(\cos\theta) \in L^2([0,\pi]; \sin^{m-2}\theta\, d\theta)$ when $m = 3, 4$.]

PROOF OF THEOREM 2.1. For each $k > 0$ we have by Schwarz's inequality and the fact that, by (1.24), Q_k is of one sign only,

$$\begin{aligned}
C_k(r,u_\mu)^2 &= \int_\Sigma \left\{\int_{\mathbf{R}^m} Q_k(r,|y|) P_k(\omega \vee y)\, d\mu(y)\right\}^2 d\omega \\
&\leqslant \int_\Sigma \left\{\int_{\mathbf{R}^m} |Q_k(r,|y|)| P_k^2(\omega \vee y)\, d|\mu|(y) \int_{\mathbf{R}^m} |Q_k(r,|y|)|\, d|\mu|(y)\right\} d\omega \\
&= \left\{\int_\Sigma P_k^2(\omega \vee e)\, d\omega\right\}\left\{\int_0^\infty Q_k(r,t)\, d\mu^*(t)\right\}^2 \\
&= \int_\Sigma \left\{\int_0^\infty Q_k(r,t) P_k(\omega \vee e)\, d\mu^*(t)\right\}^2 d\omega = C_k(r,u_\mu^*)^2,
\end{aligned}$$

as claimed. When $k = 0$ we have as in (1.9) that

$$\begin{aligned}\sigma_m^{-1/2} C_0(r, u_\mu) &= \left| \int_{|y| \leqslant r} [K(re) - K(y)]\, d\mu(y) \right| \\ &= |N(r, -u) - N(r, u)| \\ &\leqslant N(r, -u) + N(r, u) = N(r, -u_\mu^*) = \sigma_m^{-1/2} C_0(r, u_\mu^*).\end{aligned}$$

To prove Theorem 1(2), let $u \in \mathfrak{D}_m$ have order $\lambda \neq$ positive integer and put $q = [\lambda]$. Then (1.20) holds with h an harmonic polynomial of degree $\leqslant q$ and $u_\mu \in \mathfrak{D}_m$ of order λ. Further, $N(r) = N_u(r) = N(r, -u_\mu^*)$ has order λ by (1.19); thus there exists a *strong proximate order* $\lambda(t)$ in the sense of [19, p. 41], that is, $\lambda(t) \in C^2(0, \infty)$ and

$$\lambda(t) \to \lambda, \quad \lambda'(t) t \log t \to 0, \quad \lambda''(t) t^2 \log t \to 0 \qquad (t \to \infty),$$

and, if

$$N_1(t) = t^{\lambda(t)}, \quad n_1(t) = d_m^{-1} t^{m-1} N_1'(t),$$

then also

$$N(t) \leqslant N_1(t) \quad (0 < t < \infty), \qquad N(r_n) = N_1(r_n) \quad (n \geqslant 1) \tag{2.6}$$

where r_n increases to $+\infty$, and

$$n_1'(t) = \{\lambda(\lambda + m - 2)/d_m + o(1)\} t^{m-3} N_1(t) \qquad (t \to \infty). \tag{2.7}$$

For proof, see pp. 35 and 39 in [19].

In particular, $n_1(t)$ is eventually increasing, say for $t \geqslant r_1$. Define $\hat{n}$, $\hat{N}$ by

$$\begin{aligned}\hat{N}(t) &= N(t) \quad (0 < t \leqslant r_1), \\ &= N_1(t) \quad (r_1 \leqslant t < \infty); \\ \hat{N}(r) &= d_m \int_0^r \hat{n}(t) t^{1-m}\, dt.\end{aligned} \tag{2.8}$$

Clearly, $\hat{n}$ increases on $(0, \infty)$ and thus

$$\hat{u}(x) = \int_0^\infty K_q(x, te)\, d\hat{n}(t) \in \mathfrak{S}_m.$$

Further,

(2) We thank Dr. F. Abi-Khuzam for pointing out an error in a previous version of the proof of Theorem 1.

