

DIFFERENCE EQUATIONS: DISCONJUGACY, PRINCIPAL SOLUTIONS, GREEN'S FUNCTIONS, COMPLETE MONOTONICITY¹

BY

PHILIP HARTMAN

ABSTRACT. We find analogues of known results on n th order linear differential equations for n th order linear difference equations. These include the concept of disconjugacy, Pólya's criterion for disconjugacy, Frobenius factorizations, generalized Sturm theorems, existence and properties of principal solutions, signs of Green's functions, and completely monotone families of solutions of equations depending on a parameter.

1. Introduction. In the linear n th order difference equation

$$(Pu)(m) \equiv \sum_{j=0}^n \alpha_j(m)u(m+j) = 0, \quad \text{where } \alpha_n(m) = 1, \alpha_0(m) \neq 0, \quad (1.1)$$

assume that the independent (integral) variable m ranges over a finite interval $I = [a, b] = \{a, a+1, \dots, b\}$ or an infinite interval $I = [a, \infty) = \{a, a+1, \dots\}$. The coefficients $\alpha_0, \dots, \alpha_{n-1}$ are defined on I , while the solution u is defined on the set I^n , where $I^n = [a, b+n]$ if $I = [a, b]$ and $I^n = I$ if $I = [a, \infty)$.

It is known that if $n = 2$ and

$$(-1)^n \alpha_0(m) > 0, \quad (1.2)$$

then analogues of the Sturm comparison and separation theorems for linear second order differential equations are valid for (1.1); cf., e.g., [4, Chapter XVI], where the algebraic signs seem to be incorrect in Theorem IV (see §6 below).

The object of this paper is to carry analogous results on linear n th order differential equations over to (1.1). These include the formulation of the concept of disconjugacy in §1, the analogues of Pólya's [17] criterion for disconjugacy in §5, the generalized Sturm comparison theorems of [6], [7], [13] in §6, discussion of the sign of Green's function of [11], [12], [16] (cf. [2, pp. 105–109]) in §7, existence and properties of principal solutions of [6], [7] in §8, inequalities for certain solutions in §9, and completely monotone families of solutions of equations depending on a parameter [8] in §10. We illustrate the

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result of §10 with an application to Bessel functions. In §11, we consider the existence of a positive solution of a (not necessarily disconjugate) difference equation.

In order to define the concept of “disconjugacy” for (1.1), we generalize the familiar notion of “node” in the case $n = 2$; cf. [4, p. 131]. These generalizations will be chosen to yield analogues of Rolle’s theorem; cf. Proposition 5.1 below. *For a finite or infinite sequence of real numbers u : $u(a), u(a + 1), \dots$, we say that $m = a$ is a “node” for u if $u(a) = 0$ and we say that $m(> a)$ is a “node” for u if either $u(m) = 0$ or $u(m - 1)u(m) < 0$. The difference equation (1.1) is called disconjugate on I^n in the restricted sense or r -disconjugate if no solution $u \not\equiv 0$ has n nodes on I^n . When $I = [a, \infty)$, (1.1) is called nonoscillatory on I if every solution $u \not\equiv 0$ has only a finite number of nodes. [Cf. [4, p. 221], where $n = 2$, I is a finite interval, and (1.1) is called “nonoscillatory” (instead of “disconjugate”) if no solution $u \not\equiv 0$ has two nodes on I^1 (instead of I^2).]*

When $n > 2$, we can use another concept of “node” and “disconjugacy”. *For a finite or infinite sequence of real numbers u : $u(a), u(a + 1), \dots$, we say that $m = a$ is a “generalized zero” for u if $u(a) = 0$, and we say that $m(> a)$ is a “generalized zero” for u if either $u(m) = 0$ or there is an integer k , $1 \leq k \leq m - a$, such that $(-1)^k u(m - k)u(m) > 0$ and, if $k > 1$, $u(m - k + 1) = \dots = u(m - 1) = 0$. This is motivated by the fact that u should be considered to have an odd or even number of generalized zeros on $(m - k, m]$ according as $u(m - k)u(m) < 0$ or > 0 . The difference equation (1.1) is called disconjugate on I^n if no solution $u \not\equiv 0$ has n generalized zeros on I^n . Obviously, if m is a node for u , then it is a generalized zero for u , so that disconjugacy implies r -disconjugacy. It turns out that the two notions, disconjugacy and r -disconjugacy, are equivalent (§5).*

If (1.1) is disconjugate on I^n , then of course no solution $u(m) \not\equiv 0$ has n zeros on I^n . But the converse is false (even if (1.2) holds), as can be seen from the example

$$u(m + 2) - 2u(m + 1) + 3u(m) = 0 \quad \text{for } m = 0, 1. \quad (1.3)$$

One advantage of introducing generalized zeros is that the following analogue of a result [12] or [19] on differential equations becomes valid: (1.1) is disconjugate ($\equiv r$ -disconjugate) if and only if $u(m) \equiv 0$ is the only solution of (1.1) having $k (> 0)$ successive zeros at $m = a, \dots, a + k - 1$ and $n - k$ successive generalized zeros at $m = j, \dots, j + n - k - 1 \in I^n$ for some $j \geq a + k$. This is false if “generalized zero” is replaced by “node”, as can be seen from the following example:

$$\begin{aligned} u(3) - 2u(2) + u(1) - u(0) &= 0 & \text{for } m = 0, \\ u(4) - u(3) + u(2) - u(1) &= 0 & \text{for } m = 1, \end{aligned} \quad (1.4)$$

with a solution $u_1(0) = 0$, $u_1(1) = 2$, $u_1(2) = 1$, $u_1(3) = 0$, and $u_1(4) = 1$ with zeros at $m = 0, 3$ and no other node, but a generalized zero at $m = 4$, and a solution $u_2(0) = 0$, $u_2(1) = 0$, $u_2(2) = 1$, $u_2(3) = 2$, $u_2(4) = 1$ with zeros at $m = 0, 1$ and no other generalized zero.

For a finite or infinite sequence u : $u(a)$, $u(a+1)$, $\dots$, the *maximal number* S^+u [or *minimal number* S^-u] of sign changes of u is the largest [or least] number of sign changes obtainable by replacing the zero elements by arbitrary nonzero elements in the sequence; see, e.g., [5, p. 100]. S^-u is also the number of sign changes when the zero elements are deleted; cf. [5, p. 100] and [18, p. 37].

PROPOSITION 1.1. *For a finite or infinite sequence u : $u(a)$, $u(a+1)$, $\dots$, which is not identically zero, S^+u is the number of generalized zeros of the sequence.*

The concepts of “node”, “generalized zero”, and Proposition 1.1 do not seem to appear in the literature of total positivity; cf. [5], [10]. The concept of disconjugacy does appear in the following form (see [5, p. 281] for finite dimensional vectors): The set of n finite or infinite dimensional vectors u_k : $u_k(a)$, $u_k(a+1)$, $\dots$, $1 \leq k \leq n$, is said to have the property T^+ if $S^+u < n$ for all vectors $u = c_1u_1 + \dots + c_nu_n$ for all n -tuples $(c_1, \dots, c_n) \neq 0$. In view of Proposition 1.1, this is equivalent to the disconjugacy of (1.1) when $u = u_k(m)$, $m \in I^n$ and $1 \leq k \leq n$, are linearly independent solutions of (1.1).

The main results here, with the emphasis on the difference equation (1.1) and results on $m \rightarrow \infty$, are different from those in [5], although some arguments (involving standard identities on determinants) are similar. Many of the proofs for (1.1) are similar to those for differential equations and, for the sake of brevity, will be omitted or only indicated. The principal tools in [6], [7], [13], [8] for differential equations are identities for Wronskian determinants, Rolle's theorem, and results of [9]. We enumerate equivalent identities for determinants in §2, an analogue of Rolle's theorem is Proposition 5.1 below, while results of [9] are stated for both differential and difference equations.

2. Wronskian determinants. Below a , b , i , j , k , m , n , μ and ν denote integers, and m denotes the independent (integral) variable. If $I = [a, b]$ or $I = [a, \infty)$ is an interval of integers, and j is an integer (≤ 0 or ≥ 0) such that $j > -\text{card } I$, then $I^j = [a, b+j]$ or $I^j = [a, \infty)$, so that $(I^j)^k = I^{j+k}$, where these are defined, and $I^j = I$ for all j if $\text{card } I = \infty$.

Except at the end of this section, the equation (1.1) is not involved. Thus, since $u_k(a)$, $u_k(a+1)$, $\dots$ are arbitrary numbers, most of the results are or are related to standard identities on determinants, but we use the terminology

of “Wronskian determinants” for suggestiveness and economy of notation. These identities are enumerated here for convenience in the most useful form for us, generally without proofs.

For any function $u = u(m)$, we sometimes write

$$u \text{ for } u(m) \text{ and } u^* \text{ for } u(m+1). \quad (2.0)$$

As usual,

$$\begin{aligned} W(u_1, \dots, u_k) &= W(u_1, \dots, u_k)(m) \\ &= \det(u_j(m+i-1)), \text{ where } i, j = 1, \dots, k, \end{aligned} \quad (2.1)$$

is called the Wronskian of $u_1, \dots, u_k$ at m . If convenient, we write W^k in place of W to indicate that it is a $k \times k$ determinant in which the j th column is $u_j(m), \dots, u_j(m+k-1)$. We put $\Delta^0 u(m) = u(m)$,

$$\begin{aligned} \Delta u(m) &= u(m+1) - u(m) = u^* - u \text{ and} \\ \Delta^k u(m) &= \sum_{j=0}^k (-1)^j \binom{k}{j} u(m+j), \end{aligned} \quad (2.2)$$

so that $\Delta = \Delta^1$, $\Delta^k = \Delta^{k-1}\Delta$,

$$W(u_1, \dots, u_k) = \det(\Delta^{i-1}u_j(m)), \text{ where } i, j = 1, \dots, k, \quad (2.3)$$

$$\Delta(u/v) = W(v, u)/v v^*. \quad (2.4)$$

We have the following obvious identity

$$W(u_1, \dots, u_k) = W(u_1/v, \dots, u_k/v) \prod_{\mu=0}^{k-1} v(m+\mu), \quad (2.5)$$

when $v \neq 0$. On replacing k by $k+1$, $(u_1, \dots, u_k)$ by $(u_0, \dots, u_k)$ and v by u_0 , (2.5) implies

$$W^{k+1}(u_0, \dots, u_k) = W^k(\Delta(u_1/u_0), \dots, \Delta(u_k/u_0)) \prod_{\mu=0}^k u_0(m+\mu). \quad (2.6)$$

By (2.4) and the case $v(m) = u_0 u_0^*$ of (2.5), (2.6) gives

$$W^k[W^2(u_0, u_1), \dots, W^2(u_0, u_k)] = W^{k+1}(u_0, \dots, u_k) \prod_{\mu=1}^{k-1} u_0(m+\mu). \quad (2.7)$$

The relations (2.6), (2.7) and an induction on j lead to a variant of Sylvester's theorem (cf. [5, p. 15]),

$$\begin{aligned} W^k[W^{j+1}(u_1, \dots, u_j, v_1), \dots, W^{j+1}(u_1, \dots, u_j, v_k)] \\ = W^{j+k}(u_1, \dots, u_j, v_1, \dots, v_k) \prod_{\mu=1}^{k-1} W^j(u_1, \dots, u_j)(m+\mu). \end{aligned} \quad (2.8)$$

By (2.4) and the case $k = 2$ of (2.8),

$$\Delta \left\{ \frac{W^{j+1}(u_1, \dots, u_j, w)}{W^{j+1}(u_1, \dots, u_j, v)} \right\} = \frac{W^{j+2}(u_1, \dots, u_j, v, w) W^j(u_1, \dots, u_j)^*}{W^{j+1}(u_1, \dots, u_j, v) W^{j+1}(u_1, \dots, u_j, v)^*}. \quad (2.9)$$

Arguments similar to those used for the proof of Corollary 2.2 of [6, p. 311] (cf. also [5, pp. 297–298]) give

$$W^k(x, u_1, \dots, \hat{u}_j, \dots, u_k) \omega_{k-1} = W^{k-1}(x, u_1, \dots, \hat{u}_j, \dots, u_{k-1}) \omega_k + W^k(x, u_1, \dots, u_{k-1}) W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k), \quad (2.10)$$

where $\hat{u}_j$ indicates the omission of u_j , $x = x(m)$ is arbitrary, and

$$\omega_k = \omega_k(m) = W(u_1, \dots, u_k). \quad (2.11)$$

PROPOSITION 2.1. *Let $k \geq 2$ and*

$$\omega_0 = 1 \quad \text{and} \quad \omega_1 > 0, \dots, \omega_k > 0. \quad (2.12)$$

Then, for $1 \leq j \leq k-1$,

$$W^k(x, u_1, \dots, \hat{u}_j, \dots, u_k) = \omega_k \sum_{\mu=j}^k W^{\mu-1}(u_1, \dots, \hat{u}_j, \dots, u_\mu) W^\mu(x, u_1, \dots, u_{\mu-1}) / \omega_{\mu-1} \omega_\mu, \quad (2.13)$$

where $W(x, u_1, \dots, u_{\mu-1}) = x$ for $\mu = 1$ and $W^{j-1}(u_1, \dots, \hat{u}_j, u_\mu) = \omega_{j-1}$ for $\mu = j$.

