THE CONJUGACY PROBLEM FOR BOUNDARY LOOPS IN 3-MANIFOLDS

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ABSTRACT. A geometric solution of the word problem for fundamental groups of compact, orientable, irreducible, sufficiently large 3-manifolds has been given by F. Waldhausen. We present here a solution of a restricted version of the conjugacy problem for this same class of 3-manifolds; however, the conjugacy problem for 3-manifolds remains in general unsolved. The main results is that there is an algorithm that will determine for any two loops L_1, L_2 in the boundary of a compact, orientable, irreducible sufficiently large 3-manifold M if L_1 is freely homotopic in M to L_2.

1. Waldhausen's results on the word problem [6] are used in proving Theorem 3.2; however, the techniques used for solving the word problem are not generally effective in studying free homotopies. In particular, it is difficult to detect the existence of free homotpies in the presence of high genus cutting surfaces. This problem can be avoided in the case of boundary loops, thanks to a second theorem of Waldhausen [7], [2] which assures us that if an essential disk or annulus does not exist in a 3-manifold M, then all free homotopies of boundary loops in M occur in the boundary of M.

In order to prove Theorem 3.2, a second algorithm is produced that is of interest. If l is a loop in a 2-manifold F, we use the notation (l, F) to denote the collection of all loops in F that are freely homotopic in F to l.

THEOREM. *There is an algorithm BA that will construct for any loop l in the boundary of a compact, orientable, irreducible 3-manifold M and for any incompressible subset F of the boundary of M one representative from each class (k, F) such that l is freely homotopic in M to a loop in (k, F).*

Implicit in the statement that BA is an algorithm is the fact that there are only finitely many classes (k, F) whose members are freely homotopic to a fixed loop l. Thus as a corollary we obtain the following result of W. Jaco [5].

THEOREM (W. JACO). *Let F be an incompressible subset of the boundary of a 3-manifold M. If l is a loop in M then there are only finitely many classes (k, F)*

such that l is freely homotopic in M to a member of (k, F).

2. **Preparations.** For definitions of such terms as *incompressible 2-manifold, boundary incompressible 2-manifold, irreducible 3-manifold, and sufficiently large 3-manifold*, the reader is referred to [8]. A disk D properly embedded in a 3-manifold M is *essential* in M if bd D is not contractible in bd M. An annulus properly embedded in a 3-manifold M is *essential* if it is incompressible and boundary incompressible in M. A *cutting surface* for a 3-manifold M is an incompressible, boundary incompressible 2-manifold F properly embedded in M such that bd $F = \varnothing$ if and only if bd $M = \varnothing$. If F is a 2-manifold properly embedded in M, we shall refer to the 3-manifold that is the closure of the complement of a regular neighborhood of F in M as M *cut at* F. An *arc* is an embedding of the unit interval. A *path* is an immersion of the unit interval.

The first step is to produce an algorithm that will construct special types of cutting surfaces. The importance of this procedure will become apparent as the paper develops. The following algorithm of W. Haken [4] will be useful.

2.1 THEOREM (HAKEN). *Let M be a compact, orientable, irreducible 3-manifold and J a finite graph in* bd M. *For any value of n, there is an algorithm H for finding out if there exists in M an essential disk meeting J in fewer than n points. If such a disk exists, the algorithm will find one.*

A collection of disks $C = \{D_1, \ldots, D_r\}$ properly embedded in a compact, orientable, irreducible 3-manifold M is a *complete set of cutting disks* for M if:

(i) For $i \neq j$, $D_i \cap D_j = \varnothing$.

(ii) D_j is essential in M cut at $\bigcup_{i \neq j} D_i$.

(iii) No component of M cut at C contains an essential disk.

Let F denote an incompressible submanifold of bd M and C a complete set of cutting disks for M. Then C is *irreducible with respect to F* if whenever a path l in C is homotopic with endpoints fixed to a path k in F, then l is homotopic with endpoints fixed to k in $F \cup C$.