(2.9) $$\liminf_{n\to\infty}\frac{N(r_n)}{m_2(r_n,u)} \geqslant \liminf_{n\to\infty}\frac{N(r_n)}{m_2(r_n,u_\mu)+m_2(r_n,h)} \geqslant \liminf_{n\to\infty}\frac{N(r_n)}{m_2(r_n,u_\mu^*)}$$

where we have used (2.4) and $m_2(r_n,h) = O(r_n^q) = o(N(r_n))$, by (2.6).

We proceed to estimate $m_2(r_n,u_\mu^*)$. For each $k \geqslant 1$, we have from the proof of Theorem 2.1 that

$$\sigma_m^{-1} C_k(r,u_\mu^*)^2 = I_k^2\left\{\int_0^\infty Q_k(r,t)\,d\mu^*(t)\right\}^2$$

where [11, pp. 15, 33, 4]

$$I_k^2 = \sigma_m^{-1}\int_\Sigma P_k^2(\omega \vee e)\,d\omega$$

(2.10) $$= \frac{(m-2)\Gamma(k+m-2)}{\Gamma(m-2)\Gamma(k+1)(2k+m-2)} \qquad (k \geqslant 1, m \geqslant 3),$$

$$I_k^2 = \frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2 k\theta}{k^2}\,d\theta = \frac{1}{2k^2} \qquad (k \geqslant 1, m = 2).$$

By (1.24), (1.8) and two integrations by parts,

$$\left|\int_0^\infty Q_k(r,t)\,d\mu^*(t)\right| = \int_0^\infty |Q_k(r,t)|\,d\mu^*(t)$$

$$= \beta_k N(r) + \frac{k(k+m-2)}{d_m}\int_0^\infty N(t)\left|Q_k\left(\frac{r}{t},1\right)\right|\frac{dt}{t}$$

where

$$d_m\beta_k = 2k+m-2 \quad (1 \leqslant k \leqslant q), \qquad = -(2k+m-2) \quad (k > q).$$

Thus at the r_n, by (2.6) and (2.7),

$$C_k(r_n,u_\mu^*) \leqslant C_k(r_n,\hat{u}) \qquad (k \geqslant 0).$$

It is easy to see from elementary properties of proximate orders that

(2.11) $$\liminf_{n\to\infty}\frac{\hat{N}(r_n)}{m_2(r_n,\hat{u})} = \lim_{r\to\infty}\frac{\hat{N}(r)}{m_2(r,\hat{u})} = K_2(\lambda,m),$$

where

$$K_2(\lambda,m) = r^\lambda/m_2(r,U_\lambda) \qquad (0 < r < \infty)$$

and

$$U_\lambda(r\omega) = \frac{\lambda(\lambda + m - 2)}{d_m}\int_0^\infty K_q(r\omega, te)t^{\lambda+m-3}\,dt \qquad (= J_\lambda(\omega)r^\lambda)$$

is the subharmonic function with $N(r, -U_\lambda) \equiv r^\lambda$. (A proof of (2.11) is sketched below.)

For $U_\lambda(x)$, clearly

$$\sigma_m^{-1/2} C_k(1, U_\lambda) = I_k\left\{\beta_k + \frac{k(k + m - 2)}{d_m}\int_0^\infty t^\lambda \left|Q_k\left(\frac{1}{t}, 1\right)\right|\frac{dt}{t}\right\},$$

and a direct calculation using (1.24) and

$$K_2(\lambda, m)^{-2} = m_2(1, U_\lambda)^2 = 1 + \sigma_m^{-1}\sum_{k=1}^\infty C_k(1, U_\lambda)^2$$

shows that $K_2(\lambda, m)$ coincides with $C(\lambda, m)$ defined in (14). In view of (2.9) and (2.11), the proof of Theorem 1 (for general $u \in \mathfrak{D}_m$) is complete.