If $j = k-1$, (2.13) follows from (2.10). In particular, the proposition holds for $k = 2$, and the general case of (2.13) follows from an induction on k by substituting the case $k-1$ of (2.13) into (2.10); cf. [7, p. 442].

We can verify

$$\omega_{k-1} W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k)^* = \omega_k W^{k-2}(u_1, \dots, \hat{u}_j, \dots, u_{k-1})^* + W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k) \omega_{k-1}^* \quad (2.14)$$

for $1 \leq j \leq k$ and $k \geq 2$ by writing (2.14) in the form

$$W^2[\omega_{k-1}, W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k)] = \omega_k W^{k-2}(u_1, \dots, \hat{u}_j, \dots, u_{k-1})^*,$$

and applying (2.8) when $k = 2$ and $j+1 = k-1$. An induction on k then gives the following proposition (in which (2.15) for $j = k-1$ is implied by (2.14)).

PROPOSITION 2.2. *Let $k \geq 2$ and (2.12) hold. Then, for $1 \leq j \leq k$,*

$$W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k)^* = \omega_k \sum_{\mu=j}^k W^{\mu-1}(u_1, \dots, \hat{u}_j, \dots, u_\mu) \omega_{\mu-1}^* / \omega_{\mu-1} \omega_\mu. \quad (2.15)$$

DEFINITION. We say that the set of functions $u_1, \dots, u_{n-1}$ is a $w_n(I)$ -system if they are defined on I , $\text{card } I \geq n - 1$, and

$$\omega_k(m) = W(u_1, \dots, u_k)(m) > 0 \quad \text{for } m \in I^{1-k}, 1 \leq k < n. \quad (2.16)$$

Also, $u_1, \dots, u_{n-1}$ is a $W_n(I)$ -system if

$$W^k(u_{i(1)}, \dots, u_{i(k)})(m) > 0 \quad \text{for } m \in I^{1-k}, 1 \leq k < n, \quad (2.17)$$

for every set of indices $1 \leq i(1) < \dots < i(k) < n$.

DEFINITION. We say that a set of functions $u_1, \dots, u_{n-1}$ is a $\tilde{W}_n(I)$ -system if they are defined on I , form a $w_n(I)$ -system, and satisfy

$$W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k)(m) \geq 0 \quad \text{for } m \in I^{2-k}, 1 \leq j \leq k < n. \quad (2.18)$$

Introduce the notation,

$$D(i(1), \dots, i(k); \mu(1), \dots, \mu(k)) = \det(u_{i(p)}(\mu(q)), p, q = 1, \dots, k), \quad (2.19)$$

is the determinant in which the p th column is $(u_{i(p)}(\mu(1)), \dots, u_{i(p)}(\mu(k)))$, where $i(1) < \dots < i(k)$ and $\mu(1) < \dots < \mu(k)$. Also, let

$$D_k(\mu(1), \dots, \mu(k)) = D(1, \dots, k; \mu(1), \dots, \mu(k)); \quad (2.20)$$

so that, in particular, $W(u_1, \dots, u_k)(m) = D_k(m, \dots, m + k - 1)$.

DEFINITION. We say that the set of functions $u_1, \dots, u_{n-1}$ is a $Dw_n(I)$ -system if they are defined on I , $\text{card } I \geq n - 1$, and

$$D_k(\mu(1), \dots, \mu(k)) > 0 \quad (2.21)$$

for $1 \leq k < n$ and all sets of indices $\mu(1) < \dots < \mu(k)$, $\mu(j) \in I$. Also, $u_1, \dots, u_{n-1}$ is a $DW_n(I)$ -system if

$$D(i(1), \dots, i(k); \mu(1), \dots, \mu(k)) > 0 \quad (2.22)$$

for $1 \leq k < n$ and all sets of indices $1 \leq i(1) < \dots < i(k) < n$, $\mu(1) < \dots < \mu(k)$, $\mu(j) \in I$.

In [6], we introduced the terminology of w_n - and W_n -systems of functions as we were not aware of earlier terminology for these notions. We have continued this notation here to facilitate the adaptation of the proofs of [6], [7] to the situation here. We should remark that the theorem of [5, p. 296] shows that a $Dw_n(I)$ -system is a Markov system in the sense of [5]. Also, $u_1, \dots, u_{n-1}$ is a $Dw_n(I)$ -system if and only if it is a Descartes system [10, p. 25].

PROPOSITION 2.3. *Let $u_1, \dots, u_{n-1}$ be a $w_n(I)$ -system. Then a necessary and sufficient condition that $u_1, \dots, u_{n-1}$ be a $W_n(I)$ -system is that*

$$W(u_1, \dots, u_k)(m) > 0 \quad \text{for } 1 \leq j \leq k < n \quad (2.23)$$

hold for $m = a$,

This is a slight generalization of a theorem of Fekete [3] in which the assumption is that (2.23) hold for all admissible m .

PROOF. The necessity of (2.23) is clear. In order to prove the converse, we first show that (2.23), for $m = a$, and (2.16) imply that, for $m = a$,

$$W(u_{i(1)}, \dots, u_{i(k)})(m) > 0 \quad \text{for } 1 \leq k < n \quad (2.24)$$

and arbitrary sets of indices $1 \leq i(1) < \dots < i(k) < n$. This is clear if $k = 1$ or $k = 2$. And if we assume the set of inequalities in (2.24) for a given k at $m = a$, then they follow for $k + 1$ from Proposition 2.1.

Thus the proof will be complete if we show that (2.23), for $m = a$, and (2.16) imply (2.23) for $m \in I^{j-k}$. The inequalities (2.23) hold for $1 = j \leq k < n$, by (2.16). Assume their validity if $j > 1$ and j is replaced by $j - 1$. Note that $\Delta(u_j/u_{j-1}) = W(u_{j-1}, u_j)/u_{j-1}u_j^* > 0$ for $m \in I^{-1}$. Thus $u_j(m) > 0$, by (2.23) at $m = a$, implies that $u_j(m) > 0$ for $m \in I$. This gives (2.23) for $k = j$. Assume its validity if k is replaced by $k - 1$, when $1 < j \leq k - 1 < n - 1$. By (2.9),

$$\Delta \left\{ \frac{W(u_j, \dots, u_k)}{W(u_{j-1}, \dots, u_{k-1})} \right\} = \frac{W(u_{j-1}, \dots, u_k)W(u_j, \dots, u_{k-1})^*}{W(u_{j-1}, \dots, u_{k-1})W(u_{j-1}, \dots, u_{k-1})^*}.$$

By a similar argument, $W(u_j, \dots, u_k) > 0$ for $m \in I^{j-k}$. This completes the proof.

As an immediate corollary, we have

PROPOSITION 2.4. Let $u_1, \dots, u_n$ be a $W_{n+1}(I)$ -system, $\text{card } I \geq n + 1$. Let $u_0(m)$, $m \in I$, satisfy $W(u_0, \dots, u_k) > 0$ for $m \in I^{1-k}$, $0 \leq k \leq n$. Then $u_0, \dots, u_n$ is a $W_{n+2}(I)$ -system.

An analogue of Proposition 2.3 is

PROPOSITION 2.5. Let $u_1, \dots, u_{n-1}$ be a $w_n(I)$ -system and satisfy

$$W(u_1, \dots, \hat{u}_j, \dots, u_k)(a) \geq 0 \quad \text{for } 1 \leq j \leq k < n. \quad (2.25)$$

Then $u_1, \dots, u_{n-1}$ is a $\tilde{W}_n(I)$ -system, i.e., (2.18) holds.

PROOF. It is clear that (2.18) holds for $m \in I^{2-k}$ for $k = 2$ with $1 \leq j \leq k$. Assume the validity of (2.18) if $k > 2$ and k is replaced by $k - 1$, $1 \leq j \leq k - 1 < n - 1$. By (2.8), with $k = 2$ and $j + 1 = k - 1$,

$$\begin{aligned} & W^2[W^{k-1}(u_1, \dots, u_{k-1}), W^{k-1}(u_1, \dots, \hat{u}_j, \dots, u_k)] \\ &= W(u_1, \dots, u_k)W^{k-2}(u_1, \dots, \hat{u}_j, \dots, u_{k-1}) \geq 0. \end{aligned}$$

Hence by the definition of W^2 ,

$$\begin{aligned} & W(u_1, \dots, \hat{u}_j, \dots, u_k)^* \\ & \geq W(u_1, \dots, u_{k-1})^* W(u_1, \dots, \hat{u}_j, \dots, u_k) / W(u_1, \dots, u_{k-1}); \end{aligned}$$

and so, the inequality (2.18) is obtained successively for $m = a + 1, a + 2, \dots$

PROPOSITION 2.6. *Let I be a finite interval and let (1.2) hold in (1.1). Let $u_1, \dots, u_{n-1}$ be a $w_n(I)$ -system [and/or a $W_n(I)$ -system]. Then there exist functions $u_0(m)$ and $u_n(m)$, $m \in I$, such that $u_0, \dots, u_{n-1}$ and $u_1, \dots, u_n$ are $w_{n+1}(I)$ -systems [and/or $W_{n+1}(I)$ -systems], also $(-1)^n Pu_0 > 0$ and $Pu_n > 0$ for $m \in I$.*

It suffices to choose $u_0(m) = \varepsilon^m$ and $u_n(m) = \lambda^m$ for sufficiently small $\varepsilon > 0$ and sufficiently large $\lambda > 1$.

Recall that a standard result on Wronskians of solutions of (1.1) is

PROPOSITION 2.7. *Let $u_1, \dots, u_n$ be solutions of (1.1). Then*

$$W(u_1, \dots, u_n)(m+1) = (-1)^n a_0(m) W(u_1, \dots, u_n)(m) \quad \text{for } m \in I. \quad (2.26)$$

3. Reduction to first order systems. In order to make the results of [9] available, it will be convenient to write the n th order linear difference equation (1.1) for the scalar u as a suitable first order difference equation for a vector $y = (y_1, \dots, y_n)$; cf. [7, pp. 443–445].

PROPOSITION 3.1. *Let $u_1, \dots, u_n$ be a $w_{n+1}(I^n)$ -system. Then (1.1), for $m \in I$, is equivalent to a first order system*

$$\Delta y = -Ay, \quad \text{i.e., } y(m+1) = (I - A(m))y(m) \quad \text{for } m \in I, \quad (3.1)$$

where $A(m)$ is an $n \times n$ matrix function for $m \in I$,

$$y_k = W^k(u, u_1, \dots, u_{k-1})/\omega_k \quad \text{for } 1 \leq k \leq n \quad (\text{so that } y_1 = u/u_1), \quad (3.2)$$

and (3.1) can be written as

$$\Delta y_k = -y_{k+1}\omega_{k+1}\omega_{k-1}^*/\omega_k\omega_k^* \quad \text{for } 1 \leq k < n, \quad (3.3)$$

$$\Delta y_n = -(\omega_{n-1}^*/\omega_n^*) \sum_{j=1}^n \sum_{k=j}^n (-1)^{n+j} (Pu_j) W(u_1, \dots, \hat{u}_j, \dots, u_k) y_k / \omega_{k-1}. \quad (3.4)$$

Also, we have

$$\det(I - A) = (-1)^n \alpha_0 \omega_n / \omega_n^*. \quad (3.5)$$

If $u_1, \dots, u_n$ is a set of solutions of (1.1) or is a $\tilde{W}_{n+1}(I^n)$ -system and

$$(-1)^{n+j} (Pu_j) > 0 \quad \text{for } m \in I, 1 \leq j < n, \quad (3.6)$$

then the entries of the matrix A are nonnegative on I and if, in addition, $(-1)^n \alpha_0 > 0$, then the entries of $(I - A)^{-1}$ are also nonnegative on I .