2.2 LEMMA. *Let M denote a compact, orientable, irreducible 3-manifold and F an incompressible submanifold of* bd M. *There is an algorithm that will construct a complete set C of cutting disks for M that is irreducible with respect to F.*

PROOF. Haken's algorithm H [4] can be applied to determine if there is an essential disk in M. If no such disk exists, put $C = \varnothing$; otherwise, the algorithm will produce one such disk D_1. If an essential disk exists in M cut at D_1, Haken's algorithm will produce a disk D_2 that meets bd D_1 minimally. Clearly if any such disk exists, there is one that misses D_1 altogether. Hence $D_1 \cap D_2 = \varnothing$. Haken's algorithm is applied repeatedly in this fashion to

construct a mutually exclusive collection $C+ = \{D_1, \ldots, D_s\}$ of cutting disks for M with the property that M cut at $C+$ contains no essential disks. (Certainly this stage must be reached in no more than 2 rank $\pi_1(M)$ steps.)

At this point, it may be so that some D_j is inessential in M cut at $\bigcup_{i \neq j} D_i$. (It is, for example, possible via the above construction to arrive at five cutting disks for a genus three cube-with-handles.) If this is the case, then D_j cobounds a 3-cell in M cut at $\bigcup_{i \neq j} D_i$ with a disk E in the boundary of M cut at $\bigcup_{i \neq j} D_i$. It follows that each component of M cut at $\bigcup_{i \neq j} D_i$ is homeomorphic with a component of M cut at $C+$ so that no component of M cut at $\bigcup_{i \neq j} D_i$ contains an essential disk. Thus D_j can be deleted from the collection $C+$ without sacrificing property (iii) in the definition of a complete set of cutting disks. After deleting all such extra disks from $C+$, the remaining collection $C = \{D_1, \ldots, D_r\}$ is clearly a complete set of cutting disks for M.

The *complexity* of a complete set of cutting disks C (denoted by $\#C$) is the total number of points in the intersection $C \cap$ bd F. Let K denote the boundary of $F \cap (M$ cut at $C)$. If α is an arc in bd D_j that is homotopic in M cut at C with endpoints fixed to an arc β in K, and if α is not homotopic in $F \cup D_j$ to β, then we shall describe an operation Σ that will produce a new complete set of cutting disks C^* with $\#C^* < \#C$. The idea is quite simple. Its description is unfortunately a bit awkward. Note first of all that the incompressibility of F assures us that (int $\alpha) \cap$ (bd $F) \neq \varnothing$. Let L denote the component of M cut at C that contains β. Since bd L is incompressible, the loop theorem assures us that α is homotopic to β in bd L and hence α cobounds a disk E with β in bd L. If E contains D_j, then note that cl((bd $D_j) - \alpha$) is not homotopic in $F \cup D_j$ to β since α is homotopic in D_j to cl(bd($D_j) - \alpha$). So, if $D_j \subset E$, replace α in what follows by cl((bd $D_j) - \alpha$) and replace E by cl($E - D_j$). We proceed assuming $E \cap D_j = \alpha$. By a small deformation of E, we obtain a disk E^* with the following properties.

(i) bd $E^* = \alpha^* \cup \beta^*$ is freely homotopic in bl L with bd E.

(ii) int $E^* \subset$ int L.

(iii) int $\alpha^* \subset$ int D_j.

(iv) $\beta^* \subset F$.

The disk E^* cobounds a 3-cell R in L with a disk in the boundary of L. The arc α^* separates D_j into two disks $P \subset R$ and Q. Put $D_i^* = D_i$ if $i \neq j$, $D_j^* = Q \cup E^*$, and $C^* = \{D_1^*, D_2^*, \ldots, D_r^*\}$. This alteration has the effect of replacing α in bd C by β^* so that since $\alpha \cap$ bd $F \neq \varnothing$ and $\beta^* \subset F$, it is clear that $\#C^* < \#C$.