The truth of (2.11) can be seen easily from the integral representation for $\hat{u}(x)$, together with (2.7), (1.14) and properties of proximate orders; we deduce

$$\lim_{r\to\infty}\frac{\hat{u}(r\omega)}{r^{\lambda(r)}} = \frac{\lambda(\lambda + m - 2)}{d_m}\int_0^\infty K_q(\omega, se)s^{\lambda+m-3}\,ds$$

for all $\omega \in \Sigma_m$, $\omega \neq (1, 0, \ldots, 0)$. Further, if $\cos\theta = \omega \vee e$ and $\delta > 0$ is given, the limit holds uniformly for θ in $\delta \leqslant |\theta| \leqslant \pi$ and, if $m \leqslant 4$,

$$\int_{\{\omega:|\theta|<\delta\}} |\hat{u}(r\omega)|^2\,d\omega \leqslant C(\delta)r^{2\lambda(r)} \qquad (r \geqslant r_0)$$

where $C(\delta) \to 0$ as $\delta \to 0$; this last can be seen from the estimate

$$\begin{aligned} C_1|\hat{u}(x)| &\leqslant \int_{2r}^\infty \left(\frac{r}{t}\right)^{q+1} t^{\lambda(t)-1}\,dt + \int_{r/2}^{2r} |K_q(x, te)|t^{\lambda(t)+m-3}\,dt \\ &\quad + \int_{r_1}^{r/2}\left(\frac{r}{t}\right)^q t^{\lambda(t)-1}\,dt + \int_0^{r_1} |K_q(x, te)|\,dn(t) \\ &\leqslant r^{\lambda(r)}\left\{C_2 + 2^{\lambda+1}\int_{r/2}^{2r} |K(x - te)|t^{m-3}\,dt\right\} \end{aligned}$$

for all large r, where the last integral is of the type considered in (2.5). (An obvious modification is needed in the estimate for $[r_1, r/2]$ when $m = 2$, $q = 0$; cf. Lemma 1.1.)

From

$$\sum_{k=1}^\infty \log\left(1 - \frac{\lambda^2}{k^2}\right) = \log\left(\frac{\sin\pi\lambda}{\pi\lambda}\right)$$

we observe, after two differentiations with respect to λ, that

$$C(\lambda,2)^{-2}=\frac{1}{2}\left(\frac{\pi\lambda}{\sin\pi\lambda}\right)^2\left\{1+\frac{\sin 2\pi\lambda}{2\pi\lambda}\right\}. \tag{2.12}$$

A convenient expression for $C(\lambda,3)$ is given by

$$\frac{2\lambda+1}{\lambda^2(\lambda+1)^2}C(\lambda,3)^{-2}=\sum_{k=-\infty}^{\infty}\frac{1}{(k-\lambda)^2}-\sum_{k=1}^{\infty}\frac{1}{(k+\lambda)^2}-\sum_{k=0}^{\infty}\frac{1}{(k+\lambda+1)^2}$$

$$=\left(\frac{\pi}{\sin\pi\lambda}\right)^2-2\sum_{k=1}^{\infty}\frac{1}{(k+\lambda)^2},$$

$$C(\lambda,3)^2=\left(\frac{\sin\pi\lambda}{\pi\lambda}\right)^2\frac{2\lambda+1}{(\lambda+1)^2}\left\{1-\frac{2}{\pi^2}(\sin^2\pi\lambda)\sum_{k=1}^{\infty}\frac{1}{(k+\lambda)^2}\right\}^{-1}. \tag{2.13}$$

$C(\lambda,4)$ can be summed explicitly in terms of elementary functions:

$$\frac{4(\lambda+1)}{\lambda^2(\lambda+2)^2}C(\lambda,4)^{-2}$$

$$=\frac{1}{\lambda^2}+\sum_{k=1}^{\infty}\frac{k-\lambda+(\lambda+1)}{(k-\lambda)^2}-\sum_{k=1}^{\infty}\frac{k+\lambda+1-(\lambda+1)}{(k+\lambda+1)^2}$$

$$=\frac{1}{\lambda^2}+\sum_{k=1}^{\infty}\left\{\frac{1}{k-\lambda}-\frac{1}{k+\lambda+1}\right\}$$

$$+(\lambda+1)\left\{\sum_{k=1}^{\infty}\frac{1}{(k-\lambda)^2}+\sum_{k=1}^{\infty}\frac{1}{(k+\lambda+1)^2}\right\}$$