PROOF. By the definition of y_k in (3.2), (2.9) implies

$$\Delta y_k = -W^{k+1}(u, u_1, \dots, u_k)\omega_{k-1}^*/\omega_k\omega_k^* \quad \text{for } 1 \leq k \leq n; \quad (3.7)$$

in particular, (3.3) holds. When u is a solution of (1.1), we obtain

$$W^{n+1}(u, u_1, \dots, u_n) = \sum_{j=1}^n (-1)^{n+j}(Pu_j)W^n(u, u_1, \dots, \hat{u}_j, \dots, u_n)$$

by adding α_i times the i th row of the matrix in W^{n+1} to the last for $i = 0, \dots, n-1$, and expanding along the last row; cf. [7, p. 444]. Hence (2.13), with $k = n$, and (3.2) imply (3.4). The arguments leading to (3.3)–(3.4) are reversible, so that (1.1) is equivalent to (3.3)–(3.4) by virtue of (3.2).

If $x = (x_1, \dots, x_n)$ is a vector, then we can write the equation $(I - A)x = y$ as

$$\begin{aligned} x_k &= \lambda_k x_{k+1} + y_k \quad \text{for } 1 \leq k < n, \text{ where } \lambda_k = \omega_{k+1}\omega_{k-1}^*/\omega_k\omega_k^*, \\ x_n &= \sum_{k=1}^n A_{nk}x_k + y_n, \end{aligned} \quad (3.8)$$

where, by (3.4),

$$A_{nk} = (\omega_{n-1}^*/\omega_n^*) \sum_{j=1}^k (-1)^{n+j}(Pu_j)W(u_1, \dots, \hat{u}_j, \dots, u_k)/\omega_{k-1}. \quad (3.9)$$

Thus, we have

$$x_k = x_n \left(\prod_{\mu=k}^{n-1} \lambda_\mu \right) + \sum_{j=k}^{n-1} y_j \left(\prod_{\mu=k}^{j-1} \lambda_\mu \right) \quad \text{for } 1 \leq k < n. \quad (3.10)$$

Hence

$$x_n \left[1 - \sum_{k=1}^n A_{nk} \left(\prod_{\mu=k}^{n-1} \lambda_\mu \right) \right] = \sum_{k=1}^n A_{nk} \sum_{j=k}^{n-1} \left(\prod_{\mu=k}^{j-1} \lambda_\mu \right) y_j + y_n. \quad (3.11)$$

A simple induction on n shows that

$$\det(I - A) = 1 - \sum_{k=1}^n A_{nk} \left(\prod_{\mu=k}^{n-1} \lambda_\mu \right). \quad (3.12)$$

By the definition of $\lambda_1, \dots, \lambda_{n-1}$ in (3.8), we have

$$\prod_{\nu=k}^{j-1} \lambda_\nu = \omega_j\omega_{k-1}^*/\omega_k\omega_{j-1}^* \quad \text{for } 2 \leq j \leq n, \quad (3.13)$$

so that $\omega_n^* \det(I - A)$ is given by

$$\omega_n^* + (-1)^n \omega_n \sum_{k=1}^n \sum_{j=1}^k (-1)^{j+1}(Pu_j)W(u_1, \dots, \hat{u}_j, \dots, u_k)\omega_{k-1}^*/\omega_{k-1}\omega_k. \quad (3.14)$$

If we interchange the order of summation over j, k and note that $Pu_j = \sum \alpha_i(m)u_j(m+i)$, (3.14) becomes

$$\begin{aligned} \omega_n^* + (-1)^n \omega_n \sum_{i=0}^n \alpha_i \sum_{j=1}^n (-1)^{j+1} u_j(m+i) \\ \cdot \sum_{k=j}^n W(u_1, \dots, \hat{u}_j, \dots, u_k) \omega_{k-1}^* / \omega_{k-1} \omega_k. \end{aligned}$$

Applying (2.15), with $k = n$, to the inner sum, this becomes

$$\omega_n^* + (-1)^n \sum_{i=0}^n \alpha_i \sum_{j=1}^n (-1)^{j+1} u_j(m+i) W(u_1, \dots, \hat{u}_j, \dots, u_n)^*. \quad (3.15)$$

The inner sum over j represents the expansion along the first row of the $n \times n$ determinant in which the first row is $(u_1(m+i), \dots, u_n(m+i))$ and the k th row is $(u_1(m+k-1), \dots, u_n(m+k-1))$ for $k = 2, \dots, n$. Thus the inner sum is 0 for $i = 1, \dots, n-1$, and is ω_n and $(-1)^{n-1} \omega_n^*$ for $i = 0$ and $i = n$. Hence (3.15) is $(-1)^n \alpha_0 \omega_n$ (since $\alpha_n = 1$). This proves (3.5).

The last assertion in Proposition 3.1 is clear from (3.3)–(3.4) and from (3.10)–(3.11).

PROPOSITION 3.2. *Assume (1.2) for (1.1). Let $u_1, \dots, u_n$ be a $w_{n+1}(I^n)$ -system, $u_1, \dots, u_{n-1}$ a $\tilde{W}_n(I^n)$ -system, and*

$$(-1)^{n+j} P u_j \geq 0 \quad \text{on } I \text{ for } 1 \leq j \leq n. \quad (3.16)$$

Then, for $m \in I$, (1.1) is equivalent to a first order system for a vector $z = (z_1, \dots, z_n)$,

$$\Delta z = -Bz, \quad \text{i.e., } z(m+1) = (I - B(m))z(m) \text{ for } m \in I, \quad (3.17)$$

where $B(m)$ is an $n \times n$ matrix function for $m \in I$,

$$z_k = W^k(u, u_1, \dots, u_{k-1}) / \omega_k \quad \text{for } 1 \leq k < n, \quad (3.18)$$

$$z_n = \tau W^n(u, u_1, \dots, u_{n-1}) / \omega_n, \quad (3.19)$$

$\tau = \tau(m) > 0$ for $m \in I^n$, and (3.17) can be written as

$$\Delta z_k = -z_{k+1} \omega_{k+1} \omega_{k-1}^* / \omega_k \omega_k^* \quad \text{for } 1 \leq k < n-1, \quad (3.20)$$

$$\Delta z_{n-1} = -z_n \omega_n \omega_{n-2}^* / \omega_{n-1} \omega_{n-1}^* \tau, \quad (3.21)$$

$$\Delta z_n = -(\tau^* \omega_{n-1}^* / \omega_n^*) \sum_{k=1}^{n-1} \sum_{j=1}^k (-1)^{n+j} (P u_j) W(u_1, \dots, \hat{u}_j, \dots, u_k) z_k / \omega_{k-1}; \quad (3.22)$$

also, $\det(I - B) > 0$, and the entries in B and in $(I - B)^{-1}$ are nonnegative for $m \in I$.

PROOF. We indicate how to obtain this from Proposition 3.1. Make the change of dependent variables given by $z_k = y_k$ for $1 \leq k < n$ and $z_n = \tau y_n$. Thus (3.20)–(3.21) follow from (3.3). Write (3.4) as

$$y_n^* - (1 - A_{nn}) y_n = - \sum_{k=1}^{n-1} A_{nk} y_k = - \sum_{k=1}^{n-1} A_{nk} z_k.$$

If we multiply this equation by $\tau^* = \tau(m+1)$, we obtain (3.22) provided that $\tau(m) = (1 - A_{nn})\tau(m+1)$. Note that, by (3.5) and (3.12),

$$1 - A_{nn} > \sum_{k=1}^{n-1} A_{nk} \left(\prod_{\nu=k}^{n-1} \lambda_{\nu} \right) \geq 0.$$

Thus, we can choose $\tau(m)$, $m \in I$, by letting $\tau(a) = 1$ and

$$\tau(m+1) = \tau(m) / (1 - A_{nn}) = 1 / \prod_{\nu=1}^m (1 - A_{nn}(\nu)).$$

The equation $(I - B)x = z$ can be written

$$x_k = \lambda_k x_{k+1} + z_k \quad \text{for } 1 \leq k < n-1, \quad x_{n-1} = \lambda_{n-1} x_n / \tau + z_{n-1},$$

where λ_k is defined in (3.8), and

$$x_n = \tau^* \sum_{k=1}^{n-1} A_{nk} x_k + z_n.$$

By the arguments in the last proof, we see that

$$\det(I - B) = 1 - (\tau^* / \tau) \sum_{k=1}^{n-k} A_{nk} \left(\prod_{\nu=k}^{n-1} \lambda_{\nu} \right) = (1 - A_{nn})^{-1} \det(I - A),$$

and that the entries in B and $(I - B)^{-1}$ are nonnegative.

4. A factorization theorem. Standard textbooks give a “variation of constants” formula for the solution of linear inhomogeneous difference equations, but do not state it in the following form. Let $U(m, \nu)$, defined on $I^n \times I$, be the Cauchy function for (1.1); i.e., for fixed $\nu \in I$, $u(m) = U(m, \nu)$ is the solution of (1.1) satisfying the initial conditions

$$U(m, \nu) = 0 \quad \text{for } m = \nu + 1, \dots, \nu + n - 1 \quad \text{and} \quad U(\nu + n, \nu) = 1. \quad (4.1)$$

The condition $\alpha_0 \neq 0$ is needed to assure the definition of $U(m, \nu)$ for $a \leq m \leq \nu$. If $u_1, \dots, u_n$ are linearly independent solutions of (1.1), then

$$U(m, \nu) = \sum_{j=1}^n (-1)^{n+j} u_j(m) W(u_1, \dots, \hat{u}_j, \dots, u_n)(\nu + 1) / \omega_n(\nu + 1), \quad (4.2)$$

where the sum is the $n \times n$ determinant in which the j th column is $(u_j(\nu + 1), \dots, u_j(\nu + n - 1), u_j(m))$. In particular,

$$U(m, m) = (-1)^{n-1} \omega_n(m) / \omega_n(m + 1). \quad (4.3)$$

PROPOSITION 4.1. *The unique solution of*

$$(Pv)(m) = f(m) \quad \text{for } m \in I, v(a) = \dots = v(a + n - 1) = 0, \quad (4.4)$$

is given by

$$v(m) = \sum_{\nu=a}^{m-1} U(m, \nu) f(\nu) \quad \text{for } m \in I^n. \quad (4.5)$$

Note that if $I = [a, b]$ is finite, then (4.1) implies that $U(m, \nu) = 0$ in (4.5) for $m \in I^n - I$, $\nu > b$, so that we should replace the sum by one over the range $a < \nu < \max(m-1, b)$ or define $f(m)$ arbitrarily for $m > b$.

