TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 240, June 1978

CONTINUOUS MAPS OF THE INTERVAL WITH FINITE NONWANDERING SET

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ABSTRACT. Let f be a continuous map of a closed interval into itself, and let $\Omega(f)$ denote the nonwandering set of f. It is shown that if $\Omega(f)$ is finite, then $\Omega(f)$ is the set of periodic points of f. Also, an example is given of a continuous map g, of a compact, connected, metrizable, one-dimensional space, for which $\Omega(g)$ consists of exactly two points, one of which is not periodic.

1. Introduction and statement of results. This paper is concerned with an analysis of the nonwandering set and periodic points (see §2 for definitions) of continuous maps of a closed interval onto itself. Most of the paper deals with maps with finite nonwandering set.

Let I be a closed interval and $f \in C^0(I, I)$. Let $\Omega(f)$ denote the nonwandering set of f, and let $P(f)$ denote the set of positive integers which occur as the period of some periodic point of f. Our main results are the following (see §2 for definitions):

THEOREM A. *If $\Omega(f)$ is finite, then $\Omega(f)$ is the set of periodic points of f.*

THEOREM B. *If f has finitely many periodic points then for some positive integer n, $P(f) = \{2^k: k = 0, 1, \ldots, n\}$.*

THEOREM C. *$\Omega(f)$ is contained in the closure of the set of eventually periodic points of f.*

EXAMPLE D. *There is a continuous map g of a compact, connected, metrizable, one-dimensional space, for which $\Omega(g)$ is finite, but $\Omega(g)$ is not the set of periodic points of g.*

A major portion of this paper is devoted to proving Theorem A. We remark that since $f(\Omega(f)) \subset \Omega(f)$, it follows that if $\Omega(f)$ is finite, then for any point $x \in \Omega(f)$, the orbit of x is finite. This implies that x is eventually periodic (i.e. some point in the orbit of x is periodic) but does not imply that x is periodic. It is possible for $f \in C^0(I, I)$ to have points $x \in \Omega(f)$ which are eventually periodic but not periodic. In proving Theorem A, we show this cannot

Presented to the Society, January 27, 1977; received by the editors September 15, 1976.
AMS (MOS) subject classifications (1970). Primary 54H20.
[1]Partially supported by NSF grant MCS 76-05822.

happen when $\Omega(f)$ is finite. The proof uses the idea of the unstable manifold which we define in §2. Of course, the unstable manifold defined here is a modification of the unstable manifold of a hyperbolic periodic point of a differentiable map (see [5] or [6]).

The unstable manifold is a very familiar object in the context of one-to-one maps. However, other researchers have encountered difficulties in properly defining and using the unstable manifold in the context of endomorphisms (differentiable maps, not necessarily one-to-one). This is why, in §2 of this paper, we are very careful in defining and proving elementary properties of the unstable manifold.

Theorem B is contained in a theorem of Sharkovskiy (see [7]). We include the theorem in this paper, because the proof given here is short and elementary and [7] has not been translated from the Russian.

It is true that for any positive integer n, there is a map $f \in C^0(I, I)$ with $P(f) = \{2^k: k = 0, \ldots, n\}$. For a proof, see Lemma 16 of [3].

The proof of Theorem C (given in §5) is valid if the interval I is replaced by the circle S^1, with the additional hypothesis that f has a periodic point. This is not true of Theorems A and B.

Finally, we note that for $f \in C^0(I, I)$, $\Omega(f)$ may not be the closure of the set of periodic points of f. See [1] for an example. Although the example is given as a mapping of the circle, it can easily be modified to a mapping of an interval.

2. Preliminary definitions and results. Let X be a compact topological space, and let f be a continuous map of X into itself. Let n be a positive integer. We define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let f^0 denote the identity map.

A point $x \in X$ is said to be periodic if for some $n > 0$, $f^n(x) = x$. In this case the minimum of $\{n > 0: f^n(x) = x\}$ is called the period of x.

For any $x \in X$ we define the orbit of x by $\mathrm{orb}(x) = \{f^n(x): n = 0, 1, 2, \ldots\}$. The orbit of any periodic point will be called a periodic orbit. We say a point $x \in X$ is eventually periodic if $\mathrm{orb}(x)$ is finite (or equivalently if some element of $\mathrm{orb}(x)$ is periodic).