$$=(\lambda+1)\sum_{k=-\infty}^{\infty}\frac{1}{(k-\lambda)^2}-\frac{1}{\lambda}-\sum_{k=1}^{\infty}\left\{\frac{1}{\lambda-k}+\frac{1}{\lambda+k}\right\}$$

$$=(\lambda+1)\left(\frac{\pi}{\sin\pi\lambda}\right)^2-\pi\cot\pi\lambda,$$

$$C(\lambda,4)^2=\left(\frac{\sin\pi\lambda}{\pi\lambda}\right)^2\left(\frac{2}{\lambda+2}\right)^2\left\{1-\frac{\sin 2\pi\lambda}{2\pi(\lambda+1)}\right\}^{-1}. \tag{2.14}$$

We deduce easily

THEOREM 2.2. *Let $u\in\mathfrak{D}_m$ have finite order λ. Then*

$$\limsup_{r\to\infty}\frac{N(r)}{T(r,u)}\geqslant\frac{|\sin\pi\lambda|}{\pi\lambda(\lambda+1)^{\frac{1}{2}m-1}}\qquad(0\leqslant\lambda<\infty;m\leqslant 4). \tag{2.15}$$

For, by Schwarz's inequality and (1.10),

$$\begin{aligned} m_2(r,u) &\geqslant \sigma_m^{-1}\int_\Sigma u(r\omega)^+\,d\omega + \sigma_m^{-1}\int_\Sigma \{-u(r\omega)\}^+\,d\omega \\ &= T(r,u) - N(r,u) + T(r,-u) - N(r,-u) = 2T(r,u) - N(r) \end{aligned} \tag{2.16}$$

and thus

$$\frac{k(u)}{2-k(u)} = \limsup_{r\to\infty} \frac{N(r)}{2T(r,u)-N(r)} \geqslant C(\lambda, m).$$

Solving this inequality for $k(u)$ and using simple estimates with (2.12)–(2.14), we obtain (2.15).

3. **Bounds for $k(u)$ when $m \geqslant 5$.** Theorem 2 is contained in

THEOREM 3.1. *Let $u \in \mathfrak{D}_m$ have finite order λ. Then*

$$k(u) \geqslant A_m|\sin \pi\lambda|/(\lambda+1)^{\frac{1}{2}(m+1)} \qquad (0 < \lambda < \infty)$$

where we may take $A_m = m^{-m}$ $(m \geqslant 5)$.

We assume λ is not a positive integer, and let $q = [\lambda]$. By Lemma 1.2, (1.20) holds with h of degree at most q. Then

$$\begin{aligned} \sigma_m^{-1}\int_\Sigma |u_\mu(r\omega)|\,d\omega &\leqslant \int_{\mathbf{R}^m}\left\{\sigma_m^{-1}\int_\Sigma |K_q(r\omega,y)|\,d\omega\right\} d|\mu|(y) \\ &= \int_0^\infty B_q(r/t)t^{2-m}\,dn(t) \end{aligned}$$

where

$$B_q(r) = \sigma_m^{-1}\int_\Sigma |K_q(r\omega,e)|\,d\omega;$$

here e denotes the unit vector in the positive x_1-direction.

LEMMA 3.1. *When $0 < r < \infty$,*

$$B_q(r) \leqslant 2e(m-2)^{\frac{1}{2}(m-2)}(q+1)^{\frac{1}{2}(m-3)}r^{q+1}/(r+1). \tag{3.1}$$

Assuming the validity of (3.1), we put

$$S(r) = r^{q+1}/(r+1)$$

and use $rS'(r) \leqslant (q+1)S(r)$ to get

$$\int_0^\infty S\left(\frac{r}{t}\right)t^{2-m}\,dn(t) = d_m^{-1}\int_0^\infty \left\{d_m S\left(\frac{r}{t}\right) + S'\left(\frac{r}{t}\right)\frac{r}{t}\right\}dN(t)$$
$$\leqslant d_m^{-1}(q+m-1)(q+1)\int_0^\infty S\left(\frac{r}{t}\right)N(t)\frac{dt}{t}.$$