PROPOSITION 4.2. *Let $1 < k < n$ and $u_1, \dots, u_k$ be solutions of (1.1) such that $\omega_k = W(u_1, \dots, u_k) \neq 0$ on I^{n-k+1} . Then there exists a unique difference equation of order $n - k$,*

$$P_1 v \equiv \sum_{j=0}^{n-k} \beta_j(m) v(m+j) = 0, \quad m \in I, \quad (4.6)$$

such that

$$\beta_{n-k} = 1 \quad \text{and} \quad \beta_0 = (-1)^k \alpha_0 \omega_k / \omega_k^* \neq 0, \quad (4.7)$$

and v is a solution of (4.6) if and only if there is a solution u of (1.1) such that, on I^{n-k} ,

$$v = W(u, u_1, \dots, u_k). \quad (4.8)$$

This corresponds to the Mammana [15] type of factorization for linear differential equations (and an obvious analogue of the proof of Proposition 4.2 below can be used to give a new proof of Mammana's theorem). Let

$$P_2 w \equiv W(w, u_1, \dots, u_k) \equiv \sum_{j=0}^k \gamma_j(m) w(m+j), \quad m \in I^{n-k}, \quad (4.9)$$

where $\gamma_0(m) = \omega_k^* \neq 0$ and $\gamma_k(m) = (-1)^k \omega_k \neq 0$. Then Proposition 4.2 implies the factorization

$$P = P_1 P_2. \quad (4.10)$$

PROOF. Let $W(m, \nu)$, defined in $I^n \times I^{n-k}$, be the Cauchy function for (4.9), i.e., for fixed ν , $w(m) = W(m, \nu)$ is the solution of

$$P_2 w = 0, \quad w = 0 \quad \text{for } \nu < m < \nu + k \quad \text{and} \quad w = 1 \quad \text{if } m = \nu + k. \quad (4.11)$$

Define $u = P_2^{-1} v$ as the unique solution of

$$P_2 u = v \quad \text{and} \quad u = 0 \quad \text{for } m = a, \dots, a + k - 1,$$

so that

$$u = P_2^{-1} v = \sum_{\nu=a}^{m-1} W(m, \nu) v(\nu) \quad \text{for } m \in I^n. \quad (4.12)$$

We define $P_1 v$ by $P_1 v = P P_2^{-1} v$, and show that it has the desired form (4.6). To this end, note that, for $i = 0, \dots, k - 1$,

$$u(m+i) = \sum_{\nu=a}^{m-1} W(m+i, \nu)v(\nu)$$

and that, for $i = k, \dots, n$,

$$u(m+i) = \sum_{\nu=m}^{m+i-k} W(m+i, \nu)v(\nu) + \sum_{\nu=a}^{m-1} W(m+i, \nu)v(\nu).$$

Since $w(\cdot) = W(\cdot, \nu)$ is a solution of $P_2 w = 0$ and hence of $Pw = 0$, it is seen that $P_1 v = Pu$ is

$$P_1 v = \sum_{i=k}^n \alpha_i(m) \sum_{\nu=m}^{m+i-k} W(m+i, \nu)v(\nu) \equiv \sum_{j=0}^{n-k} \beta_j(m)v(m+j),$$

where the coefficients

$$\beta_j(m) = \sum_{i=j+k}^n \alpha_i(m) W(m+i, j+m) \quad \text{for } j = 0, \dots, n-k,$$

are defined for $m \in I$. It follows that $\beta_{n-k} = \alpha_n = 1$ and, since $PW(\cdot, \nu) = 0$,

$$\beta_0 = - \sum_{i=0}^{k-1} \alpha_i(m) W(m+i, m) = -\alpha_0(m) W(m, m).$$

As $W(m, m) = (-1)^{k-1} \omega_k / \omega_k^*$ by the analogue of (4.3), the proof is complete.

5. Disconjugacy. In this section, we prove most of the following result. The proof is completed in §8 where we obtain the implication (c) $\Rightarrow$ (e) for infinite intervals.

THEOREM 5.1. *The following are equivalent:*

- (a) (1.1) has a $w_{n+1}(I^n)$ -system of solutions;
- (b) there exist positive functions $p_k(m)$, $m \in I^{n-k+1}$, for $1 \leq k \leq n$ such that if $\Delta_k u = \Delta(p_k u) = p_k(m+1)u(m+1) - p_k(m)u(m)$, then

$$Pu = p_{n+1} \Delta_n \cdots \Delta_1 u = p_{n+1} \Delta \{ p_n \Delta [\cdots \Delta(p_1 u)] \}, \quad (5.1)$$

where $p_{n+1} = 1/p_1 \cdots p_n$ for $m \in I$;

- (c) (1.1) is disconjugate on I^n ;
- (d) (1.1) is r -disconjugate on I^n ;
- (e) (1.1) has a $DW_{n+1}(I^n)$ -system of solutions;
- (f) $u(m) \equiv 0$ is the only solution of (1.1) having k (> 0) successive zeros at $m = a, \dots, a+k-1$ and $n-k$ successive generalized zeros at $m = j, \dots, j+n-k-1$ for some $j, a+k \leq j \in I^{k+1}$;
- (g) if $u = u_k(m)$, $m \in I^n$ and $1 \leq k \leq n$, are linearly independent solutions of (1.1), then the determinants $D_n(\mu(1), \dots, \mu(n))$ do not vanish and are of the same sign for all $\mu(1) < \cdots < \mu(n), \mu(j) \in I^n$.

The factorization (5.1) of P is the analogue of the Frobenius factorization

(cf. Pólya [17]) for disconjugate differential equations. By virtue of Proposition 1.1, the equivalence of (c) and (g) is a theorem of Krein and Gantmacher [5, Satz 1, p. 283] (where “ $m \geq n$ ” should be replaced by “ $m > n$ ”).

We can also formulate necessary or sufficient conditions for the disconjugacy of (1.1) in terms of its coefficients. To this end, let $\Phi = (\phi_{jm})$ be the rectangular matrix, with $j \in I$ as row index and $m \in I^n$ as column index, defined by

$$\phi_{jm} = 0, \quad \phi_{jm} = \alpha_{m-j}(m) \quad \text{or} \quad \phi_{jm} = 0 \quad (5.2)$$

according as $a \leq m < j$, $j \leq m \leq j + n$, $m > j + n$. Thus if one considers $u = u(m)$ as a vector $(u(a), u(a+1), \dots)$, then $Pu = \Phi u$. Put

$$\Phi(j(1), \dots, j(k); \mu(1), \dots, \mu(k)) = \det(\phi_{j(i), \mu(m)}; i, m = 1, \dots, k) \quad (5.3)$$

for $a \leq j(1) < \dots < j(k) \in I$, $a \leq \mu(1) < \dots < \mu(k) \in I^n$, also

$$\Phi_k(\mu(1), \dots, \mu(k)) = \Phi(a, \dots, a + k - 1; \mu(1), \dots, \mu(k)). \quad (5.4)$$

The Krein-Gantmacher criterion (g) will be used to prove the first part of the following theorem in which there is no loss of generality in assuming that I is finite. We shall use an induction on n and the factorization of Proposition 4.2 to obtain the second.

THEOREM 5.2. *Let $I = [0, b]$, $d = \text{card } I = b + 1$. (i) A necessary and sufficient condition for (1.1) to be disconjugate on I^n is that*

$$(-1)^{nd + d(d-1)/2 + \sum \mu(m)} \Phi_d(\mu(1), \dots, \mu(d)) > 0 \quad (5.5)$$

for $0 \leq \mu(1) < \dots < \mu(d) \leq b + n$. (ii) A necessary condition for (1.1) to be disconjugate on I^n is that

$$(-1)^{kn + \sum j(i) + \sum \mu(m)} \Phi(j(1), \dots, j(k); \mu(1), \dots, \mu(k)) > 0 \quad (5.6)$$

for $0 \leq j(1) < \dots < j(k) \leq d$, $0 \leq \mu(1) < \dots < \mu(k) \leq b + n$, except in trivial cases resulting from $\phi_{jm} = 0$ if $0 \leq m < j$ and $j + n < m \leq b + n$; in particular,

$$(-1)^{n+j} \alpha_j(m) > 0 \quad \text{for } 0 \leq j \leq n, m \in I. \quad (5.7)$$

In particular, a necessary condition for disconjugacy is that the rectangular matrix $(-1)^n \Phi$ be “sign regular”; cf. [5, p. 86]. The necessary conditions (5.7) are not sufficient; cf. the example in (1.4).

We first prove Theorem 5.1 and then Theorem 5.2.

PROOF OF THEOREM 5.1. In this section, we prove the implications $a \Leftrightarrow (b)$ and $(a) \Rightarrow (c)$ for arbitrary intervals I , and the implications $(d) \Rightarrow (a)$, $(a) \Rightarrow (e)$, and $(f) \Rightarrow (a)$ for finite intervals, while $(c) \Rightarrow (d)$, $(c) \Rightarrow (f)$ and $(e) \Rightarrow (a)$ are trivial. This gives a complete proof of Theorem 5.1 for finite I . Theorem 8.1 below gives $(c) \Rightarrow (e)$ for infinite intervals. As $(d) \Rightarrow (a)$ and $(f) \Rightarrow (a)$ on finite intervals imply $(d) \Rightarrow (c)$ and $(f) \Rightarrow (c)$ on arbitrary intervals, the validity

of Theorem 5.1 for infinite intervals follows.

On (a) $\Rightarrow$ (b). This can be obtained from Proposition 4.2 and an induction on n or, as we shall proceed, directly from Proposition 3.1. Let $u_1, \dots, u_n$ be a $w_{n+1}(I^n)$ -system of solutions. Then (1.1) is equivalent to the first order system (3.3)–(3.4), where (3.4) is merely $\Delta y_n = 0$. Hence (5.1) holds with $p_1 = 1/u_1$, $p_k = \omega_{k-1}\omega_{k-1}^*/\omega_k\omega_{k-2}^*$ for $2 \leq k \leq n$, and $p_{n+1} = 1/p_1 \cdots p_n = \omega_n\omega_{n-2}^*\omega_n$.

On (b) $\Rightarrow$ (a). Define u_k successively for $k = 1, \dots, n$ as solutions of $\Delta_k \cdots \Delta_1 u_k = 0$ such that $\omega_k > 0$; cf. Proposition 2.7.

We shall use here (and later in the paper) the following analogue of Rolle's theorem.

PROPOSITION 5.1. *Suppose that the finite sequence $v(1), \dots, v(j)$ has N_j nodes [or generalized zeros] and that $\Delta v(1), \dots, \Delta v(j-1)$ has M_j nodes [or generalized zeros]. Then $M_j \geq N_j - 1$.*

PROOF. The proposition is obvious if $j = 2$. Assume $j > 2$ and that the proposition is valid if j is replaced by an integer $i < j$. If $N_j = N_{j-1}$, then the proposition holds. Suppose therefore that $m = j$ is a node, so that $N_j = N_{j-1} + 1$. We can also suppose $N_{j-1} \geq 1$.

Case 1. $v(j) = v(j-1) = 0$. Clearly, $M_j = M_{j-1} + 1$.

Case 2. $v(j) = 0$, $v(j-1) \neq 0$, say $v(j-1) > 0$. Let $m = i$ be the largest node for $v(1), v(2), \dots$, with $i \leq j-1$. Hence $N_i = N_{j-1}$ and $M_i \geq N_{j-1} - 1$. We consider the alternatives $v(i) = 0$ or $v(i)v(i-1) < 0$. In the first alternative, $i < j-1$ and $v(i+1) > 0$, so that $\Delta v(i) > 0$, while $\Delta v(j-1) < 0$, so that $M_j \geq M_i + 1 \geq N_{j-1} = N_j - 1$. In the second alternative, $v(j-1) > 0$ implies that $v(i) > 0$, $v(i-1) < 0$, so that $\Delta v(i-1) > 0$. Again $\Delta v(j-1) < 0$ implies the desired result.

Case 3. $v(j)v(j-1) < 0$, say $v(j) < 0$, $v(j-1) > 0$. The arguments here are as in the last case.

This proves the proposition for the case of "nodes". Similar arguments are valid in the case of "generalized zeros".

On (a) $\Rightarrow$ (c). In order to verify (a) $\Rightarrow$ (c), assume (a) and that $u_1, \dots, u_n$ are defined as in the proof (a) $\Rightarrow$ (b). Suppose, if possible, that (c) does not hold, so that (1.1) has a solution $u \not\equiv 0$ which has at least n generalized zeros on I^n . Then there is a k , $1 < k \leq n$, such that $u = c_1 u_1 + \cdots + c_k u_k$, $c_k \neq 0$, and so $\Delta_{k-1} \cdots \Delta_1 u = \text{const } \omega_k \neq 0$ does not change signs on I^{n-k+1} . But this contradicts Proposition 5.1 which implies that $\Delta_{k-1} \cdots \Delta_1 u$ has at least $n - k + 1 > 0$ generalized zeros on I^{n-k+1} .

We next verify the following result which seems of interest in itself.