A point $x \in X$ is said to be wandering if for some neighborhood V of x, $f^n(V) \cap V = \varnothing$ for all $n > 0$. The set of points which are not wandering is called the nonwandering set and denoted by $\Omega(f)$. $\Omega(f)$ is a nonempty closed set and $f(\Omega(f)) \subset \Omega(f)$.

Throughout this paper we let I denote a closed interval, and $C^0(I, I)$ denote the space of continuous maps of I into itself. Let $f \in C^0(I, I)$ and let p be a periodic point of f. We define the unstable manifold $W^u(p, f)$ as follows. Let $x \in W^u(p, f)$ if for any neighborhood V of x, $x \in f^n(V)$ for some positive integer n. If p is a fixed point of f we define $W^u(p, f, +)$ and

$W^u(p, f, -)$ as follows. Let $x \in W^u(p, f, +)$ if for every interval K with left endpoint p, $x \in f^n(K)$ for some positive integer n. Let $x \in W^u(p, f, -)$ if for every interval K with right endpoint p, $x \in f^n(K)$ for some positive integer n.

We will now prove some basic results concerning $W^u(p, f)$. It will be helpful for the reader, in following most of the proofs in this paper, to draw an interval and label points in the correct order.

LEMMA 1. *Let $f \in C^0(I, I)$. If p is a fixed point of f, then $W^u(p, f)$ is connected.*

PROOF. Let b and c be points in $W^u(p, f)$ and suppose $b < x < c$. Without loss of generality we may assume that $x \geq p$. Let V be any open interval about p. Then for some $n > 0$, $c \in f^n(V)$. Since $f^n(V)$ is an interval containing p and c, we have $x \in f^n(V)$. Hence $x \in W^u(p, f)$. Q.E.D.

LEMMA 2. *Let $f \in C^0(I, I)$ and let $\{p_1, \ldots, p_n\}$ be a periodic orbit of f. Then*

$$W^u(p_1, f) = W^u(p_1, f^n) \cup \cdots \cup W^u(p_n, f^n).$$

PROOF. By renumbering we may assume that $f(p_i) = p_{i+1}$ for $i = 1, \ldots, n - 1$ and $f(p_n) = p_1$. First we show that

$$W^u(p_1, f) \subset W^u(p_1, f^n) \cup \cdots \cup W^u(p_n, f^n).$$

Suppose $z \in W^u(p_1, f)$ and

$$z \notin W^u(p_1, f^n) \cup \cdots \cup W^u(p_n, f^n).$$

For each $i = 1, \ldots, n$, there is a neighborhood V_i of p_i such that $z \notin \bigcup_{m=0}^{\infty} f^{nm}(V_i)$. Let $W_1 = V_1$ and for each $j = 2, \ldots, n$ let W_j be a neighborhood of p_1, with $f^{j-1}(W_j) \subset V_j$. Let $W_0 = W_1 \cap \cdots \cap W_n$. Then $z \notin \bigcup_{m=0}^{\infty} f^m(W_0)$. This contradicts $z \in W^u(p_1, f)$. Hence

$$W^u(p_1, f) \subset W^u(p_1, f^n) \cup \cdots \cup W^u(p_n, f^n).$$

We now show that

$$W^u(p_1, f^n) \cup \cdots \cup W^u(p_n, f^n) \subset W^u(p_1, f).$$

Let $z \in W^u(p_k, f^n)$ for some $k = 1, \ldots, n$. Let V be any neighborhood of p_1. If $k = 1$, let $N = V$. If $k > 1$, let N be a neighborhood of p_k with $f^{n-k+1}(N) \subset V$. Since $z \in W^u(p_k, f^n)$, $z \in f^{nm}(N)$ for some $m > 0$. Hence $z \in f^r(V)$ where $r = nm$ if $k = 1$, and $r = nm - (n - k + 1)$ if $k > 1$. Thus $z \in W^u(p_1, f)$. Q.E.D.

LEMMA 3. *Let $f \in C^0(I, I)$ and let p be a periodic point of f. Let $J = W^u(p, f)$. Then $f(J) = J$.*

PROOF. First we show $f(J) \subset J$. Let $x \in J$. Then for any neighborhood W

of p, $x \in f^m(W)$ for some positive integer m. Hence $f(x) \in f^{m+1}(W)$. Thus $f(x) \in J$ and $f(J) \subset J$.