By (2.16), (1.20) and Lemma 3.1,

$$2T(r,u) \leqslant N(r) + C_m(q)\int_0^\infty S\left(\frac{r}{t}\right)N(t)\frac{dt}{t} + O(r^q) \tag{3.2}$$

where

$$C_m(q) = 4e(m-2)^{\frac{1}{2}(m-2)}(q+1)^{\frac{1}{2}(m+1)}.$$

For given $\varepsilon > 0$, there exists [6, p. 101] a sequence $r_n \to \infty$ with $N(t) \leqslant (t/r_n)^{\lambda-\varepsilon}N(r_n)$ $(0 < t \leqslant r_n)$, $N(t) \leqslant (t/r_n)^{\lambda+\varepsilon}N(r_n)$ $(t > r_n)$. Thus

$$\limsup_{n\to\infty}\frac{T(r_n,u)}{N(r_n)} \leqslant \frac{1}{2}\left\{1 + C_m(q)\int_0^\infty S(t)t^{-\lambda-1}\,dt\right\},$$
$$k(u) \geqslant (4\pi e)^{-1}(m-2)^{\frac{1}{2}(2-m)}\frac{|\sin\pi\lambda|}{(q+1)^{\frac{1}{2}(m+1)}}$$

and Theorem 3.1 follows.

Proof of Lemma 3.1. We first suppose $0 < r < 1$. Then (1.14) implies

$$K_q(r\omega, e) = -\sum_{k=q+1}^{\infty} P_k(\omega \vee e)r^k.$$

Since the P_k are orthogonal on Σ,

$$B_q(r)^2 \leqslant \sigma_m^{-1}\int_\Sigma K_q(r\omega,e)^2\,d\omega = \sum_{k=q+1}^{\infty} I_k^2 r^{2k}$$

where the I_k are given in (2.10). By simple estimates,

$$I_k^2 \leqslant (m-2)^2 k^{m-4} \qquad (k \geqslant 1).$$

Put $r_0 = \exp\{-1/(q+1)\}$. Then

$$B_q(r)^2 \leqslant \{e(m-2)r^{q+1}\}^2 \sum_{k=q+1}^{\infty}\psi(k) \qquad (0 \leqslant r \leqslant r_0)$$

where $\psi(x) = x^p r_0^{2x}$ $(p = m-4)$ increases on $0 < x < x_0 = (q+1)p/2$ and then decreases, so that

$$\sum_{k=q+1}^{\infty} \psi(k) \leqslant \int_{q+1}^{\infty} \psi(x)\,dx + \psi(x_0)$$

$$= e^{-2}\left(\frac{q+1}{2}\right)^{p+1}\{2^p + p2^{p-1} + p(p-1)2^{p-2} + \cdots + p!\}$$

$$+ \left(\frac{p(q+1)}{2e}\right)^p$$

$$< p^p(q+1)^{p+1} \qquad (p = m-4),$$

(3.3) $$B_q(r) \leqslant e(m-2)^{\frac{1}{2}(m-2)}(q+1)^{\frac{1}{2}(m-3)}r^{q+1} \qquad (0 < r \leqslant r_0).$$

For $r > r_0$, (1.13) yields

$$K_q(r\omega, e) = K(r\omega - e) + \sum_{k=0}^{q} P_k(\omega \vee e)r^k,$$

$$B_q(r) \leqslant \min\{1, r^{2-m}\} + \sigma_m^{-1}\int_\Sigma \left|\sum_{k=0}^{q} P_k(\omega \vee e)r^k\right| d\omega$$

where the second term is dominated by

$$Q = \left\{\sigma_m^{-1}\int_\Sigma \sum_{k=0}^{q} (P_k(\omega \vee e)r^k)^2\,d\omega\right\}^{1/2} = \left\{\sum_{k=0}^{q} I_k^2 r^{2k}\right\}^{1/2}.$$

Thus

$$Q^2 \leqslant (r/r_0)^{2q}\sum_{k=0}^{q} I_k^2 \qquad (r_0 < r < \infty)$$

and by (2.10)

$$I_k^2 \leqslant \frac{(m-2)^{m-3}}{2\Gamma(m-2)}(q+1)^{m-4} \qquad (1 \leqslant k \leqslant q).$$

We deduce

$$B_q(r) \leqslant e(m-2)^{\frac{1}{2}(m-3)}(q+1)^{\frac{1}{2}(m-3)}r^q \qquad (r_0 < r < \infty),$$

and (3.1) follows.