It must be shown that C^* is again a complete set of cutting disks for M. To this end, let L_1 denote a second component of M cut at C that meets D_j. (Possibly $L_1 = L$.) Observe first of all that M cut at C^* is obtained from M

cut at C by cutting away the 3-cell R from L along E^* and attaching R to L_1 along the disk P. In particular, each component of M cut at C^* is homeomorphic with a component of M cut at C so that no component of M cut at C^* contains an essential disk. Also, D_j^* is isotopic to D_j in M cut at $\bigcup_{i \neq j} D_i$ and so is essential in M cut at $\bigcup_{i \neq j} D_i^*$. Suppose $k \neq j$, and bd D_k^* bounds a disk S in the boundary of M cut at $\bigcup_{i \neq k} D_i^*$. Then S must contain D_j^* in its interior; otherwise, D_k is inessential in M cut at $\bigcup_{i \neq k} D_i$. The disk S meets the 3-cell R along E^*. But then E^* can be replaced by its complement in bd R to produce a disk S^* in the boundary of M cut at $\bigcup_{i \neq k} D_i$ whose boundary is bd D_k. This is, of course, not consistent with the fact that C is a complete set of cutting disks for M.

In view of the fact that the word problem is solvable for 2-manifolds [1], it is possible to determine if any arc α in bd C is homotopic in M cut at C with endpoints fixed to an arc β in K, but α is not homotopic to β in $F \cup C$. Then as shown above, the operation Σ can be applied to C to produce a new complete set of cutting disks C^* with $\# C^* < \# C$. As $\# C$ is finite, the above algorithm can be applied repeatedly to produce a complete set of cutting disks (which we again call C) with the following property.

(A) *If α is any arc in* bd C *that is homotopic in M cut at C with endpoints fixed to an arc β in K, then α is homotopic to β in $F \cup C$.*

It remains to show that C is irreducible with respect to F. In order to avoid a mass of notation, we shall accomplish this in three steps. We establish first the property:

(A1) *If α is any arc in* bd C *that is homotopic in the boundary of M cut at C with endpoints fixed to a path k in F, then α is homotopic with endpoints fixed to k in $F \cup C$.*

Suppose then that $f \colon E \to M$ is a homotopy satisfying the hypothesis of (A1). (E is a 2-cell, bd E consists of two arcs σ, λ with $f(\sigma) = \alpha, f(\lambda) = k$.) The map f is taken to be transverse with respect to bd F and among all such homotopies of this type which cannot be deformed into F, f is chosen so that the number of components of $f^{-1}(\text{bd } F)$ is a minimum. The incompressibility of F assures us then that $f^{-1}(\text{bd } F)$ consists of spanning arcs. Let E_1 denote the closure of the component of $E - f^{-1}(\text{bd } F)$ that contains λ, and let τ denote the component of $f^{-1}(\text{bd } F)$ that lies in the boundary of E_1 and has its initial point adjacent to the initial point of σ. The endpoints of τ divide σ into three arcs $\sigma_1, \sigma_2, \sigma_3$. σ_1 has the same initial points as σ, the endpoints of σ_2 coincide with those of τ, and the terminal point of σ_3 is the terminal point of σ. The arc τ cobounds a disk E_2 with σ_2. Let E_3 denote the closure of $E - E_2$, $f_2 = f | E_2$, $f_3 = f | E_3$. Making use of property (A) of the collection C, it is possible to deform f_2 into $F \cup C$. The map f_3 can be regarded as a homotopy of the path $f(\tau^{-1}) f(\sigma_1^{-1}) k \subset F$ to the arc $f(\sigma_3) \subset$ bd C. Furthermore, after a small deformation of f_3 to make it transverse with respect to

bd F the number of components of f_3^{-1}(bd F) is smaller than the number of components of f^{-1}(bd F). The minimality assumption for f then insures that f_3 can also be deformed into $F \cup C$. This establishes property (A1) for C.

(A2) *If α is any path in $D_j \subset C$ that is homotopic in M cut at C with endpoints fixed to a path k in F, then α is homotopic with endpoints fixed to k in $F \cup C$.*

Let $f: E \to M$ be a homotopy satisfying the hypothesis of (A2). Since M cut at C has incompressible boundary, the map f can be deformed into the boundary of M cut at C. This deformation can clearly be chosen so that f^{-1}(bd D_j) is a single spanning arc τ in E. Then τ separates E into two disks E_1 ($f(E_1) \subset D_j$) and E_2. Observe that $f|E_2$ satisfies the hypothesis of property (A1) so that it can be deformed into $F \cup C$.

