

ANALYTIC EQUATIONS AND SINGULARITIES OF PLANE CURVES

BY

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ABSTRACT. Theorems (Artin, Wavrik) exist which show that sufficiently good approximate (power series) solutions to a system of analytic equations may be approximated by convergent solutions. This paper considers the problem of explicitly determining the order, β , to which an approximate solution must solve the system of equations.

The paper deals with the case of one equation, $f(x, y) = 0$, in two variables. It is shown how β depends on the singularities of the curve $f(x, y) = 0$. A method for obtaining the minimal β is given. A rapid way of finding β using the Newton Polygon for f applies in special cases.

Introduction. Let k be a field of characteristic 0 complete with respect to a nontrivial valuation. We use $k\{\}$ to denote convergent power series; $k[[\]]$ to denote formal power series; and $k[\]$ to denote polynomials. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_p)$, $z = (z_1, \dots, z_r)$, $f = (f_1, \dots, f_m)$ with $f_i \in k\{x, y\}[[z]]$.

The following theorem was proved in [7]:

THEOREM. $\forall \alpha \exists \beta$ such that if $\bar{y}(x) \in k[[x]]^p$ with $\bar{y}(0) = 0$ and $\bar{z}(x) \in k[[x]]^r$ satisfy

$$f(x, \bar{y}(x), \bar{z}(x)) \equiv 0 \pmod{(x)^\beta}.$$

Then $\exists y(x) \in k\{x\}^p$ with $y(0) = 0$ and $z(x) \in k\{x\}^r$ such that

- (1) $f(x, y(x), z(x)) = 0$,
- (2) $y(x) \equiv \bar{y}(x)$, $z(x) \equiv \bar{z}(x) \pmod{(x)^\alpha}$.

This result may be paraphrased by saying that sufficiently good approximate solutions to a system of analytic equations can be approximated by actual solutions. The corresponding theorem in the case in which the f_i are polynomials was obtained by Artin in [1]. The theorem has applications in deformation theory (see [7])—it is the genesis of results asserting the existence of only finitely many obstructions to deformation. The proof given in [7] is

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purely existential. No calculation, estimate, or bound for β is obtained from the proof. If the theorem is to be applied to a deformation theory problem, no means is provided for determining when the last obstruction is encountered. The purpose of the present paper is to explore means for explicitly calculating β .

The number β must depend on the nature of the singularity determined by $f(x, y, z) = 0$. To clarify this connection we restrict to the case $n = p = m = 1, r = 0, f$ a polynomial. Thus we seek to find solutions of a single polynomial equation, $f(x, y) = 0$, in two variables.

The proof in [7] uses a complex multiple induction. An approach to the calculation of β would be a more detailed analysis of that proof. In §1 we present a drastically simplified version of that proof. This simplified proof yields an explicit value for β which, unfortunately, is not usually close to the minimal value.

In §2 an alternative proof is provided which is based on the resolution of singularities by quadratic transformations. This proof provides a criterion for determining when the last obstruction is encountered.

In §3 the resolution scheme is used to find the minimum β in specific cases.

In §4 a method for rapidly determining β from the Newton Polygon for f is discussed.

1. A proof of the theorem.

THEOREM I. *Given $\alpha > 0$ and $f(x, y) \in k[x, y] \exists \beta$ such that if $\bar{y}(x) \in k[[x]]$ satisfies $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ then $\exists y(x) \in k[[x]]$ with*

- (1) $f(x, y(x)) = 0$,
- (2) $y(x) \equiv \bar{y}(x) \pmod{(x)^\alpha}$.

THEOREM II. *Given $f, g \in k[x, y] \forall \gamma \exists \beta$ such that if $\bar{y}(x) \in k[[x]]$ satisfies*
 (i) $f(x, \bar{y}(x)) \equiv 0 \pmod{(g(x, \bar{y}(x)) + (x)^\beta)}$,
 (ii) $g(x, \bar{y}(x)) \not\equiv 0 \pmod{(x)^\gamma}$,
then $f(x, \bar{y}(x)) \equiv 0 \pmod{(g(x, \bar{y}(x)))}$.

PROOF OF THEOREM II. We show that $\beta = \gamma - 1$ satisfies the requirements. Let $\bar{y}(x)$ satisfy the hypothesis. If $g(x, \bar{y}(x))$ is a unit in $k[[x]]$ then the conclusion is obvious. If $g(x, \bar{y}(x))$ is not a unit then we write $g(x, \bar{y}(x)) = x^s u(x)$ where $u(0) \neq 0$ and $1 \leq s < \gamma$.

