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# THE PL GRASSMANNIAN AND PL CURVATURE

BY

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ABSTRACT. A space $\mathcal{G}_{n,k}$ is constructed, together with a block bundle over it, which is analogous to the Grassmannian $G_{n,k}$ in that, given a PL manifold $M^n$ as a subcomplex of an affine triangulation of $R^{n+k}$, there is a natural "Gauss map" $M^n \to \mathcal{G}_{n,k}$ covered by a block-bundle map of the PL tubular neighborhood of $M^n$ to the block bundle over $\mathcal{G}_{n,k}$. Certain subcomplexes of $\mathcal{G}_{n,k}$ are then studied in connection with immersion problems, the chief result being that a connected manifold $M^n$ (nonclosed) PL immerses in $R^{n+k}$ satisfying certain "local" conditions if and only if its stable normal bundle is represented by a map to the subcomplex of $\mathcal{G}_{n,k}$ corresponding to the condition. An important example of such a condition is a restriction on PL curvature, e.g., nonnegative or nonpositive, PL curvature having been defined by D. Stone.

**0.** The purpose of this paper is to construct and examine a certain space, which we shall call $\mathcal{G}_{n,k}$, which is related to BPL and to the classification of normal block bundles of locally-flat PL embeddings of $n$-manifolds in $R^{n+k}$ much as the finite Grassmannian $G_{n,k}$ is related to BO and to the classification of normal vector bundles of smooth embeddings of $n$-manifolds in $R^{n+k}$. That is:

(a) The family of spaces $\{\mathcal{G}_{n,k}\}$ forms a double sequence

$$\begin{array}{ccc} \mathcal{G}_{n,k} & \to & \mathcal{G}_{n+1,k} \\ \downarrow & & \downarrow \\ \mathcal{G}_{n,k+1} & \to & \mathcal{G}_{n+1,k+1} \end{array}$$

(b) There is a canonical $k$-block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$.

(c) Given a triangulated combinatorial manifold $M^n$ embedded as a subcomplex of a piecewise-linear triangulation of $R^{n+k}$, there is a natural "Gauss" map $\nu$: $M^n \to \mathcal{G}_{n,k}$ so that $\nu^*(\gamma_{n,k})$ is the normal block bundle of the embedding.

Specifying further what is meant by "natural" in (c) above, we mean that the Gauss map $\nu$ is immediately determined pointwise by the geometric data of the situation. One does not need Brown's Theorem **[B]**, or even Rourke and Sanderson's construction of universal $k$-block bundles **[RS]**, which, in

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any case, determine classifying maps only up to homotopy. The situation is thus analogous to the case where a smooth manifold $M^n$ embedded in $R^n$ (with $R^n$ having its usual linear structure) automatically acquires a map, i.e., the classical Gauss map, to $G_{n,k}$. This justifies, in part at least, our calling $\mathcal{G}_{n,k}$ a polyhedral analogue of the finite Grassmannian and our reference to $\nu$ as the Gauss map.

In §3 we shall use the Grassmannian, and its subcomplexes, to study the problem of polyhedrally immersing a manifold $M^n$ in $R^{n+n}$ so that its curvature (in the sense of D. Stone [$\mathbf{S_1}$], [$\mathbf{S_2}$]), is everywhere nonnegative or nonpositive. We reduce this problem to a homotopy problem (i.e., the usual kind of lifting problem), in the case of manifolds each component of which is open or has nonvoid boundary.

We also take note of the fact that in a preliminary version of this paper, the Grassmannian and Gauss map were used to prove a close approximation to the statement that there exist "local combinational formulae" for cocycles representing rational characteristic classes of PL manifolds. A suggestion of Rourke improved both the result (which is now no longer an approximation, but the statement itself), and the method of proof. A separate, joint paper will appear with this result [**LR**].

We shall need some preliminary definitions. "Manifold" shall always mean PL manifold and "triangulated manifold" shall always mean combinatorially triangulated. By a $j$-plane in $R^{n+k}$ we mean a $j$-dimensional linear subspace. An affine $j$-plane is the translate of a $j$-plane by some fixed vector. If $U_1$ is a $j_1$-plane, $U_2$ a $j_2$-plane, with $j_1 < j_2$ we say that $U_1$ is in $U_2$ when we mean that $U_1$ is a linear subspace of $U_2$. If $U_1$ is an affine $j_1$-plane, we say that $U_1$ is in $U_2$ when it is an affine subspace of $U_2$. If $U$ is a $j$-plane we use $D_U$ to designate the unit disc of $U$ and $S_U$ to denote the unit $(j - 1)$-sphere in $U$.

A piecewise linear triangulation of $R^{n+k}$ means a triangulation such that every $j$-simplex $\sigma$ is a subset of some affine $j$-plane.

If $M^n$ is a triangulated manifold and $\sigma$ is a closed simplex, then $\mathrm{st}(\sigma, M^n)$ is the union of all closed simplices having $\sigma$ as a face; $\mathrm{lk}(\sigma, M^n)$ is the union of all simplices $\tau$ such that $\tau \subseteq \mathrm{st}(\sigma, M^n)$, $\tau \cap \sigma = \varnothing$. The dual cell $\sigma^*$ is the union of all simplices $\tau$ of the first barycentric subdivision of the given triangulation such that $\tau \cap \sigma = \{\text{barycenter of } \sigma\}$.

**1. Formal links.** We need to define the notion of "formal link" which will play a central role in the subsequent constructions.

To begin with, let $U$ be a $(j + k)$-plane of $R^{n+k}$. A triangulation $T$ of $S_U$ shall be called "nice" if and only if:

(a) For each $r$-simplex $\sigma$ of $T$ there is a unique $(r + 1)$-plane in $U$ which contains $\sigma$.

(b) If $c(\sigma)$ is the convex hull in $U$ of the vertices of $\sigma$, then $c(\sigma)$ is contained in the union of all those segments from the origin to points in $\sigma$.