PROPOSITION 5.2. *Suppose that (1.1) is r -disconjugate on I^n . Let $u = u_k^0(m)$, $m \in I^n$ and $1 \leq k \leq n$, be a solution of (1.1) satisfying the partial set of initial*

conditions

$$u_k(m) = 0 \quad \text{for } a \leq m < a + n - k \quad \text{and} \quad (-1)^{k-1} u_k(a + n - k) > 0. \quad (5.8)$$

Then, for $1 \leq k \leq n$,

$$D_k(\mu(1), \dots, \mu(k)) > 0 \quad \text{for } a + n - k \leq \mu(1) \in I^{n-k+1} \quad (5.9)$$

and all sets $\mu(1) < \dots < \mu(k), \mu(j) \in I^n$. In particular,

$$\omega_k^0(m) \equiv W(u_1^0, \dots, u_k^0) > 0 \quad \text{for } a + n - k \leq m \in I^{n-k+1}. \quad (5.10)$$

PROOF. It is clear that (5.9) holds for $k = 1$. Suppose that $1 < k \leq n$ and that (5.9) holds if k is replaced by $k - 1$. Suppose first that, if possible, $D_k(\mu^0(1), \dots, \mu^0(k)) = 0$ for some set of indices $a + n - k \leq \mu^0(1) < \dots < \mu^0(k)$. Then, since $a_0(m) \neq 0$, there is a solution $u = c_1 u_1^0 + \dots + c_k u_k^0 \not\equiv 0$ of (1.1) such that $u(m) = 0$ for k values of $m = \mu^0(1), \dots, \mu^0(k)$. But $u(m) = 0$ also for the $n - k$ values of $m = a, \dots, a + n - k - 1 (< \mu(1))$, so that $u \not\equiv 0$ has n nodes. Hence $D_k(\mu^0(1), \dots, \mu^0(k)) = 0$ cannot hold.

Arrange the possible sets of indices $(\mu(1), \dots, \mu(k))$ lexicographically. It is clear, from (5.8), that the inequality in (5.9) holds for the first set of indices $(\mu(1), \dots, \mu(k)) = (a + n - k, \dots, a + n - 1)$. Suppose that there is a first set of indices $(\mu^1(1), \dots, \mu^1(k))$ such that $D_k(\mu^1(1), \dots, \mu^1(k)) < 0$. Let $(\mu^0(1), \dots, \mu^0(k))$ be a set of indices preceding $(\mu^1(1), \dots, \mu^1(k))$ which has $k - 1$ elements in common with it, and which has the property that, for some index J , $\mu^0(J) \neq \mu^1(j)$ and $\mu^0(J) + 1 = \mu^1(J) \neq \mu^0(j)$ for $1 \leq j \leq k$. Since (1.1) is r -disconjugate, there exists a solution u such that

$$\begin{aligned} u(m) &= 0 \quad \text{for } m = a, \dots, a + n - k - 1 \text{ and } m = \mu^0(j), j \neq J, \\ u(m) &\neq 0 \quad \text{for } m = \mu^0(J). \end{aligned}$$

Thus u is a linear combination $u = c_1 u_1^0 + \dots + c_k u_k^0$ of $u_1^0, \dots, u_k^0$ and $c_k \neq 0$. We can therefore suppose that $c_k = 1$ and, in fact, we can consider u to be u_k (without affecting any of the considerations above). Thus $D_k(\mu^0(1), \dots, \mu^0(k)) > 0$ implies that

$$(-1)^{k+J} u_k(\mu^0(J)) D_{k-1}(\mu^0(1), \dots, \hat{\mu}^0(J), \dots, \mu^0(k)) > 0,$$

while $D_k(\mu^1(1), \dots, \mu^1(k)) < 0$ gives

$$(-1)^{k+J} u_k(\mu^1(J)) D_{k-1}(\mu^1(1), \dots, \hat{\mu}^1(J), \dots, \mu^1(k)) < 0,$$

so that $u_k(\mu^0(J)) u_k(\mu^1(J)) < 0$. Hence $u = u_k$ has a node at $m = \mu^0(J) + 1 = \mu^1(J) \neq \mu^0(j)$. But then u_k has n nodes which is impossible. This completes the proof of Proposition 5.2.

On (d) $\Rightarrow$ (a), I finite. For $t \geq 0$, let $u = u_k^t(m)$, $1 \leq k \leq n$, be the solution of (1.1) determined by the initial conditions

$u_k^t(m) = (-1)^{k-1} t^{n-k+a-m} / (n-k+a-m)!$ for $m = a, \dots, a+n-k$,
 $u_k^t(m) = 0$ for $m = a+n-k+1, \dots, a+n-1$,
 where $0^0 = 1$, so that u_k^t satisfies (5.8) for $t = 0$. Hence $\omega_k^t = W(u_1^t, \dots, u_k^t)$ satisfies

$$\omega_k^t(m) > 0, \quad 1 \leq k \leq n, \quad (5.11)$$

for $t = 0, a+n-k \leq m \in I^{n-k}$, by (5.9). It can also be shown that (5.11) holds for $t > 0$ and $a \leq m \leq a+n-k$ by noting that $\omega_k^t(m)$ on this m -range is a Wronskian determinant $[= \det(d^{i-1}x_j/dt^{i-1})]$ of k of the functions $x_j = \pm t^j/j!$ for $j = 0, 1, \dots$; cf. [5, p. 322] or [4, Lemma 8, p. 95].

Since I is a finite interval, it follows from the case $t = 0$ in (5.11) and from continuity, that (5.11) holds for small $t > 0$ and $a+n-k \leq m \in I^{n-k}$. Consequently, for small $t > 0$, $u_1^t, \dots, u_n^t$ is a $w_n(I^n)$ -system.

On (a) $\Rightarrow$ (e), I finite. This can be proved by an induction on n . We omit the proof as it can be modeled on the proof [6, pp. 319–320] of the necessity of $B_n(I)$ in Theorem 1.1_n, p. 309; cf. the proof of Theorem 6.1 below.

On (f) $\Rightarrow$ (a), I finite. It is clear that the last part of the proof (d) $\Rightarrow$ (a) can be adapted here if we verify

PROPOSITION 5.3. *Suppose that (1.1) satisfies (f). Let $u = u_k^0(m)$, $m \in I$ and $1 \leq k \leq n$, be as in Proposition 5.2. Then (5.10) holds.*

PROOF. It is clear that (5.10) holds for $k = 1$. Suppose that $1 < k \leq n$ and that (5.10) holds when k is replaced by $i < k$.

Note that (f) implies that (5.10) holds if “ > 0 ” is replaced by “ $\neq 0$ ”. For if $\omega_k^0(m_0) = 0$ for some m_0 , $a+n-k \leq m_0 \in I^{n-k+1}$, then a linear combination ($\neq 0$) of $u_1^0, \dots, u_k^0$ has $n-k$ zeros at $m = a, \dots, a+n-k-1$ and k zeros at $m = m_0, \dots, m_0+k-1$. Note also that $\omega_k^0(a+n-k) = 1 > 0$.

Suppose, if possible, that there is at least m_0 , $a+n-k \leq m_0 \in I^{n-k+1}$, such that $\omega_k^0(m_0+1) < 0$. By virtue of (f), (1.1) has a solution $u = u(m)$ satisfying

$$u(m) = 0 \quad \text{for } m = a, \dots, a+n-k-1,$$

$$u(m) = 0 \quad \text{for } m = m_0+1, \dots, m_0+k-1 \text{ and } u(m_0) \neq 0.$$

Thus u is a linear combination $u = c_1 u_1^0 + \dots + c_k u_k^0$ with $c_k \neq 0$. We can suppose first that $c_k = 1$ and second that $u = u_k^0$. Then $\omega_k^0(m_0) > 0$ and $\omega_k^0(m_0+1) < 0$ give

$$(-1)^{k+1} u_k^0(m_0) \omega_{k-1}^0(m_0+1) > 0 \quad \text{and} \quad u_k^0(m_0+k) \omega_{k-1}^0(m_0) < 0.$$

Since $\omega_{k-1}^0 > 0$ by the induction hypothesis, we have $(-1)^k u_k^0(m_0) u_k^0(m+k) > 0$ and hence $m = m_0+k$ is a generalized zero for $u = u_k^0$. This is a contradiction, and completes the proof.

PROOF OF THEOREM 5.2(i). Let $u_1, \dots, u_n$ be linearly independent solutions

of (1.1) and $\omega_n = W(u_1, \dots, u_n)$. Then, by (g) in Theorem 5.1, (1.1) is disconjugate on I^n if and only if

$$D_n(\nu(1), \dots, \nu(n))/\omega_n(b+1) > 0 \quad \text{for } 0 \leq \nu(1) < \dots < \nu(n) \leq b+n. \quad (5.12)$$

In particular, (1.2) is necessary for disconjugacy; cf. Proposition 2.7. Thus, the desired result follows if we show that

$$\Phi_d(\mu(1), \dots, \mu(d)) = (-1)^{0+1+\dots+(d-1)+\sum \mu(m)} \left[\prod_{m \in I} \alpha_0(m) \right] \cdot D_n(\nu(1), \dots, \nu(n))/\omega_n(b+1), \quad (5.13)$$

where the sets of integers $\{\mu(1), \dots, \mu(d)\}$ and $\{\nu(1), \dots, \nu(n)\}$ are complementary sets in $I^n = [0, b+n]$.

In order to verify (5.13), let $W^{(n)}$ be the $n \times n$ matrix having the determinant $\omega_n(b+1)$, i.e., $W^{(n)} = (u_j(m+b))$, $j, m = 1, \dots, n$, and let $\Phi^{(d)}$ be the $d \times d$ matrix consisting of the first d columns of Φ , so that $\det \Phi^{(d)} = \alpha_0(0) \cdot \dots \cdot \alpha_0(b)$, since the elements below the main diagonal are 0. Let $\tilde{\Phi}$ be the $(d+n) \times (d+n)$ matrix in which Φ is the set of first d rows, the $n \times n$ matrix $(W^{(n)})^{-1}$ is in the lower right corner, and the other elements in the last n rows are 0. Also, let $\tilde{M}$ be the $(d+n) \times (d+n)$ matrix in which the last n columns are $(u_k(0), \dots, u_k(b+n))$, $k = 1, \dots, n$; $(\Phi^{(d)})^{-1}$ is in the upper left corner, and the other elements in the first d columns are 0. As $Pu_k = 0$, Φ annihilates the last n columns of $\tilde{M}$. Hence $\tilde{\Phi}$ and $\tilde{M}$ are inverse matrices, and (5.13) follows; cf., e.g., [5, p. 15].

REMARK. Before beginning the proof of part (ii), we might mention that certain cases of (5.6) are immediate consequences of part (i). For example, if we replace $I = [0, b]$ by a subinterval $[a_1, a_1 + k - 1]$, then (5.6) follows in the nontrivial cases of $(j(1), \dots, j(k)) = (a_1, \dots, a_1 + k - 1)$. Also, since $Pu = W(u_1, \dots, u_n, u)/\omega_n$, (5.7) follows from

$$(-1)^{n+j} \alpha_j(m) = D_n(m, \dots, \widehat{m+j}, \dots, m+n)/\omega_n. \quad (5.14)$$

The proof of (ii) can be completed by the use of a Fekete type of theorem; cf. Proposition 2.3. We indicate another type of proof.

PROOF OF THEOREM 5.2(ii). If (1.1) is disconjugate on I^n , then there exists a solution $u = u_0(m) > 0$ such that if one applies Proposition 4.2, with $k = 1$ and $u_0 = u_1$, then the resulting difference equation (4.6) of order $n-1$ is disconjugate on I ; cf. the proof of Theorem 6.1 below. In this case, (4.9) corresponds to

$$-P_2 w = W(u_0, w) = u_0(m)w(m+1) - u_0(m+1)w(m).$$

If Ψ_1, Ψ_2 are the matrices belonging to $P_1, -P_2$ as Φ belongs to P , then (4.10) implies $\Phi = \Psi_1 \Psi_2$. Thus one can easily prove Theorem 5.2(ii) by an induction

on n , using the Cauchy-Binet identities for subdeterminants of matrix products; cf. [5, p. 12].