Now

$$\begin{aligned} f(x, \bar{y}(x)) &= A(x)g(x, \bar{y}(x)) + B(x)x^\beta \\ &= (A(x) + (u(x))^{-1}B(x)x^{\beta-s})g(x, \bar{y}(x)) \\ &\equiv 0 \pmod{(g(x, \bar{y}(x)))}, \end{aligned}$$

since $\beta = \gamma - 1 \geq s$.

PROPOSITION 1. Given $f(x, y) \in k[x, y] \forall \alpha, \gamma \exists \beta \geq \gamma$ such that if $\bar{y}(x) \in k[[x]]$ satisfies

$$\begin{aligned} f(x, \bar{y}(x)) &\equiv 0 \pmod{(x)^\beta}, \\ f_y(x, \bar{y}(x)) &\not\equiv 0 \pmod{(x)^\gamma}, \end{aligned}$$

then $\exists y(x) \in k[[x]]$ with

- (1) $f(x, y(x)) = 0$,
- (2) $y(x) \equiv \bar{y}(x) \pmod{(x)^\alpha}$.

PROOF. Let $g = (f_y(x, y))^2$. We will show that $\beta = \alpha + 2\gamma$ satisfies the requirements.

Since $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ we have $f(x, \bar{y}(x)) \equiv 0 \pmod{(g(x, \bar{y}(x)) + (x)^\beta)}$. Also $\beta = \alpha + 2\gamma \geq 2\gamma - 1$ and $g(x, \bar{y}(x)) \not\equiv 0 \pmod{(x)^{2\gamma}}$. By the previous theorem we see that $f(x, \bar{y}(x)) \equiv 0 \pmod{(g(x, \bar{y}(x)))}$.

Let $F(x, Y) = f(x, \bar{y}(x) + Y)$ so that $F(x, 0) = f(x, \bar{y}(x))$ and

$$\partial F(x, 0)/\partial Y = f_y(x, \bar{y}(x)).$$

Let J be the ideal in $k[[x]]$ generated by $\partial F(x, 0)/\partial Y$ and $J' = (x)^\alpha$. We have $f(x, \bar{y}(x)) = B(x)(f_y(x, \bar{y}(x)))^2$. The left side is $\equiv 0 \pmod{(x)^\beta}$ while $(f_y(x, \bar{y}(x)))^2 \not\equiv 0 \pmod{(x)^{2\gamma}}$ thus $B(x) \equiv 0 \pmod{(x)^t}$ with $t \geq \beta - 2\gamma = \alpha$. This shows that $F(x, 0) \in J^2 \cdot J'$. We now apply the Tougeron Implicit Function Theorem ([5, III.3.2], or [7, Theorem 3]): there is a $\hat{y}(x) \in J \cdot J'$ such that $F(x, \hat{y}(x)) = 0$. Set $y(x) = \bar{y}(x) + \hat{y}(x)$. Then $f(x, y(x)) = 0$ and $y(x) \equiv \bar{y}(x) \pmod{(x)^\alpha}$ as required.

PROOF OF THEOREM I. The task is to reduce the proof to a case of Proposition 1.

Write $f(x, y)$ and $f_y(x, y)$ in powers of y . Let $D(x)$ denote the discriminant of f (as a polynomial in y). We can find [6, Theorem 9.6] polynomials $A(x, y)$, $B(x, y) \in k[x, y]$ so that

$$D(x) = A(x, y)f(x, y) + B(x, y)f_y(x, y). \quad (*)$$

Case 1. f is irreducible.

In this case $D(x) \neq 0$. Let $d = \text{ord}(D(x))$. If $\bar{y}(x) \in k[[x]]$ satisfies $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^d}$ then we see from (*) that $f_y(x, \bar{y}(x)) \not\equiv 0 \pmod{(x)^d}$. Let α be given and set $\beta = 2d + \alpha$. By Proposition 1 we find a solution, $y(x)$, to $f(x, y(x)) = 0$ such that $y(x) \equiv \bar{y}(x) \pmod{(x)^\alpha}$.

Case 2. f is reducible.