(c) For any simplex $\sigma^i$ of $T$, the convex structure is the same as that obtained by considering $\sigma^i$ as a subset of $S_U$ and defining, for $x, y \in \sigma^i$, $0 \leqslant t \leqslant 1$, the convex combination $t \cdot x + (1 - t) \cdot y$ as $tx + (1 - t)y/\|tx + (1 - t)y\|$ where multiplication by scalars in the latter expression has its usual meaning for the vector space $U$.

1.1 DEFINITION. A formal link $L$ of dimension $(n, k, j)$ is a triple $(U_L, T_L, \Sigma_L)$ where $U_L$ is a $(j + k)$-plane of $R^{n+k}$, $T_L$ is a nice triangulation of $S_{U_L}$, $\Sigma_L$ is a subcomplex of $T_L$ which is a combinatorial $(j - 1)$-sphere.

If $k = 2$ we shall further assume that the sphere pair $(T_L, \Sigma_L)$ is unknotted.

Let $L = (U_L, T_L, \Sigma_L)$ be a formal link of dimension $(n, k, j)$ and let $v$ be a vertex of $\Sigma_L$. We define a new formal link $L_v$ of dimension $(n, k, j - 1)$ as follows: Let $R$ denote the segment from 0 to $v$ in $U_L$; then $U_v$ is the $(j + k - 1)$-plane of $U_L$ orthogonal to $R$. Let $U'$ be an affine $(j + k - 1)$-plane of $U_L$ parallel to $U_v$ and passing through the midpoint $m$ of $R$. Let $S'$ be a small $(j + k - 2)$-sphere of radius $\lambda$ in $U'$ centered at $m$. If $\sigma$ is a simplex of $\mathrm{lk}(v, T_L)$ then let $\tau(\sigma) = \sigma * v$ be the corresponding simplex of $\mathrm{st}(v, T_L)$. Let $P_\sigma$ be the union of all segments in $U$ from the origin to points in $\tau(\sigma)$. We claim that if we set $\sigma_1 = S' \cap P_\sigma$, then $\sigma_1$ is homeomorphic to $\sigma$, and thus letting $\sigma$ range over all the simplices of $\mathrm{lk}(v, T_L)$ we obtain a triangulation of $S'$ isomorphic to that of $\mathrm{lk}(v, T_L)$. Now consider the similarity transformation on $U$ given by $u \to \lambda^{-1} \cdot (u - m)$ which carries $U'$ onto $U_v$ and $S'$ onto $S_{U_v}$. This induces a triangulation of $S_{U_{L_v}}$ with one simplex $\bar{\sigma}$ for each simplex $\sigma$ of $\mathrm{lk}(v, T_L)$. Call this triangulation $T_{L_v}$. Clearly $T_{L_v}$ is a nice triangulation of $S_{U_v}$. Let $\Sigma_{L_v} = \bigcup_{\sigma \subseteq \mathrm{lk}(v, \Sigma_L)} \bar{\sigma}$. $\Sigma_{L_v}$ is clearly a subcomplex of $T$ combinatorially equivalent to $S^{j-2}$. We thus set $L_v = (U_v, T_v, \Sigma_v)$.

Furthermore, we may consider a formal link $L = (U_L, T_L, \Sigma_L)$ of dimension $(n, k, j)$ and an arbitrary $r$-simplex $\sigma$ of $\Sigma_L$. Let $v_0, \ldots, v_r$ be its vertices, arbitrarily ordered. Let $L_0 = L_{v_0}$. Clearly there are vertices $v_1^1, v_2^1, \ldots, v_r^1$ of $\Sigma_{L_0}$ corresponding to $v_1, \ldots, v_r$. Then set $L_1 = L_{v_1^1}$ and obtain vertices $v_2^2, \ldots, v_r^2$ of $\Sigma_{L_1}$ corresponding to $v_2^1, \ldots, v_r^1$. Continuing in this fashion, we may define $L_{i+1} = (L_i)_{v_{i+1}^{i+1}}$ for $i < r$, finally obtaining $L_r$, a formal link of dimension $(n, k, j - r - 1)$.

1.2 LEMMA. *$L_r$ is independent of the ordering $v_0, \ldots, v_r$.*

We sketch the proof. Let $b$ be the barycenter of $\sigma$, and let $m$ be the midpoint of the ray in $U_L$ from the origin to $b$. Let $X_\sigma$ denote the unique $(r + 1)$-plane of $U_L$ in which $\sigma$ lies and let $U_\sigma$ be the $k + j - r - 1$ plane of $U_L$ orthogonally complementary to $X_\sigma$. Let $U'$ denote the affine $(k + j - r - 1)$-plane of $U_L$ parallel to $U_\sigma$ and passing through $m$. Given a simplex $\tau$ of $\mathrm{lk}(\sigma, T_L)$, let $\rho(\tau)$ be the simplex $\tau * \sigma$ of $\mathrm{st}(\sigma, T_L)$. Let $P_\tau$ be the union of all rays in $U_L$ from the origin to points in $\rho(\tau)$. If $S'$ is a small sphere of radius $\lambda$ in $U'$, centered at $m$, let $\tau_1 = S' \cap P_\tau$. $\tau_1$ is homeomorphic to $\tau$, and letting $\tau$ range over all the simplices of $\mathrm{lk}(\sigma, T_L)$, we obtain a triangulation of $S'$. We note that the similarity transformation $u \to \lambda^{-1}(u - m)$ carries $S'$ onto $S_{U_\sigma}$ thereby providing $S_{U_\sigma}$ with a nice triangulation $T_\sigma$ with one simplex $\bar{\tau}$ for each simplex $\tau$ of $\mathrm{lk}(\sigma, T_L)$. We let $\Sigma_\sigma = \bigcup_{\tau \subseteq \mathrm{lk}(\sigma, \Sigma_L)} \bar{\tau}$, and thereby obtain a formal link $L_\sigma = (U_\sigma, T_\sigma, \Sigma_\sigma)$ of dimension $(n, k, j - r - 1)$. We now claim that $L_\sigma$ is the same as the $L_r$ constructed above from the ordering $v_0, \ldots, v_r$ of the vertices of $\sigma$, irrespective of which ordering was used. Q.E.D.