Finally we show that C is irreducible with respect to F. Let $f: E \to M$ denote a homotopy where bd E consists of two arcs α, β with $f(\alpha) \subset C$ and $f(\beta) \subset F$. The map f is taken to be transverse with respect to C, and among all such homotopies that cannot be deformed into $F \cup C$, f is chosen so that the number of components of $f^{-1}(C)$ is minimal. It is then immediate that no component of $f^{-1}(C)$ is a simple closed curve. Let τ denote an innermost spanning arc in $f^{-1}(C)$. The endpoints of τ divide β into three arcs $\beta_1, \beta_2, \beta_3$ where the endpoints of β_2 coincide with those of τ. Now τ divides E into two disks E_1 containing β_2 and E_2. Put $f_1 = f|E_1$ and $f_2 = f|E_2$. Property (A2) is available to deform f_1 into $F \cup D_j$. ($f(\tau) \subset D_j$.) Thus $f(\tau)$ is homotopic to an arc in the boundary of D_j that is contained in F. Regarding f_2 as a homotopy of $f(\alpha) \subset C$ to $f(\beta_1 \tau \beta_2) \subset F$, the minimality assumption on f allows f_2 to be deformed into $F \cup C$. This completes the proof of Lemma 2.2.

S^1 shall denote the unit sphere in the complex plane with the real number 1 as base point. If F is a 2-manifold properly embedded in M, and l a loop in M meeting F at a point P, then l *meets F essentially at P* if given any map f: $S^1 \times [0, 1] \to M$ such that $f(1, 0) = P, f|S^1 \times 0 = l$, and f is transverse with respect to F, then the component of $f^{-1}(K)$ meeting $(1, 0)$ is an arc joining $S^1 \times 0$ with $S^1 \times 1$. A loop meets a 2-manifold essentially if it meets it essentially at some point.

If l is a loop in a 3-manifold M and F an incompressible subset of the boundary of M, then there are possibly many free homotopies of l into F. Our next goal is to show that in order to find all classes (k, F) that are freely homotopic in M to l, it suffices to consider special types of homotopies. Let K denote an essential disk or annulus for M. Let $f: S^1 \times [0, 1] \to M$ be a map such that $f|S^1 \times 0 = l$, and suppose f is transverse with respect to K. The type of the homotopy f with respect to K is determined according to the following criteria.

Type 0. $f^{-1}(K) = \varnothing$.

Type 1. $f^{-1}(K)$ consists of a single nontrivial simple closed curve on $S^1 \times [0, 1]$.

Type 2. $f^{-1}(K)$ consists of two nontrivial curves k_1, k_2, and $f(k_1)$ is homotopic on K to $f(k_2)^{-1}$.

Type 3. l meets K essentially.

2.3 LEMMA. *Let l be a noncontractible loop in the boundary of a compact, orientable, irreducible 3-manifold M, F an incompressible submanifold of bd M, and k a loop in F that is freely homotopic in M to l. Suppose C is a complete set of cutting disks for M that is irreducible with respect to F and that $l \cap C = \emptyset$. Then there is a loop $k_1 \in (k, F)$ and a type 0 homotopy with respect to C of l to k_1.*

PROOF. Among all elements of (k, F) that are freely homotopic in M to l, choose k_1 so that the number of points in $k_1 \cap C$ is minimal. Let f: $S^1 \times [0, 1] \to M$ be a map such that $f(S^1 \times 0) = l, f(S^1 \times 1) = k_1$, and f is transverse with respect to C. Since l is not contractible, the simple closed curve components of $f^{-1}(C)$ can be removed by a simple cutting and pasting operation. Further, $l \cap C = \emptyset$ so that no spanning arc in $f^{-1}(C)$ has an endpoint on $S^1 \times 0$. Let β denote a spanning arc in $f^{-1}(C)$ with endpoints on $S^1 \times 1$. Then β cobounds a disk with a subarc λ of $S^1 \times 1$. Since C is irreducible with respect to F, $f(\lambda)$ is homotopic in $F \cup C$ with endpoints fixed to an arc η in bd C. Observe then that either η or its complementary arc in the boundary of C must lie in F. We choose the arc that lies in F and continue to call it η. Note also that $\eta^{-1}f(\lambda)$ is a loop in F that contracts in M and hence in F. It follows that η is homotopic in F with endpoints fixed to $f(\lambda)$. But then replacing the subpath $f(\lambda)$ of k_1 by η, a loop $k_2 \in (k, F)$ is obtained with the property that k_2 is homotopic in M to l, and k_2 meets C fewer times than does k_1.