4. **Examples.** For $m \geqslant 3$, let $q < \lambda < q+1$ for some integer $q \geqslant 0$ and consider

(4.1) $$U_\lambda(x) = \frac{\lambda(\lambda+m-2)}{m-2}\int_0^\infty K_q(x, te)t^{\lambda+m-3}\,dt,$$

a subharmonic function whose Riesz mass is distributed along the positive x_1-axis with

$$N(r) = N(r, -U_\lambda) = r^\lambda \qquad (0 < r < \infty). \tag{4.2}$$

Then

$$U_\lambda(-x) = \frac{\lambda(\lambda + m - 2)}{m - 2} I_\lambda(\cos\theta) r^\lambda \tag{4.3}$$

where $x = r\omega$, $\cos\theta = -\omega \vee e$ and

$$\begin{aligned} I_\lambda(\cos\theta) &= \int_0^\infty K_q(\tau\omega, -e)\tau^{-\lambda-1}\, d\tau \\ &= \int_0^\infty \left\{ \sum_{k=0}^{q} P_k(\omega \vee e)(-1)^k \tau^{-k-m+2} - \frac{1}{(1 + \tau^2 + 2\tau\cos\theta)^\nu} \right\} \tau^{\lambda+m-3}\, d\tau. \end{aligned}$$

Here and below, $\nu = (m - 2)/2$.

We have the representation

$$I_\lambda(\cos\theta) = \frac{1}{e^{2\pi\lambda i} - 1} \int_\Gamma \frac{z^{\lambda+m-3}\, dz}{(1 + z^2 + 2z\cos\theta)^\nu}, \tag{4.4}$$

where Γ consists of the circles $|z| = R$ and $|z| = \varepsilon$ $(0 < \varepsilon < 1 < R)$ respectively oriented positively and negatively, joined by segments along the upper and lower edges of the real axis between ε and R. To see this, use

$$(1 + z^2 + 2z\cos\theta)^{-\nu} = \sum_{k=0}^{\infty} (-1)^k P_k(\cos\theta) z^{-k-m+2} \qquad (|z| = R)$$

in (4.4) with Cauchy's theorem and let $\varepsilon \to 0$, $R \to \infty$. Thus we can evaluate I_λ by residues when m is even. (This procedure is used by Hayman [8, Chapter 4] for orders $\lambda < 1$.)

We deduce

$$I_\lambda(\cos\theta) = \frac{2\pi i}{e^{2\pi\lambda i} - 1} \left\{ \frac{g^{(\nu-1)}(\bar{a})}{(\nu - 1)!} + \frac{\bar{g}^{(\nu-1)}(a)}{(\nu - 1)!} \right\} \tag{4.5}$$

where $g(z) = z^{\lambda+m-3}(z - a)^{-\nu}$, $\bar{g}$ is the similar expression with $\bar{a}$ in place of a, and $a = -e^{i\theta}$. By direct calculation,

$$I_\lambda(\cos\theta) = \frac{\pi}{\sin\pi\lambda} \frac{\sin(\lambda + 1)\theta}{\sin\theta} \qquad (\nu = 1)$$

and for $\nu > 1$,

$$I_\lambda(\cos\theta)$$

(4.6) $$= \frac{\pi}{\sin\pi\lambda}\left\{\frac{(\lambda+m-3)\cdots(\lambda+m-\nu-1)}{2^{\nu-1}(\nu-1)!}\cdot\frac{\cos[(\lambda+\nu)\theta-\pi\nu/4]}{(\sin\theta)^\nu}+R\right\}$$

where

(4.7) $$|R| \leqslant C(\nu)(\lambda+1)^{\nu-2}(\sin\theta)^{3-m} \qquad (0<\theta<\pi)$$

and $C(\nu)$ does not depend on θ or λ. This follows easily from (4.5) and

$$g^{(\nu-1)}(z) = \sum_{j=0}^{\nu-1}\binom{\nu-1}{j}D^{(\nu-j-1)}(z^{\lambda+m-3})D^{(j)}((z-\bar{a})^{-\nu})$$

where $D = d/dz$, and the similar expression for $\bar{g}^{(\nu-1)}$.