Consider once more an arbitrary formal link $L = (U_L, T_L, \Sigma_L)$ of dimension $(n, k, j)$ and let $v^*$ denote the dual $(j-1)$-cell of $v$ in $\Sigma_L$ (as a subcomplex of the first barycentric subdivision of $\Sigma_L$). Let $\hat{v}^*$ denote the dual $(j + k - 1)$-cell of $v$ in $T_L$ (as a subcomplex of the first barycentric subdivision of $T_L$). We claim that there is an obvious homeomorphism $(\mathrm{st}(v, T_L), \mathrm{st}(v, \Sigma_L)) \to \hat{v}^*, v^*$. This is obtained by "radially projecting" $\mathrm{lk}(v, T_L)$ onto $\mathrm{bdy}(\hat{v}^*, v^*)$ and then linearly extending to convex combinations $\lambda v + (1 - \lambda)u$, $u \in \mathrm{lk}(v, T_L)$ (which is the generic form of a point in $\mathrm{st}(v, T_L)$).

We now adopt the convention that, if $L$ is a formal link of dimension $(n, k, 0)$, making $\Sigma_L = \varnothing$, we shall interpret $c\Sigma_L$ as the set consisting of a single "cone" point; otherwise $c$ has its usual meaning, i.e., unreduced cone. (Alternatively, we may read $cX$ as the reduced cone on $X^+ = X \cup \{*\}$ where $*$ is a disjoint base point.)

Now consider some formal link $L$, $v$ a vertex of $\Sigma_L$, and $L_v = (U_v, T_v, \Sigma_v)$. We may identify $D_{U_v}$ with $cT_v$, and, since $\mathrm{st}(v, T_L) = c\,\mathrm{lk}(v, T_L)$, we obtain a homeomorphism $h_{(L,v)}\colon cT_v, c\Sigma_v \to \hat{v}^*, v^*$.

If $\sigma$ is a simplex of $\Sigma_L$ and $v_0, \ldots, v_r$ an ordering of its vertices, then, by constructing $L_i$, $i = 0, 1, \ldots, r$, as before we have a composite of 1-1 maps

$$
\begin{array}{cccc}
cT_{L_r} & \to & T_{L_{r-1}} & \\
 & & \cap & \\
 & & cT_{L_{r-1}} & \to \quad T_{L_{r-2}} \\
 & & & \vdots \\
 & & & T_{L_0} \\
 & & & \cap \\
 & & & cT_{L_0} \to T_L
\end{array}
$$

which restricts to

$$\begin{array}{cccccc}
c\Sigma_{L_r} & \to & \Sigma_{L_{r-1}} & & & \\
 & & \cap\!\mid & & & \\
 & & c\Sigma_{L_{r-1}} & \to & \Sigma_{L_{r-2}} & \\
 & & & & \vdots & \\
 & & & & \Sigma_{L_0} & \\
 & & & & \cap\!\mid & \\
 & & & & c\Sigma_{L_0} \to \Sigma_L &
\end{array}$$

Here each of the horizontal maps is of the form $h_{(L_{i-1}, v_i)}$ for some vertex $v_i$ of $\Sigma_{L_i}$ corresponding to $v_i$.

Since $L_r = L_\sigma$, we have a homeomorphism $cT_{L_\sigma}$, $c\Sigma_{L_\sigma}$ into $T_L$, $\Sigma_L$ and it is easy to see that the image is $\hat{\sigma}^*$, $\sigma^*$ where $\hat{\sigma}^*$ is the dual cell of $\sigma$ in the first derived of $T_L$ and $\sigma^*$ is the dual cell of $\sigma$ in the first derived of $\Sigma_L$.

1.3 LEMMA. *This homeomorphism is independent of the ordering $v_0, \ldots, v_r$ of the vertices of $\sigma$.*

We leave the proof to the reader. In the light of this lemma we are justified in calling this homeomorphism $h_{(L,\sigma)}$.

Note that if $\sigma$ is a simplex of $\Sigma_L$, and of the form $\sigma = \tau * \rho$, then there is a simplex $\bar{\rho}$ of $\Sigma_L$ corresponding to $\rho$ and, furthermore, $(L_\tau)_{\bar{\rho}} = L_\sigma$. We claim, in extension of Lemma 1.3 above, that $h_{(L,\sigma)} = h_{(L,\tau)} \circ h_{(L_\tau, \bar{\rho})}$.

Given a formal link $L = (U_L, T_L, \Sigma_L)$ of dimension $(n, k, j)$, let $X_L$ be the plane of dimension $(n - j)$ in $R^{n+k}$ orthogonal to $U_L$, thus decomposing $R^{n+k}$ as $U_L \oplus X_L$. Given a simplex $\sigma$ in $T_L$ let $Q_\sigma$ be the union of all those infinite rays in $U_L$ from the origin through points in $\sigma$. Let $V_\sigma$ denote $Q_\sigma \times X_L$ (as a subset of $R^{n+k}$ $(= U_L \oplus X_L)$.) Let $V_L = \bigcup_{\sigma \subseteq \Sigma_L} V_\sigma$. $V_L$ is thus an $n$-dimensional piecewise-linear submanifold of $R^{n+k}$. Note that if $\dim \sigma = r$, $V_\sigma$ is a manifold of dimension $(n + r - j + 1)$, which is contained in some $(n + r - j + 1)$-plane of $R^{n+k}$. Henceforth, we shall call $X_L$ the *axis* of $V_L$.