Let $f = f_1 \cdots f_t$ be a factorization of f into irreducible factors. Let β_i satisfy the condition of Theorem I for the equation $f_i(x, y) = 0$. Let $\beta = \beta_1 + \cdots + \beta_t$. If $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ then $\exists i$ s.t. $f_i(x, \bar{y}(x)) \equiv 0 \pmod{(x)^{\beta_i}}$. By Theorem I, applied to f_i , we find $y(x)$ such that $f_i(x, y(x)) = 0$ hence also $f(x, y(x)) = 0$. Moreover $y(x) \equiv \bar{y}(x) \pmod{(x)^\alpha}$.

Notice that we have not only proved the theorem, but we even have a formula for β that satisfies the conditions:

If $f(x, y) \in k[x, y]$ is irreducible and $d = \text{ord}(D(x))$, where $D(x)$ is the discriminant of f , then $\beta = 2d + \alpha$ satisfies the conditions of Theorem I. If $f(x, y)$ factors, $f = f_1 \cdots f_i$ with f_i irreducible, then $\beta = \beta_1 + \cdots + \beta_i$ with β_i as above for f_i .

2. β and the resolution of singularities. In this section an outline is given of an alternative proof of Theorem I.

We will always assume that $f(x, y) = 0$ has no multiple solutions. This will be the case, for example, when $f(x, y)$ is irreducible.

PROPOSITION 2. *If $f(x, y) = 0$ is regular at $(0, a_0)$ then there is a solution to $f(x, y) = 0$ with $y(0) = a_0$ if and only if there is a solution to $f(x, y) \equiv 0 \pmod{(x)^2}$ with $y(0) = a_0$.*

PROOF. Apply the Implicit Function Theorem.

We will refer to this as the "IFT case".

REMARKS. (1) In the IFT case, $\beta = \max(\alpha, 2)$ satisfies the conditions of Theorem I.

(2) The solution is unique in this case and the a_i ($i > 1$) can be calculated by recursion.

(3) Kung and Traub [4] provide an algorithm for computing the a_i ($i > 1$) with lower "cost" by using Newton Iteration.

Now suppose that $f(x, y) = 0$ has an r_0 -fold point at $(0, a_0)$ with $r_0 > 1$. We write $f(x, a_0 + xy) = x^{r_0} f_1(x, y)$.

DEFINITION 1. $f_1(x, y) = 0$ is called the proper transform of $f(x, y) = 0$ under the standard quadratic transformation with center $(0, a_0)$.

DEFINITION 2. The points $(0, a_1)$ satisfying $f_1(0, a_1) = 0$ are said to lie over $(0, a_0)$.

The process is repeated using f_1 and the $(0, a_1)$.

PROPOSITION 3. *A plane curve singularity may be resolved by a finite sequence of standard quadratic transformations.*

REMARK. The proof of Proposition 3 is found in [3]. Bounds for the number of quadratic transformations required can be given in terms of the "apparent genus" of the curve and, ultimately, in terms of the degree of $f(x, y)$.

PROPOSITION 4. *If $y = a_0 + a_1x + \cdots + a_kx^k$ satisfies $f(x, y) \equiv 0 \pmod{(x)^\beta}$ for β sufficiently large, then the $(0, a_i)$ are centers of successive quadratic transformations and conversely.*

PROOF. If $(0, a_0)$ is an r_0 -fold point then $y = a_0 + a_1x + \cdots$ satisfies $f(x, y) \equiv 0 \pmod{(x)^\beta}$ if and only if $y_1 = a_1 + a_2x + \cdots$ satisfies $f_1(x, y_1) \equiv$

$0 \bmod (x)^{\beta - r_0}$. The proof follows from this.

REMARKS. (1) Since a finite (bounded in advance) sequence of quadratic transformations leads to regular points, the IFT case will apply after a finite number of steps.

(2) The last obstruction is encountered when $f_k(x, y) = 0$ is regular at $(0, a_k)$.

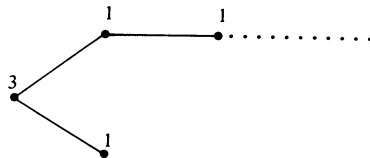
(3) Theorem I follows from Propositions 3 and 4.

(4) Since the multiplicity of a singularity does not increase under quadratic transformation, a bound for β can be obtained. This bound is usually excessively large.

