TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 242, August 1978

THE COHOMOLOGY OF THE SYMMETRIC GROUPS

BY

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ABSTRACT. Let $\mathfrak{S}_n$ be the symmetric group on n letters and SG the limit of the sets of degree $+1$ homotopy equivalences of the $n-1$ sphere. Let p be an odd prime. The main results of this paper are the calculations of $H^*(\mathfrak{S}_n, Z/p)$ and $H^*(SG, Z/p)$ as algebras, determination of the action of the Steenrod algebra, $\mathcal{A}(p)$, on $H^*(\mathfrak{S}_n, Z/p)$ and $H^*(SG, Z/p)$ and integral analysis of $H^*(\mathfrak{S}_n, Z, p)$ and $H^*(SG, Z, p)$.

0. Introduction. Let K and L be discrete groups with L abelian. The groups $H^n(K, L)$ have been of interest for years. **[12]** and **[11]** first considered these cohomology groups algebraically and their relation with topological problems. The algebraic groups $H^n(K, L)$ are isomorphic to $H^n(B_K, L)$ where B_K is the topological classifying space for the group K.

Suppose K is $\mathfrak{S}_n$, the symmetric group on n letters. Then $H^*(\mathfrak{S}_n, L)$ is especially important. In the 1950's, work on cohomology operations, **[29]** and **[30]**, showed the necessity for knowledge of $H^*(\mathfrak{S}_{p^i}, Z/p)$. The construction of the mod p Steenrod operations depends on properties of $\mathfrak{S}_p$. Furthermore the Adem relations were derived using the structure of $H^*(\mathfrak{S}_{p^2}, Z/p)$.

If L is a ring then $H^*(K, L)$ is a graded ring. The homology of symmetric products, **[9]**, **[17]**, **[20]**, **[21]**, and **[28]**, computed the groups $H^i(\mathfrak{S}_n, Z/p)$ as Z/p vector spaces. The graded ring structure, which was not analyzed, becomes important in later problems.

There is an interesting link that ties $\mathfrak{S}_n$ to SG. Recall $Q(S^0) = \text{dir lim } \Omega^n S^n$ is the space of "infinite loops of S^∞" and $SG = \text{dir lim } SG_n$ where SG_n is the space of degree $+1$ homotopy equivalences of S^{n-1}. SG is homotopy equivalent to the $+1$ component of $Q(S^0)$.

THEOREM. (1) *There is a canonical map* ω: $B_{\mathfrak{S}_\infty} = \text{dir lim } B_{\mathfrak{S}_n} \to Q(S^0)_0$ *inducing integral and* mod p *homology isomorphisms.*

(2) *The inclusions* $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ *give* $H_*(\mathfrak{S}_\infty)$ *the structure of an algebra.* ω_* *is an algebra isomorphism and a Hopf algebra isomorphism* mod p *where* $H_*(Q(S^0)_0)$ *is an algebra under the loop sum product.*

Received by the editors October 21, 1975 and, in revised form, March 29, 1977.
AMS (*MOS*) *subject classifications* (1970). Primary 18H10; Secondary 55F40.
Key words and phrases. Cohomology of groups, classifying spaces, Steenrod algebra.

The above theorem is contained in the work of many people including [10], [16], [22], [24], [25].

Thus $B_{\mathbb{S}_\infty}$ properly interpreted is a model for SG.

In all that follows let p be an *odd* prime. We will write $H^*(K)$ for $H^*(K, Z/p)$. $H^*(K, Z, p)$ is, by definition, [5], the p-primary component of $H^*(K, Z)$. In [4] the algebra structure of $H^*(\mathbb{S}_{p^2})$ is computed but the arguments do not generalize to $\mathbb{S}_{p^i}$; $i \geqslant 3$. The main results of this paper are the calculations of $H^*(\mathbb{S}_n)$ and $H^*(SG)$ as algebras, determination of the action of the Steenrod algebra, $\mathcal{Q}(p)$, on $H^*(\mathbb{S}_n)$ and $H^*(SG)$ and integral analysis of $H^*(\mathbb{S}_n, Z, p)$ and $H^*(SG, Z, p)$.

This paper is essentially my Stanford University Ph. D. thesis written under the direction of R. James Milgram, whom I would like to thank for his advice and encouragement. I would also like to thank the referee for his numerous helpful comments including shorter proofs for two of the propositions in §II. In addition after submission of this paper I learned that Benjamin Cooper [35] and Hùynh Mùi [36] have also studied $H^*(\mathbb{S}_{p^n})$.

I. Statement of results. It is well known that a p-Sylow subgroup K_p of a finite group K contains all the p-primary homology information; more precisely, $H^*(K)$ and $H^*(K, Z, p)$ are isomorphic to subrings of $H^*(K_p)$ and $H^*(K_p, Z, p)$ respectively, which are invariant under the action of certain automorphisms. It is also well known, [6], that a p-Sylow subgroup of $\mathbb{S}_{p^i}$ is isomorphic to $\mathrm{wr}^i Z/p$, the i-fold wreath product of Z/p. In the next section we examine a specific embedding of $\mathrm{wr}^i Z/p$ in $\mathbb{S}_{p^i}$ and show the existence of an $H^*(\ )$ detecting family consisting of subgroups of the form $\times^m Z/p$. In fact we have the following subgroups and natural inclusions: $k_{j,i}\colon T_{j,i} \to \mathbb{S}_{p^i}$ for $1 \leqslant j \leqslant i$ and the map $k_i^* = \prod_{j=1}^i k_{j,i}^*\colon H^*(\mathbb{S}_{p^i}) \to \prod_{j=1}^i H^*(T_{j,i})$, where $T_{j,i} = \times^{p^{i-j}}(\times^j Z/p)$.

The first theorems compute the images of $k_{j,i}^*$'s and the map k_i^*. We show that k_i^* detects a set of multiplicative generators for $H^*(\mathbb{S}_{p^i})$ whose relations are trivial to compute. Hence the map k_i^* determines $H^*(\mathbb{S}_{p^i})$. Later for simplicity we will want to identify $u \in H^*(\mathbb{S}_{p^i})$ with its natural image $k_{j,i}^*(u) \in H^*(T_{j,i})$ but we must wait until Theorems A–D have been stated to avoid possible confusion.

Recall $H^*(\times^k Z/p) = E(e_1, \ldots, e_k) \otimes P(b_1, \ldots, b_k)$ with degree $(e_m) = 1$, degree $(b_m) = 2$ for all m. Furthermore $\beta_p(e_m) = b_m$, where β_p is the Bockstein operator associated with the exact coefficient sequence $0 \to Z/p \to Z/p^2 \to Z/p \to 0$.

Consider the following classes in $H^*(\times^i Z/p)$: (a matrix cohomology class will always mean the cohomology class given by the formal determinant of that matrix)

$$L_i = \begin{vmatrix} b_1^{p^{i-1}} & \cdots & b_i^{p^{i-1}} \\ \vdots & \vdots & \vdots \\ b_1^{p^r} & \cdots & b_i^{p^r} \\ \vdots & \vdots & \vdots \\ b_1 & \cdots & b_i \end{vmatrix}$$

i.e. the k, j entry of L_i is $b_j^{p^k}$ $(0 \leqslant r \leqslant i-1)$.

$$Q_{j,i} = \frac{\begin{vmatrix} b_1^{p^i} & \cdots & b_i^{p^i} \\ \vdots & \vdots & \vdots \\ \widehat{b_1^{p^j}} & \cdots & \widehat{b_i^{p^j}} \\ \vdots & \vdots & \vdots \\ b_1 & \cdots & b_i \end{vmatrix}}{L_i}$$

i.e. the b^{p^j} row of the numerator is omitted $(1 \leqslant j \leqslant i-1)$.

$$\underline{L}_i = \begin{vmatrix} b_1^{p^{i-1}} & \cdots & b_1^{p^{i-1}} \\ \vdots & & \vdots \\ b_1^p & \cdots & b_i^p \\ e_1 & \cdots & e_i \end{vmatrix}$$

i.e. $\underline{L}_i$ is the L_i determinant with the $b_1 \cdots b_i$ row replaced by the row $e_1 \cdots e_i$.

$$M_{j,i} = \begin{vmatrix} b_1^{p^{i-1}} & \cdots & b_i^{p^{i-1}} \\ \vdots & \cdots & \vdots \\ \widehat{b_1^{p^j}} & \cdots & \widehat{b_i^{p^j}} \\ \vdots & \vdots & \vdots \\ b_1 & \cdots & b_i \\ e_1 & \cdots & e_i \end{vmatrix}$$

i.e. the b^{p^j} row is omitted $(1 \leqslant j \leqslant i-1)$.

Note. (i) If $i = 1$ then $L_1 = b_1$ and $\underline{L}_1 = e_1$ are the only two classes defined.
(ii) [19] proved $Q_{j,i}$ is integral, not merely rational, mod p. See appendix for proof.

$\mathfrak{S}_{p^i}$ can be thought of as the permutations of the point set $\Pi^i Z/p$. Let $k_{i,i}$: $T_{i,i} = \times^i Z/p \to \{\text{permutations of } \Pi^i Z/p\}$ be defined by: $k_{i,i}(a_1, \ldots, a_i)$ sends $(b_1, \ldots, b_i)$ to $(a_1 + b_1, \ldots, a_i + b_i)$ where Z/p is written additively. Then $k_{i,i}$ is seen to be equivalent to the adjoint representation (2.5) and includes $T_{i,i}$ in $\mathfrak{S}_{p^i}$. The normalizer N of $k_{i,i}(T_{i,i})$ in $\mathfrak{S}_{p^i}$ maps onto $\mathrm{GL}(i, Z/p)$ (2.10) and induces an action on $H^*(T_{i,i})$ as follows. If $\cup_x$ in $\mathrm{GL}(i, Z/p)$ represents the coset $xT_{i,i}$ in N then the homomorphism ad_x: $H^*(T_{i,i}) \to H^*(T_{i,i})$ operates as follows: $\mathrm{ad}_x(e_m) = \cup_x\, e_m$, $\hat{\mathrm{ad}}_x(b_m) = \cup_x\, b_m$ where e_m, b_m are treated as the vectors $(0, \ldots, e, \ldots, 0)$ and $(0, \ldots, b, \ldots, 0)$ in $H^*(T_{i,i})$ with nonzero entries in the mth place. Hence ad_x operates on the above determinant classes via the determinant function; that is, $\mathrm{ad}_x(L_i) = \det(\cup_x\, L_i)$. By 2.13 image $k_{i,i}^*$ is contained in $H^*(T_{i,i})^{\mathrm{GL}(i,Z/p)}$.

Let $\mathscr{W}_1$ be the algebra $E(\underline{L}_1 L_1^{p-2}) \otimes P(L_1^{p-1})$. For i greater than 1 let $\mathscr{W}_i$ be the subalgebra of $H^*(T_{i,i})$ generated by: 1, L_i^{p-1}, $Q_{j,i}$, $\underline{L}_i L_i^{p-2}$, $M_{j,i}L_i^{p-2}$, $M_{j,i}\underline{L}_i L_i^{p-3}$, $M_{j,i}M_{h,i}L_i^{p-3}$ with $1 \leqslant j,\ h \leqslant i-1$ and $j < h$. $\mathscr{W}_i$ is contained in $H^*(T_{i,i})^{\mathrm{GL}(i,Z/p)}$ (2.12). Then $\mathscr{W}_i$ contains the polynomial algebra $P(L_i^{p-1}, Q_{1,i}, Q_{2,i}, \ldots, Q_{i-1,i})$ and all other generators of $\mathscr{W}_i$ are exterior. However the algebra they generate is not an exterior subalgebra as there are zero products. The multiplication of these exterior products is determined by the relations:

(1) $\underline{L}_i^2 = M_{j,i}^2 = 0$, $1 \leqslant j \leqslant i-1$,

(2) $\underline{L}_i M_{1,i} M_{2,i} \cdots M_{i-1,i} \neq 0$.

For example $(M_{2,i}M_{3,i}L_i^{p-3})(M_{2,i}M_{5,i}L_i^{p-3}) = 0$.

THEOREM A. image $k_{i,i}^* \cong \mathscr{W}_i$.

EXAMPLES. (i) If $i = 1$ then $0 \to H^*(\mathfrak{S}_p) \to^{k_{1,1}^*} H^*(Z/p)$ where $H^*(Z/p) \cong E(\underline{L}_1) \otimes P(L_1)$ and $H^*(\mathfrak{S}_p) \cong E(\underline{L}_1 L_1^{p-2}) \otimes P(L_1^{p-1})$.

(ii) If $i = 2$ the results of [4] are obtained.

(iii) Let $p = 3$, $i = 3$ then $k_{3,3}^*$: $H^*(\mathfrak{S}_{27}) \to H^*(Z/3 \times Z/3 \times Z/3)$ and image $k_{3,3}^*$ is generated by:

(1) polynomial generators

$$L_3^2 = \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \end{vmatrix}^2, \qquad Q_{1,3} = \frac{\begin{vmatrix} b_1^{27} & b_2^{27} & b_3^{27} \\ b_1^9 & b_2^9 & b_3^9 \\ b_1 & b_2 & b_3 \end{vmatrix}}{L_3},$$

$$Q_{2,3} = \frac{\begin{vmatrix} b_1^{27} & b_2^{27} & b_3^{27} \\ b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \end{vmatrix}}{L_3}.$$

(2) exterior generators

$$M_{1,3}M_{2,3} = \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1 & b_2 & b_3 \\ e_1 & e_2 & e_3 \end{vmatrix} \begin{vmatrix} b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \\ e_1 & e_2 & e_3 \end{vmatrix},$$

$M_{1,3}\underline{L}_3$, $M_{1,3}L_3$, $M_{2,3}\underline{L}_3$, $M_{2,3}L_3$, $\underline{L}_3L_3$.