It is now appropriate to take note of the relationship between formal links and submanifolds of $R^{n+k}$. Let $M^n$ be a combinatorial submanifold which is a locally flat subcomplex of a piecewise-linear triangulation of $R^{n+k}$. Let $\sigma$ be an $r$-simplex of $M^n$, $\sigma \subsetneq \partial M^n$. The formal link $L\langle\sigma, M^n\rangle$ is defined as follows:

Let $Y_\sigma$ be the affine $r$-plane containing $\sigma$ and let $U_\sigma$ be the $(n + k - r)$-plane orthogonal to $Y_\sigma$. Let $U'$ be an affine $(n + k - r)$-plane parallel to $U_\sigma$ and passing through the barycenter $b_\sigma$ of $\sigma$. Let $S'$ be a small $(n + k - r - 1)$-sphere of radius $\lambda$ in $U'$ centered at $b_\sigma$. Given a simplex $\tau$ of $\mathrm{lk}(\sigma, R^{n+k})$, let $\rho(\tau) = \tau * \sigma \subseteq \mathrm{st}(\sigma, R^{n+k})$. Let $\tau' = S' \cap \rho(\tau)$. Thus $S'$ acquires a triangulation having one simplex $\tau'$ for each $\tau$ of $\mathrm{lk}(\sigma, R^{n+k})$. Consider $u \to 1/\lambda \cdot (u - b_\sigma)$ taking $U'$ onto $U_\sigma$ and $S'$ onto $S_{U_\sigma}$. Let $T_\sigma$ be the induced triangulation of $S_{U_\sigma}$ under this homeomorphism. $T_\sigma$ has one simplex $\bar{\tau}$ for each simplex $\tau$ of $\mathrm{lk}(\sigma, R^{n+k})$. Moreover, if $\bar{\tau}$ is of dimension $i$ it lies in an $(i + 1)$-plane of $U_\sigma$ and, furthermore, $\bar{\tau}$ lies in the union of all rays through $c(\bar{\tau})$. Thus $T_\sigma$ is a nice triangulation of $S_{U_\sigma}$.

Now let $\Sigma_\sigma = \bigcup_{\tau \subseteq \mathrm{lk}(\sigma, M^n)} \bar{\tau}$. $\Sigma_\sigma$ is a triangulated sphere of dimension $n - r - 1$. We now define the formal link of $\sigma$ to be the $(n, k, n - r)$-dimensional formal link $L\langle \sigma, M^n \rangle = (U_\sigma, T_\sigma, \Sigma_\sigma)$.

We note the following property:

Given $M^n \subseteq R^{n+k}$ as above, let $\tau \subseteq \sigma$ be simplices of $M^n$ so that $\sigma = \tau * \rho$. Thus, corresponding to $\rho \subseteq \mathrm{lk}(\tau, M^n) \subseteq \mathrm{lk}(\tau, R^{n+k})$ there is a simplex $\bar{\rho}$ of $\Sigma_\tau$. We claim that $(L\langle \tau, M \rangle)_{\bar{\rho}} = L\langle \sigma, M \rangle$. The reader is invited to verify this.

**2. The complex $\mathcal{G}_{n,k}$ and the Gauss map.** We wish to define a certain space $\mathcal{G}_{n,k}$, a $k$-block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$, and a "natural" Gauss map $\nu$: $M^n \to \mathcal{G}_{n,k}$ for every combinatorial $n$-manifold given as a subcomplex of a piecewise-linear triangulation of $R^{n+k}$. This map $\nu$ will be seen to be covered in a natural way by a block bundle map $E \to E_{n,k}$ where $E$ is the normal block bundle of the embedding $M^n \subseteq R^{n+k}$ and $E_{n,k}$ is the total space of $\gamma_{n,k}$.

For the remainder of this definitional material, fix $n$ and $k$. Consider the set of formal links $L$ of dimension $(n, k, j)$. Consider further a set $\{e_L\}$ of $j$-dimensional cells indexed by these formal links. Intuitively, it is convenient to think of $e_L$ as the topological $j$-cell $c\Sigma_L$. We now recall that for a simplex $\sigma$ of $\Sigma_L$ we have a homeomorphism $h_{(L,\sigma)}$ from $cT_{L_\sigma}$ onto the dual cell $\hat{\sigma}^*$ of $\sigma$ in $T_L$, and that $h_{(L,\sigma)}$ takes $c\Sigma_{L_\sigma}$ homeomorphically onto the dual cell $\sigma^*$ of $\sigma$ in $\Sigma_L$. We denote by $h^0_{(L,\sigma)}$ the restriction $h_{(L,\sigma)}|c\Sigma_{L_\sigma} \to \sigma^*$. If we continue to think of $e_L$ as $c\Sigma_L$, we may define $\mathcal{G}_{n,k}$ as the cell complex obtained by taking $\bigcup_L e_L$ ($L$ a formal link of dimension $(n, k, j)$, $0 \leqslant j \leqslant n$) and then identifying $e_{L_\sigma}$ with a subspace of $\dot{e}_L$ via $h^0_{(L,\sigma)}$. $\mathcal{G}_{n,k}$ is thus seen to be a C-W complex with one $j$-cell for every formal link of dimension $(n, k, j)$.

To build the block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$, we think first of the block bundle over the disjoint union $\bigcup_L e_L$, where the "block" over the $j$-cell $e_L \cong c\Sigma_L$ is the $j + k$-cell $b_L \cong cT_L$. If we then divide out $\bigcup_L b_L$ by the identifications obtained from the maps $h_{(L,\sigma)}$: $b_{L_\sigma} \to b_L$, we obtain a space $E_{n,k}$ which is the total space of a block bundle $\gamma_{n,k}$ over $\mathcal{G}_{n,k}$.