Since $I_\lambda(\cos\theta)$ is even in θ,

(4.8) $$r^{-\lambda}T(r,U_\lambda) = \frac{\lambda(\lambda+m-2)}{m-2}2\sigma_m^{-1}\int_0^\pi I_\lambda(\cos\theta)^+\,d\omega(\theta) \equiv K(\lambda,m)^{-1}$$

where

$$d\omega(\theta) = \sigma_{m-1}(\sin\theta)^{m-2}\,d\theta.$$

Thus

$$T(1,U_\lambda) = \frac{\pi\lambda}{|\sin\pi\lambda|}\,\frac{(\lambda+m-2)\cdots(\lambda+m-\nu-1)}{\nu!\,2^{\nu-1}}\left(\frac{\sigma_{m-1}}{\sigma_m}\right)H_\lambda$$

where

$$H_\lambda = \int_0^\pi\left\{(-1)^q\overset{+}{\cos}\left[(\lambda+\nu)\theta-\frac{\pi\nu}{4}\right](\sin\theta)^\nu\right\}d\theta+\varepsilon_\lambda,$$

with $|\varepsilon_\lambda| \leqslant C_1(\nu)/(\lambda+1)$ by (4.7). On the other hand, since

$$\lim_{\beta\to\infty}\int_a^b f(\theta)\overset{+}{\cos}(\beta\theta+\gamma)\,d\theta$$

$$= \lim_{\beta\to\infty}\int_a^b f(\theta)\{\cos(\beta\theta+\gamma)\}^-\,d\theta = \frac{1}{\pi}\int_a^b f(\theta)\,d\theta$$

for any $f \in L^1(a,b)$ and γ real, we obtain

$$H_\lambda = \frac{1}{\pi}\int_0^\pi \sin^\nu\theta\,d\theta + o(1)$$

on letting $\lambda \to \infty$ so that first $q = [\lambda]$ is even, then odd.

We deduce that the U_λ satisfy

$$\frac{N(r)}{T(r, U_\lambda)} \equiv \frac{\sigma_m(m-2)}{2\lambda(\lambda+m-2)}\left\{\int_0^\pi I_\lambda(\cos\theta)^+\, d\omega(\theta)\right\}^{-1}$$

$$= \alpha_m |\sin \pi\lambda| \lambda^{-\frac{1}{2}m}\{1 + o(1)\} \qquad (0 < r < \infty; \lambda \to \infty)$$

where α_m depends only on the dimension; this proves (16) for m even.

In fact, from (4.4) $I_\lambda(\cos\theta)$ can be seen to satisfy a differential equation of hypergeometric type [17, p. 178], thus [17, pp. 175, 104]

$$I_\lambda(\cos\theta) = \beta\, {}_2F_1\left(\lambda + 2\nu, -\lambda; \nu + \frac{1}{2}; \frac{1+\cos\theta}{2}\right), \tag{4.9}$$

$$\beta = I_\lambda(1)\Gamma(\tfrac{1}{2} + \nu + \lambda)\Gamma(\tfrac{1}{2} - \nu - \lambda)/\Gamma(\tfrac{1}{2} + \nu)\Gamma(\tfrac{1}{2} - \nu)$$

where the ${}_2F_1$ has a known asymptotic expansion [17, p. 77] for large λ like that in (4.6), but valid for all real ν. Further, our analysis giving (4.6) from (4.4), when ν is integral, remains valid for half-integral ν in the case $\theta = 0$, and we can asymptotically evaluate the factor $I_\lambda(1)$ in (4.9). (The ${}_2F_1$ in (4.9) is essentially a Gegenbauer function [17, p. 175].)

We conclude that the functions U_λ satisfy (16) for any $m \geqslant 3$, by known asymptotic results. When $m = 2$, (4.1) gives $U_\lambda(x) = \pi\lambda \csc \pi\lambda(\cos\theta\lambda)r^\lambda$ for all $\lambda \neq$ positive integer, $|\theta| \leqslant \pi$, $r > 0$.