(3) the relations that any product of exterior generators is zero except

(a) $(M_{1,3}M_{2,3})(\underline{L}_3L_3) = -(M_{1,3}\underline{L}_3)(M_{2,3}L_3) = (M_{2,3}\underline{L}_3)(M_{1,3}L_3)$,

(b) $(M_{1,3}L_3)(\underline{L}_3L_3) = (M_{1,3}\underline{L}_3)L_3^2$,

(c) $(M_{2,3}L_3)(\underline{L}_3L_3) = (M_{2,3}\underline{L}_3)L_3^2$,

(d) $(M_{1,3}L_3)(M_{2,3}L_3) = (M_{1,3}M_{2,3})L_3^2$.

The proof of Theorem A depends, in part, on [17] and a counting argument. As noted above the classes in image $k_{i,i}^*$ are $\mathrm{GL}(i, Z/p)$ invariant. A calculation and [8] show $P(b_1, \ldots, b_i)^{\mathrm{GL}(i,Z/p)}$ is isomorphic to the polynomial subalgebra of image $k_{i,i}^*$. For $i = 2$, [4] shows

$$(E(e_1, e_2) \otimes P(b_1, b_2))^{\mathrm{GL}(2,Z/p)} \cong H^*(Z/p \times Z/p)^{\mathrm{GL}(2,Z/p)} \cong \text{image } k_{2,2}^*.$$

If $p \geqslant 5$, $i \geqslant 3$ then $(E(e_1, \ldots, e_i) \otimes P(b_1, \ldots, b_i))^{\mathrm{GL}(i,Z/p)}$ properly contains image $k_{i,i}^*$; for example, $M_{1,i}M_{2,i}\underline{L}_iL_i^{p-4}$ is not in image $k_{i,i}^*$. For $p = 3$, $i \geqslant 3$, it is unknown if image $k_{i,i}^*$ equals the ring of invariants.

Consider the inclusion $\times_{m=1}^{p}(\mathfrak{S}_{p^{i-1}})_m \to^{I_{i-1}} \mathfrak{S}_{p^i}$ where $(\mathfrak{S}_{p^{i-1}})_m$ permutes the p^{i-1} letters $((m-1)p^{i-1} + 1, \ldots, mp^{i-1})$. Then let $k_{i-1,i}: T_{i-1,i} \to \mathfrak{S}_{p^i}$ be the composition $I_{i-1}(\times_{m=1}^{p}(k_{i-1,i-1})_m)$. More generally let $k_{j,i}: T_{j,i} \to \mathfrak{S}_{p^i}$ be the composition $I_j(\times_{m=1}^{p^{i-j}}(k_{j,j})_m)$ where I_j is the inclusion $\times_{m=1}^{p^{i-j}}(\mathfrak{S}_{p^j})_m \to \mathfrak{S}_{p^i}$ given by letting $(\mathfrak{S}_{p^j})_m$ permute the mth block of p^j letters.

Let $1 \leqslant j \leqslant i$, then $\mathfrak{S}_{p^{i-j}}$ operates on $T_{j,i}$ and on the algebra $\otimes_{m=1}^{p^{i-j}}(\mathcal{W}_j)_m$ contained in $H^*(T_{j,i}) \cong \otimes_{m=1}^{p^{i-j}}(H^*(\times^j Z/p))_m$ by permuting the p^{i-j} copies of $\times^j Z/p$.

THEOREM B. *For* $1 \leqslant j \leqslant i$ image $k_{j,i}^*$ *is isomorphic to the algebra of* $\mathfrak{S}_{p^{i-j}}$ *invariant classes of* $\otimes_{m=1}^{p^{i-j}}(\mathcal{W}_j)_m$.

Notation. Let $u_m \in (\mathcal{W}_j)_m$ then $\mathfrak{S}\langle u_1, u_2, \ldots, u_{p^{i-j}}\rangle$ is the $\mathfrak{S}_{p^{i-j}}$ invariant class generated by $u_1u_2 \cdots u_{p^{i-j}}$ (u_m is allowed to be $1 \in H^0(\times^j Z/p)$). If u_1 is odd dimensional then $\mathfrak{S}\langle u_1, u_1, \ldots, u_{p^{i-j}}\rangle = 0$.

EXAMPLES. (i) image $k_{1,1}^*$ is generated by:

$$A_{k,i} = \sum_{m=1}^{p^{i-1}} \left(\underline{L}_1 L_1^{(p-2)+k(p-1)}\right)_m$$
$$= \mathfrak{S}\langle\left(\underline{L}_1 L_1^{(p-2)+k(p-1)}\right), 1, \ldots, 1\rangle, \quad \text{for } 0 \leqslant k \leqslant p^{i-1} - 1,$$

and

$$B_{k,i} = \sum (L_1^{p-1})_{m_1}(L_1^{p-1})_{m_2} \cdots (L_1^{p-1})_{m_k},$$

where $1 \leqslant k \leqslant p^{i-1}$ and the sum runs over all sequences $1 \leqslant m_1 < m_2$

$< \cdots < m_k \leqslant p^{i-1}$. Thus $B_{k,i} = \mathbb{S}\langle L_1^{p-1}, L_1^{p-1}, \ldots, L_1^{p-1}, 1, \ldots, 1\rangle$ where L_1^{p-1} appears k times.

(ii) Let $p = 3$, then $k_{2,3}^*$: $H^*(\mathbb{S}_{27}) \to H^*(T_{2,3})$ and image $k_{2,3}^*$ is generated by:

$\mathbb{S}\langle \text{ext}, 1, 1\rangle$ $\quad$ $\mathbb{S}\langle \text{poly}, 1, 1\rangle$ $\quad$ $\mathbb{S}\langle \text{ext}, \text{poly}, 1\rangle$

$\mathbb{S}\langle M_{1,2}\underline{L}_2, M_{1,2}\underline{L}_2, M_{1,2}\underline{L}_2\rangle$ $\quad$ $\mathbb{S}\langle \text{ext}, \text{poly}, \text{poly}\rangle$

$\mathbb{S}\langle \text{poly}, \text{poly}, 1\rangle$ $\quad$ $\mathbb{S}\langle \text{poly}, \text{poly}, \text{poly}\rangle$ $\quad$ $\mathbb{S}\langle \text{ext}, \text{ext}, \text{poly}\rangle$

where

(a) ext runs through $M_{1,2}\underline{L}_2$, $M_{1,2}L_2$, and $\underline{L}_2L_2$.

(b) poly runs through L_2^2 and $Q_{1,2}$.

(c) As $M_{1,2}L_2$ and $\underline{L}_2L_2$ are odd dimensional neither can appear twice in any $\mathbb{S}\langle -, -, -\rangle$. For example $\mathbb{S}\langle \underline{L}_2L_2, \underline{L}_2L_2, 1\rangle = 0$. Note that $\mathbb{S}\langle M_{1,2}\underline{L}_2, 1, 1\rangle$ has height 3 while $\mathbb{S}\langle M_{1,2}L_2, 1, 1\rangle$ is exterior.

(iii) In image $k_{2,i}^*$ the classes

$$\mathbb{S}\langle M_{1,2}\underline{L}_2L_2^{p-3}, 1, \ldots, 1\rangle$$

and

$$\mathbb{S}\langle (M_{1,2}\underline{L}_2L_2^{p-3})_1, \ldots, (M_{1,2}\underline{L}_2L_2^{p-3})_p, 1, \ldots, 1\rangle$$

have height p while $\mathbb{S}\langle M_{1,2}\underline{L}_2L_2^{p-3}, \ldots, M_{1,2}\underline{L}_2L_2^{p-3}\rangle$ is exterior. This pattern generalizes to image $k_{j,i}^*$, $3 \leqslant j \leqslant i-1$, in the obvious way.

Note. Example (iii) shows how all even dimension exterior generators in $\mathcal{W}_j$ build classes in $H^*(T_{j,i})$ which are the images under $k_{j,i}^*$ of classes $u \in H^*(\mathbb{S}_{p^i})$ where each u generates a truncated polynomial algebra of height p in $H^*(\mathbb{S}_n)$. These are the truncated polynomial algebras described in **[22]**.

Let $u \in H^*(\mathbb{S}_{p^i})$ then $k_i^*(u) = (k_{1,i}^*(u), \ldots, k_{i,i}^*(u))$ and the algebra structure restricted to these detecting groups is compatible with component-wise projection. Clearly to calculate $H^*(\mathbb{S}_{p^i})$ we must know when a class $u \in H^*(\mathbb{S}_{p^i})$ has nontrivial image under more than one $k_{i,j}^*$.

DEFINITION. *$u \in H^*(\mathbb{S}_{p^i})$ is a multiple image class if and only if $k_{j,i}^*(u) \neq 0$ for at least two different values of j.*

Given $u_1, u_2 \in H^*(\mathbb{S}_{p^i})$ with u_1 detected only by $k_{j_1,i}^*$ and u_2 detected only by $k_{j_2,i}^*$ with $j_1 \neq j_2$ then $u_1 + u_2$ is a multiple image class. However this type of multiple image class is decomposable as a sum of classes and thus is a "trivial" multiple image class. The next three definitions and following theorem give all "nontrivial"; i.e., indecomposable, multiple image classes.

DEFINITION. *$\mathcal{M}_i$ is the subalgebra contained in $\mathcal{W}_i$ generated by* 1, *$M_{g,i}M_{h,i}L_i^{p-3}$, $Q_{h,i}$, $1 \leqslant g, h \leqslant i-1, g < h$.*

DEFINITION. *Given $x_{m,j} \in \mathcal{M}_j$ we define $x_{m,j-1} \in \mathcal{W}_{j-1}$ as follows*:

(a) *If $x_{m,j} = 1$ then $x_{m,j-1} = 1$.*

(b) *If* $x_{m,j} = Q_{h,j}$ *then* $x_{m,j-1} = Q_{h-1,j-1}$, *for* $2 \leqslant j \leqslant i$ *and* $1 \leqslant h \leqslant j-1$ *with the convention* $Q_{0,j-1} = L_{j-1}^{p-1}$.

(c) *If* $x_{m,j} = M_{g,j}M_{h,j}L_j^{p-3}$ *then* $x_{m,j-1} = -M_{g-1,j-1}M_{h-1,j-1}L_{j-1}^{p-3}$, *for* $3 \leqslant j \leqslant i$, $0 < g, h < j$ *and* $g < h$ *with the convention* $M_{0,j-1} = \underline{L}_{j-1}$.

(d) *If* $x_{m,j} = x'_{m,j}x''_{m,j}$ *then* $x_{m-1,j-1} = x'_{m-1,j-1}x''_{m-1,j-1}$.

Note. (a) through (d) define a unique class $x_{m,j-1}$ for every $x_{m,j} \in \mathfrak{M}_j$.

DEFINITION. $u \in H^*(\mathbb{S}_n)$ *is sum indecomposable if and only if* $u = u_1 + u_2$ *for* $u_1, u_2 \in H^*(\mathbb{S}_n)$ *implies* u_1 *or* u_2 *is zero.*

THEOREM C. *Suppose* $u \in H^*(\mathbb{S}_{p^i})$ *is both sum indecomposable and a multiple image class. Further suppose j is the largest integer such that* $k_{j,i}^*(u) \neq 0$. *Then*

$$k_{j,i}^*(u) = \mathbb{S}\langle x_{1,j}, \ldots, x_{p^{i-j},j}\rangle$$

with $x_{m,j} \in \mathfrak{M}_j$ *for* $1 \leqslant m \leqslant p^{1-j}$, *and*

$$k_{j-1,i}^*(u) = \mathbb{S}\langle x_{1,j-1}, \ldots, x_{1,j-1}, \ldots, x_{p^{i-j},j-1}, \ldots, x_{p^{i-j},j-1}\rangle$$

where each $x_{m,j-1}$ *is as defined above and appears p times in* $k_{j-1,i}^*(u)$. *If* $j-1 \geqslant 2$ *and each* $x_{m,j-1} \in \mathfrak{M}_{j-1}$ *(not just* $\mathfrak{W}_{j-1}$*) then* $k_{j-2,i}^*(u) \neq 0$ *and may be obtained from* $k_{j-1,i}^*(u)$ *precisely as* $k_{j-1,i}^*(u)$ *was obtained from* $k_{j,i}^*(u)$. *In fact this iteration continues r times until either* $j - r = 2$ *or* $x_{m,j-r} \notin \mathfrak{M}_{j-r}$ *when* $k_{j-(r+t),i}^*(u) = 0$ *for all* $t > 0$. *Thus u has* $r + 1$ *nontrivial images in the detecting groups*: $k_{j-s,i}^*$ *for* $0 \leqslant s \leqslant r$.

EXAMPLE. For $H^*(\mathbb{S}_{27}, Z/3)$ the only sum-indecomposable multiple image classes of k_0^* occurring as generators in the examples after Theorems A and B are:

$(B_9, (Q_{1,2})_1(Q_{1,2})_2(Q_{1,2})_3, Q_{2,3})$,
$(0, (L_2^2)_1(L_2^2)_2(L_2^2)_3, Q_{1,3})$,
$(0, (M_{1,2}\underline{L}_2)_1(M_{1,2}\underline{L}_2)_2(M_{1,2}\underline{L}_2)_3, -M_{1,3}M_{2,3})$,
$(B_3, \mathbb{S}\langle Q_{1,2}, 1, 1\rangle, 0)$,
$(B_6, \mathbb{S}\langle Q_{1,2}, Q_{1,2}, 1\rangle, 0)$.