Now let $M^n$ be a locally flat submanifold of $R^{n+k}$, given as a subcomplex in some piecewise-linear triangulation. We wish to define the Gauss map $\nu$: $M^n \to \mathcal{G}_{n,k}$. Let $\sigma$ be a simplex of $M^n$ ($\sigma \not\subseteq \partial M^n$ if $M^n$ has boundary), and let

$\hat{\sigma}^*$, $\sigma^*$ denote, respectively, the dual cell of $\sigma$ in the triangulation of $R^{n+k}$ and in the triangulation of $M^n$. Specifically, $\hat{\sigma}^*$ and $\sigma^*$ are subcomplexes of the first barycentric subdivision. If $M^n$ has no boundary the dual cells are, of course, the cells of the dual cell decomposition of $M^n$. If $M^n$ has a nonvoid boundary, then $\bigcup_{\sigma \not\subseteq \partial M^n} \sigma^*$ is itself an $n$-manifold which is a deformation retract of $M^n$.

We define $\nu$ explicitly on $M_0^n = \bigcup_{\sigma \not\subseteq \partial M^n} \sigma^*$. We may quickly describe $\nu$ as a map of this cell structure to the given cell structure on $\mathcal{G}_{n,k}$ as follows. First, we get a map $\nu$: $M_0^n \to \mathcal{G}_{n,k}$ by assigning $\sigma^*$ to the cell $e_{L\langle\sigma,M^n\rangle}$. It is easily seen that this is compatible with incidence relations.

If fact we may be even more specific; that is, the map $\nu$ is easily specified pointwise. In particular, if $\sigma$ is an $(n-j)$-simplex then the dual cell $\sigma^*$ of $\sigma$ as a subcomplex of the first barycentric subdivision of $M^n$, is isomorphic in an obvious way to the cone on the first subdivision of $\Sigma_{L\langle\sigma,M^n\rangle}$. This gives an obvious homeomorphism $\nu_\sigma$: $\sigma^* \to c\Sigma_{L\langle\sigma,M^n\rangle} = e_{L\langle\sigma,M^n\rangle}$. Moreover, if $\tau$ is a face of $\sigma$, $\sigma^*$ is a face of $\tau^*$ and we then have a strictly commutative diagram

$$\begin{array}{ccc} \sigma^* & \xrightarrow{\subseteq} & \tau^* \\ \downarrow \nu_\sigma & & \downarrow \nu_\tau \\ e_{L\langle\sigma,M^n\rangle} & \xrightarrow{i} & e_{L\langle\tau,M^n\rangle} \end{array}$$

where $i$ is the map defined by first noting that $L\langle\sigma, M^n\rangle$ is $[L\langle\tau, M^n\rangle]_\rho$ for some simplex $\rho$ of $\Sigma_{L\langle\tau,M^n\rangle}$ thus allowing $i$ to be defined as the composition

$$e_{L\langle\sigma,M^n\rangle} = c\Sigma_{L\langle\sigma,M^n\rangle} \xrightarrow{h(L\langle\sigma,M^n\rangle,\rho)} \Sigma_{L\langle\tau,M^n\rangle} \subseteq c\Sigma_{L\langle\tau,M^n\rangle} = e_{L\langle\tau,M^n\rangle}.$$

The family $\{\nu_\sigma\}$ thus defines a map $\nu$: $M_0^n \to \mathcal{G}_{n,k}$.

Now by Rourke and Sanderson [**RS**], the normal block bundle of $M^n$ in $R^{n+k}$ may be specified as follows (at least over $M_0^n$). The total space $E$ is the union of $\hat{\sigma}^*$ over all the simplices $\sigma$ of $M^n$ with $\sigma \not\subseteq \partial M^n$. Here $\hat{\sigma}^*$ is the "block" over $\sigma^*$. We shall define a map $N$ from $E$ to $E_{n,k}$. This is done by noting that $\hat{\sigma}^*$ is, as a subcomplex of the first subdivision of $R^{n+k}$, isomorphic to $cT_{L\langle\sigma,M^n\rangle} = b_{L\langle\sigma,M^n\rangle}$ in a natural way. Let $N_\sigma$ denote this homeomorphism. Again, if $\tau$ is a face of $\sigma$, we have a commutative diagram

$$\begin{array}{ccc} \hat{\sigma}^* & \xrightarrow{\subseteq} & \hat{\tau}^* \\ \downarrow N_\sigma & & \downarrow N_\tau \\ b_{L\langle\sigma,M^n\rangle} & \xrightarrow{j} & b_{L\langle\tau,M^n\rangle} \end{array}$$

where $j$ is defined similarly to $i$ above. The family $\{N_\sigma\}$ defines a map $N$: $E \to E_{n,k}$, with $N|M_0^n = \nu$.

Thus it is evident that the map $\nu$, which depends only on the triangulation of $M^n$ and the ambient $R^{n+k}$, is entitled to be called the Gauss map. (There is no essential difficulty resulting from $M_0^n$ being strictly smaller than $M^n$ when $\partial M^n \neq 0$.)

We also point out that, by an easy extension of this observation, if $M^n$ is immersed in a triangulated $R^{n+k}$ so that the immersion is an embedding on all subspaces of the form $\mathrm{st}(\sigma, M^n)$, $\sigma$ a simplex of $M^n$ (that is, $M^n$ is immersed as a subcomplex of a triangulated $\mathbf{R}^{n+k}$), then the notion of a Gauss map, classifying the normal block bundle of the immersion is likewise naturally defined.