This contradiction assures us that in fact $k_1 \cap C = \emptyset$, and the proof is complete.

2.4 LEMMA. *Let l be a simple closed curve in the boundary of a compact, orientable, irreducible 3-manifold M with incompressible boundary. Let K be an essential annulus in M such that $K \cap l = \emptyset$. If k is a loop in bd M that is freely homotopic in M to l, then there is a loop $k_1 \in (k, \text{bd } M)$ that is freely homotopic to l via a homotopy of type 0, 1, or 2 with respect to K.*

PROOF. Let $F: S^1 \times [0, 1] \to M$ be a map such that $f|S^1 \times 0 = l, f|S^1 \times 1 = k$, and f is transverse with respect to K. Using the incompressibility of K and bd M, the fact that K is essential, and that $l \cap K = \emptyset$, it is easy to alter f so that $f^{-1}(K)$ consists entirely of nontrivial simple closed curves $\gamma_1, \ldots, \gamma_r$ on $S^1 \times [0, 1]$. If $r \leq 1$, then f is a type 0 or 1 homotopy as required.

If $r > 2$, then by a theorem of Jaco [5], it is true that $f(\gamma_i)$ is homotopic in K to $f(\gamma_1)^{\pm 1}$ for each i ($1 \leqslant i \leqslant r$). Furthermore, if $f(\gamma_i)$ is homotopic in K to $f(\gamma_j)$ then γ_i and γ_j can be removed from the preimage of $f^{-1}(K)$ by a simple cutting and pasting operation on $S^1 \times [0, 1]$. If as many of the γ_i as possible are removed from $f^{-1}(K)$ by this process, then f is clearly a type 0, 1, or 2 homotopy with respect to K.

2.5 LEMMA. *Let M be a compact, orientable, irreducible 3-manifold with incompressible boundary. Let K be an essential annulus in M and l a loop in bd M that meets K essentially at a point P. Let λ be any path on K joining P to a point q on the other boundary component of K. If there is a class $(k, \text{bd } M)$ such that $l \notin (k, \text{bd } M)$ but l is freely homotopic in M to k, then $\lambda^{-1}l\lambda$ is homotopic with q fixed to a loop in $(k, \text{bd } M)$. Furthermore, if such a class exists, it is unique.*

PROOF. Suppose k is a loop in bd M that is freely homotopic in M to l, but $l \notin (k, \text{bd } M)$. In order to prove the theorem, it suffices to show that $\lambda^{-1}l\lambda$ is homotopic in M with q fixed to a member of $(k, \text{bd } M)$. To this end, let f: $S^1 \times [0, 1] \to M$ be a map such that $f|S^1 \times 0 = l, f|S^1 \times 1 = k, f(1, 0) = P$, and f is transverse with respect to K. The component α of $f^{-1}(K)$ that meets $(1, 0)$ is a spanning arc on $S^1 \times [0, 1]$. Since P is an essential intersection point, α must have 1 endpoint on each boundary component of $S^1 \times [0, 1]$. With no loss of generality $(1, 1)$ is taken to be an endpoint of α. There are two cases to be considered.

Case 1. $f(1, 1)$ lies on the same boundary component of K as q. The map f can be corrected along bd K so that $f(1, 1) = q$. Observe that the loop $f(\alpha)^{-1}\lambda$ is homotopic with q fixed to a loop σ in bd K. In particular, $\sigma^{-1}k\sigma \in (k, \text{bd } M)$. Let f_1: $S^1 \times [1, 2] \to M$ be a map such that $f_1|S^1 \times 1 = k, f_1|S^1 \times 2 = \sigma^{-1}k\sigma$ and $f_1|1 \times [1, 2] = f(\alpha)^{-1}\lambda$.