Consider u_1u_2 in $H^*(\mathbb{S}_{p^3})$ where $k_3^*(u_1) = (\mathbb{S}\langle L_1^{p-1}, 1, \ldots, 1\rangle, 0, 0)$ and $k_3^*(u_2) = (0, \mathbb{S}\langle L_2^{p-1}, 1, \ldots, 1\rangle, 0)$. Then $k_3^*(u_1u_2) = 0$ but in fact $u_1u_2 \neq 0$ in $H^*(\mathbb{S}_{p^3})$ and u_1u_2 is detected by subgroups of the form $T_1 \times T_2 \times \cdots \times T_p$ where $T_n = T_{1,2}$ or $T_{2,2}$ and both $T_{1,2}$ and $T_{2,2}$ must occur at least once. These detecting groups are included in $\mathbb{S}_{p^3}$ through $\times^p(\mathbb{S}_{p^2})$. More generally a nonsymmetric detecting group, $\times_{n=1}^p(\times_{m=1}^t(T_{r_m,s_m}))_n$ of $\mathbb{S}_{p^i}$ is a product of detecting groups of $\mathbb{S}_{p^{i-1}}$ included in $\mathbb{S}_{p^i}$ through $\times^p(\mathbb{S}_{p^{i-1}})$ where $T_{r_1,s_1} \neq T_{r_2,s_2}$ for some r_1, r_2, s_1 and s_2. These nonsymmetric detecting groups detect all classes $u \in H^*(\mathbb{S}_{p^i})$ not detected by the map k_i^* as stated in Theorem D. First we need

DEFINITION. *Let* $u \in H^*(\mathbb{S}_{p^i})$ *and* $n < p^i$. *Then we have the natural inclusion* $I_{p^i,n}\colon \mathbb{S}_n \hookrightarrow \mathbb{S}_{p^i}$. *We say u restricts nonzero to* $\mathbb{S}_n$ *if and only if* $I^*_{p^i,n}(u) \neq 0$. *For notational convenience we write u for both the class in* $H^*(\mathbb{S}_{p^i})$ *and the restriction in* $H^*(\mathbb{S}_n)$.

THEOREM D. (1) *The classes in* $H^*(\mathbb{S}_{p^i})$ *not detected by* k_i^* *are products of classes that are detected by* k_i^*.

(2) *Let* $u_m \in H^*(\mathbb{S}_{p^i})$. *Suppose* $k_i^*(u_m) \neq 0$, $\Pi^r_{m=1} k_i^*(u_m) = 0$ *and let* n_m *be the smallest power of p such that* u_m *restricts nonzero to* $H^*(\mathbb{S}_{n_m})$. *Then* $\Pi^r_{m=1} u_m \neq 0$ *in* $H^*(\mathbb{S}_{p^i})$ *unless*:

(a) $u_{m_1} = u_{m_2}$ *is an odd dimensional exterior class in* $H^*(\mathbb{S}_{n_{m_1}})$, *for some* $1 \leqslant m_1 < m_2 \leqslant r$.

(b) $u_{m_1} = u_{m_2} = \cdots = u_{m_p}$ *is an even dimensional exterior class in* $H^*(\mathbb{S}_{n_{m_1}})$ *for some* $1 \leqslant m_1 < m_2 < \cdots < m_p \leqslant r$ *or*

(c) $\mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_r}$ *is not contained in* $\mathbb{S}_{p^i}$.

Note. The classes u_{m_1} appearing in condition (b) are the generators for the truncated polynomial algebras described in example (iii) after Theorem B.

Thus every $u \in H^*(\mathbb{S}_{p^i})$ is expressible as a sum of monomials $\Sigma a(u_1, \ldots, u_r) u_1 \otimes \cdots \otimes u_r$ where $a(u_1, \ldots, u_r) \in Z/p$, $u_t \in H^*(\mathbb{S}_{p^i})$ with $k_i(u_t) \neq 0$ for all t.

DEFINITION. $u \in H^*(\mathbb{S}_{p^i})$ *is proper if and only if* $u = \Sigma a(u_1, \ldots, u_r) u_1 \otimes \cdots \otimes u_r$ *with* $k_i^*(u_1 \otimes \cdots \otimes u_r) \neq 0$ *for each monomial in the sum*.

Thus Theorems A through D compute $H^*(\mathbb{S}_{p^i})$ and from this point on we will identify elements of $H^*(\mathbb{S}_{p^i})$ with their image under k_i^*. That is $L_i^{p-1} Q_{j,i} \in H^*(\mathbb{S}_{p^i})$ is the unique proper class $u \in H^*(\mathbb{S}_{p^i})$ such that $k_i^*(u) = (0, \ldots, 0, L_i^{p-1} Q_{j,i})$. Care must be taken with multiple image classes under this identification. Notice, by Theorem C, that $Q_{1,i} \in H^*(\mathbb{S}_{p^i})$ is the unique proper class $u \in H^*(\mathbb{S}_{p^i})$ such that $k_i^*(u) = (0, \ldots, 0, \mathbb{S}\langle L_{i-1}^{p-1}, \ldots, L_{i-1}^{p-1}\rangle, Q_{1,i})$.

Since

$$\mathcal{P}^j(b^{p^k}) = \begin{cases} b^{p^k} & \text{if } j = 0, \\ b^{p^{k+1}} & \text{if } j = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

it is easy to determine the action of the Steenrod algebra $\mathcal{Q}(p)$ on $H^*(\mathbb{S}_{p^i})$. Consider $M_{1,3}L_3$ in $H^{47}(\mathbb{S}_{27}, Z/3)$. Then

$$\mathcal{P}^1 \left\{ \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1 & b_2 & b_3 \\ e_1 & e_2 & e_3 \end{vmatrix} \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right\} = \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1^3 & b_2^3 & b_3^3 \\ e_1 & e_2 & e_3 \end{vmatrix} \begin{vmatrix} b_1^9 & b_2^9 & b_3^9 \\ b_1^3 & b_2^3 & b_3^3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{L}_3 L_3.$$

This computation involved use of the Cartan formula; however, all terms except the first are zero. The next theorem describes the $\mathcal{A}(p)$ action on $\mathcal{W}_i$. Note the polynomial subalgebra of $\mathcal{W}_i$ is closed under the $\mathcal{A}(p)$ action while a class in the ideal generated by the exterior generators of $\mathcal{W}_i$ may be "bocksteined" into the polynomial algebra; e.g., $\beta\mathcal{P}^1(M_{1,i}L_i^{p-2}) = L_1^{p-1}$ for $i > 1$. Using the Cartan formula and the following theorem it is trivial to compute the $\mathcal{A}(p)$ action on all the detecting groups.

THEOREM E. *The following relations and the Cartan formula describe the $\mathcal{A}(p)$ action on $\mathcal{W}_i$.*

(1) $\mathcal{P}^{p^{h-1}}(M_{j,i}M_{h,i}L_i^{p-3}) = M_{j,i}M_{h-1,i}L_i^{p-3}$, *$j > h$ and $M_{0,i} = \underline{L}_i$,*

(2) $\mathcal{P}^{p^{j-1}}(M_{j,i}M_{h,i}L_i^{p-3}) = M_{j-1,i}M_{h,i}L_i^{p-3}$, *$j > h$ and $M_{0,i} = \underline{L}_i$,*

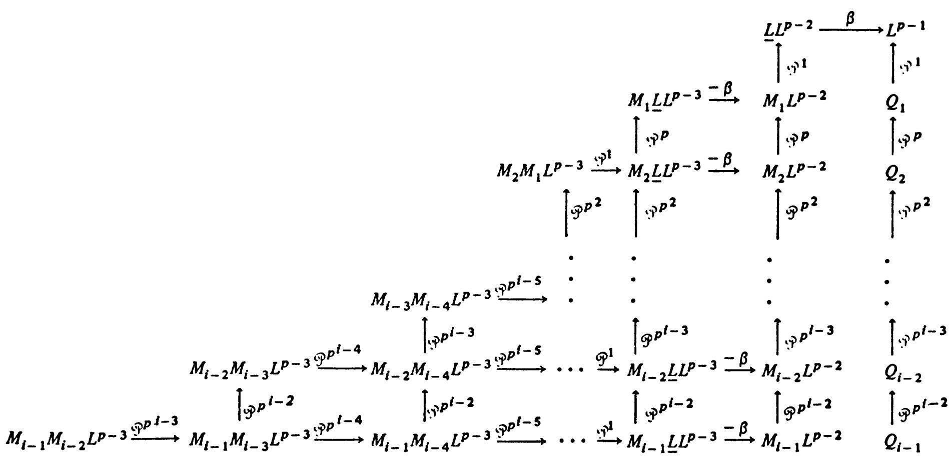
(3) $\beta(\underline{L}_i) = L_i$,

(4) $\mathcal{P}^{p^{h-1}}(Q_{h,i}) = Q_{h-1,i}$, *with* $Q_{0,i} = L_i^{p-1}$,

(5) $\mathcal{P}^{p^{i-1}}(L_i^{p-1}) = -Q_{i-1,i}L_i^{p-1}$ *for $i > 1$ while* $\mathcal{P}^j(L_1^{p-1}) = \binom{p-1}{j}L_1^{(p-1)(j+1)}$ *for* $j \leqslant p-1$.

(6) $\mathcal{P}^{p^{i-1}}(M_{i-1,i}\underline{L}_iL_i^{p-3}) = (p-2)(M_{i-1,i}\underline{L}_iL_i^{p-3})(Q_{i-1,i})$, $\mathcal{P}^{p^{i-1}}(M_{i-1,i}L_i^{p-2}) = (p-2)(M_{i-1,i}L_i^{p-2})(Q_{i-1,i})$.

The following diagram is conceptually helpful.



THE ACTION OF $\mathcal{A}(p)$ ON THE GENERATORS OF W_i

EXAMPLES. (i) Consider $A = (0, \mathfrak{S}\langle M_{1,2}\underline{L}_2, M_{1,2}\underline{L}_2, M_{1,2}\underline{L}_2\rangle, -M_{2,3}M_{1,3})$ in $H^{30}(\mathfrak{S}_{27}, Z/3)$. Then

$$\mathcal{P}^1\beta(A) = (0, -\mathfrak{S}\langle \underline{L}_2L_2, M_{1,2}\underline{L}_2, M_{1,2}\underline{L}_2\rangle, 0)$$

while

$$\beta\mathcal{P}^1(A) = (0, 0, M_{2,3}L_3).$$

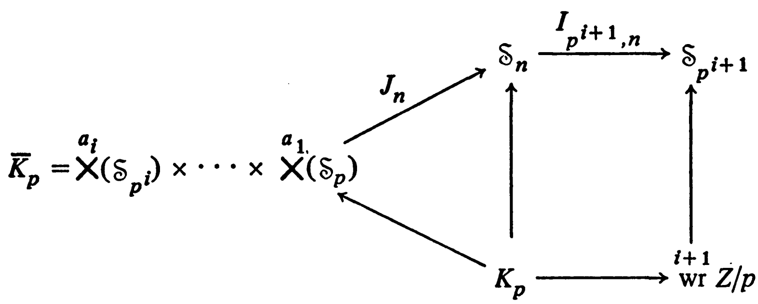
(ii)

$$\mathcal{P}^{p^{i-2}}\mathcal{P}^{p^{i-1}}\left(M_{i-1,i}L_i^{p-2}\right) = \mathcal{P}^{p^{i-2}}\left((p-2)\left(M_{i-1,i}L_i^{p-2}\right)Q_{i-1,i}\right)$$
$$= (p-2)\left[\left(M_{i-2,i}L_i^{p-2}\right)Q_{i-1,i} + \left(M_{i-1,i}L_i^{p-2}\right)Q_{i-2,i}\right].$$

Let n be an arbitrary integer. Then n may be written uniquely as follows: $n = \sum_{j=0}^{i} a_j p^j$ with $0 \leqslant a_j \leqslant p-1$, $a_i \neq 0$. A p-Sylow subgroup K_p of S_n is isomorphic to

$$K_p = \overset{a_i}{\times}\left(\overset{i}{\mathrm{wr}}\ Z/p\right) \times \overset{a_{i-1}}{\times}\left(\overset{i-1}{\mathrm{wr}}\ Z/p\right) \times \cdots \times \overset{a_1}{\times}\ (Z/p).$$

To compute $H^*(\mathbb{S}_n)$ consider the following diagram of inclusions



THEOREM F. (1) *$I^*_{p^{i+1},n}$ is surjective.*

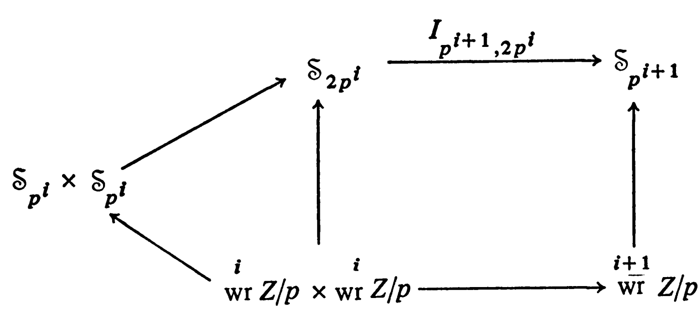
(2) *J^*_n is injective.*

(3) *$v \in$ Image J^*_n if and only if there exists a $u \in H^*(\mathbb{S}_{p^{i+1}})$ such that*

$$(I_{p^{i+1},n} \circ J_n)^*(u) = v$$
$$= \sum \mathbb{S}\langle u_{i,1}, \ldots, u_{i,a_i}\rangle \otimes \cdots \otimes \mathbb{S}\langle u_{1,1} \cdots u_{i,a_1}\rangle \in H^*\left(\bar{K}_p\right)$$

with $u_{t,r} \in H^*(\mathbb{S}_{p^t})$ for each r.