We observe that there are natural maps $\alpha$: $\mathcal{G}_{n,k} \to \mathcal{G}_{n,+1}$ $\beta$: $\mathcal{G}_{n,k} \to \mathcal{G}_{n+1,k}$ thus yielding a double sequence

$$\begin{array}{ccccccc}
 & & \downarrow\beta & & \downarrow\beta & & \\
\xrightarrow{\alpha} & & \mathcal{G}_{n,k} & \xrightarrow{\alpha} & \mathcal{G}_{n,k+1} & & \xrightarrow{\alpha} \\
 & & \downarrow\beta & & \downarrow\beta & & \\
\xrightarrow{\alpha} & & \mathcal{G}_{n+1,k} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k+1} & & \xrightarrow{\alpha} \\
 & & \downarrow\beta & & \downarrow\beta & &
\end{array}$$

The definition of $\alpha$ and $\beta$ is as follows:

Let $L$ be a formal link of dimension $(n, k, j)$, $L = (U_L, T_L, \Sigma_L)$. Let $P$ denote the line in $R^{n+k+1}$ orthogonal to $R^{n+k}$. Set $U_K = U_L \oplus P$. Then $S_{U_K} \cap P$ consists of two points $n$ and $s$. We choose a triangulation $T_K$ of $S_{U_K}$ by letting the simplices be those of $T_L$, together with additional simplices $\sigma^+$ and $\sigma^-$ for each simplex $\sigma$ of $T_L$. $\sigma^+$ is the union of all 90° arcs from $n$ to the points of $\sigma$, and $\sigma^-$ is the union of all 90° arcs from $s$ to points of $\sigma$. We also include $n$ and $s$ as vertices. Thus, $T_K$ is isomorphic as a complex to the unreduced suspension of $T_L$. We set $\Sigma_K = \Sigma_L$. We let $K(L)$ be the formal link of dimension $(n, k + 1, j)$ given by $K = (U_K, T_K, \Sigma_K)$. The set map $L \to K(L)$ from $(n, k, j)$-links to $(n, k + 1, j)$-links is consistent with incidence relations among the cells of $\mathcal{G}_{n,k}$, $\mathcal{G}_{n,k+1}$ because $K(L_\sigma) = [K(L)]_\sigma$ for $\sigma$ a simplex of $\Sigma_L = \Sigma_K$. Thus we obtain a cellular map $\alpha$: $\mathcal{G}_{n,k} \to \mathcal{G}_{n,k+1}$.

On the other hand, given the $(n, k, j)$-link $L$, we may regard $U_L$ as a $(j + k)$-plane of $R^{n+k+1}$, and a fortiori, $L$ may be regarded as a formal link of dimension $(n + 1, k, j)$. This allows us to view $\mathcal{G}_{n,k}$ as a subcomplex of $\mathcal{G}_{n+1,k}$ and it is seen, again without difficulty that this induces an inclusion $\beta$: $\mathcal{G}_{n,k} \overset{\subseteq}{\to} \mathcal{G}_{n+1,k}$.

It is clear that $\alpha^* \gamma_{n,k+1} = \gamma_{n,k} \oplus \varepsilon^1$ and that $\beta^* \gamma_{n+1,k} = \gamma_{n,k}$.

**3. Subcomplexes of $\mathcal{G}_{n,k}$–applications to D. Stone's theory of polyhedral curvature.** We shall use our construction of the polyhedral Grassmannian to study some immersion problems. In particular, we shall study the question of deciding when a manifold $M^n$ may be immersed piecewise linearly in $R^{n+k}$

subject to constraints on the induced polyhedral sectional curvature in the sense of D. Stone [$\mathbf{S_1}$], [$\mathbf{S_2}$].

Recall that the Hirsch immersion theorem, as applied in the PL case, [**H**], [**HP**] tells us that the classifying map $M^n \to \mathrm{BPL}$ for the stable normal bundle of $M^n$ lifts to $\mathrm{BPL}(k)$ if and only if $M^n$ immerses in $R^{n+k}$ (provided either $k \neq 0$ or $M^n$ has a handlebody decomposition with no $n$-handles). The same, of course, holds true if $\mathrm{BPL}(k)$ be replaced by $\mathcal{G}_{n,k}$.

We seek to generalize this result.

We shall say that a subcomplex $B$ of $\mathcal{G}_{n,k}$ is "geometric" if it has the property

(i) If $e_L \subseteq B$ and $V_J = V_L$ then $e_J \subseteq B$.

For the remainder of this discussion, let $M^n$ be a manifold admitting a handle decomposition with no $n$-handles, i.e., every component of $M^n$ is either open or has nonvoid boundary.

3.1 THEOREM. *Let $B$ be a geometric subcomplex of $\mathcal{G}_{n,k}$. $M^n$ immerses in $R^{n+k}$ with a Gauss map $\nu$: $M^n \to B \subseteq \mathcal{G}_{n,k}$ if and only if the stable normal classifying map $M^n \to BPL$ lifts to $B$.*

Before proceeding to the proof of 3.1, we state a corollary which is the prime motivating example. David Stone has defined an analog of "sectional curvature" for polyhedral manifolds embedded, or immersed, in Euclidean space. (See [$\mathbf{S_1}$], [$\mathbf{S_2}$] for details.) In particular, one may talk of two parameters $K_-(x, d)$, $K_+(x, d)$ for each point $x \in M^n$ and each "tangent direction" $d$. One may call $M$ "nonnegatively" curved at $x$ if $K_-(x, d) \geqslant 0$ for all choices $d$ and "nonpositively" curved at $x$ if $K_+(x, d) \leqslant 0$ for all $d$. (The meaning of strictly positive or negative curvature is somewhat elusive, since $K_+(x, d) = K_-(x, d) = 0$ for any $x$ in the interior of an $n$-simplex or $(n-1)$-simplex.) However, if $M^n$ is polyhedrally immersed in $R^{n+k}$, and $x \in \operatorname{int} \sigma$, with st $\sigma$ embedded, it is immediately clear from Stone's definition that the curvature properties of the immersion at $x$ depend solely on the formal link of $\sigma$, in our sense. To put it another way, if $L$ is a formal link of dimension $(n, k, j)$, then Stone curvatures $K_-(L, d)$, $K_+(L, d)$ are defined for all "directions" $d$, i.e., all points $d \in \Sigma_L$. We may define $K_-(L) = \min K_-(L, d)$, $K_+(L) = \max K_+(L, d)$.