We now show that f maps J onto J. Suppose $f(J)$ is a proper subset of J. Let $z \in J - f(J)$, and let n be the period of p. By Lemma 2, $z \in W^u(p_0, f^n)$ for some $p_0 \in \text{orb}(p)$. Let $K = W^u(p_0, f^n)$.

First suppose that K is a neighborhood of p_0. Then $z \in f^{nm}(K)$ for some positive integer m. Note that since $f(J) \subset J$, $f^r(J) \subset J$ for every positive integer r. Hence

$$f^{nm}(K) \subset f^{nm}(J) \subset f\left(f^{nm-1}(J)\right) \subset f(J).$$

Thus $z \in f(J)$, a contradiction.

Now suppose that K is not a neighborhood of p_0. Then K must be an interval with one endpoint p_0. Without loss of generality we may assume $K = [p_0, b]$ for some $b \in I$.

Choose $c < p_0$ such that $\forall x \in [c, p_0]$, $f^n(x) \neq z$. This can be done by continuity of f^n, since $f^n(p_0) = p_0$. Since $c < p_0$, $c \notin K$. Hence, there is a neighborhood $V = (a, d)$ of p_0, with $c < a < p_0 < d < z$, such that $c \notin \bigcup_{m=0}^{\infty} f^{nm}(V)$. Note that for any positive integer m, $z \notin f^{nm}(K)$ because $f^{nm}(K) \subset f(J)$. Now $c \notin f^n(V)$ by choice of V. Also $z \notin f^n(V)$. This is true since $V = (a, p_0) \cup [p_0, d)$, and $z \notin f^n(a, p_0)$ by choice of c, while $z \notin f^n([p_0, d))$ because $[p_0, d) \subset K$. Since $f^n(V)$ is an interval containing p_0, $f^n(V) \subset (c, z)$. By repeating the above argument inductively, it follows that for any positive integer m, $z \notin f^{nm}(V)$. This is a contradiction, since $z \in K$ and $K = W^u(p_0, f^n)$. Q.E.D.

Lemma 4. *Let $f \in C^0(I, I)$ and let p be a periodic point of f. Let $J = W^u(p, f)$, and let $\bar{J}$ denote the closure of J. Then any element of $\bar{J} - J$ is periodic.*

Proof. Let $x \in \bar{J} - J$. By Lemma 3, $f(\bar{J}) = \bar{J}$, so x must have an inverse image $y \in \bar{J}$. Since $f(J) = J$, $y \notin J$. Hence $y \in \bar{J} - J$. Thus $f(\bar{J} - J) \supset \bar{J} - J$.

It follows from Lemmas 1 and 2 that $\bar{J} - J$ is a finite set. Hence f maps $\bar{J} - J$ homeomorphically onto itself. This implies that every point in $\bar{J} - J$ is periodic. Q.E.D.

Lemma 5. *Let $f \in C^0(I, I)$. Let $K \subset I$ be a closed interval with $K \subset f(K)$. Then f has a fixed point in K.*

Proof. For some points x and y in K, $f(x)$ is the left endpoint of $f(K)$, and $f(y)$ is the right endpoint of $f(K)$. Hence $f(x) \leq x$ and $f(y) \geq y$. By continuity, for some z in the closed interval joining x and y, $f(z) = z$. Q.E.D.

3. Proof of Theorem A.

LEMMA 6. *Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points, and p is a fixed point of f. Let $x \in W^u(p,f)$. If $x > p$, then $x \in W^u(p,f,+)$. If $x < p$, then $x \in W^u(p,f,-)$.*

PROOF. Since the two assertions are analogous we will prove only the first. Let $x \in W^u(p,f)$ and $x > p$.

Suppose $x \notin W^u(p,f,+)$. Then $x \in W^u(p,f,-)$. Let p_1 be the closest fixed point to p which is less than p (or let p_1 be the left endpoint of I if there are no fixed points less than p). Then $\forall y \in (p_1, p)$, $f(y) < y$ (because if $f(y) > y$ $\forall y \in (p_1, p)$, then $W^u(p,f,-) \subset W^u(p,f,+)$ and $x \in W^u(p,f,+)$ a contradiction).

Let z be the supremum of $\{y < p_1 : f(y) = p\}$. This set is nonempty since $x \in W^u(p,f,-)$. Note that $f(z) = p$ and $z < p_1$.