IMPORTANT EXAMPLE. Let $n = 2p^i$. We have



Recall the definition of $A_{k,i}$ and $B_{k,i}$ (see example (i) after Theorem B). Then $I^*_{p^{i+1},2p^i}(A_{k,i+1}) = A_{k,i} \otimes 1 + 1 \otimes A_{k,i} = \mathbb{S}\langle A_{k,i}, 1\rangle$ for $1 \leqslant k \leqslant p^i$, while for

$p^i < k \leq p^{i+1}$, $I^*_{p^{i+1},2p^i}(A_{k,i+1}) = A'_{k,i} \otimes 1 + 1 \otimes A'_{k,i}$ where $A'_{k,i}$ is expressible in terms of $A_{r,i}$ and $B_{r,i}$ for $r < p^i$.

$$I^*_{p^i+p,2p^i}(B_{k,i+1}) = \sum_{n+m=k} B_{n,i} \otimes B_{m,i} = \sum_{n=0}^{p^i} \mathbb{S}\langle B_{n,i}, B_{2p^i-n,i}\rangle,$$

where $0 \leq n, m \leq p^i$, $0 \leq k \leq 2p^i$, and $B_{0,i} = 1$. Similar restrictions occur on the other detecting groups. Thus the natural inclusions $\mathbb{S}_n \to \mathbb{S}_{n+1} \to \cdots \to \text{dir lim } \mathbb{S}_n$ are easily analyzed. Clearly

$$\mathbb{S}_{p^i} \to \mathbb{S}_{p^{i+1}} \to \cdots \to \text{dir lim } \mathbb{S}_{p^i}$$

is a cofinal direct limit and we have $H^*(\text{dir lim } \mathbb{S}_n) \cong H^*(\text{dir lim } \mathbb{S}_{p^i}) \cong \text{inv lim } H^*(\mathbb{S}_{p^i})$. Notice Theorem F implies inv lim $H^t(\mathbb{S}_{p^i})$ is attained for each t at a finite stage.

Recall the theorem stated in the introduction that ties dir lim $B_{\mathbb{S}_n}$ to $Q(S^0) = \text{dir lim } \Omega^n S^n$. Furthermore, if G_n is the set of homotopy equivalences of S^{n-1} then $G = \text{dir lim } G_n$ is homotopy equivalent to the union of the $+1$ and -1 components of $Q(S^0)$. Thus dir lim $B_{\mathbb{S}_n}$ properly interpreted is a model for G and we have:

$$\text{inv lim } H^*(\mathbb{S}_{p^i}) \cong H^*(Q(S^0)_0) \cong H^*(SG)$$

as algebras. Thus $H^*(SG)$ can be identified with "infinite symmetric sums" in the $\mathcal{W}_i$ algebras with the proper identifications; i.e., $\mathbb{S}\langle Q_{j,i}, 1, \ldots\rangle \leftrightarrow \mathbb{S}\langle Q_{j-1,i-1}, \ldots, Q_{j-1,i-1}, 1, \ldots\rangle$. The $\mathcal{Q}(p)$ action on $H^*(SG)$ restricts to that on $B_{\mathbb{S}_{p^i}}$ for each i and there is a unique action which has this property. Theorem E describes the restriction of this action. Recall, **[22]** and **[24]**, $H^*(\text{dir lim } \mathbb{S}_{p^i})$ is a Hopf algebra isomorphic to $H^*(Q(S^0)_0)$ with the coalgebra product on $H^*(\text{dir lim } \mathbb{S}_{p^i})$ induced by the inclusions $\mathbb{S}_{p^i} \times \mathbb{S}_{p^i} \to \mathbb{S}_{2p^i}$. Thus Theorem F gives the loop sum coalgebra map on $H^*(Q(S^0)_0)$.

As $Q(S^0)_0$ is an H-space it is possible to obtain integral information about $H^*(SG, Z, p)$ on $H^*(\mathbb{S}_{p^i}, Z, p)$ (see **[14]**). **[2]** gives a Hopf algebra Bockstein spectral sequence with

$$E_1 \cong H^*(\text{dir lim } \mathbb{S}_{p^i}, Z/p),$$
$$E_\infty \cong H^*(\text{dir lim } \mathbb{S}_{p^i}, Z, p)/\text{Torsion}.$$

Let $x, y \in \mathcal{W}_j$ and let

$$L_{n,j}(x: y_{n+1}, \ldots, y_m, 1, \ldots) = \mathbb{S}\langle xL_j^{p-1}, \ldots, xL_j^{p-1}, y_{n+1}, \ldots, y_m, 1, \ldots\rangle$$

and

$$\underline{L}_{n,j}(x: y_{n+1}, \ldots, y_m, 1, \ldots) = \mathbb{S}\langle x\underline{L}_j L_j^{p-2}, xL_j^{p-1}, \ldots, xL_j^{p-1}, y_{n+1}, \ldots, y_m, 1, \ldots\rangle$$

where $y_r \neq xL_j^{p-1}$ or $x\underline{L}_j L_j^{p-2}$. Note a class in $H^*(\text{dir lim } \mathbb{S}_{p^i})$ may have more than one representation as $L_{n,j}(\cdots)$ or $\underline{L}_{n,j}(\cdots)$; for example,

$$\mathbb{S}\langle xL_j^{p-1}, xL_j^{p-1}, yL_j^{p-1}, 1, \ldots\rangle = L_{2,j}(x{:}\, y, 1, \ldots) = L_{1,j}(y{:}\, x, 1, \ldots).$$

THEOREM G. *Let* $k_{j,\infty}^* = \operatorname{dir}\lim_i k_{j,i}^*$ *and let* $u \in H^*(\operatorname{dir}\lim \mathbb{S}_{p^i})$ *be a proper class. Then there exists a smallest positive integer j such that* $k_{j,\infty}^*(u) \neq 0$. *Then* $k_{j,\infty}^*(u) = \mathbb{S}\langle x_1, \ldots, x_m, 1, \ldots\rangle$ *and*

(1) *If some* x_n *contains an odd number of* $M_{g,j}$ *factors or if* $k_{j,\infty}^*(u) = \underline{L}_{n,j}(\cdots)$ *or* $L_{n,j}(\cdots)$ *for n not divisible by p then u is in the image or domain of* β_p.

(2) *Let* $r \geq 2$. *If* $d_{r-1}(v) = u$ *in* E_{r-1} *of the Bockstein spectral sequence and* $k_{j,\infty}^*(u) = \mathbb{S}\langle x_1, \ldots, x_m, 1, \ldots\rangle$ *with no* x_n *containing an odd number of* $M_{g,h}$ *terms or the factor* $\underline{L}_j$ *then there exist* v' *and* u' *such that* $d_r(v') = u'$ *where* $k_{j,\infty}^*(u') = \mathbb{S}\langle x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m, 1, \ldots\rangle + \Sigma u''$. *Each* x_h *appears p times in* $\mathbb{S}\langle x_1, \ldots, x_1, \ldots, x_m, \ldots, x_m, 1, \ldots\rangle$ *and each* $u'' = \mathbb{S}\langle x_1, \ldots, x_t, 1, \ldots\rangle$ *with* $t < pm$.

COROLLARY 1. *Let* $r \geq 2$ *then*

$$d_r\big(\underline{L}_{p^{r-1},j}(x{:}\, 1, \ldots)\big) = L_{p^{r-1},j}(x{:}\, 1, \ldots)$$

where x satisfies the same conditions as the x_n*'s in* (2) *of Theorem* G.

Let R_i be the inclusion $\mathbb{S}_{p^i} \to \operatorname{dir}\lim \mathbb{S}_{p^i}$ then R_i^* gives the Bockstein structure of $H^*(\mathbb{S}_{p^i}, Z, p)$.

COROLLARY 2. $Q_{j,i} \in H^*(\mathbb{S}_{p^i}, Z, p)$ *has order* p^{j+1}.

EXAMPLES. (i) $L_{p^r,j}(M_{1,j}\underline{L}_j L_j^{p-3}{:}\, 1, \ldots)$ is a class of order p in $H^*(SG, Z, p)$, while $L_{p^r,j}(M_{1,j}M_{2,j}L_j^{p-3}{:}\, 1, \ldots)$ is a class of order p^{r+1}.

(ii) $(B_6, \mathbb{S}\langle Q_{1,2}, Q_{1,2}, 1\rangle, 0) \in H^{24}(\mathbb{S}_{27}, Z, 3)$ has order 9.

Finally the results of this paper have an application to cobordism theory. Although [3], [13] and [18] completely compute the PL and TOP cobordism ring at the prime 2, the odd case still has unanswered questions, notably the odd torsion in Ω^{PL}. Using results of [3], [15], [26], [27], [32], [34], [37], [38], [39] and this paper one may calculate the E^2 term of the Adams spectral sequence converging to $\Omega^{PL} \otimes Z_{(p)}$. Current joint work with H. Ligaard, J. P. May and R. J. Milgram computes this E^2 term and gives infinite families of nontrivial differentials of all orders in the spectral sequence.

II. The embedding and the detecting family.

2.1. DEFINITION. *Let K be a finite group and L a subgroup of* $\mathbb{S}_n$ *then K* wr *L is defined to be the group whose elements are*

$$\{(f, g){:}\ f \text{ is a mapping of } (1, 2, \ldots, n) \text{ into } K, g \in L\}$$

and whose multiplication is given by $(f, g)(f', g') = (ff'_g, gg')$, *where* $f_g(g(i)) = f(i)$ *and* $ff'(i) = f(i)f'(i)$.

2.2. Definition. *Let X be a space and $\{A_i\}$ a collection of subspaces of X. $\{A_i\}$ is a Z/p cohomology detecting family for X if the inclusion map $H^*(X) \to \prod H^*(A_i)$ is an injection.*

2.3. Lemma. *Let K_p be a p-Sylow subgroup of K, then the transfer $t(K, K_p)$: $H^*(K_p) \to H^*(K)$ is an epimorphism and the inclusion $i(K_p, K)$: $H^*(K) \to H^*(K_p)$ is a monomorphism whose image consists of stable elements of $H^*(K_p)$. Furthermore we have the direct sum decomposition* $H^*(K_p) \cong \operatorname{Im} i(K_p, K) \oplus \operatorname{Ker} t(K, K_p)$.

Proof. See [5, Chapter XII, p. 257] for the definition of stable and p. 259 for a proof of the lemma.

Recalling that a p-Sylow subgroup of $\mathfrak{S}_{p^i}$ is isomorphic to $\mathrm{wr}^i Z/p$, [6] gives

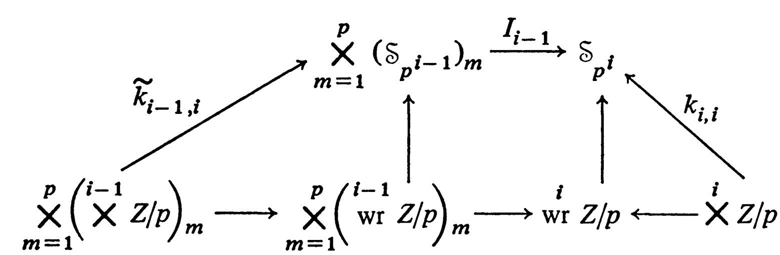
2.4. Corollary. *If $\{A_j\}$ is a Z/p detecting family for $\mathrm{wr}^i Z/p$ then it is one for $\mathfrak{S}_{p^i}$ also.*

2.5. Definition. *Let G be a finite group of order n. Then the adjoint representation A: $G \to \mathfrak{S}_n$ is defined as follows: Let $A(g)$ be the permutation $\{g_i \mapsto gg_i\}$ where $\mathfrak{S}_n$ is thought of as the permutations on the n elements of G.*

The adjoint representation is obviously a monomorphism and includes G in $\mathfrak{S}_n$. Let $G = \times^i Z/p$, then the adjoint representation of $\times^i Z/p$ in $\mathfrak{S}_{p^i}$ is clearly equivalent to the map $k_{i,i}$: $\times^i Z/p \to \mathfrak{S}_{p^i}$ defined in §I. (The two maps differ by at most a reordering of the elements of $\times^i Z/p$; that is, an inner automorphism of $\mathfrak{S}_{p^i}$.)

Again considering $\mathfrak{S}_{p^i}$ as the permutations on the set $\prod^i Z/p$ the map I_{i-1}: $\times_{m=1}^{p} (\mathfrak{S}_{p^{i-1}})_m \to \mathfrak{S}_{p^i}$ defined in the introduction is realized by letting $(\mathfrak{S}_{p^{i-1}})_m$ permute the set $\prod^{i-1} Z/p \times \{m\}$ contained in $\prod^i Z/p$.

Note that under the specific embeddings $k_{i,i}$ and I_{i-1} the subgroup $\times^{i-1} Z/p \times \{0\} \to \times^i Z/p \to^{k_{i,i}} \mathfrak{S}_{p^i}$ is contained in the subgroup $\times_{m=1}^{p} (\mathfrak{S}_{p^{i-1}})_m \to^{I_{i-1}} \mathfrak{S}_{p^i}$. Any p-Sylow subgroup of $\times_{m=1}^{p} (\mathfrak{S}_{p^{i-1}})_m$ that contains $\times^{i-1} Z/p \times \{0\}$ is isomorphic to $\times_{m=1}^{p} (\mathrm{wr}^{i-1} Z/p)_m$. Then $\times_{m=1}^{p} (\mathrm{wr}^{i-1} Z/p)_m$ and $\times^i Z/p$ generate a p-Sylow subgroup of $\mathfrak{S}_{p^i}$ which must be isomorphic to $\mathrm{wr}^i Z/p$. Thus we have the following commutative diagram with the above mentioned inclusions:



where $\tilde{k}_{i-1,i} = \times_{m=1}^{p}(k_{i-1,i-1})_m$. The specific form of $k_{i,i}$ and $\tilde{k}_{i-1,i}$ guarantees $\times_{m=1}^{p}(\times^{i-1}Z/p)_m$ factors through $\times_{m=1}^{p}(\mathrm{wr}^{i-1}Z/p)_m$.