3. Computing the minimum β . We record the results of the resolution process of §2 in the form of a "tree diagram" showing the points that arise in successive quadratic transformations and their multiplicities. This diagram is similar to the classical diagram of "infinitely near points" to a singularity.

EXAMPLE 1. $f(x, y) = y^5 + 2xy^4 - xy^2 - 2x^2y + (x^4 - x^3)$ [6, Chapter IV, §3.3] has the tree diagram $\overset{3}{\text{---}}\overset{1}{\text{---}}\overset{1}{\text{---}}$ at $(0, 0)$. Thus $(0, 0)$ is a 3-fold point with a simple point lying over it. The termination of the diagram means that the simple point has no points lying over it (it has a vertical tangent!).

EXAMPLE 2. $f(x, y) = y^4 - y^3 - 2xy^2 + (2x^4 - x^2)y + x^5$ (this is the preceding example with x and y interchanged) has tree diagram



at $(0, 0)$. $(0, 0)$ is a triple point with two simple points lying over it. One of the simple points has no points over it . . . the other has a sequence of simple points over it.

The tree diagram will always have a finite number of branches some of which terminate and some which continue indefinitely with a line of simple points. We attach to the i th branch in the tree the number $\beta_i = \beta_i(\alpha)$.

DEFINITION 3. (a) For a terminating branch with multiplicities r_0, \dots, r_k set $\beta_i = r_0 + \dots + r_k + 1$.

(b) For a nonterminating branch with multiplicities r_i set $\beta_i = \sum_{i=0}^{\infty} r_i + 1$.

THEOREM 1. $\beta = \max(\beta_i)$ is the minimum β which satisfies the conditions of Theorem I.

PROOF. This follows from the observation used in the proof of Proposition 4.

REMARK. For a nonterminating branch there is a smallest k so that $r_i = 1$

when $i \geq k$. If $\alpha > k$ we obtain $\beta_i = r_0 + \dots + r_{k-1} - k + \alpha$. We can always use this procedure for the case of a nonterminating branch: Take the sum of the multiplicities of all vertices corresponding to singular points, subtract the number of such vertices and add α .

- EXAMPLES. (1) $f(x, y) = y^5 + 2xy^4 - xy^2 - 2x^2y + (x^4 - x^3)$. $\beta = 5$.
 (2) $f(x, y) = y^4 - y^3 - 2xy^2 + (2x^4 - x^2)y + x^5$. $\beta = \max(5, \alpha + 2)$.
 (3) $f(x, y) = y^5 - x^2y^2 + x^5$ [2, p. 48, Example 2]. $\beta = 6$.
 (4) $f(x, y) = y^n - x^m$ ($n, m > 0, n \nmid m$). $\beta = m + 1$. (We may compare this with the value $\beta = 2m(n - 1) + \alpha$ found in §1.)
 (5) $f(x, y) = y^n - x^{kn}$ ($n, k > 0$). $\beta = kn - k + \alpha$ (minimum if $\alpha > k$).

4. β and the Newton Polygon. In this section we show how β may easily be obtained from the Newton Polygon for $f(x, y)$ in special cases.

These methods apply only to solutions that begin with $a_0 = 0$. For $a_0 \neq 0$ the origin of the coordinate system must be translated.

We will also assume that y is not a factor of $f(x, y)$. The case in which it is a factor is treated at the end of the section.

Let $f(x, y) = \sum a_{ij}x^i y^j$. Let $I = \{(i, j) \mid a_{ij} \neq 0\}$. We define a partial order on \mathbf{R}^2 by $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. Let $S = \{(x, y) \in \mathbf{R}^2 \mid (x, y) \geq (i, j) \text{ for some } (i, j) \in I\}$ and let \hat{S} be the convex closure of S .

DEFINITION 4. The *Newton Polygon* of f at $(0, 0)$ is that part of the boundary of \hat{S} which is not contained in the coordinate axes.

EXAMPLES. The Newton Polygons for Examples 1-5 are given in Figures 1-4.