6. Sturm comparison theorem. The Sturm comparison theorem for $n = 2$ has the following generalization:

THEOREM 6.1. *Assume (1.2). Then a sufficient condition for (1.1) to be disconjugate on I^n is that there exist a set of functions $u_1(m), \dots, u_{n-1}(m)$ for $m \in I^n$ which is a $W_n(I^n)$ -system satisfying (2.23) at $m = a$ (so that $u_1, \dots, u_{n-1}$ is a $W_n(I^n)$ -system) and $(-1)^{n+j}Pu_j \geq 0$ for $m \in I$, $1 \leq j < n$.*

If $n = 2$, these sufficient conditions mean that there exists a positive function $u_1(m)$, $m \in I^n$, such that $Pu_1 < 0$ on I ; cf. this with [4, p. 223], where the condition seems to be given incorrectly as $Pu_1 > 0$ [and where $u_1(a) \geq 0$ is permitted, but I^2 is replaced by I^1]. It will remain undecided if the condition "(2.23) at $m = a$ " can be relaxed to "(2.25)"; cf. [7, Theorem 18.1_n, p. 449].

COROLLARY 6.1. *Assume that (1.2) holds and that, for fixed $m \in I$, the polynomial*

$$P(m, \lambda) = \lambda^n + \alpha_{n-1}(m)\lambda^{n-1} + \dots + \alpha_0(m) \quad (6.1)$$

has positive real roots $(0 <) \lambda_1(m) \leq \dots \leq \lambda_n(m)$, and let there exist constants $\sigma_1, \dots, \sigma_{n-1}$ satisfying

$$0 < \lambda_1(m) \leq \sigma_1 \leq \lambda_2(m) \leq \dots \leq \sigma_{n-1} \leq \lambda_n(m) \quad \text{for } m \in I; \quad (6.2)$$

in particular, $(-1)^{n+k}P(m, \sigma_k) \geq 0$. Then (1.1) is disconjugate on I^n .

If $0 < \sigma_1 < \dots < \sigma_{n-1}$, this is a consequence of Theorem 6.1 with the choice $u_k(m) = \sigma_k^m$ for $m \in I^n$, $1 \leq k < n$. The proof in the case $\sigma_k = \sigma_{k+1}$ for some k can be modeled on that of [6, Theorem 6.1, pp. 320–321], and will be omitted.

PROOF OF THEOREM 6.1. There is no loss of generality in assuming that I is a finite interval. In this case, there exists a function $u_n(m)$, $m \in I^n$, such that $\omega_n = W(u_1, \dots, u_n) > 0$ on I^1 , $Pu_n > 0$ on I ; cf. Proposition 2.6. Thus $u_1, \dots, u_n$ satisfies the conditions of Proposition 3.2, so that this proposition and the arguments of [9] imply that there exists a solution $u = u_0(m)$ of (1.1) such that $x_1 = u_0 > 0$ and $x_k = W(u_0, \dots, u_{k+1}) > 0$ for $m \in I^{n-k+1}$, $2 \leq k \leq n$. Thus $u_0, \dots, u_{n-1}$ is a $W_{n+1}(I^n)$ -system; Proposition 2.3. The remainder of the proof can now be modeled on that of first part of Theorem 1.1_n of [6, p. 309].

7. Green's functions. If (1.1) is disconjugate on $I = [a, b]$, then for any function $f = f(m)$, $m \in I$, the boundary value problem,

$$(Pv)(m) \equiv \sum_{j=0}^m \alpha_j(m)v(m+j) = f(m), \quad m \in I, \quad (7.1)$$

$$v(m) = 0 \quad \text{for } m = m(1), \dots, m(n), \quad (7.2)$$

where $a \leq m(1) < \dots < m(n) \leq b + n$, has a unique solution. Furthermore, there exists a Green's function $G(m, \nu)$, defined on $I^n \times I$, such that the solution is given by

$$v(m) = \sum_{\nu=a}^b G(m, \nu) f(\nu) \quad \text{for } m \in I^n. \quad (7.3)$$

The Green's function is characterized as follows: for a fixed $\nu \in I$, $v(m) = G(m, \nu)$ is unique solution of the boundary value problem, (7.2) and

$$(Pv)(m) = \delta_{m\nu} \quad \text{for } m \in I^n, \quad (7.4)$$

where $\delta_{m\nu}$ is 0 or 1 according as $m \neq$ or $= \nu$. See Bôcher [1, pp. 83–88] (or [4, Chapter IX], for a reproduction of this part of [1]).

THEOREM 7.1. *Let $I = [a, b]$, let (1.1) be disconjugate on I^n , and let $G(m, \nu)$ be the Green's function for (7.1)–(7.2). For $m \in I^n$, put*

$$\sigma(m) = \text{card } \{j: 1 \leq j \leq n, m(j) < m\}. \quad (7.5)$$

Then we have, for $1 \leq k \leq \text{card } I = b - a + 1$,

$$(-1)^{nk + \sum \sigma(\mu(i))} \det(G(\mu(i), \nu(j)); i, j = 1, \dots, k) \geq 0, \quad (7.6)$$

for $a \leq \mu(1) < \dots < \mu(k) \leq b + n$, $a \leq \nu(1) < \dots < \nu(k) \leq b$; strict inequality holds in (7.6) when $k = d$ and $\mu(i) \neq m(1), \dots, m(n)$. Also if $m(1) = a$ and $m(n) = b + n$, then

$$(-1)^{n + \sigma(m)} G(m, \nu) > 0 \quad \text{for } m \neq m(1), \dots, m(n); \quad (7.7)$$

in particular, if $0 < k < n$ and $m(j) = a + j - 1$ for $1 \leq j \leq k$ and $m(j) = b + j$ for $k \leq j \leq n$, then

$$(-1)^{n+k} G(m, \nu) > 0 \quad \text{for } a + k \leq m \leq b + k. \quad (7.8)$$

It will be clear from the proof (cf. (7.9) below) that we can easily determine when strict inequality holds in (7.6).

PROOF. For convenience, let $a = 0$, $I = [0, b]$, $d = b + 1$. Let Φ be the $d \times (d + n)$ matrix defined in connection with (5.2). Consider $G = (G(m, \nu))$ to be a rectangular matrix with $m \in I^n = [0, b + n]$ as row index and $\nu \in I = [0, b]$ as column index. Hence, the characterization of the Green's function involving (7.4) is equivalent to the matrix equation $\Phi G = E$, where E is $d \times d$ identity matrix, and to be requirement that the entries in the m th row of G are 0 for $m = m(1), \dots, m(n)$.

Let $\Phi_{(d)}$ be the $d \times d$ matrix obtained from Φ by deleting the m th column for $m = m(1), \dots, m(n)$, and $G_{(d)}$ the $d \times d$ matrix obtained from G by deleting its m th row for $m = m(1), \dots, m(n)$, consisting of zeros. Then $\Phi_{(d)}$ and $G_{(d)}$ are inverse matrices. Hence, if $\mu(i) \neq m(1), \dots, m(n)$, then

$$\det(G(\mu(i), \nu(j))) = (-1)^{\sum [\mu(i) + \sigma(\mu(i))] + \sum \nu(j)} \det(\phi_{\lambda(p), \rho(q)}) / \det \Phi_{(d)}, \quad (7.9)$$

where $\{\mu(i)\}$, $\{\rho(q)\}$ are complementary sets in $I^n - \{m(1), \dots, m(n)\}$ and $\{\nu(j)\}$, $\{\lambda(p)\}$ are complementary sets in I ; cf. [5, p. 15]. By (5.6),

$$(-1)^{(d-k)n + \sum \lambda(p) + \sum \rho(q)} \det(\phi_{\lambda(p), \rho(q)}) \geq 0$$

and, by (5.5),

$$(-1)^{nd + d(d-1)/2 + \sum \mu(i) + \sum \rho(q)} \det \Phi_{(d)} > 0.$$

Since $\sum \nu(j) + \sum \lambda(p) = d(d-1)/2$, (7.6) follows. When $k = d$, so that the set of indices $\{\lambda(p)\}$, $\{\rho(q)\}$ are vacuous, (7.9) should be replaced by $\det(G(\mu(i), \nu(j))) = 1/\det \Phi_{(d)}$.

If the first and last column of Φ are deleted, then any $(d-1) \times (d-1)$ subdeterminant of the resulting matrix is not 0. Hence (7.7) follows from (7.9).

In the next theorem, we let $I = [0, \infty)$ and use Theorem 8.3 below.

THEOREM 7.2. *Let (1.1) be disconjugate on $I = [0, \infty)$. Let $0 \leq k < n$, $b > 0$ and $G(m, \nu) = G^k(m, \nu; b)$ be the Green's function associated with*

$$(Pv)(m) = f(m) \quad \text{for } m \in [0, b], \quad (7.10)$$

$$v(m) = 0 \quad \text{for } 0 \leq m < k \text{ and } b + k < m \leq b + n. \quad (7.11)$$

Then the limit

$$\gamma^k(m, \nu) = \lim_{b \rightarrow \infty} G^k(m, \nu; b) \quad \text{exists on } I \times I; \quad (7.12)$$

$v(m) = \gamma^k(m, \nu)$ satisfies (7.4),

$$\gamma^k(m, \nu) = 0 \quad \text{for } 0 \leq m < k, \quad (7.13)$$

$$(-1)^{n+k} \gamma^k(m, \nu) \geq 0 \quad \text{for } 0 < k < n, \quad (7.14)$$

$$\gamma^k(m, \nu) = \gamma^0(m, \nu) - \sum_{j=1}^k \gamma^{j-1}(j-1, \nu) \eta_j(m), \quad (7.15)$$

where $u = \eta_j(m)$ is the unique j th principal solution of (1.1) satisfying $\eta_j(m) = 0$ for $0 \leq m < j-1$, $\eta_j(j-1) = 1$ in Theorem 8.3.

PROOF. We begin with the case $k = 0$. It is easy to see from the characterization of Green's functions involving (7.4) that $G^0(m, \nu; b) = \gamma^0(m, \nu)$ is independent of b . In fact, $\gamma^0(m, \nu)$ for $m = 0, \dots, \nu$ is the last column of the inverse of the $(\nu+1) \times (\nu+1)$ triangular matrix (ϕ_{jm}) , $j, m = 0, \dots, \nu$; so that

$$\gamma^0(m, \nu) = (-1)^{m+\nu} \det(\phi_{ij}; 0 \leq i < \nu, i \neq m, 0 \leq j < \nu) / \prod_{i=0}^{\nu} \alpha_0(i) \quad (7.16)$$

for $0 \leq m \leq \nu$, while $\gamma^0(m, \nu) = 0$ for $m > \nu$.

Note that, by (7.8), $(-1)^{n+k}G^k > 0$ for $0 < k \leq m \leq b + k$, so that (7.14) follows when the limit (7.12) exists.

We complete the proof by an induction on k . Let $k > 1$. Then, if $G^{k-1}(k-1, \nu; b) \neq 0$,

$$u_k(m, b) = [G^{k-1}(m, \nu; b) - G^k(m, \nu; b)]/G^{k-1}(k-1, \nu; b)$$

is independent of ν and is the unique solution of (1.1) satisfying $u(k-1) = 1$, and $u(m) = 0$ for $0 \leq m < k-1$ and $b+k < m \leq b+n$. Thus, whether or not $G^{k-1}(k-1, \nu; b) = 0$,

$$G^k(m, \nu; b) = G^{k-1}(m, \nu; b) - G^{k-1}(k-1, \nu; b)u_k(m, b). \quad (7.17)$$

By Theorem 8.3 below, $\eta_k(m) = \lim u_k(m, b)$ exists as $b \rightarrow \infty$. Thus, by the induction hypothesis, (7.12) holds and

$$\gamma^k(m, \nu) = \gamma^{k-1}(m, \nu) - \gamma^{k-1}(k-1, \nu)\eta_k(m). \quad (7.18)$$

This completes the proof.

8. Principal solutions, $I = [a, \infty)$. The proof of Theorem A in [6, Appendix A, pp. 352–355], implies

THEOREM 8.1. *Let (1.1) be nonoscillatory on $I = [a, \infty)$; cf. §1. Then (1.1) has solutions $u = u_k(m) \not\equiv 0$, $1 \leq k \leq n$, such that*

$$u_{k-1}(m) = o(u_k(m)) \quad \text{as } m \rightarrow \infty, \quad 1 < k \leq n. \quad (8.1)$$

If $u_1, \dots, u_n$ is a family of solutions of (1.1) satisfying (8.1), we call u_k a k th principal solution. It is clear that a first principal solution is unique up to a constant ($\neq 0$) factor and that, for $1 < k \leq n$, a k th principal solution is unique up to a constant ($\neq 0$) factor and the sum of a linear combination of the first $k-1$ principal solutions.