More generally if $I_{m_1,\dots,m_n}\colon \mathfrak{S}_{p^{m_1}} \times \cdots \times \mathfrak{S}_{p^{m_n}} \to \mathfrak{S}_{p^i}$ is defined by letting $\mathfrak{S}_{p^{m_r}}$ permute the p^{m_r} letters $(p^{m_1} + \cdots + p^{m_{r-1}} + 1, \dots, p^{m_1} + \cdots + p^{m_r})$ then the map $I_{m_1,\dots,m_n} \circ (\prod_{r=1}^{n} k_{m_r,m_r})$ includes $\prod_{r=1}^{n}(\times^{m_r}Z/p)$ in $\mathfrak{S}_{p^i}$.

If $m_1 = m_2 = \cdots = m_{p^{i-j}} = j$ then $\prod_{r=1}^{p^{i-j}}(\times^j Z/p) \to \mathfrak{S}_{p^i}$ has the form

$$k_{j,i} = I_{j,\dots,j} \circ \prod_{r=1}^{p^{i-j}} (k_{j,j})_r\colon \times^{p^{i-j}} \left(\times^{j} Z/p\right) \to \mathfrak{S}_{p^i}.$$

2.6. Definition. *Let* $T_{j,i} = \times^{p^{i-j}}(\times^j Z/p)$. *Let* $k_{j,i}\colon T_{j,i} \to \mathfrak{S}_{p^i}$ *be the above inclusion. Then* $T_{j,i}$ *is called a totally symmetric detecting group.*

Notice $T_{j,i}$ and $k_{j,i}$ are defined for $1 \leq j \leq i$. The following lemmas are established in the proofs of Theorems A through D:

2.7. Lemma. *The set* $\{I_{m_1,\dots,m_n} \circ (\prod_{r=1}^{n}(k_{m_r,m_r}))\colon \prod_{r=1}^{n} \times^{m_r}(Z/p) \to \mathfrak{S}_{p^i}\}$ *forms a* Z/p *detecting family for* $\mathfrak{S}_{p^i}$.

2.8. Lemma. *The totally symmetric detecting groups* $T_{j,i}$, $1 \leq j \leq i$, *detect a set of multiplicative generators for* $H^*(\mathfrak{S}_{p^i})$. *(This is the first part of Theorem* D.)

2.9. Lemma. *In* Z/p *cohomology*, $\operatorname{Ker} k_{i,i}^* \cap \operatorname{Ker} I_{i-1}^* = 0$.

These lemmas may be proved directly using [27], induction on i, and 3.1.

We now examine the normalizers of the detecting subgroups in $\mathfrak{S}_{p^i}$. Consider $k_{i,i}\colon T_{i,i} \to \mathfrak{S}_{p^i}$. Let $a_r \in \mathfrak{S}_{p^i}$ generate $k_{i,i}(0 \times 0 \times \cdots \times (Z/p)_r \times \cdots \times 0)$ and let N_i be the normalizer of $k_{i,i}(T_{i,i})$ in $\mathfrak{S}_{p^i}$. Define a homomorphism $\psi\colon N_i \to \mathrm{GL}(i, Z/p)$ as follows: If $x \in N_i$ then $xa_rx^{-1} = a_1^{s_{1,r}}a_2^{s_{2,r}} \cdots a_i^{s_{i,r}}$. Then let $\psi(x)$ be the matrix whose (m, n)th entry is $s_{m,n}$. Clearly $\psi(x)$ is nonsingular.

2.10. Proposition. *The sequence* $1 \to k_{i,i}(T_{i,i}) \to N_i \to^{\psi} \mathrm{GL}(i, Z/p) \to 1$ *is exact.*

Proof. Preceding $k_{i,i}$ by any automorphism $\varphi\colon T_{i,i} \to T_{i,i}$ is just a reordering of the underlying set of $T_{i,i}$. This reordering, considered as an element of $\mathfrak{S}_{p^i}$, conjugates $k_{i,i}$ to $k_{i,i} \circ \varphi$. This implies ψ is onto. The remainder of the proposition follows trivially.

For $x \in \mathfrak{S}_{p^i}$ the homomorphism $\mathrm{ad}_x\colon H^*(T_{i,i}) \to H^*(xT_{i,i}x^{-1})$ is induced by the inner automorphism $y \to xyx^{-1}$. Let $E = \sum_{m=1}^{i} a_m e_m$ and $B = \sum_{m=1} a'_m b_m$ in $H^*(T_{i,i})$ then it follows directly from the definition of ψ that

2.11. Proposition. *For* $x \in N_i$, $\mathrm{ad}_x(E) = \psi(x)E$ *and* $\mathrm{ad}_x(B) = \psi(x)B$.

Since ad_x is a ring homomorphism 2.11 determines ad_x on all of $H^*(T_{i,i})$.

Since the pth power homomorphism, $a \mapsto a^p$, is the identity on Z/p we have $P(x_1^p, \ldots, x_i^p) = (P(x_1, \ldots, x_i))^p$ for all polynomials P. This fact and direct computation yield

2.12. PROPOSITION. ad_x *operates on the classes* L_i, $Q_{j,i}$, $M_{j,i}$, $\underline{L}_i$ *via multiplication by the determinant function.*

2.13. COROLLARY. *The algebra* $\mathcal{U}_i$ *is contained in* $H^*(T_{i,i})^{\mathrm{GL}(i,Z/p)}$.

2.14. LEMMA. *If G is a finite group, K a subgroup, and $N_{K,G}$ the normalizer of K in G then the image of $H^*(G)$ in $H^*(K)$ is contained in $H^*(K)^{N_{K,G}}$.*

PROOF. Any inner automorphism of G induces the identity on $H^*(G)$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} H^*(G) & \xrightarrow{\mathrm{id}} & H^*(G) \\ i(K,G)\downarrow & & \downarrow i(xKx^{-1},G) \\ H^*(K) & \xrightarrow{\mathrm{ad}_x} & H^*(xKx^{-1}) \end{array}$$

Allowing x to run through $N_{K,G}$ gives the lemma.

2.15. COROLLARY. *Let* $u \in H^*(\mathfrak{S}_{p^i})$ *then* $k_{i,i}^*(u) \in H^*(T_{i,i})^{\mathrm{GL}(i,Z/p)}$.

PROOF. Immediate from 2.10 and 2.14.

Let $N_{j,i}$ be the normalizer of $k_{j,i}\colon T_{j,i} \to \mathfrak{S}_{p^i}$ in $\mathfrak{S}_{p^i}$.

2.16. PROPOSITION. *The sequence*

$$1 \to \overset{p^{i-j}}{\times} N_j \to N_{j,i} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathfrak{S}_{p^{i-j}} \to 1$$

is exact.

PROOF. Both $N_{j,i}$ and $\times^{p^{i-j}} N_j$ act on $T_{j,i}$ via conjugation. But $x \in N_{j,i}$ permutes the p^{i-j} orbits of $\times^{p^{i-j}} N_j$. This gives a homomorphism $\varphi\colon N_{j,i} \to \mathfrak{S}_{p^{i-j}}$ which is clearly onto and has an obvious section ψ. Notice $\psi(\varphi(x)^{-1}) \cdot x \in \times^{p^{i-j}} N_j$ as $\psi(\varphi(x)^{-1}) \in N_{j,i}$ and $\psi(\varphi(x)^{-1}) \cdot x \in \times^{p^{i-j}} \mathfrak{S}_{p^j}$. The proposition follows.

Let $N_{m_1, \ldots, m_n}$ be the normalizer of $I_{m_1, \ldots, m_n}(\prod_{r=1}^n (k_{m_r, m_r}))\colon \prod_{r=1}^n (\times^{m_r} Z/p) \to \mathfrak{S}_{p^i}$ in $\mathfrak{S}_{p^i}$ and let $\mathfrak{S}_{(m_1, \ldots, m_n)}$ be the subgroup of $\mathfrak{S}_n$ generated by the transpositions (a, c) where $m_a = m_c$. Minor modification of 2.16 yields the following three propositions.

2.17. PROPOSITION. *The sequence* $1 \to \times_{r=1}^n N_{m_r} \to N_{m_1, \ldots, m_n} \rightleftarrows \mathfrak{S}_{(m_1, \ldots, m_n)} \to 1$ *is exact.*

2.18. PROPOSITION. *Let* $\overline{N}_j$ *be the normalizer of* $I_j\colon \times^{p^{i-j}} \mathfrak{S}_{p^j} \to \mathfrak{S}_{p^i}$ *in* $\mathfrak{S}_{p^i}$. *Then the sequence* $1 \to \times^{p^{i-j}} \mathfrak{S}_{p^j} \to \overline{N}_j \rightleftarrows \mathfrak{S}_{p^{i-j}} \to 1$ *is exact.*

2.19. PROPOSITION. *Let* $\overline{N}_{m_1,\ldots,m_n}$ *be the normalizer of* $I_{m_1,\ldots,m_n}$: $\times_{r=1}^n \mathfrak{S}_{m_r} \to \mathfrak{S}_{p^l}$ *in* $\mathfrak{S}_{p^l}$. *Then the sequence* $1 \to \times_{r=1}^n \mathfrak{S}_{m_r} \to \overline{N}_{m_1,\ldots,m_n} \rightleftarrows \mathfrak{S}_{(m_1,\ldots,m_n)} \to 1$ *is exact.*

2.20. LEMMA. *If* G *is a finite group and* K *a subgroup then* $i(K, G)^*t(G, K) = \Sigma_{x \in G/K} t_x i_x \mathrm{ad}_x$ *where* ad_x: $H^*(K) \to H^*(xKx^{-1})$ *is the homomorphism induced by* $y \mapsto xyx^{-1}$ *for* $y \in K$, i_x *is the inclusion map* $H^*(xKx^{-1}) \to H^*(xKx^{-1} \cap K)$ *and* t_x *is the transfer* $H^*(xKx^{-1} \cap K) \to H^*(K)$.

PROOF. [**5**, XII. 9.1, p. 257].

2.21. PROPOSITION. *If* K *is a proper subgroup of* $\times^m Z/p$ *then the transfer* t: $H^*(K) \to H^*(\times^m Z/p)$ *is zero.*

PROOF. [**4**, I.2.1].

III. Some properties of $\mathcal{Q}(p)$ and the proof of Theorem E. In this section we state facts about the Steenrod algebra needed to prove Theorems A through D and give a proof of Theorem E.

First recall the construction of the Steenrod pth powers ([**31**] gives the complete treatment and we quote it frequently in what follows). Let X be a finite regular cell complex then we have the following spaces and maps:

$$X^p \xrightarrow{j} W_{Z/p} \times_{Z/p} X^p \xleftarrow{1 \times \Delta} W_{Z/p} \times_{Z/p} X = B_{Z/p} \times X$$

where j is the inclusion and Δ is the diagonal map. Given any $u \in H^*(X)$ there exists a unique natural class $\mathcal{P}(u)$ in $H^*(W_{Z/p} \times_{Z/p} X^p)$ such that:

(1) $j^*(\mathcal{P}(u)) = u \otimes \cdots \otimes u = u^{\otimes p}$.

(2) $(1 \times \Delta)^*(\mathcal{P}(u))$ in $H^*(B_{Z/p} \times X)$ can be expanded by the Künneth theorem. $(1 \times \Delta)^*(\mathcal{P}(u)) = \Sigma w_k \otimes D_k(u)$ where w_k generates $H^k(Z/p)$ and D_k: $H^q(X) \to H^{pq-k}(X)$ are homomorphisms which define the elements of $\mathcal{Q}(p)$.

(3) $\beta D_{2k}(u) = D_{2k-1}(u)$, $\beta D_{2k-1}(u) = 0$ and $\beta D_0(u) = 0$.

3.1. THEOREM [**31**]. *If* $z \in H^*(W_{Z/p} \times_{Z/p} X^p)$, *then* z *is of the form* $z = tz_1 + z_2 \cdot \mathcal{P}(z_3)$ *with* $z_1 \in H^*(X^p)$, $z_2 \in H^*(B_{Z/p})$ *and* $z_3 \in H^*(X)$, *where* t *is the transfer. Furthermore the sequence*

$$H^*(X^p) \xrightarrow{t} H^*(W_{Z/p} \times_{Z/p} X^p) \xrightarrow{(1\times\Delta)^*} H^*(B_{Z/p} \times X)$$

is exact.

PROOF. [**31**, VII. 4.1, p. 104 and VIII. 3.6, p. 126].

3.2. DEFINITION [**31**]. *Let* $u \in H^q(X)$ *then*

$$\mathcal{P}^j(u) = a_{j,q} D_{(q-2j)(p-1)}(u),$$

$$\beta \mathcal{P}^j(u) = a_{j,q} D_{(q-2j)(p-1)-1}(u),$$

where $a_{j,q}$ *is a nonzero constant in* Z/p *dependent on* j *and* q. *If* $k \neq (q-2j)(p-1)$ *or* $(q-2j)(p-1)-1$ *for some* j *then* $D_k(u) = 0$.

3.3. PROPOSITION. *If* q *is even, say* $q = 2n$, *then* $a_{j,2n} = (-1)^{j+n}$.

PROOF. Follows directly from [**31**, VII. 6.1 and VII. 6.3] (note correction of the formula in VII. 6.1 on the first page of the appendix to [**31**]).