The maps f and f_1 can be pasted together at the $S^1 \times 1$ level to produce a map g: $S^1 \times [0, 2] \to M$. Observe that $g|\alpha \cup 1 \times [1, 2]$ is homotopic with endpoints fixed to λ. Hence $\lambda^{-1}l\lambda$ is homotopic with q fixed to $\sigma^{-1}k\sigma \in (k, \text{bd } M)$.

Case 2. $f(1, 1)$ lies in the boundary component of K containing P. An argument similar to that used in Case 1 shows that l is homotopic in M with P fixed to a loop $k_1 \in (k, \text{bd } M)$. But bd M is incompressible so that in fact l is homotopic to k_1 in bd M. Thus $l \in (k, \text{bd } M)$ contrary to our assumption.

2.6 LEMMA. *Let M be a compact, orientable, irreducible 3-manifold and F an incompressible subset of bd M. Let D be an essential disk in M and l a loop in bd M that meets D essentially at a point P. Let R be a component of $F \cap D$ and λ a path in D joining P to a point q in R. There is a class (k, F) that is freely homotopic in M to l such that some element of (k, F) meets R essentially*

only if $\lambda^{-1}l\lambda$ *is homotopic with q fixed to a loop in F. Furthermore, if this class exists, it is unique.*

The proof is similar to the argument in Lemma 2.5 (but less complex) and is omitted.

Two other known algorithms are required for our constructions.

THEOREM 2.7 (HAKEN [4]). *Let M be a compact, orientable, irreducible 3-manifold. For any value of χ, there is an algorithm H for finding out if there exists in M a cutting surface F with $\chi(F) > \chi$. If such a surface exists, the algorithm will find one.*

THEOREM 2.8 (WALDHAUSEN [7]). *Let M be a compact, orientable, irreducible 3-manifold and F an incompressible subset of* bd *M. There is an algorithm W for determining if any path k with endpoints in F is homotopic in M with endpoints fixed to a path in F.*

Finally we shall require involving free homotopies in 2-manifolds. The proofs of the next two lemmas are straightforward (especially in view of the fact that the conjugacy problem is solved for 2-manifold groups [1]) and are omitted.

2.9 LEMMA. *There is an algorithm S to determine if two loops in a compact, orientable 2-manifold are freely homotopic.*

2.10 LEMMA. *Let G be a compact, orientable 2-manifold and F an incompressible submanifold. There is an algorithm SA that will construct for any loop l in G a representative from each class (k, F) whose members are freely homotopic in G to l.*

3. The algorithms. In [4], Haken observes that with each compact, orientable, irreducible 3-manifold with boundary M, there is associated an integer $n(M)$ with the property that the result of cutting M along any sequence of $n(M)$ non-disk-cutting surfaces is a cube-with-handles. (The reader is cautioned that this is a stronger theorem that Waldhausen's more often quoted Hierarchy theorem [8].) We shall refer to the number $n(M)$ as *the length of M.*

3.1 THEOREM. *There is an algorithm BA that will construct for any loop l in the boundary of a compact, orientable, irreducible, sufficiently large 3-manifold M and for any incompressible submanifold F of* bd *M one representative from each class (k, F) such that l is freely homotopic in M to k.*

PROOF. The algorithm BA will in fact be presented as a series of algorithms. In particular, the algorithm BA n is the algorithm BA for 3-manifolds of length no greater than n. BA n, 0 (resp. BA n, 1) refers to 3-manifolds of length no greater than n that are known to contain no essential disks (resp.

that are known to contain at least one essential disk). The algorithm BA 0, 0 is, of course, trivial. Waldhausen's solution of the word problem [7] is applied to l to determine if l is contractible in M. If l is contractible, then 1 point in each component of F will suffice for the classes (k, F). In what follows, l is taken to be noncontractible.

Step 1. *Given* BA $n - 1$, *the algorithm* BA n, 0 *can be constructed*. At this stage the boundary of M is by assumption incompressible. We shall obtain those classes $(k, \text{bd } M)$ such that l is freely homotopic in M to k. The SA algorithm is then available to complete this step.