When (1.1) is disconjugate and (1.2) holds, we can obtain other properties of principal solutions. As a basis for an induction, we begin with properties of the first principal solution.

THEOREM 8.2. *Let (1.1) be disconjugate on $I = [a, \infty)$. (i) Let $b > a$ and $u = u_1(m, b)$, $m \in I$, be the solution of (1.1) determined by*

$$u(b+1) > 0 \quad \text{and} \quad u(m) = 0 \quad \text{for } b+1 < m \leq b+n, \quad (8.2)$$

$$|u(a)| + \dots + |u(a+n-1)| = 1, \quad (8.3)$$

then the limit

$$\eta_1(m) = \lim_{b \rightarrow \infty} u_1(m, b) \quad \text{exists on } I \quad (8.4)$$

and is, of course, a solution of (1.1). (ii) Also,

$$\eta_1(m) > 0 \quad \text{on } I; \quad (8.5)$$

so that (i) is valid if the normalization (8.3) on $u_1(m, b)$ is replaced by

$$u_1(a, b) = 1, \quad (8.6)$$

in which case, $\eta_1(a) = 1$. (iii) If $U_1, \dots, U_n$ is a $w_{n+1}(I)$ -system of solutions of (1.1), then $W(\eta_1, U_1, \dots, U_k) \geq 0$ for $1 \leq k \leq n$. (iv) If $U_1, \dots, U_n$ is a $W_{n+1}(I)$ -system of solutions of (1.1), then $\eta_1, U_2, \dots, U_n$ is a $W_{n+1}(I)$ -system.

We omit the proof which can be modeled on that of Theorem 7.1_n in [6]. As in that theorem, we could enumerate other properties of η_1 . See also [2, Theorem 12, pp. 110–111] for an analogue of a proof of a simpler theorem, similar to Theorem 8.2 above.

THEOREM 8.3. Let (1.1) be disconjugate on $I = [a, \infty)$. Then there exists a unique set of n solutions of (1.1) characterized by (i) $\eta_1, \dots, \eta_n$ is an ordered set of principal solutions; (ii) $u = \eta_k(m)$ satisfies

$$u(m) = 0 \quad \text{for } a \leq m < a + k - 1 \text{ and } u(a + k - 1) = 1, \quad (8.7)$$

$$u(m) > 0 \quad \text{for } m \geq a + k - 1; \quad (8.8)$$

(iii) if $u = u_k(m, b)$ is the solution of (1.1) satisfying (8.7) and

$$u(m) = 0 \quad \text{for } b + k < m \leq b + n, \quad (8.9)$$

then

$$\eta_k(m) = \lim_{b \rightarrow \infty} u_k(m, b); \quad (8.10)$$

finally (iv) $\eta_1, \dots, \eta_n$ is a $DW_{n+1}(I)$ -system.

The analogous results (i), (ii), (iv) for differential equations are given in [6], while (iii) was given in [8, Appendix 3].

PROOF. The existence of $u = \eta_1(m)$ satisfying (ii) and (iii) is clear from Theorem 8.2. Assume $n > 1$ and the validity of Theorem 8.3 for difference equations of order $n - 1$. Make the "variations of constants" $v = \Delta(u/\eta_1) = W(\eta_1, u)/\eta_1\eta_1^*$ to obtain a difference equation of order $n - 1$,

$$Qv = \sum_{j=0}^{n-1} \delta_j(m)v(m+j) \quad \text{for } m \in I, \quad (8.11)$$

with $\delta_{n-1}(m) = \eta_1(m+n-1)\eta_1(m+n) > 0$ and $(-1)^{n-1}\delta_0(m) > 0$; cf. Proposition 4.2 with $k = 1$ and $\delta_j(m) = \beta_j(m)\eta_1(m+j)\eta_1(m+j+1)$. The equation (8.11) is disconjugate on I^{n-1} by virtue of (iv) in Theorem 8.2; cf. the proof of Theorem 6.1. Thus, by the induction hypothesis, (8.11) has solutions $v_1, \dots, v_{n-1}$ satisfying the analogues of (i)–(iv) in Theorem 8.3. Thus, there are solutions $\eta_2, \dots, \eta_n$ of (1.1) such that

$$v_{k-1} = \text{const } \Delta(\eta_k/\eta_1), \quad \text{where } \text{const} = 1/\eta_1(a+k-1), \quad 1 < k \leq n, \quad (8.12)$$

and η_k is unique up to the addition of a constant multiple of η_1 . We can determine η_k uniquely by putting

$$\eta_k(m) = \eta_1(m) \sum_{\nu=a}^{m-1} v_{k-1}(\nu)/\eta_1(a+k-1), \quad (8.13)$$

where "const" was inserted in (8.12) to achieve the normalization in the last part of (8.7).

It follows that $\eta_1, \dots, \eta_n$ satisfy assertion (ii). Also, they satisfy (iii) for

$$v_{k-1}(m, b) = \text{const } \Delta(u_k(m, b)/\eta_1(m)), \quad \text{const} = \eta_1(a+k-1),$$

or equivalently,

$$u_k(m, b) = \eta_1(m) \sum_{\nu=a}^{m-1} v_{k-1}(\nu, b)/\eta_1(a+k-1).$$

We can verify (i), i.e., that $\eta_1, \dots, \eta_n$ are ordered principal solution as in [6] or in [2, p. 113].

In order to show that $\eta_1, \dots, \eta_n$ is a $DW_{n+1}(I)$ -system, consider a determinant (2.19). Divide the first row by $\eta_1(\mu(1))/\eta_1(a+i(1)-1) > 0$, the second by $\eta_1(\mu(2))/\eta_1(a+i(1)-1) > 0, \dots$, so that we obtain a determinant of the matrix in which the m th element in the j th column is

$$\sum_{\nu=a}^{\mu(m)-1} v_{i(j)-1}(\nu) \quad \text{if } i(j) > 1.$$

If we subtract the first row from the second, the second from the third, etc., the m th element in the j th column is

$$\sum_{\nu=\mu(m-1)}^{\mu(m)-1} v_{i(j)-1}(\nu) \quad \text{for } m > 1, i(j) > 1.$$

Thus, according as $i(1) > 1$ or $i(1) = 1$, the resulting determinant is

$$\sum_{\nu_1=a}^{\mu(1)-1} \sum_{\nu_2=\mu(1)}^{\mu(k)-1} \cdots \sum_{\nu_k=\mu(k-1)}^{\mu(k)-1} \det(v_{i(j)-1}(\nu_m); j, m = 1, \dots, k) > 0$$

or is

$$\sum_{\nu_2=\mu(1)}^{\mu(2)-1} \cdots \sum_{\nu_k=\mu(k-1)}^{\mu(k)-1} \det(v_{i(j)-1}(\nu_m); j, m = 2, \dots, k) > 0.$$

This completes the proof.

9. Inequalities, $I = [a, \infty)$. An analogue of Theorem 14.1_n of [7, p. 440] on differential equations holds for (1.1). Recall the notation: $u^* = u(m+1)$.

THEOREM 9.1. *Let (1.2) hold and let $I = [a, \infty)$. Let $u_0, \dots, u_{n-1}$ be functions on $m \in I$ satisfying*

$$(-1)^{n+k} P u_k > 0 \quad \text{for } 0 \leq k < n, \quad (9.1)$$

$$u_1, \dots, u_{n-1} \text{ is a } W_n(I)\text{-system}, \quad (9.2)$$

$$W(u_0, \dots, u_k) \geq 0 \quad \text{for } 0 \leq k < n. \quad (9.3)$$

Then (1.1) has positive linear independent solutions $u = x_k(m)$, $m \in I$ and $1 \leq k \leq n$, satisfying

$$x_1^*/x_1 \leq u_1^*/u_1 < x_2^*/x_2 < \dots < u_{n-1}^*/u_{n-1} < x_n^*/x_n, \quad (9.4)$$

$$W(x_1, u_0) < 0, \quad \text{so that } u_0^*/u_0 \leq x_1^*/x_1 \text{ where } u_0 > 0, \quad (9.5)$$

and the inequalities

$$W(x_1, \dots, x_k) > 0 \quad \text{for } 1 \leq k \leq n,$$

$$W(x_1, \dots, x_k, u_p, \dots, u_q) > 0 \quad [\text{or } \geq 0]$$

$$\text{for } 1 \leq k < p \leq q < n \quad [\text{or } 1 \leq k \leq p \leq q < n], \quad (9.6)$$

$$W(x_1, \dots, x_k, u_{k-1}) \leq 0, \quad W(x_1, \dots, x_k, u_{k-1}, u_p, \dots, u_q) \leq 0$$

for $1 \leq k \leq n$ and $k < p \leq q < n$.

If, in addition, $u = u_n(m)$, $m \in I$, satisfies

$$Pu_n \geq 0, \quad (9.7)$$

$$W(u_k, \dots, u_n) > 0 \quad \text{for } 1 \leq k \leq n, \quad (9.8)$$

then the solution $u = x_n(m)$ can be chosen to satisfy

$$x_n^*/x_n < u_n^*/u_n, \quad (9.9)$$

and $q = n$ is permitted in (9.6).

We omit the proof which can be modeled on that in [7, pp. 442–448]. Theorem 9.1 remains correct if there are no assumptions or assertions concerning u_0 . (But if no u_0 is specified, $u_0 \equiv 0$ is a possible (trivial) choice.)

COROLLARY 9.1. Let $I = [a, \infty)$ and let (1.2) hold. Let there exist positive constants $\sigma_0 < \sigma_1 < \dots < \sigma_n$ such that the polynomial (6.1) satisfies $(-1)^{n+k}P(m, \sigma_k) \geq 0$ for $m \in I$, $0 \leq k \leq n$. Then (1.1) has positive linearly independent solutions $u = x_k(m)$, $m \in I$ and $1 \leq k \leq n$, satisfying

$$0 < \sigma_0 \leq x_1^*/x_1 \leq \sigma_1 < x_2^*/x_2 < \dots < x_n^*/x_n < \sigma_n.$$

This assertion is valid if there are no assumptions or conclusions concerning σ_0 and/or σ_n .

This follows from Theorem 9.1 by the choice $u_k(m) = \sigma_k^m$, $m \geq a$ and $0 \leq k \leq n$.

10. Complete monotonicity. In (1.1), let $\alpha_j(m) = \alpha_j(m, \lambda)$ be defined on $I \times \Lambda$, where Λ is a (bounded on unbounded) interval on the real axis, such that $\alpha_j(m, \cdot) \in C^\infty(\Lambda)$ and, for a fixed k , $0 \leq k < n$, $(-1)^{n+k}d\alpha_j(m, \lambda)/d\lambda$ are completely monotone functions of $\lambda \in \Lambda$ for $0 \leq j \leq n$, i.e.,

$$(-1)^{n+k+i+1}d^i\alpha_j(m, \lambda)/d\lambda^i \geq 0 \quad \text{for } i = 1, 2, \dots \quad (10.1)$$

PROPOSITION 10.1. *In (1.1), assume (10.1) for some k , $0 < k < n$, and that, for every fixed $\lambda \in \Lambda$, (1.1) is disconjugate on I^n , $I = [0, b]$. For fixed λ , let $G^k(m, \nu) = G^k(m, \nu; \lambda)$ be the Green's function for (7.10)–(7.11) and let $u_k(m, b) = u_k(m, b; \lambda)$ be the solution of (1.1) defined in Theorem 8.3. Then $(-1)^{n+k}G^k(m, \nu; \cdot)$ and $u_k(m, b; \cdot)$ are completely monotone functions of $\lambda \in \Lambda$ (for fixed $m \in I^n$, $\nu \in I$).*

PROOF. This follows as in [8]. In (7.4), let $v(m) = G^k(m, \nu; \lambda)$. Differentiation with respect to λ gives

$$PG^{k'}(m, \nu; \lambda) = - \sum_{j=0}^n \alpha'_j(m, \lambda) G^k(m + j, \nu; \lambda), \quad \text{where } ' = d/d\lambda. \quad (10.2)$$

Thus, by (7.3),

$$\begin{aligned} (-1)^{n+k+1}G^{k'}(m, \nu; \lambda) &= \sum_{\mu=0}^b (-1)^{n+k}G^k(m, \mu; \lambda) \\ &\quad \cdot \sum_{j=0}^n (-1)^{n+k} \alpha'_j(\mu, \lambda) (-1)^{n+k}G^k(\mu + j, \nu; \lambda). \end{aligned} \quad (10.3)$$

Since $(-1)^{n+k}G^k \geq 0$ by (7.8), it follows from (10.1) that $(-1)^{n+k+1}G^{k'} \geq 0$. Successive differentiations of (10.4) show that $(-1)^{n+k+i}d^iG^k/d\lambda^i \geq 0$ for $i = 1, 2, \dots$.