Let $B^+_{n,k}$ denote the subcomplex of $\mathcal{G}_{n,k}$ consisting of the union of all $e_L$ such that $K_-(L) \geqslant 0$ and $K_-(J) \geqslant 0$ for all faces $J$ of $L$. Similarly, let $B^-_{n,k}$ be the union of $e_L$ such that $K_+(L) \leqslant 0$, $K_+(J) \leqslant 0$ for faces $J$ of $L$. We thus have, as an immediate corollary of 3.1 for manifolds $M^n$ having, as usual, no top-dimensional handle.

3.2 COROLLARY. *$M^n$ immerses in $R^{n+k}$ with everywhere nonnegative (resp. nonpositive) Stone curvature if and only if the stable normal-bundle classifying map $M^n \to$* BPL *lifts to $B^+_{n,k}$ (resp., $B^-_{n,k}$).*

We now proceed to the proof of 3.1. I should like to thank the referee for some suggestions which make some of the constructions to be used simpler and more elegant.

We supplement the given cell decomposition of $\mathcal{G}_{n,k}$ via the cells $e_L$ by a new decomposition into contractible subspaces $\bar{e}_L$, one for each formal link $L$. $\bar{e}_L$ will neither contain nor be contained in $e_L$, in general. However, if $C$ is any subcomplex of $\mathcal{G}_{n,k}$, and for some indexing set $\mathcal{I}$, $\{e_i\}_{i\in\mathcal{I}}$ is the set of all cells of $C$, then $\bigcup_{i\in\mathcal{I}}\bar{e}_{L_i} = \bar{C}$ will contain $C$ as a deformation retract.

To define $\bar{e}_L$, we shall first define spaces $e_{L,J} \subseteq e_J$ where either $L = J$ or $J = L_\sigma$, for some simplex $\sigma$ of $\Sigma_L$. Consider, therefore $c\Sigma_L$, which is the pre-image of $e_L$ before indentifications. We may barycentrically subdivide $e_L$ once to get a complex isomorphic to the cone $c\Sigma'_L$ on the first barycentric subdivision of $\Sigma_L$. We subdivide once more obtaining a complex which we call $c_L$. We let $c_{L,\sigma}$ be the simplicial regular neighborhood in $c_L$ of the vertex $\beta_\sigma$ of $c\Sigma'_L$ where $\beta_\sigma$ is the barycenter of the simplex $\sigma$ of $\Sigma_L$. We let $c_{L,L}$ be the regular neighborhood of the cone point. Now we let $e_{L,L} \subseteq e_L$ be the (homeomorphic) image of $c_{L,L}$ in $e_L$. We let $e_{L,J} = \bigcup_{J=L_\sigma}\text{image}(c_{L,\sigma}) \subseteq e_L$.

Finally, we let $\bar{e}_L = e_{K,L}$ where the union is taken over all $K$ having $L = K_\tau$, $\tau$ a simplex of $\Sigma_K$, together with $K = L$.

We claim that the decomposition $\{\bar{e}_L\}$ has the property required of it, i.e. $\bar{C} = \bigcup_{e_L\subseteq C}\bar{e}_L$ contains $C$ as a deformation retract for any subcomplex $C$ of $\mathcal{G}_{n,k}$.

We wish to define a certain map $F$: $\mathcal{G}_{n,k} \to \mathbf{R}^{n+k}$ having the property that $F(\bar{e}_L) \subseteq V_L$ for all formal links $L$. Moreover, if we set $\gamma_L = \gamma_{n,k}|\bar{e}_L$ there will be block-bundle maps $\phi_L$: $\gamma_L \to \nu(V_L)$, where $\nu(V_L)$ denotes the normal block bundle of $V_L$. The set $\{\phi_L\}$ is to be consistent in the sense that for any two links $L, J$, $\phi_L|\gamma_L \cap \gamma_J = \phi_J|\gamma_L \cap \gamma_J$.

To facilitate the construction, it is convenient to work with a slightly smaller "copy" of $E(\gamma_{n,k})$ embedded within $E(\gamma_{n,k})$. Consider once more a link $L$, and the cell pair $(cT_L, c\Sigma_L)$. We take the second subdivision of this pair, i.e. the barycentric subdivision of the simplicial pair $(cT'_L, c\Sigma'_L)$ to obtain the triangulated cell pair $(d_L, c_L)$ (where $c_L$ obviously is the triangulation used earlier in the construction of $\bar{e}_L$). We let $d_{L,L}$ be the simplicial regular neighborhood of $c_{L,L}$, i.e. the union of all closed simplices of $d_L$ incident to $c_{L,L}$. We let $d_{L,\sigma}$ = the simplicial regular neighborhood of $c_{L,\sigma}$.

Now, we let

$$A_{L,L} = \text{image } d_{L,L}, \quad \text{and} \quad A_{L,J} = \bigcup_{J=L_\sigma} (\text{image } d_{L,\sigma}) \subseteq E(\gamma_{n,k}).$$

Finally, we let $A_L = \bigcup_K A_{K,L}$ where the union is over all $K$ with $L = K_\tau$, together with $K = L$.

It is obvious that $\bar{E} = \bigcup_L A_L \subseteq E(\gamma_{n,k})$ is a block bundle over $\mathcal{G}_{n,k}$, i.e. it is a regular neighborhood of $\mathcal{G}_{n,k}$ in $E(\gamma_{n,k})$ and thus isomorphic, as a block bundle to $\gamma_{n,k}$ itself. Moreover, $A_L$ is the "block" over $\bar{e}_L$. Thus in order to define $F$, and the maps $\phi_L$: $\gamma_L \to \nu(V_L)$, we claim that it will suffice to specify a certain map $G$: $\bar{E} \to \mathbf{R}^{n+k}$, where $F = G|\mathcal{G}_{n,k}$.