Haken's algorithm is first applied to construct a maximal Euler characteristic cutting surface K for M. If $\chi(K) < 0$, then, according to a theorem of Waldhausen [7], each free homotopy of l with a loop k in bd M can be deformed into bd M. The loop l itself is taken as the required representative.

Suppose then that K is an essential annulus. By repeated application of Waldhausen's algorithm W, the number of points in $l \cap K$ can be minimized. Let N denote M cut at K and K_0, K_1 annuli in the boundary of N that are identified to obtain M.

Case 1. $K \cap l = \varnothing$. Then according to Lemma 2.4, only homotopies of type 0, 1, or 2 need be considered. The algorithm BA $n - 1$ is available to construct homotopies of type 0. In order to construct possible homotopies of types 1 and 2, BA $n - 1$ is applied to produce representatives $\alpha_1, \ldots, \alpha_r$ for the classes $(k, K_0 \cup K_1)$ that are freely homotopic in N to l. Appealing once more to a theorem of Jaco [5], we see that $r \leqslant 4$. If $r = 0$, then all homotopies must be of type 0 and these have already been constructed.

If $r = 1$, then say $\alpha_1 \subset K_0$ and α_1 is identified in M with a loop β_1 in K_1. Type 1 homotopies in M can now be constructed by applying BA $n - 1$ to N to find representatives for those classes $(k, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_1. Type 2 homotopies may also occur. Now β_1 is not freely homotopic in N to α_1 or to α_1^{-1} since β_1 is not freely homotopic in N to l or to l^{-1}; however, β_1 may be homotopic in N to β_1^{-1}. (This can be determined by applying BA $n - 1$.) If β_1 is not freely homotopic in N to β_1^{-1}, then no type 2 homotopies occur, and the required classes have been constructed. If β_1 is freely homotopic in N to β_1^{-1}, then the type 2 homotopies can be constructed by adding the inverse of each representative in (bd M) $\cap N$ that occurs as a type 0 homotopy.

For $r = 2$, the situation becomes a bit more complex. If α_1 and α_2 both lie on K_0, then $\alpha_1 = \alpha_2^{-1}$ on K_0 (again by [5]). Let α_1 be identified with the loop β_1 on K_1 in M. Type 1 homotopies are then constructed by appealing to BA $n - 1$ to produce the classes $(k_1, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_1 and the classes $(k_1, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_1^{-1}. Type 2 homotopies can in this case always be replaced by type 0

homotopies since α_1 is freely homotopic in N to $\alpha_1^{-1} = \alpha_2$. The second possibility is that $\alpha_1 \subset K_0$ and $\alpha_2 \subset K_1$. Let α_1 be identified with $\beta_1 \subset K_1$ and α_2 identified with $\beta_2 \subset K_0$ in M. Type 1 homotopies can be produced by using BA $n - 1$ to construct the classes $(k_1, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_1, and the classes $(k_2, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_2. Now β_1 may be homotopic in N to β_1^{-1} in which case type 2 homotopies occur as elements of $(k, (\text{bd } M) \cap N)$ that are freely homotopic in N to α_1^{-1}. A second possibility is that β_1 is freely homotopic in N to α_1^{-1}. In this case, we obtain still more type 2 homotopies by adding the classes $(k, (\text{bd } M) \cap N)$ that are freely homotopic in N to β_1^{-1}. A similar analysis must be carried out for β_2.

The value $r = 3$ is not possible, for if $r = 3$, then say $\alpha_1 \in K_0$, α_2, $\alpha_3 \in K_1$. Since α_2 is freely homotopic in N to α_3, it must be that $\alpha_3 = \alpha_2^{-1}$ in K_1. But α_2 is freely homotopic in N to α_1 so that α_1^{-1} is freely homotopic in N to $\alpha_2^{-1} = \alpha_3$ and hence to l. Hence if $r \geqslant 3$ then in fact $r = 4$.

For $r = 4$, let α_1, $\alpha_2 \in K_0$ and α_3, $\alpha_4 \in K_1$. Then $\alpha_1 = \alpha_2^{-1}$ and $\alpha_3 = \alpha_4^{-1}$. It follows immediately that type 2 homotopies can always be replaced by type 0 homotopies. The construction of type 1 homotopies is similar to constructions in earlier cases and is left to the reader.