5. **Proof of Theorem 3.** We can assume all the Riesz mass of $u(x)$ is on the negative x_1-axis, so that

$$u(x) = \int_0^\infty K_q(x, -te)\, d\mu(t) + h(x) = u_\mu(x) + h(x) \tag{5.1}$$

where u has order $\lambda \in (q, q+1)$ and the degree of $h(x)$ is at most q. For any $\gamma \in (\lambda, q+1)$,

$$\begin{aligned} \int_0^\infty u_\mu(r\omega) r^{-\gamma-1}\, dr &= \int_0^\infty d\mu(t) \int_0^\infty K_q(r\omega, -te) r^{-\gamma-1}\, dr \\ &= \int_0^\infty t^{-\gamma-m+2}\, d\mu(t) \int_0^\infty K_q(\tau\omega, -e)\tau^{-\gamma-1}\, d\tau \\ &= \frac{\gamma(\gamma+m-2)}{m-2} I_\gamma(\cos\theta) \int_0^\infty N(t) t^{-\gamma-1}\, dt \end{aligned} \tag{5.2}$$

where I_γ is defined in (4.3).

Let $\mathcal{E} \subset \Sigma \cap \{x_m \geqslant 0\}$ be measurable $d\omega$, and define $E \subset [0, \pi]$ by $E = \{\theta \colon \omega \vee e = \cos\theta, \omega \in \mathcal{E}\}$, and

$$T(r, u_\mu; \mathfrak{S}) = 2\sigma_m^{-1} \int_{\mathfrak{S}} u_\mu(r\omega)\, d\omega.$$

Thus $T(r, u_\mu; \mathfrak{S}) \leqslant T(r, u_\mu)$ and by (5.2)

$$\int_0^\infty T(r, u_\mu; \mathfrak{S}) r^{-\gamma-1}\, dr$$

$$= \frac{\gamma(\gamma+m-2)}{m-2}\left\{2\sigma_m^{-1}\int_E I_\gamma(\cos\theta)\, d\omega(\theta)\right\}\int_0^\infty N(t)t^{-\gamma-1}\, dt$$

where $d\omega(\theta)$ was defined in §4.

Using a theorem of Pólya [13] just as in [9, pp. 225–227], we deduce

$$\liminf_{r\to\infty} \frac{A(\lambda)N(r) + r^\tau}{T(r, u_\mu)} \leqslant 1 \tag{5.3}$$

where $\tau < \lambda$ is arbitrary and

$$A(\gamma) = \frac{\gamma(\gamma+m-2)}{m-2} 2\sigma_m^{-1} \int_E I_\gamma(\cos\theta)\, d\omega(\theta). \tag{5.4}$$

Since $N(r) \leqslant T(r, u_\mu)$, it follows from (5.3) that there exists $\{r_n\} \to \infty$ with

$$A(\lambda) \liminf_{n\to\infty} \frac{N(r_n)}{T(r_n, u_\mu)} \leqslant 1$$

and

$$\lim_{n\to\infty} \frac{\log T(r_n, u_\mu)}{\log r_n} = \lambda.$$

Thus by (5.1)

$$A(\lambda) \liminf_{r\to\infty} \frac{N(r)}{T(r, u)} \leqslant 1. \tag{5.5}$$

Since E is an arbitrary subset of $[0, \pi]$ and I_γ is independent of r, we can take

$$E = \{\theta : I_\lambda(\cos\theta) \geqslant 0\}.$$

Then by (4.8) and (5.4), (18) follows. Assertion (19) is a simple consequence of

$$\lim_{\theta\to\pi^-} I_\lambda(\cos\theta) = -\infty \qquad (m \geqslant 3),$$

clear from (4.3). When m is even, $K(\lambda, m)$ can be computed in terms of elementary functions; in particular, (20) follows from the evaluation $I_\lambda(\cos\theta) = (\pi/\lambda \sin \pi\lambda)\cos\theta\lambda$ when $m = 2$.

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