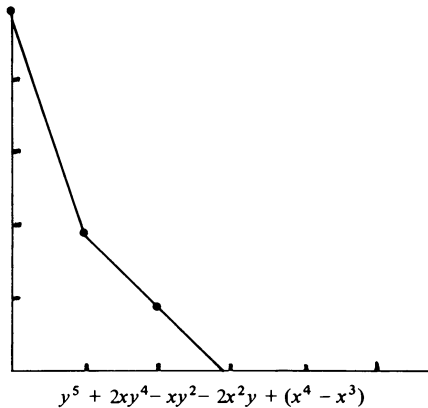


FIGURE 1

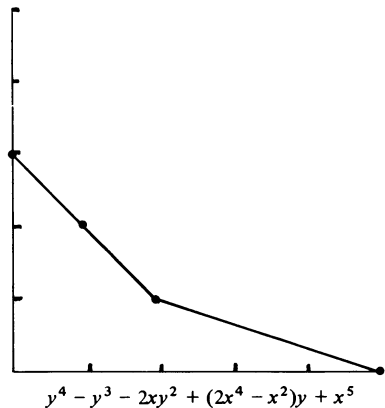


FIGURE 2

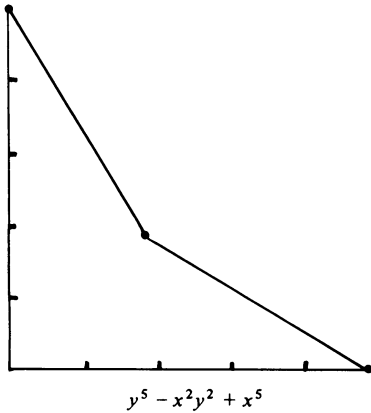


FIGURE 3

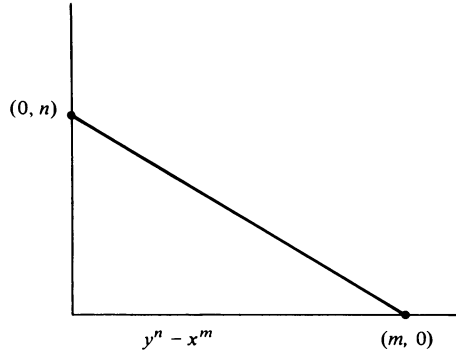


FIGURE 4

DEFINITION 5. By the *declivity* of a segment of the Newton Polygon we mean the negative of the reciprocal of its slope.

Thus if d is the declivity the segment is part of a line whose equation is $x + dy = \lambda$ for some constant λ .

DEFINITION 6. A Newton Polygon is called *simple* if every segment of integral declivity contains only two vertices.

The Newton Polygons in Figures 3 and 4 are simple. The Newton Polygons of Figures 1 and 2 are not simple.

Let $m = \text{ord}(f(x, 0))$. If $x + dy = \lambda$ for d and λ integral is the equation of a segment of the Newton Polygon we put $\lambda_d = \lambda$.

THEOREM 2. If the Newton Polygon for $f(x, y)$ is simple then $\beta = \max(m + 1, \lambda_d - d + \alpha)$ satisfies the conditions of Theorem I.

PROOF. $\bar{y}(x) \equiv 0$ can solve $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ for $\beta = m$ but not $\beta = m + 1$. Thus we have $\bar{y}(x) = a_c x^c + \dots$ with $a_c \neq 0$. Now $f(x, x^c y) = \sum a_{ij} y^j x^{i + cj} + \text{higher terms}$ (where the sum is taken over all pairs (i, j) for which $a_{ij} \neq 0$ and $i + cj = \lambda_c$ is a minimum). If there is only one such term then, since λ_c is minimal, the line $x + cy = \lambda_c$ must lie on only one side of the Newton Polygon, so $\lambda_c \leq m$. Since $a_c \neq 0$ we cannot have $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^{\lambda_c + 1}}$. However we have assumed $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ for $\beta \geq m + 1$ so this case does not occur. There must, therefore, be two pairs (i, j) in this sum. The line $x + cy = \lambda_c$ then coincides with a segment of the Newton Polygon with integral declivity and so there are exactly two terms in the sum. We write

$$f(x, x^c y) = x^\lambda f_1(x, y),$$

and note that $f_1(0, y) = \sum a_{ij} y^j$. Since $\beta \geq m + 1 \geq \lambda_c + 1$, a_c must be a (nonzero) root of $f_1(0, y) = 0$. Since there are only two terms in $f_1(0, y)$ any nonzero root is simple so $\partial f_1(0, a_c)/\partial y \neq 0$. Now clearly f_1 is the same as the

f_1 obtained by applying the method of §2 with $a_0 = \dots = a_{c-1} = 0$ and $\lambda_c = r_0 + \dots + r_{c-1}$. At this stage we are now in the IFT situation and obtain a nonterminating branch in the tree diagram. The analysis of the preceding section shows that $\beta(\alpha) = \lambda_c - c + \alpha$ works on this branch. This completes the proof.