Let y be any point with $z < y < p_1$. Then $f([z,y])$ is an interval $[f(y), p]$. Since $x \in W^u(p,f,-)$, it follows that $z \in W^u(p,f,-)$. Hence $z \in f^n([f(y), p])$ for some $n > 0$. This implies that $z \in f^{n+1}([z,y])$. Since $f^{n+1}([z,y])$ is an interval containing z and p, $f^{n+1}([z,y]) \supset [z,y]$. By Lemma 5, f has a periodic point in $[z,y]$. Since y was arbitrary, f has infinitely many periodic points, a contradiction. Q.E.D.

THEOREM 7. *Let $f \in C^0(I, I)$. Suppose f has finitely many periodic points, and p is a fixed point of f. If $x \in W^u(p,f)$ and $f(x) = p$, then $x = p$.*

PROOF. Suppose $x \in W^u(p,f)$ with $f(x) = p$, and $x \neq p$. Without loss of generality we may assume that $x > p$. By Lemma 5, $x \in W^u(p,f,+)$. This implies that $f([p,x]) \neq \{p\}$. Hence for some interval $(q,z) \subset (p,x)$, $f^{-1}(p) \cap (q,z) = \varnothing$ and $f(z) = p$. Thus for any $a \in I$ with $a < z$, $f([a,z])$ is an interval containing p.

Suppose the following is true:

(1) For any $a \in I$ with $a < z$, $f([a,z])$ contains an interval of the form $[p, b]$.

Let $a \in I$ with $p < a < z$. Then $f([a,z]) \supset [p,b]$ for some $b \in I$. Since $z \in W^u(p,f,+)$, for some $n > 0$, $z \in f^n([p,b])$. Hence $z \in f^{n+1}([a,z])$. Now $f^{n+1}([a,z])$ is an interval containing p and z. Hence $f^{n+1}([a,z]) \supset [a,z]$. By Lemma 5, f has a periodic point in $[a,z]$. Since a was an arbitrary point (with $p < a < z$) f has infinitely many periodic points, a contradiction. Hence (1) is not true.

Thus the following must be true:

(2) For any $a \in I$ with $a < z$, $f([a,z])$ contains an interval of the form $[b,p]$.

We claim that for some $y \in (p,z)$, $f(y) > p$. To prove this, suppose for all

$y \in (p, z)$, $f(y) < p$. Then $z \in W^u(p, f, -)$. Thus for any $a \in I$, with $p < a < z$, $f^n([a, z]) \supset [a, z]$ for some $n > 0$. Hence f has infinitely many periodic points, a contradiction. This establishes the claim that for some $y_0 \in (p, z), f(y_0) > p$.

Let d be the infimum of $\{v > y_0 : f(v) = p\}$. Then $f(d) = p$, and $y_0 < d < z$. Let $a \in I$ with $p < a < d$. Then $f([a, d])$ contains an interval of the form $[p, b]$ (for some $b \in I$). Since $d \in W^u(p, f, +)$ (as $W^u(p, f, +)$ is an interval containing p and x), for some $n > 0, f^n([a, d]) \supset [a, d]$. Since a was an arbitrary point with $p < a < d, f$ has infinitely many periodic points. This is a contradiction. Q.E.D.

THEOREM 8. *Let $f \in C^0(I, I)$ and suppose f has finitely many periodic points. Let $\{p_1, \ldots, p_n\}$ be a periodic orbit of f (of period n). If p_i and p_j are distinct elements of $\{p_1, \ldots, p_n\}$ then $p_j \notin W^u(p_i, f^n)$.*

PROOF. Suppose p_i and p_j are distinct elements of $\{p_1, \ldots, p_n\}$ with $p_j \in W^u(p_i, f^n)$. We claim that for each $k = 1, \ldots, n$, $W^u(p_k, f^n)$ contains an element of $\{p_1, \ldots, p_n\} - \{p_k\}$. To prove this, let V be any neighborhood of p_k. Let r be the smallest positive integer with $f^r(p_i) = p_k$. There is a neighborhood W of p_i with $f^r(W) \subset V$. Now for some $m > 0, p_j \in f^{nm}(W)$. Hence

$$f^r(p_j) \in f^r(f^{nm}(W)) = f^{nm}(f^r(W)) \subset f^{nm}(V).$$

Since V was arbitrary, $f^r(p_j) \in W^u(p_k, f^n)$. Also, $f^r(p_i) = p_k$ and $p_i \neq p_j$ imply that $f^r(p_j) \neq p_k$. This proves the claim.