The following is well known:

3.4. LEMMA. I. *Let* p *be a prime and* $a = \sum_{i=0}^{m} a_i p^i$, $c = \sum_{i=0}^{m} c_i p^i$ $(0 \leq a_i, c_i \leq p-1)$. *Then*

$$\binom{c}{a} \equiv \prod_i \binom{c_i}{a_i} \pmod{p}.$$

II. $\mathcal{P}^j(e) = 0$ *for all* $j > 0$.
III. $\mathcal{P}^j(b^k) = \binom{k}{j} b^{k+(p-1)j}$.
IV. (*Cartan formula*) $\mathcal{P}^j(uv) = \sum_{m+n=j} \mathcal{P}^m(u)\mathcal{P}^n(v)$.
V.

$$\mathcal{P}^j(b^{p^m}) = \binom{p^m}{j} b^{p^m+(p-1)j} = \begin{cases} b^{p^m} & \text{if } j = 0, \\ b^{p^{m+1}} & \text{if } j = p^m, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. [**31**, see I.2.6, V. 1, VII. 2.2 and VI. 2.3].

The proof of Theorem E follows from direct calculation and Lemma 3.4. Note: To prove relation (4) of Theorem E, just expand $\mathcal{P}^{p^{k-1}}(Q_{k,i}L_i^{p-1})$.

IV. Symmetric products and image $k_{i,i}^*$. In this chapter we summarize results of [**17**] which give $H^*(\mathfrak{S}_n)$ as Z/p vector spaces and give an upper bound on the size of image $k_{i,i}^*$.

Recall the monomial $\mathcal{P}^I = \beta^{\varepsilon_k}\mathcal{P}^{s_k} \cdots \beta^{\varepsilon_1}\mathcal{P}^{s_1} \in \mathcal{A}(p)$ is called admissible if $s_i \geq ps_{i-1} + \varepsilon_{i-1}$ for each $i \geq 1$, and the excess of $\mathcal{P}^I = 2s_k + \varepsilon_k - \sum_{j=1}^{k-1}(2s_j(p-1) + \varepsilon_j)$. The excess of any admissible monomial is nonnegative. Let $\mathcal{A}(p)_n$ be the subvector space of $\mathcal{A}(p)$ spanned by those monomials of excess $< n$.

Let $\mathrm{SP}^k(S^{2n})$ be the k symmetric product of S^{2n} (see [**17**] for the definition and properties of the symmetric products of a space).

4.1. THEOREM [**17**]. (1) $H_*(\mathrm{SP}^k(S^{2n})) = \sum_{m=1}^{k} H_*(\mathrm{SP}^m(S^{2n}), \mathrm{SP}^{m-1}(S^{2n}))$.

(2) $\mathcal{R}(S^{2n}, Z/p) = \sum_{m=1}^{\infty} H_*(\mathrm{SP}^m(S^{2n}), \mathrm{SP}^{m-1}(S^{2n}))$ *is isomorphic to* $H_*(K(Z, 2n))$.

There is a bigrading of $\mathcal{R}(S^{2n}, Z/p)$ *given by*

$$\mathcal{R}_{i,m}(S^{2n}, Z/p) = H_i(\mathrm{SP}^m(S^{2n}), \mathrm{SP}^{m-1}(S^{2n})).$$

(3) *For* $\mathcal{R}(S^{2n}, Z/p)$ *the generators* q_I *in homology are in* 1-1 *correspondence with admissible monomials* $\mathcal{P}^I = \beta^{\varepsilon_l}\mathcal{P}^{s_l} \cdots \beta^{\varepsilon_1}\mathcal{P}^{s_1}$ *in* $\mathcal{A}(p)_{2n}$ *and the bidegree*

of this generator is $(|\mathcal{P}^I| + 2n,\ p^i)$. *Moreover* $\langle q_I,\ \mathcal{P}^I(i)\rangle = 1$ *under the isomorphism in* (2).

PROOF. [17].

REMARKS. (1) is due to N. E. Steenrod. [8] and [21] also studied (1) and (2).

The next theorem follows from the fact that the singular locus of $(S^{2n})^{p^i}$ under $\mathfrak{S}_{p^i}$ has dimension $2n(p^i - 1)$.

4.2. THEOREM [17]. *For* $k < 2n - 1$, $H^k(\mathfrak{S}_{p^i}) \cong H_{2n(p^i)-k}(\mathrm{SP}^{p^i}(S^{2n}))$.

Since $H_j(\mathrm{SP}^{p^i}(S^{2n})) \cong H_j(\mathrm{SP}^{p^i}(S^{2n}),\ \mathrm{SP}^{p^i-1}(S^{2n}))$ for $j > 2n(p^i - 1) + 1$ we may identify $H^k(\mathfrak{S}_{p^i})$ with elements in $\mathfrak{R}(S^{2n}, Z/p)$ of bidegree $(2n(p^i) - k, p^i)$. Thus for $k < 2n - 1$ classes in $H^k(\mathfrak{S}_{p^i})$ correspond to classes Σa; with each $a \in \mathfrak{R}(S^{2n}, Z/p)$ having bidegree $(-, p^i)$. This gives $H^k(\mathfrak{S}_{p^i})$ as Z/p vector spaces. Recall there are two types of classes in $\mathfrak{R}(S^{2n}, Z/p)$ having bidegree $(-, p^i)$:

(1) a corresponds to $\mathcal{P}^{I_i}$ of bidegree $(|\mathcal{P}^{I_i}| + 2n, p^i)$,

(2) $a = \prod b_k$ where b_k has bidegree $(-, p^j)$, for some $j < i$ and occurs in $H_*(\mathrm{SP}^{p^j}(S^{2n}), \mathrm{SP}^{p^j-1}(S^{2n}))$.

On the other hand the multiplication map M: $\mathrm{SP}^{p^{i-1}}(S^{2n}) \times \cdots \times \mathrm{SP}^{p^{i-1}}(S^{2n}) \to \mathrm{SP}^{p^i}(S^{2n})$ and 4.2 give a map m: $\otimes^p H^*(\mathfrak{S}_{p^{i-1}}) \to H^*(\mathfrak{S}_{p^i})$.

4.3. LEMMA [21]. *m is the transfer map induced by the inclusion*

$$I_{i-1}\colon \overset{p}{\times}\ \mathfrak{S}_{p^{i-1}} \to \mathfrak{S}_{p^i}.$$

PROOF. [21].

4.4. LEMMA. *Let* $u \in H^*(\mathfrak{S}_{p^i})$ *correspond to* $a \in \mathfrak{R}(S^{2n}, Z/p)$. *If a is of type* 2 *then* $k_{i,i}^*(u) = 0$.

PROOF. Suppose a is of type 2 then a is in the image of M_*. By 4.3, u is in the image of the transfer t: $H^*(\times^p \mathfrak{S}_{p^{i-1}}) \to H^*(\mathfrak{S}_{p^i})$. But 3.1 implies $k_{i,i}^* t = 0$. Hence $k_{i,i}^*(u) = 0$.

Let $\mathfrak{R}'_{2n(p^i)-k,p^i}(S^{2n}, Z/p)$ be the subspace of $\mathfrak{R}_{2n(p^i)-k,p^i}(S^{2n}, Z/p)$ spanned by elements of type 1. Then 4.4 yields:

4.5. THEOREM [17]. *As* Z/p *vector spaces*

$$\dim((\text{image } k_{i,i}^*)_k) \leqslant \dim\left(\mathfrak{R}'_{2n(p^i)-k,p^i}(S^{2n}, Z/p)\right).$$

V. The proof of Theorem A. We now proceed with the proof of Theorem A.

5.1. LEMMA. $\mathcal{W}_i$ *is contained in* image $k_{i,i}^*$.

PROOF. By induction on i. The lemma is classically true for $i = 1$ and [4] proves the lemma for $i = 2$. Assume $\mathcal{W}_{i-1}$ is contained in image $k_{i-1,i-1}^*$. The next four lemmas establish 5.1.

5.2. LEMMA. *There exists* $u \in H^*(\mathbb{S}_{p^i})$ *such that*

$$k^*_{i-1,i}(u) = \left(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}\right)^{\otimes p} \in H^*(T_{i-1,i}).$$

PROOF. Recall the following commutative diagram containing the construction of the Steenrod powers on $\mathbb{S}_{p^{i-1}}$:

$$\begin{array}{ccccc}
 & & B_{\mathbb{S}_{p^i}} & & \\
 & & \uparrow & & \\
 & & B_{N_{\times^p(\mathbb{S}_{p^{i-1}}),\mathbb{S}_{p^i}}} & & \\
 & & \uparrow & & \\
B_{\times^p \mathbb{S}_{p^{i-1}}} & \xrightarrow{j} & W_{Z/p} \times_{Z/p} B_{\times^p \mathbb{S}_{p^{i-1}}} & = & B_{\mathbb{S}_{p^{i-1}} \mathrm{wr}\, Z/p} \\
\uparrow & & \uparrow{\scriptstyle 1 \times \Delta} & & \\
B_{T_{i-1,i}} & & W_{Z/p} \times_{Z/p} B_{\mathbb{S}_{p^{i-1}}} & = & B_{Z/p} \times B_{\mathbb{S}_{p^{i-1}}} \\
 & & \uparrow & & \\
 & & B_{T_{i,i}} & &
\end{array}$$

Of course the composition $B_{T_{i-1,i}} \to B_{\mathbb{S}_{p^i}}$ is $Bk_{i-1,i}$ and the composition $B_{T_{i,i}} \to B_{\mathbb{S}_{p^i}}$ is $Bk_{i,i}$.

Let $u' \in H^*(\mathbb{S}_{p^{i-1}})$ be such that $k^*_{i-1,i-1}(u') = M_{i-2,i-1}M_{i-3,i-1}L_i^{p-3}$ then $\mathcal{P}(u') = u'' \in H^*(\mathbb{S}_{p^{i-1}}\mathrm{wr}Z/p)$. Let $A = \mathbb{S}_{p^{i-1}}$ wr Z/p. Then 2.20 gives

$$i(A, \mathbb{S}_{p^i})^* t(\mathbb{S}_{p^i}, A) = \sum_{x \in \mathbb{S}_{p^i}/A} t_x i_x \mathrm{ad}_x$$

and we have the following commutative diagram:

$$\begin{array}{ccccccc}
H^*(A) & \xrightarrow{\mathrm{ad}_x} & H^*(xAx^{-1}) & \xrightarrow{i_x} & H^*(xAx^{-1} \cap A) & \xrightarrow{t_x} & H^*(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma H^*(T') & \xrightarrow{\Sigma\, \mathrm{ad}_x} & \Sigma H^*(xT'x^{-1}) & \xrightarrow{\Sigma i_x} & \Sigma H^*(xT'x^{-1} \cap T_{i-1,i}) & \xrightarrow{\Sigma t_x} & H^*(T_{i-1,i})
\end{array}$$

where T' runs through all inclusions $\times^m Z/p$ in A. (The last square commutes by 2.21 and [**31**, V. 7.2], as $xT_{i-1,i}x^{-1} \subset A$ implies $x \in A$.)

Thus 2.16, 2.18 and 2.21 show

$$k^*_{i-1,i}t(A, \mathbb{S}_{p^i})(u'') = \sum_x \left(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}\right)^{\otimes p}$$

where the sum runs over a coset representation $\overline{N}_{i-1} = N_{\times^p \mathbb{S}_{p^{i-1}},\mathbb{S}_{p^i}}$ mod A. As

A contains a p-Sylow subgroup of $\mathbb{S}_{p^i}$, $[\overline{N}_{i-1}: A] = c \not\equiv 0 \pmod p$. Let $u = t(A, \mathbb{S}_{p^i})(c^{-1}u'')$; then $k^*_{i-1,i}(u) = (M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1})^{\otimes p}$.

5.3. Lemma. *There exists $u \in H^*(\mathbb{S}_{p^i})$ such that*

$$k^*_{i-1,i}(u) = (Q_{i-2,i-1})^{\otimes p} \in H^*(T_{i-1,i}).$$

Proof. Identical to that of 5.2.