Simplicity of the Newton Polygon can be immediately detected. The proof of Theorem 2, however, applies in a more general situation. It should be observed that simplicity was used only to guarantee that $f_1(0, y)$ had no nonzero multiple root.

Let $(i_1, j_1), \dots, (i_k, j_k)$ be the vertices of a segment of the Newton Polygon. We call $F(y) = \sum_v a_{i,j} y^j$ the associated polynomial of this segment.

THEOREM 3. *If, for every segment with integral declivity, the associated polynomial has no nonzero multiple roots then the conclusion of Theorem 2 applies.*

EXAMPLE 6. $f(x, y) = y^3 + xy^2 - 2x^2y + x^5$ has a Newton Polygon identical to that in Figure 2. The associated polynomial for the initial segment is $F(y) = y^3 + y^2 - 2y$ which has no multiple roots. There are only two vertices on the other segment (which also has integral declivity). The two values of λ are $\lambda_1 = 3$ and $\lambda_3 = 5$. Theorem 3 applies. $\beta = \max(3 - 1 + \alpha, 5 - 3 + \alpha) = 2 + \alpha$.

By contrast, however, Theorem 3 does not apply to Examples 1 or 2.

PROPOSITION 5. *If $A(x) \in k[x]$ and $g(x, y) = f(x, A(x) + y)$ then the same β works for both f and g .*

PROOF. Obvious.

In some circumstances we can easily convert a nonsimple Newton Polygon into a simple one by applying a transformation of this type. If $F(y)$ is the associated polynomial to a segment with integral declivity d and if a is a nonzero multiple root of $F(y)$ then $A(x) = ax^d$ will at least eliminate the problem with this root.

EXAMPLE. For Example 1 the associated polynomial for the segment of declivity 1 is $F(y) = -y^2 - 2y - 1 = -(y + 1)^2$. $a = -1$ is a nonzero multiple root. Let $g(x, y) = f(x, y - x)$. Then

$$g(x, y) = y^5 - 3xy^4 + 2x^2y^3 + (2x^3 - x)y^2 - 3x^4y + (x^5 + x^4).$$

The Newton Polygon for $g(x, y)$ is

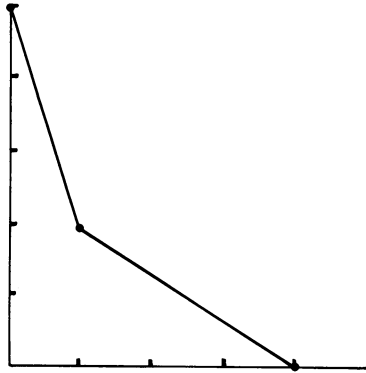


FIGURE 5

This Newton Polygon has no segment with integral declivity. Therefore $\beta = 4 + 1 = 5$ satisfies the condition of Theorem I for $g(x, y)$ and so also for $f(x, y)$.

The same transformation also works for Example 2.

Now suppose $f(x, y) = yg(x, y)$ and that Theorem 2 or 3 applies to $g(x, y)$. Let m and λ_c be as in Theorem 2 for $g(x, y)$.

THEOREM 4. $\beta = m + \alpha$ satisfies the conditions of Theorem I for $f(x, y)$.

PROOF. If $\bar{y}(x)$ satisfies $f(x, \bar{y}(x)) \equiv 0 \pmod{(x)^\beta}$ then either $\bar{y}(x) \equiv 0 \pmod{(x)^\alpha}$ (in which case we are done) or else $\text{ord}(\bar{y}(x)) = c < \alpha$.

$$f(x, \bar{y}(x)) = \bar{y}(x)g(x, \bar{y}(x)),$$

so

$$g(x, \bar{y}(x)) \equiv 0 \pmod{(x)^{\beta-c}}.$$

We have $\beta - c \geq m + \alpha - c \geq m + 1$ so $\bar{y}(x)$ coincides mod $(x)^\alpha$ with a nonterminating branch for $g(x, y) = 0$ provided $\beta - c > \lambda_c - c + \alpha$. Since $\lambda_c < m$ this is the case.

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