The proof for the assertion concerning $u_k(m, b; \cdot)$ is similar, using the result on G^k .

Since nothing is altered (except the normalization $\alpha_n(m) = 1$) if we multiply (7.4) by a factor $\alpha(m, \lambda) > 0$ before differentiating to obtain (10.2), condition (10.1) can be replaced by

$$(-1)^{n+k+i}d^i \{ [\alpha(m, \lambda)\alpha_j(m, \lambda)]' / \alpha(m, \lambda) \} / d\lambda^i \geq 0 \quad \text{for } i = 0, 1, \dots, \quad (10.4)$$

$$(-1)^i d^i [\alpha'(m, \lambda) / \alpha(m, \lambda)] / d\lambda^i \geq 0 \quad \text{for } i = 0, 1, \dots, \quad (10.5)$$

$$\alpha(m, \lambda) > 0 \quad \text{on } I \times \Lambda \text{ and } \alpha(m, \cdot), \alpha_j(m, \cdot) \in C^\infty(\Lambda). \quad (10.6)$$

REMARK. It will be clear from the proofs that Proposition 10.1 and Theorem 10.1 are valid if λ is a discrete variable and if differentiation $' = d/d\lambda$ is replaced by a difference operator.

THEOREM 10.1. *Let $I = [0, \infty)$ and (10.4)–(10.6) hold for some k , $0 < k < n$. For fixed λ , let (1.1) be disconjugate on I .*

(a) *Let $\gamma^k(m, \nu) = \gamma^k(m, \nu; \lambda)$ be defined as in Theorem 7.2 and let $\eta^k(m) = \eta^k(m; \lambda)$ be the k th principal solution of (1.1) satisfying $\eta^k(m) = 0$ for*

$0 \leq m < k - 1$ and $\eta^k(k - 1) = 1$. Then $(-1)^{n+k}\gamma^k(m, v; \lambda)$ and $\eta_k(m; \lambda)$ are completely monotone functions of $\lambda \in \Lambda$.

(b) In particular, if $k = 1$ and $u_1(m) = u_1(m; \lambda)$ is a first principal solution of (1.1), then $u_1(m; \lambda)/u_1(M; \lambda)$ is a completely monotone function of λ for $m > M \geq 0$.

Part (b) complements a result of [9] dealing with the complete monotonicity of a solution $u = u_1(m)$ as a function of $m = 0, 1, \dots$.

PROOF. Part (a) follows from the last proposition and Theorems 7.2 and 8.3. The case $M = 0$ of part (b) is contained in (a) for $\eta_1(m, \lambda) = u_1(m, \lambda)/u_1(0, \lambda)$. The cases $M > 0$ follow by replacing $I = [0, \infty)$ by $[M, \infty)$.

EXAMPLE. As a simple illustration, consider the recursion relations

$$w_{\lambda+1} - 2\lambda t^{-1}w_{\lambda} + w_{\lambda-1} = 0 \quad (10.7)$$

satisfied by the standard solutions $w_{\lambda} = J_{\lambda}(t)$ and $w_{\lambda} = Y_{\lambda}(t)$ of the Bessel equation of order λ ,

$$t^2 d^2 y / dt^2 + t dy / dt + (t^2 - \lambda^2)y = 0. \quad (10.8)$$

For a fixed (not necessarily integral) $\lambda \geq 0$, let

$$u(m) = t^{-\lambda-m}w_{\lambda+m} \quad (10.9)$$

to obtain an analogue of (1.1) with $n = 2$,

$$u(m+2) - 2(\lambda+m+1)t^{-2}u(m+1) + t^{-2}u(m) = 0. \quad (10.10)$$

Let $t = j_{\lambda k}$ be the k th positive zero of $J_{\lambda}(t)$, then $u(m) = t^{-\lambda-m}J_{\lambda+m}(t) > 0$ for $m = 0, 1, \dots$, $0 < t < j_{\lambda 1}$. Thus, by the Sturm separation theorem, (10.10) is disconjugate on $I = [0, \infty)$ for fixed (λ, t) , $\lambda \geq 0$ and $0 < t < j_{\lambda 1}$. From the Wronskian relation $Y_{\lambda}J_{\lambda+1} - Y_{\lambda+1}J_{\lambda} = 2/\pi t$ (cf. [20, p. 77]), we see that $Y_{\lambda+m}/J_{\lambda+m}$ decreases to $-\infty$ as $m \rightarrow \infty$; so that $u_1(m) = t^{-\lambda-m}J_{\lambda+m}(t)$ is a first principal solution of (10.10) for fixed λ, t . If we apply Theorem 10.1 with $\alpha(m, \lambda) = 1$, we obtain

PROPOSITION 10.2. For fixed (t, m) , $0 < t < j_{m1}$ and $m = 0, 1, \dots$,

$$J_{\lambda+m+1}(t)/J_{\lambda+m}(t) \text{ is completely monotone on } \lambda \geq 0. \quad (10.11)$$

We can apply Theorem 1.1 of [8, p. 270] to the Bessel equation (10.8) to obtain the following related result:

PROPOSITION 10.3. If $\gamma \geq 0$ and $0 < \tau < t < j_{\gamma 1}$, then $J_{\lambda+\gamma}(\tau)/J_{\lambda+\gamma}(t)$ is a completely monotone function of $\lambda^2 \geq 0$ and has a representation

$$J_{\lambda+\gamma}(\tau)/J_{\lambda+\gamma}(t) = [J_{\gamma}(\tau)/J_{\gamma}(t)] \int_0^{\infty} \exp(-\lambda^2 r) W(dr, t, \tau) \lambda \geq 0, \quad (10.12)$$

where $W(r) = W(r, t, \tau)$ is an infinitely divisible distribution function on $0 \leq r < \infty$,

$$W(\cdot, \tau_n, \tau_1) = W(\cdot, \tau_n, \tau_{n-1}) * \cdots * W(\cdot, \tau_3, \tau_2) * W(\cdot, \tau_2, \tau_1) \quad (10.13)$$

for $0 < \tau_1 < \cdots < \tau_n < j_{\gamma 1}$, $W(r, t, \tau)$ is nondecreasing in r and in τ and nonincreasing in t and, finally, $W(r, t, \tau) \rightarrow \delta_0(r)$ as $\tau \uparrow t_0$, $t \downarrow t_0$ for $0 < t_0 < j_{\gamma 1}$, and $\delta_0(r)$ is 0 or 1 according as $r = 0$ or $r > 0$.

In view of the product representation (cf. [20, p. 498]) of $J_\lambda(t)$ in terms of its zeros $j_{\lambda k}$, it might be expected that we could deduce the results (10.11) and (10.12) directly if we knew suitable monotonicity properties of $j_{\lambda k}$. But these are hard to come by. We mention the following:

PROPOSITION 10.4. *Let $\lambda \geq 0$ and $\zeta(\lambda) = \zeta_k(\lambda, \theta)$ be the k th positive zero of the solution $J_\lambda(t)\cos \theta - Y_\lambda(t)\sin \theta$ of the Bessel equation (10.8). Then*

$$(-1)^{n+1}(d/\zeta(\lambda)d\lambda)^n \zeta(\lambda) > 0 \quad \text{for } n = 1, 2, \dots \text{ and } \lambda \geq 0. \quad (10.14)$$

In particular, if $\gamma > 0$ (not necessarily an integer), then

$$(-1)^n(d/d\lambda)^n \zeta^{-\gamma}(\lambda) > 0 \quad \text{for } 0 \leq n \leq \gamma + 1, \lambda \geq 0. \quad (10.15)$$

See also [20, p. 508] and [14] for monotonicity properties of $\zeta(\lambda)$.

The relations (10.14) follow by an induction on n and the fact that $\zeta = \zeta(\lambda)$ satisfies the differential equation $d\zeta/d\lambda = \Gamma(\zeta, \lambda)\zeta$, where

$$\Gamma(\zeta, \lambda) = 2 \int_0^\infty K_0(2\zeta \sinh s) e^{-2\lambda s} ds,$$

[20, p. 508], and $(-1)^{n+j} \partial^{n+j} \Gamma / \partial \zeta^j \partial \lambda^n > 0$, since

$$K_0(x) = \int_0^\infty \exp(-x \cosh r) dr$$

(cf. [20, p. 446]). The relations (10.15) are consequences of (10.14).

11. A positive solution. This section concerns the existence of a positive solution of a (not necessarily disconjugate) difference equation of order $n + N$ of the form

$$[(P \circ \Delta^N)u](m) + (-1)^{n+N-1} \sum_{k=0}^{N-1} (-1)^k c_k(m) \Delta^k u(m) = 0 \quad (11.1)$$

on $[0, \infty)$. The result to be obtained generalizes Theorem (ii) of [9, p. 732] in the same way that Theorem (†) of [9, p. 204] generalized Theorem (i) of [9, p. 731] on differential equations.

THEOREM 11.1. *Let (1.1) be disconjugate on $I = [0, \infty)$. In (11.1), let $c_k(m)$, defined for $m \in I$ and $0 \leq k < N$, satisfy*

$$c_k(m) \geq 0, \quad (11.2)$$

$$1 - \sum_{k=0}^{N-1} c_k(m) > 0. \quad (11.3)$$

Then (11.1) has a solution $u = u(m)$ satisfying

$$u > 0 \quad \text{and} \quad (-1)^k \Delta^k u \geq 0 \quad \text{for } 0 < k \leq N, m \in I. \quad (11.4)$$

See also (11.8) below.

PROOF. We use the factorization (5.1) for Pu and write (11.1) as a first order difference equation

$$\Delta v = -Av \quad (11.5)$$

for an $(n + N)$ -vector $v = (v_1, \dots, v_{N+n})$, where

$$v_k = (-1)^{k-1} \Delta^{k-1} u \quad \text{for } 1 \leq k \leq N, \quad (11.6)$$

$$\begin{aligned} v_{N+1} &= (-1)^N p_1 \Delta^N u, \quad v_{N+2} = (-1)^{N+1} p_2 \Delta(p_1 \Delta^N u), \dots, \\ v_{N+n} &= (-1)^{N+n-1} p_{n-1} \Delta \{ p_{n-1} \Delta [\dots (p_1 \Delta^N u)] \}. \end{aligned} \quad (11.7)$$

Then (11.1) goes over into (11.5), i.e., to

$$\begin{aligned} \Delta v_k &= -v_{k+1} \quad \text{for } 1 \leq k < N, \\ \Delta v_{N+k-1} &= v_{N+K/p_k} \quad \text{for } 1 \leq k < n, \\ p_{n+1} \Delta v_{N+n} &= - \sum_{k=0}^{N-1} c_k v_{k+1}. \end{aligned}$$

In other words, the entries in the $(N + n) \times (N + n)$ matrix $A(m)$ are zero, except on the superdiagonal and the last row: the first $N - 1$ elements on the superdiagonal are 1's and the last n are $1/p_1, \dots, 1/p_n$, while the last row is $(c_0, \dots, c_{N-1}, 0, \dots, 0)/p_{n+1}$. Thus the entries in A are nonnegative. It is not difficult to verify that (11.3) assures that $I - A(m)$ is nonsingular and that the entries in $(I - A(m))^{-1}$ are nonnegative; cf. the proof of Theorem 3.1. It follows from Theorem (I) of [9, pp. 733–734] that (11.5) has a solution satisfying $v_1 > 0$ and $v_k \geq 0$ for $1 < k \leq n + N$ on I . This implies (11.4), where $u = v_1$, and the additional inequalities

$$(-1)^{N+j-1} \Delta \{ p_{j-1} \Delta [\dots (p_1 \Delta^N u)] \} \geq 0 \quad \text{for } 2 \leq j \leq n. \quad (11.8)$$

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218