By renumbering, we may assume that $p_1 < p_2 < \cdots < p_n$. Since $W^u(p_1, f^n)$ is an interval containing p_1 and some element of $\{p_1, \ldots, p_n\} - \{p_1\}, p_2 \in W^u(p_1, f^n)$. Similarly, either $p_1 \in W^u(p_2, f^n)$ or $p_3 \in W^u(p_2, f^n)$.

Suppose $p_1 \in W^u(p_2, f^n)$. By Lemma 6, $p_2 \in W^u(p_1, f^n, +)$ and $p_1 \in W^u(p_2, f^n, -)$. Since $[p_1, p_2] \subset W^u(p_1, f^n)$, it follows from Theorem 7, that for all $x \in (p_1, p_2), f^n(x) > p_1$. So $p_2 \in W^u(p_1, f^n, +)$ implies that for some $x \in (p_1, p_2), f^n(x) = p_2$. Let

$$z = \inf\{x \in (p_1, p_2) : f^n(x) = p_2\}.$$

Then $z \in (p_1, p_2)$ and $f^n(z) = p_2$. Let $p_1 < a < z$. Then $f^n([a, z])$ contains an interval of the form $[b, p_2]$. Since $p_1 \in W^u(p_2, f^n, -)$, for some $m > 0$, $f^{nm}([a, z]) \supset [a, z]$. By Lemma 5, f has a periodic point in $[a, z]$. Since a was an arbitrary point with $p_1 < a < z, f$ has infinitely many periodic points. This is a contradiction, and so $p_1 \notin W^u(p_2, f^n)$. Hence $p_3 \in W^u(p_2, f^n)$.

By the same argument, it follows that $p_{i+1} \in W^u(p_i, f^n)$ for $i = 1, \ldots, n - 1$. In particular, $p_n \in W^u(p_{n-1}, f^n)$. But since $W^u(p_n, f^n)$ is an interval containing p_n and some element of $\{p_1, \ldots, p_{n-1}\}, p_{n-1} \in W^u(p_n, f^n)$. This implies (by the same argument as the preceding paragraph) that f has infinitely many periodic points, a contradiction. Q.E.D.

THEOREM 9. *Let $f \in C^0(I, I)$ and suppose $\Omega(f)$ is finite. Let $x \in \Omega(f)$ and suppose x is not periodic. Then for some periodic point p of f, $\exists z \in W^u(p, f)$ such that $f(z) = p$ and z is not periodic.*

PROOF. $x \in \Omega(f)$ implies that $f^m(x) \in \Omega(f)$, $\forall m > 0$. Since $\Omega(f)$ is finite, this implies that x is eventually periodic. Hence $\exists z \in \text{orb}(x)$ such that $f(z) = p$ for some periodic point p, but z is not periodic. Since $z \in \text{orb}(x)$, $z \in \Omega(f)$. By Lemma 4, to prove the theorem, it suffices to show that $z \in \overline{W^u(p, f)}$. Suppose $z \notin \overline{W^u(p, f)}$.

Let (a, b) be an open interval containing z, with $[a, b] \cap W^u(p, f) = \emptyset$. Since $a \notin W^u(p, f)$ and $b \notin W^u(p, f)$, there is an open interval N containing p, such that $f^m(N) \cap \{a, b\} = \emptyset$ for every positive integer m. Now, for each positive integer m, $f^m(N)$ is an interval which contains some element of orb(p). Since $\text{orb}(p) \subset W^u(p, f)$, $\text{orb}(p) \cap (a, b) = \emptyset$. Hence $f^m(N) \cap (a, b) = \emptyset$ for every positive integer m.

By choosing N smaller if necessary, we may assume that $N \cap (a, b) = \emptyset$. Let V be a neighborhood of z with $V \subset (a, b)$ and $f(V) \subset N$. Then $f^m(V) \cap V = \emptyset$ for every positive integer m. This is a contradiction since $z \in \Omega(f)$. Q.E.D.

THEOREM A. *Let $f \in C^0(I, I)$ and suppose $\Omega(f)$ is finite. Then $\Omega(f)$ is the set of periodic points of f.*

PROOF. Suppose $x \in \Omega(f)$ and x is not periodic. By Theorem 9, for some periodic point p_1, $\exists z \in W^u(p_1, f)$ such that $f(z) = p_1$ and z is not periodic. Let n be the period of p_1 and let $\text{orb}(p_1) = \{p_1, \ldots, p_n\}$. By Lemma 2, $z \in W^u(p_k, f^n)$ for some $p_k \in \{p_1, \ldots, p_n\}$.