5.4. Lemma. *There exists $u \in H^*(\mathbb{S}_{p^i})$ such that $k^*_{i,i}(u) = M_{i-1,i}M_{i-2,i}L_i^{p-3}$.*

Proof. Let $u' \in H^*(\mathbb{S}_{p^{i-1}})$ be such that $k^*_{i-1,i-1}(u') = M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}$ and $u \in H^*(\mathbb{S}_{p^i})$ be such that $k^*_{i-1,i}(u) = (M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1})^{\otimes p} \in H^*(T_{i-1,i})$. Recall 3.1 implies image $k^*_{i,i}$ is contained in the $H^*(Z/p)$ module generated by image$(1 \times \Delta)^*\mathcal{P}$. A simple dimension check shows that the only classes in $H^*(\mathbb{S}_{p^{i-1}} \text{ wr } Z/p)$ that could project to $k^*_{i-1,i}(u)$ are $\mathcal{P}(u')$ and $b_1^x + \mathcal{P}(u')$, where $x = \frac{1}{2}$ dimension(u). By 2.15, $k^*_{i,i}(u)$ is GL$(i, Z/p)$ invariant. As $(u')^p = 0$ in $H^*(\mathbb{S}_{p^{i-1}})$ the class $b_1^x + \mathcal{P}(u')$ is not GL$(i, Z/p)$ invariant (there cannot be a pure b_r^x term in $(1 \times \Delta)^*(\mathcal{P}(u'))$ for $r \geqslant 1$). Hence $k^*_{i,i}(u) = (1 \times \Delta)^*(\mathcal{P}(u'))$. It is easy to see that dimension$(u') = 2(p^{i-1} - p^{i-2} - p^{i-3}) = 2n$. Thus

$$\begin{aligned} k^*_{i,i}(u) &= (1 \times \Delta)^*(\mathcal{P}(u')) = \sum_k w_k \otimes D_k\big(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}\big) \\ &= (-1)^n\Bigg[\sum_j w_{(2n-2j)(p-1)} \otimes (-1)^j\mathcal{P}^j\big(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}\big) \\ &\qquad + \sum_j w_{(2n-2j)(p-1)-1} \otimes (-1)^j\beta\mathcal{P}^j\big(M_{i-2,i-1}M_{i-3,i-1}L^{p-3}_{i-1}\big)\Bigg]. \end{aligned}$$

Consider $M_{i-1,i}M_{i-2,i}L_i^{p-3}$. Expanding along the e_1, b_1 columns we have

$$\begin{aligned} M_{i-1,i}M_{i-2,i}L_i^{p-3} &= \sum_{\substack{A\\B\\C_k}} (-1)^{\varphi}b_1^r(ABC_1 \cdots C_{p-3}) \\ &\quad + \sum_{\substack{D\\E\\C_k}} (-1)^{\varphi}e_1b_1^s(DEC_1 \cdots C_{p-3}) \end{aligned}$$

where A runs over all $i-1 \times i-1$ minors of $M_{i-1,i}$ eliminating the $b_1^{p^u}$ $(0 \leqslant u \leqslant i-2)$ row and column, B runs over all $i-1 \times i-1$ minors of $M_{i-2,i}$ eliminating the $b_1^{p^v}$ $(0 \leqslant v \leqslant i-3$, or $v = i-1)$ row and column, C_k $(k = 1, \dots, p-3)$ is any $i-1 \times i-1$ minor of L_i eliminating the $b_1^{p^{z_k}}$ $(0 \leqslant z_k \leqslant i-1)$ row and column, r satisfies the relation $\dim(M_{i-1,i}M_{i-2,i}L_i^{p-3}) = 2r + \dim(A) + \dim(B) + \sum_{k=1}^{p-3}\dim(C_k)$, and $\varphi \equiv u + v + \sum_{k=1}^{p-3}z_k \pmod 2$ if $v \neq i-1$, and $\equiv (i-u) + \sum_{k=1}^{p-3}z_k \pmod 2$ if

$v = i - 1$. D and E are $i-1 \times i-1$ minors of $M_{i-1,i}$ and $M_{i-2,i}$ respectively with exactly one minor eliminating the e_1 row and column, the other eliminating a $b_1^{p^t}$ row and column.

If C_k is the minor eliminating the $b_1^{p^{z_k}}$ row and column then $C_k = \mathcal{P}^{m_{z_k}}(L_{i-1})$ where $m_{z_k} = p^{z_k} + p^{z_k+1} + \cdots + p^{i-2}$ ($= 0$ if $z_k = i-1$).

Case 1. Suppose $v = i - 1$. Then the minor of $M_{i-2,i}$ eliminating the $b_1^{p^v}$ row and column is $M_{i-2,i-1}$. If A is an $i-1 \times i-1$ minor of $M_{i-1,i}$ eliminating the $b_1^{p^u}$ row and column and $AM_{i-2,i-1} \neq 0$ then $u \neq i-2$. Thus $A = \mathcal{P}^{j_1}(M_{i-3,i-1})$ where $j_1 = p^u + p^{u+1} + \cdots + p^{i-4}$ (if $u = i-3$ then $j_1 = 0$). Thus if $v = i-1$ we have

$$ABC_1 \cdots C_{p-3} = (-1)\mathcal{P}^0(M_{i-2,i-1})\mathcal{P}^{j_1}(M_{i-3,i-1})\mathcal{P}^{m_{z_1}}(L_{i-1}) \cdots \mathcal{P}^{m_{z_{p-3}}}(L_{i-1}).$$

Case 2. Suppose $0 \leqslant v \leqslant i-3$. Then $A = \mathcal{P}^{j_1}(M_{i-2,i-1})$ where $j_1 = p^u + p^{u+1} + \cdots + p^{i-3}$ unless $u = i-2$ in which case $j_1 = 0$ and $B = \mathcal{P}^{j_2}(M_{i-3,i-1})$ where $j_2 = p^v + p^{v+1} + \cdots + p^{i-4} + p^{i-2}$ unless $v = i-3$ in which case $j_2 = p^{i-2}$. Then we have

$$ABC_1 \cdots C_{p-3} = \mathcal{P}^{j_1}(M_{i-2,i-1})\mathcal{P}^{j_2}(M_{i-3,i-1})\mathcal{P}^{m_{z_1}}(L_{i-1}) \cdots \mathcal{P}^{m_{z_{p-3}}}(L_{i-1}).$$

Note. In Case 1 we have terms involving $(-1)P^0(M_{i-2,i-1})\mathcal{P}^{j_1}(M_{i-3,i-1})$ and in Case 2 if $u = i-2$ we have terms involving $P^0(M_{i-2,i-1})\mathcal{P}^{j_2}(M_{i-3,i-1})$ but it is clear that j_1 can never equal j_2 in these cases.

Thus if $ABC_1 \cdots C_{p-3} \neq 0$ we have written $ABC_1 \cdots C_{p-3}$ uniquely as $\mathcal{P}^{j_1}(M_{i-2,i-1})\mathcal{P}^{j_2}(M_{i-3,i-1})\mathcal{P}^{m_{z_1}}(L_{i-1}) \cdots \mathcal{P}^{m_{z_{p-3}}}(L_{i-1})$ for certain j_1, j_2, $m_{z_1}, \ldots, m_{z_{p-3}}$. 3.4 clearly shows if

$$Y = \mathcal{P}^{s_1}(M_{i-2,i-1})\mathcal{P}^{s_2}(M_{i-3,i-1})\mathcal{P}^{s_{z_1}}(L_{i-1}) \cdots \mathcal{P}^{s_{z_{p-3}}}(L_{i-1}) \neq 0$$

then $Y = ABC_1 \cdots C_{p-3}$ for a suitable choice of $A, B, C_1, \ldots, C_{p-3}$ and is thus analyzed in Case 1 or Case 2 above.

Let $j = j_1 + j_2 + \sum_{k=1}^{p-3} m_{z_k}$. For both $v = i-3$ and $v < i-3$ it is trivial to see that $\varphi = j \pmod 2$. Hence the Cartan formula and the above facts yield the following decomposition of $M_{i-1,i}M_{i-2,i}L_i^{p-3}$ where the first sum runs over *all* integers j.

$$M_{i-1,i}M_{i-2,i}L_i^{p-3} = \sum_j b_1^{(n-j)(p-1)} \otimes (-1)^j \mathcal{P}^j\left(M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}\right) + \sum_{\substack{D\\E\\C_k}} (-1)^{\varphi} e_1 b_1^s \otimes DEC_1 \cdots C_{p-3}.$$

Let $U = k_{i,i}^*(u) - (-1)^n(M_{i-1,i}M_{i-2,i}L_i^{p-3})$. U is clearly $\mathrm{GL}(i, Z/p)$ invariant. Any monomial term in U must contain the factor e_1e_j ($j \neq 1$) but as there is no monomial in U with an e_2e_3 factor symmetry implies $U = 0$. As

$n = p^{i-1} - p^{i-2} - p^{i-3}$ we have

$$k_{i,i}^*(u) = -M_{i-1,i}M_{i-2,i}L_i^{p-3}.$$

This proves 5.4.

Note. By keeping careful track of D, E, and $\beta(\mathcal{P}^{j_1}(M_{i-2,i-1})\mathcal{P}^{j_2}(M_{i-3,i-1}))$ it is possible to see directly that

$$\sum_{\substack{D\\E\\D_k}} (-1)^{\varphi} e_1 b_1^s \otimes DEC_1 \cdots C_{p-3}$$

$$= -\sum_j e_1 b_1^{(n-j)(p-1)-1} \otimes (-1)^j \beta \mathcal{P}^j(MML^{p-3})$$

where $MML^{p-3} = M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}$.

5.5. Lemma. *There exists $u \in H^*(\mathbb{S}_{p^i})$ such that $k_{i,i}^*(u) = Q_{i-1,i}$.*

Proof. The proof is similar to that of 5.4. We let $u' \in H^*(\mathbb{S}_{p^{i-1}})$ be such that $k_{i-1,i-1}^*(u') = Q_{i-2,i-1}$ and $u \in H^*(\mathbb{S}_{p^i})$ be such that $k_{i-1,i}^*(u) = (Q_{i-2,i-1})^{\otimes p} \in H^*(T_{i-1,i})$. Then $k_{i,i}^*(u)$ is the $\mathrm{GL}(i, Z/p)$ invariant class containing $(1 \times \Delta)^*(\mathcal{P}(u'))$. But [8] proved $Q_{i-1,i}$ is the only $\mathrm{GL}(i, Z/p)$ invariant polynomial in this dimension. Thus $k_{i,i}^*(u) = cQ_{i-1,i}$, where c is a constant. Note $(1 \times \Delta)^*(\mathcal{P}(u'))$ contains the term $w_0 \otimes D_0(Q_{i-2,i-1}) = (Q_{i-2,i-1})^p \neq 0$. Hence $c \neq 0$.

The naturality of the Steenrod algebra implies image $k_{i,i}^*$ contains $\mathcal{Q}(p)(M_{i-1,i}M_{i-2,i}L_i^{p-3}, Q_{i-1,i})$. By Theorem E any generator $\mathcal{W}_i$ is contained in $\mathcal{Q}(p)(M_{i-1,i}M_{i-2,i}L_i^{p-3}, Q_{i-1,i})$ (see the diagram after Theorem E). This completes the proof of Lemma 5.1.

By 4.5, to complete the proof of Theorem A it suffices to construct a 1-1 correspondence between nonzero monomials in $\mathcal{W}_i$ and admissible monomials in $\mathcal{Q}(p)$.

5.6. Lemma. $M_{i-1,i}M_{i-2,i} \cdots M_{1,i}\underline{L}_i \neq 0$.

Proof. The term $e_1 e_2 \cdots e_i (b_1^{p^{i-1}})^{i-1}(b_2^{p^{i-2}})^{i-1} \cdots (b_i)^{i-1}$ appears with coefficient 1 in the term-by-term expansion of $M_{i-1,i}M_{i-2,i} \cdots M_{1,i}\underline{L}_i$.

The only admissible monomials of length 1 in $\mathcal{Q}(p)_{2n}$ are $\mathcal{P}^{n-j}(u_{2n})$ and $\beta\mathcal{P}^{n-j}(u_{2n})$ which correspond to $(L_1^{p-1})^j$ and $(\underline{L}_1 L_1^{p-2})(L_1^{p-1})^{j-1}$ in $\mathcal{W}_i$. Thus we may assume, by induction, that an $i-1$ length admissible monomial in $\mathcal{Q}(p)_{2n}$ starting with $\mathcal{P}^{n-j}(u_{2n})$ corresponds to a j-fold product monomial in $\mathcal{W}_{i-1}$ $(j < n)$. Let A be an admissible monomial in $\mathcal{Q}(p)_{2n}$.

Case 1. $e_1 = 0$; that is, $A = \beta^{e_i}\mathcal{P}^{s_i} \cdots \beta^{e_2}\mathcal{P}^{s_2}\mathcal{P}^{n-j}(u_{2n})$. The dimension of $\mathcal{P}^{n-j}(u_{2n})$ is $2p(n-j) + 2j$ and hence $s_2 = p(n-j) + k$, $0 \leqslant k \leqslant j$, if $A(u_{2n})$ is nonzero and admissible. Consider

$$A' = \beta^{e_i}\mathcal{P}^{s_i} \cdots \beta^{e_2}\mathcal{P}^{s_2}(\bar{u}_{2(p(n-j)+j)}) \quad \text{where } \bar{u}_{2(p(n-j)+j)} = \mathcal{P}^{n-j}(u_{2n}).$$

A' is an admissible monomial of length $i-1$ and $s_2 = (p(n-j)+j) - (j-k)$. Thus A' corresponds to a $(j-k)$-fold product monomial in $\mathcal{W}_{i-1}$, call it U_{j-k}. Identify A with $\bar{U}_{j-k}(Q_{i-1,i})^k$ in $\mathcal{W}_i$. $\bar{U}_{j-k}$ comes from U_{j-k} by changing the detecting index from $i-1$ to i; i.e., $Q_{m,i-1} \to Q_{m,i}$.

Case 2. $e_1 = 1$; that is, $A = \beta^{e_i}\mathcal{P}^{s_i} \cdots \beta^{e_2}\mathcal{P}^{s_2}\beta\mathcal{P}^{n-j}(u_{2n})$. Then consider that part of A until a second Bockstein occurs.

$$A = \beta^{e_i}\mathcal{P}^{s_i} \cdots \beta\mathcal{P}^{s_k}\mathcal{P}^{s_{k-1}} \cdots \mathcal{P}^{p(p(n-j)+m_1)+m_2}\mathcal{P}^{p(n-j)+m_1}\beta\mathcal{P}^{n-j}(u_{2n})$$

with $m_1 \geqslant 1$.