Case 2. $l \cap K \neq \varnothing$. In this case we are requred to construct type 3 homotopies of l. Let P be a point of $l \cap K$ and choose an arc λ on K joining P to a point q on the opposite boundary component of K. Waldhausen's algorithm W can be applied to determine if $\lambda^{-1} l \lambda$ is homotopic in M with q fixed to a loop in bd M. According to Lemma 2.5, this is sufficient to determine all classes $(k, \text{bd } M)$ such that l is freely homotopic in M to k. This completes the proof of Step 1.

Step 2. *Given* BA n, 0, *it is possible to construct* BA n, 1.

PROOF. By assumption, M contains an essential disk. The algorithm produced in Lemma 2.2 is applied to produce a complete set of cutting disks C that is irreducible with respect to F. Waldhausen's algorithm W is then applied to minimize the number of points in $l \cap C$.

Case 1. $l \cap C = \varnothing$. Then l lies in a component N of M cut at $\bigcup D_i$. According to Lemma 2.3, it suffices to consider type 0 homotopies with respect to $\bigcup D_i$. Since bd N is incompressible, BA n, 0 is available to construct these homotopies.

Case 2. $l \cap D_1 \neq \varnothing$. Let $D_1 \in C$ and P a point on $D_1 \cap l$, and let $R_1, \ldots, R_r$ denote the components of $F \cap D_1$. Choose arcs $\lambda_1, \ldots, \lambda_r$ on D_1 joining P to points $q_1 \in R_1, \ldots, q_r \in R_r$. For each i apply Waldhausen's algorithm W to determine if $\lambda_i^{-1} l \lambda_i$ is homotopic with q fixed to a loop in F. According to Lemma 2.6, this process will produce the proper classes in F. This procedure is repeated for each D_i such that $l \cap D_i \neq \varnothing$. This completes Step 2.

At this point, Steps 1 and 2 together with Haken's algorithm H establishes the existence of the BA n algorithm for each n. All that remains to establish BA is a procedure for determining an upper bound for the length of a 3-manifold given in terms of a triangulation. Haken notes in [4] that the length of M is bounded above by the maximum number of disjoint, mutually nonparallel, closed incompressible 2-manifolds that can be embedded in M, and that this number is in turn bounded above by a number that can be computed from a triangulation of M [3]. This completes the proof of Theorem 3.1.

3.2 THEOREM. *There is an algorithm* A *that will determine for any pair of loops l_1 and l_2 in the boundary of a compact, orientable, irreducible, sufficiently large 3-manifold M if l_1 is freely homotopic in M to l_2.*

PROOF. Haken's algorithm is first applied to determine if M contains an essential disk.

Case 1. M contains no essential disk. The algorithm BA can be applied to construct those classes $(k_1, \text{bd } M), \ldots, (k_r, \text{bd } M)$ such that l_1 is freely homotopic in M to each k_i. Then the conjugacy problem for 2-manifolds can be applied to determine if l_2 lies in any of the classes $(k_1, \text{bd } M), \ldots, (k_r, \text{bd } M)$.

Case 2. M contains an essential disk. Lemma 2.2 is applied to produce a complete set of cutting disks C for M. Waldhausen's algorithm W is then applied to minimize the intersections of $l_1 \cup l_2$ with C. If l_1 does not meet some disk $D_i \in C$ in exactly the same number of points as does l_2 then l_1 is not freely homotopic in M to l_2. If $l_1 \cap C = \varnothing$, then determination of conjugacy can be made as in Case 1 above. Suppose then that P is a point of $l_1 \cap D_1$, and let $l_2 \cap D_1 = \{q_1, \ldots, q_s\}$. For each i, let λ_i be an arc in D_1 joining P to q_i. Then l_1 is freely homotopic to l_2 in M if and only if one of the loops $\lambda_i l_2 \lambda_i^{-1} l_1^{-1}$ is contractible in M. Waldhausen's word problem algorithm [6] is available to determine the contractibility of these curves.

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