Note that $f^n(z) \in \{p_1, \ldots, p_n\}$ and (by Lemma 3) $f^n(z) \in W^u(p_k, f^n)$. Hence, by Theorem 8, $f^n(z) = p_k$. Since $\Omega(f)$ is finite, f^n has only finitely many periodic points. Also, $z \in W^u(p_k, f^n)$ and $f^n(z) = p_k$. This implies, by Theorem 7, that $z = p_k$. This is a contradiction, because z is not periodic. Q.E.D.

4 Proof of Theorem B.

THEOREM 10. *Let $f \in C^0(I, I)$, suppose f has a periodic point which is not fixed. Then f has a periodic point of period* 2.

PROOF. We may assume f has a periodic point of period greater than 2, or else the theorem is proved. Let n be the smallest element of $\{m \geqslant 3 : f$ has a periodic point of period $m\}$. Let $\{x_1, \ldots, x_n\}$ be a periodic orbit of period n, with $x_i < x_{i+1}$ for $i = 1, \ldots, n - 1$.

Let $I_k = [x_k, x_{k+1}]$ for $k = 1, \ldots, n - 1$. Note that for each I_k, $f(I_k) \supset I_j$ for some $j \neq k$. Hence for some set of distinct I_k's, $\{I_{k_1}, \ldots, I_{k_m}\}$, $f(I_{k_i}) \supset$

$I_{k_{i+1}}$ for $i = 1, \ldots, m-1$ and $f(I_{k_m}) \supset I_{k_1}$, where $2 < m < n-1$.

Let J_{k_m} be a closed interval with $J_{k_m} \subset I_{k_m}$ and $f(J_{k_m}) = I_{k_1}$. Also, for $i = 1, \ldots, m-1$ let J_{k_i} be a closed interval with $J_{k_i} \subset I_{k_i}$ and $f(J_{k_i}) = J_{k_{i+1}}$. Then $f^m(J_{k_1}) = I_{k_1}$. By Lemma 5, f^m has a fixed point $y \in J_{k_1}$. Since $m < n$, $\mathrm{orb}(y) \cap \{x_1, \ldots, x_n\} = \varnothing$. Hence $f^i(y)$ is in the interior of $I_{k_{i+1}}$ for $i = 1, \ldots, m-1$. Since the i_k's have pairwise disjoint interiors, y is a periodic point of f of period m. It follows from the choice of n, and the fact that $m < n$, that $m = 2$. Q.E.D.

THEOREM 11. *Let $f \in C^0(I, I)$. Suppose f has a periodic point whose period is not a power of 2. Then for each positive integer k, f has a periodic point of period 2^k. (In particular, f has infinitely many periodic points.)*

PROOF. Let k be a positive integer and $n = 2^{(k-1)}$. Then f^n has a periodic point which is not fixed. By Theorem 10, f^n has a periodic point x of period 2. Since $n = 2^{(k-1)}$, x is a periodic point of f of period 2^k. Q.E.D.

THEOREM B. *Let $f \in C^0(I, I)$ and suppose f has finitely many periodic points. Let $P(f)$ denote the set of positive integers which occur as the period of some periodic point of f. Then for some nonnegative integer n, $P(f) = \{2^k\colon k = 0, 1, \ldots, n\}$.*

PROOF. By Theorem 11, there is a nonnegative integer n, such that $P(f) \subset \{2^k\colon k = 0, 1, \ldots, n\}$, and $2^n \in P(f)$. If $n \in \{0, 1\}$, the theorem follows immediately, so we may assume $n > 2$.

Let $j = 2^{(n-2)}$. Then f^j has a periodic point of period 4. By Theorem 10, f^j has a periodic point of period 2. Hence f has a periodic point of period 2^{n-1}.

Repeating the above argument, it follows that $P(f) = \{2^k\colon k = 0, 1, \ldots, n\}$. Q.E.D.

5. Proof of Theorem C.

THEOREM C. *Let $f \in C^0(I, I)$. Then $\Omega(f)$ is contained in the closure of the set of eventually periodic points of f.*

PROOF. Suppose the statement is false. Let V be the complement in I of the closure of the set of eventually periodic points of f. Then $V \cap \Omega(f) \neq \varnothing$.

Let $x \in V \cap \Omega(f)$. Let W be the component of V with $x \in W$. Since V is open in I, W is an interval and W is a neighborhood of x.