Further suppose $k < i$. Then

$$s_k = p(p(p(\cdots(p(n-j)+m_1)+m_2)+\cdots+m_{k-2})+m_{k-1}$$

and $\mathcal{P}^{s_k} \cdots \beta\mathcal{P}^{n-j}(u_{2n})$ has dimension $2p^k(n-j) + 2p^{k-1}m_1 + 2p^{k-2}m_2 + \cdots + 2pm_{k-1} + 2(j - m_1 - m_2 - \cdots - m_{k-1}) + 1$. For A to be admissible and nonzero we must also have $j - m_1 - m_2 - \cdots - m_{k-1} \geqslant 0$ and $j - m_1 - m_2 - \cdots - m_{k-1} + 1 \geqslant 0$. Then

$$A' = \beta^{e_i}\mathcal{P}^{s_i} \cdots \mathcal{P}^{s_{k+1}}(\beta\mathcal{P}^{s_k} \cdots \beta\mathcal{P}^{n-j}(u_{2n})) = A''(\beta\mathcal{P}^{s_k} \cdots \beta\mathcal{P}^{n-j}(u_{2n}))$$

and A'' corresponds to a $j - m_1 - m_2 - \cdots - m_k + 1$ fold product monomial in $\mathcal{W}_{i-k}$, call it $U_{A''}$. Identify A with the monomial

$$\bar{U}_{A''}(M_{i-k,i}M_{i-1,i}L_i^{p-3})(Q_{i-k,i})^{m_{k-1}-1}(Q_{i-k-1,i})^{m_{k-2}} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1}$$

where $\bar{U}_{A''}$ comes from $U_{A''}$ by changing the detecting index from $i-k$ to i; i.e., $Q_{m,i-k} \to Q_{m,i}$. If $k = i$ or no second Bockstein occurs assign to A the monomial

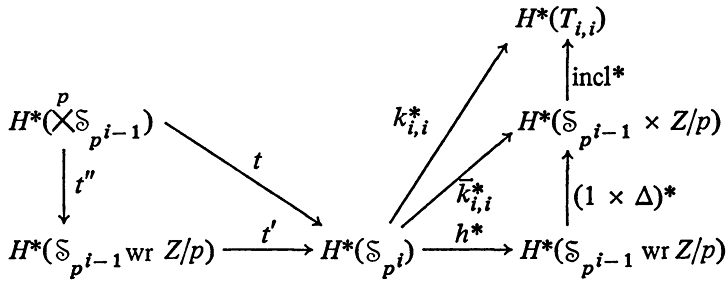
$$(M_{i-1,i}\underline{L}_iL_i^{p-3})(L_i^{p-3})^{m_i}(Q_{1,i})^{m_{i-1}} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \quad \text{or}$$

$$(M_{i-1,i}L_i^{p-2})(L_i^{p-3})^{m_i}(Q_{1,i})^{m_{i-1}} \cdots (Q_{i-2,i})^{m_2}(Q_{i-1,i})^{m_1-1} \quad \text{respectively}$$

where $m_i = j - m_1 - m_2 - \cdots - m_{i-1}$.

Let $U_{A(u_{2n})}$ be the above constructed monomial in $\mathcal{W}_i$ corresponding to $A(u_{2n})$. It is routine to verify that for $U_{A''}$ in $\mathcal{W}_{i-k}$ and $\bar{U}_{A''}$ in $\mathcal{W}_i$ constructed above we have $\dim(U_{A''}) + 2j(p^i - p^{i-k}) = \dim(U_{A(u_{2n})})$. This fact and induction on i show that if $A(u_{2n})$ has dimension $2n(p^i) - k$ then $U_{A(u_{2n})}$ has dimension k. Lemma 5.6 shows $U_{A(u_{2n})} \neq 0$. Hence, by Theorem 4.5, $(\mathcal{W}_i)_k$ must fill out (image $k_{i,i}^*)_k$ for $k \ll n$. This finishes the proof of Theorem A.

VI. Proof of Theorems B, C, D, and F. Consider the following commutative diagram:



where $h = i(\mathbb{S}_{p^{i-1}} \operatorname{wr} Z/p, \mathbb{S}_{p^i})$.

6.1. Proposition. *Let* $u \in H^*(\mathbb{S}_{p^i})$. *If* $k^*_{i,i}(u) = 0$ *then there exists* $z \in H^*(\times^p \mathbb{S}_{p^{i-1}})$ *such that* $t(z) = u$.

Proof. By 4.4 and Theorem A, $k^*_{i,i}(u) = 0$ implies $\bar{k}^*_{i,i}(u) = 0$. Hence $(1 \times \Delta)^* h^*(u) = 0$ and $h^*(u) \in \ker(1 \times \Delta)^*$. By 3.1 there exists $z \in H^*(\times^p(\mathbb{S}_{p^{i-1}}))$ such that $t''(z) = h^*(u)$. Then $t(z) = t't''(u) = t'(h^*(u)) = [\mathbb{S}_{p^i}: \mathbb{S}_{p^{i-1}} \operatorname{wr} Z/p]u = u \,(\operatorname{mod} p)$.

Let $u_{s,i-1} \in H^*(\mathbb{S}_{p^{i-1}})$, then, by induction, $u_{s,i-1}$ pulls back to a $\mathbb{S}_{p^{i-1}}$ detecting subgroup $\prod_{t=1}^q T_{s_t,s_t} \to \mathbb{S}_{p^{i-1}}$ (recall §II gives these subgroups and their inclusions into $\mathbb{S}_{p^{i-1}}$). Thus to complete the computation of $H^*(\mathbb{S}_{p^i})$ it suffices to compute the map $I^*_{i-1}t$. First consider the maps $\Phi_{m_1,\ldots,m_n} = (I_{m_1,\ldots,m_n} \circ (\prod_{r=1}^n (k_{m_r,m_r})))^* t_{m_1,\ldots,m_n}$: $H^*(\times_{r=1}^n \mathbb{S}_{p^{m_r}}) \to H^*(\mathbb{S}_{p^i}) \to \bigotimes_{r=1}^n H^*(T_{m_r,m_r})$ for all $(m_1, \ldots, m_r)$ such that $\sum_{r=1}^n p^{m_r} = p^i$, with $n \geqslant 2$ and $t_{m_1,\ldots,m_n}$ the transfer $H^*(\times_{r=1}^n \mathbb{S}_{p^{m_r}}) \to H^*(\mathbb{S}_{p^i})$.

6.2. Lemma. *Let* $u = u_{1,m_1} \otimes \cdots \otimes u_{n,m_n} \in H^*(\times_{r=1}^n \mathbb{S}_{p^{m_r}})$ *and* $k^*_{m_r,m_r}(u_{r,m_r}) = v_r$. *Then*

$$\Phi_{m_1,\ldots,m_n}(u) = \sum_{\sigma \in \mathbb{S}_{(m_1,\ldots,m_n)}} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.$$

Proof. As in the proof of 5.2, 2.16 through 2.21 and the following commutative diagram give the proposition:

$$\begin{array}{ccccccc}
H^*(A) & \xrightarrow{\operatorname{ad}_x} & H^*(xAx^{-1}) & \xrightarrow{i_x} & H^*(xAx^{-1} \cap A) & \xrightarrow{t_x} & H^*(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma H^*(T') & \xrightarrow{\Sigma \operatorname{ad}_x} & \Sigma H^*(xT'x^{-1}) & \xrightarrow{\Sigma i_x} & \Sigma H^*(xT'x^{-1} \cap T) & \xrightarrow{\Sigma t_x} & H^*(T)
\end{array}$$

where $A = \times_{r=1}^n(\mathbb{S}_{p^{m_r}})$, T' runs through all inclusions of $\times^m Z/p$ in A and $T = \times_{r=1}^n T_{m_r,m_r}$.

The only $\mathbb{S}_{(m_1,\ldots,m_n)}$ invariant classes not in image $\Phi_{m_1,\ldots,m_n}$ are classes $u' = u_{1,m_1} \otimes \cdots \otimes u_{n,m_n}$ containing $(u_{r_0,m_{r_0}})^{\otimes p} \in \otimes^p H^*(T_{m_{r_0},m_{r_0}})$ as a factor. Recall $u^{\otimes p} \xleftarrow{j^*} \mathcal{P}(u) \xrightarrow{(1\times\Delta)^*} (1 \times \Delta)^*(\mathcal{P}(u))$. Thus u' is in the

$$\text{image}\left(I_{m_1,\dots,m_n} \circ \left(\prod_{r=1}^{n} k_{m_r,m_r}\right)\right)^* : H^*(\mathfrak{S}_{p^l}) \to \bigotimes_{r=1}^{n} H^*(T_{m_r,m_r}).$$

Hence we have

6.3. LEMMA. Image$(I_{m_1,\dots,m_n} \circ (\prod_{r=1}^n k_{m_r,m_r}))^* \cong \mathfrak{S}_{(m_1,\dots,m_n)}$ *invariant classes of* $\otimes_{r=1}^n H^*(T_{m_r,m_r})$.

This proves Theorems B and D. A trivial modification of 6.2 and 6.3 proves Theorem F. As 3.1 shows the only multiple image classes are generated by the $\mathcal{P}(\ )$'s, Theorem C follows, up to constants. Using the notation of Theorem C if $x_{m,i-1} = M_{i-2,i-1}M_{i-3,i-1}L_{i-1}^{p-3}$ then 5.4 gives $x_{m,i} = -1(M_{i-1,i}M_{i-2,i}L_i^{p-3})$. If $x_{m,i-1} = Q_{i-2,i-1}$ then direct computation shows the constant c in 5.5 is 1 hence $x_{m,i} = Q_{i-1,i}$. It is easy to see that application of the Steenrod pth powers or direct computation yield that the constant is $+1$ for multiple image polynomial generators and -1 for even dimensional multiple image exterior generators.

VII. Proof of Theorem G.

PROOF OF (1). Let $k_{j,\infty}^*(u) = \mathfrak{S}\langle x_1, \dots, x_m, 1, \dots\rangle$. As j is the smallest integer such that $k_{j,\infty}^*(u) \neq 0$ it follows that at least one x_h contains a factor equal to L_j^{p-1}, $\underline{L}_j L_j^{p-2}$, $M_{g,j}\underline{L}_j L_j^{p-3}$, or $M_{g,j}L_j^{p-2}$. If $k_{j,\infty}^*(u)$ has at least one representative of the form $\underline{L}_{n,j}(\cdots)$ with p not dividing n then $\beta_p(k_{j,\infty}^*(u)) = \Sigma n L_{n,j}(\cdots) + B \neq 0$ (where B cannot contain terms in the first sum). Similarly if some $x_h = M_{g,j}\underline{L}_j L_j^{p-3}Y$ and no $x_{h'} = M_{g,j}L_j^{p-2}Y$ then $\beta_p(k_{j,\infty}^*(u)) \neq 0$. Suppose every time the term $M_{g,j}\underline{L}_j L_j^{p-3}Y$ appears the term $M_{g,j}L_j^{p-2}Y$ also appears; then if $k_{j,\infty}^*(u) \neq \underline{L}_{n,j}(\cdots)$ Y must be a product of $Q_{h,j}$'s. It is then easy to construct a class u' such that $\beta_p(u') = u$ (just replace one $M_{g,j}L_j^{p-2}Y$ by $M_{g,j}\underline{L}_j L_j^{p-3}Y$). If $\beta_p(u) = 0$ and $M_{g,j}L_j^{p-2}Y$ appears a similar construction yields u' such that $\beta_p(u') = u$. The only possibility left is $\beta_p(u) = 0$, and $k_{j,\infty}^*(u) = L_{n,j}(\cdots)$. Then $\beta_p(u') = u$ where $k_{j,\infty}^*(u') = \underline{L}_{n,j}$.

PROOF OF (2). We need the following

THEOREM [2]. *Let* $r \geqslant 2$. *In homology with the loop sum multiplication if* $d^{r-1}(a) = b$ *then* $d^r(a^p) = a^{p-1}b$.

PROOF. Theorem 5.4 of [2].

The homology and cohomology Bockstein spectral sequences are Hopf algebra duals and Theorem F gives the loop sum coalgebra map in cohomology. If a, b in $H_*(Q(S^0)_0)$ are dual to u, v respectively then Theorem F gives $\langle u', a^p\rangle = 1$. Now u' is not dual to a^p on the E_1 level; in fact $(u')^* = a^p + \Sigma a_i$. It is easy to see however that the a_i are all dual to classes u'' where $k_{j,\infty}^*(u'') = \mathfrak{S}\langle x_1, \dots, x_t, 1, \dots\rangle$ with $t < pm$.

Many times it is easy to see that the a_i classes do not live to E_r. Such is the case with Corollary 1 as induction on r and the fact that $\{L_{p^m,j}(x\colon 1,\dots)\}_{m=1}^{r-1}$ generate the subalgebra $\{L_{n,j}(x\colon 1,\dots)\}$ (where $n = 1,\dots, p^r - 1$) prove the corollary.

PROOF OF COROLLARY 2. The reduction homomorphism $j_r\colon H^*(\ , Z/_{p^r}) \to E_r$ is onto and if $k_{i,i}^*(u) = Q_{j,i}$ then $k_{j,i}^*(u) = R_i^*(L_{p^j,j}(1\colon 1,\dots))$.

Appendix. We give a proof that the quotient determinants, $Q_{j,i} \in \mathcal{W}_i$ are integral mod p. L_i has an explicit factorization first discovered by E. H. Moore in 1896

LEMMA [**19**]. *$L_i = \prod_{(m_1,\dots,m_i)}(m_1b_1 + \cdots + m_ib_i)$ where $(m_1,\dots,m_i)$ runs over all elements of $T_{i,i}$ with first nonzero coefficient equal to one.*

PROOF. (Compare with [**8**, p. 76].) L_i is invariant under the special linear group $\mathrm{SL}(i, Z/p)$ which acts transitively on the nonzero elements of $T_{i,i}$. Since b_1 is a factor of L_i it follows that $\alpha(b_1) = m_1b_1 + \cdots + m_ib_i$ is a factor as well. Hence the product above divides L_i (the factors are all relatively prime). But both sides have the same degree, hence they differ only up a constant factor. But the diagonal term $b_1^{p^{i-1}}b_2^{p^{i-2}} \cdots b_i$ occurs in both sides only once and each time with coefficient 1.

More generally b_1 is a factor of the numerator of $Q_{j,i}$ for every j, so L_i is also a factor of the numerator of $Q_{j,i}$ by the above argument. This gives:

LEMMA. *$Q_{j,i}$ is a nontrivial polynomial invariant under* $\mathrm{GL}(i, Z/p)$.

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