Let n be the smallest element of $\{m > 0\colon f^m(W) \cap W \neq \varnothing\}$. This set is nonempty since $x \in \Omega(f)$. Since $f^n(W) \cap W \neq \varnothing$, and no point of W is eventually periodic, it follows that $f^n(W) \subset \overline{W}$. This fact and the choice of n, imply that $x \in \Omega(f^n)$.

Since there are no periodic points in W, either $\forall y \in W$, $f^n(y) > y$ or

$\forall y \in W, f^n(y) < y$. Without loss of generality, we may assume that $\forall y \in W, f^n(y) > y$.

Let K be a closed interval which contains a neighborhood of x (in I) with $K \subset W$. Let d be the minimum value of the function g defined on K by $g(y) = f^n(y) - y$. Then $d > 0$.

Let N be an interval of length smaller than d and a neighborhood of x, with $N \subset K$. Then $f^n(N) \cap N = \varnothing$. Since $f^n(y) > y$, $\forall y \in W, f^{nm}(N) \cap N = \varnothing$, $\forall m > 0$. This is a contradiction, since $x \in \Omega(f^n)$. Q.E.D.

6. Example D.

EXAMPLE D. *There is a continuous map g of a compact, connected, metrizable, one-dimensional space X, for which $\Omega(g)$ is finite, but $\Omega(g)$ is not the set of periodic points of g.*

PROOF. Let S be any circle in the plane, and let A be any arc in the plane joining two distinct points on S such that $S \cap A$ consists of exactly two points. Let $X = S \cup A$ (see Figure 1). Then X is a compact, connected, metrizable, one-dimensional space (with the topology induced by the usual topology on the plane).

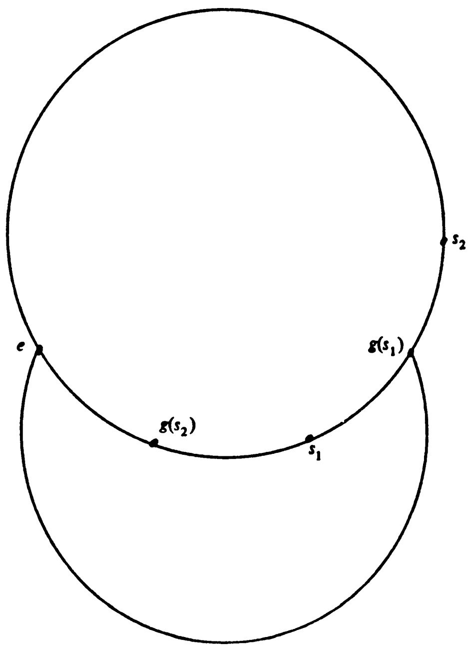

FIGURE 1

Let (a, b) (respectively $[a, b]$) denote the open (respectively closed) arc on S from a counterclockwise to b. Let g be a continuous map of X into itself, as pictured in Figure 1, with the following properties.

(1) g has exactly one fixed point, e.

(2) $S \cap A = \{e, g(s_1)\}$.

(3) g maps the interval $[e, s_1]$ homeomorphically onto $[e, g(s_1)]$.

(4) g maps the interval $[s_1, g(s_1)]$ homeomorphically onto the arc A, with $g(g(s_1)) = e$.

(5) g maps the interval $[g(s_1), s_2]$ homeomorphically onto $[e, g(s_2)]$.

(6) g maps the interval $[s_2, e]$ homeomorphically onto $[e, g(s_2)]$.

(7) $g(x) = e, \forall x \in A$.

We will show that $\Omega(g) = \{e, g(s_1)\}$. All points in $(g(s_1), e)$ are wandering, because these points are not in the image of g. Also, all points in $A - \{e, g(s_1)\}$ are wandering by Property (7) above.

For any $x \in (e, g(s_1))$ there is a neighborhood V of x and a positive integer n with $g^n(V) = \{e\}$. Thus, all points in $(e, g(s_1))$ are wandering. However, $g(s_1)$ is nonwandering. To see this let V be a neighborhood of $g(s_1)$. Then $g(V)$ contains an interval on S of the form $[e, b]$. Hence $g^n(V) \cap V \neq \emptyset$, for some $n > 0$.

Thus $\Omega(g) = \{e, g(s_1)\}$. So $\Omega(g)$ is finite, but $\Omega(g)$ contains a point which is not periodic. Q.E.D.

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