Now consider once more the cell $cT_L$ triangulated as $d_L$. Let $\bar{d}_L = d_{L,L} \cup \bigcup_\sigma d_{L,\sigma}$. To define a map $\Gamma_L$: $\bar{d}_L \to \mathbf{R}^{n+k}$ it will obviously suffice to take an affine extension of a map on vertices. So we note first that any vertex of $\bar{d}_L$ must either be a barycenter $\beta_\sigma$ of some simplex $\sigma$ of $\Sigma_L$, or else the cone point $x$ of $c\Sigma_L$, or must be contiguous to $x$ or some $\beta_\sigma$ (i.e. contiguous in the sense of being connected by a 1-simplex). Now for such a vertex $v$, we denote $l(v) = L$ if $v$ is contiguous to $x$. Let $l(v) = L_\sigma$ if $\nu = \beta_\sigma$ or if $v$ is contiguous to $\beta_\sigma$, but not to $x$, nor to $b_\tau$ for any proper face $\tau$ of $\sigma$. Let $P_K = \{v|l(v) = K\}$ where $K = L$ or $L_\sigma$, thus partitioning the vertices of $\bar{d}_L$ into disjoint families. Note that the largest subcomplex of $\bar{d}_L$ containing the vertices $P_L$ (resp. $P_{L_\sigma}$) is naturally isomorphic to the cone on the second barycentric subdivision of $T_L$ (resp. $T_{L_\sigma}$). So let $\Gamma_L(v)$ be the natural image of $v$ in $|T_L| = S_{U_L}$ for $v \in P_L$ (resp., $\Gamma_{L_\sigma}(v)$ = natural image of $v$ in $|T_L| = S_{U_{L_\sigma}}$, $v \in P_{L_\sigma}$). By linear extension $\Gamma_L$: $\bar{d}_L \to \mathbf{R}^{n+k}$ is defined. It is easily seen that this is compatible with identifications in $E(\gamma_{n,k})$, $\mathcal{G}_{n,k}$, so that we obtain a map $G$: $\bar{E} \to \mathbf{R}^{n+k}$, which we may think of as the "tautologous" map on $\bar{E}$. (Note: $\bar{E} = \bigcup_L$ image $\bar{d}_L$.)

We now assert that, if $G_L = G|A_L$, $G_L^{-1}(V_L) \supseteq \bar{e}_L$. Moreover, the reader may easily check that $G_L$ is PL transverse regular to $V_L$. Thus $G_L$ may be regarded as a block-bundle map $\gamma_L \to \nu(V_L)$. That is, we have a bundle locally defined over $\bar{e}_L$ as $F^*(\nu(V_L))$, and it is easily seen that a tube representing this bundle embeds in $\bar{E} \subseteq E(\gamma_{n,k})$. Thus this "local" block bundle pieces together to form a bundle isomorphic to $\gamma_{n,k}$ and the set $\{\phi_L\}$ is defined in the obvious way.

We now consider a manifold $M^n$ with a reduction of its stable normal block-bundle to a $k$-block bundle $\nu_k$, such that there is a bundle map $\theta$: $\nu_k \to \gamma_{n,k}|B$ where $B$ is a geometric subcomplex. We replace $B$ by $\hat{B}$, the union of all $\bar{e}_L$ for $e_L$ a cell of $B$. Let $h$: $M^n \to \hat{B}$ be the map covered by $\theta$. By general position considerations (i.e., a series of codimension-one transversality arguments) it is trivial to show that, for $e_L \subset B$, i.e., $\bar{e}_L \subseteq \hat{B}$, $h^{-1}(\bar{e}_L)$ may be assumed to be a codimension-0 submanifold of $M^n$. Denote this manifold by $M_L$. For $e_L \not\subset B$, $M^{-1}(\bar{e}_L) = \varnothing$. It is also easy to see that $M_L \cap (\bigcup_{e_J \subset e_L} M_J)$ is a codimension-0 submanifold of $\partial M_L$.

Now we consider $g = F \cdot h$: $M \to R^{n+k}$. Note that $g(M_L) \subseteq V_L$, $g(M_L \cap M_J) \subseteq V_{(J,L)}$ for $e_L$ a face of $e_J$. Clearly, by composing $\phi_L$ with $\theta|M_L$ we obtain maps $G_L$: $\nu_k/M_L \to \nu(V_L)/V_L$ covering $g|M_L$. Moreover, on $M_L \cap M_J \neq \varnothing$, $G_L$ and $G_J$ coincide as maps to $V_J \cap V_L$. But, by stability considerations for bundles, we must also then have maps $G'_L\colon \tau(M_L^\circ) \to \tau(V_L)$ where $\tau$ now denotes the PL tangent bundle, and where $M_L^\circ$ denotes $M_L$ minus a disc in the interior of each component. Now we apply the PL version of Hirsch's immersion theorem **[HP]**, stepwise, to deform $g$ so that each $g|M_L^\circ$ is an immersion (codimension 0) of $M_L$ in $V_L$.

Inductively, one does this by starting with $M_L^\circ$ for $\dim e_L = 0$, and deforming $g$ to an immersion, keeping $M_L^\circ \cap M_J^\circ$ in $V_{(J,L)}$ for each $J$ such that $e_J \supset e_L$. Then, one immerses $M_L^\circ$ in $V_L$, for $\dim e_L = 1$, extending the given one on $M_L^\circ \cap (\bigcup_{\dim e_K = 0} M_K^\circ)$, again using a relative version of the immersion theorem. We proceed, inductively, to immerse $M_L^\circ$ corresponding to successively higher-dimension $e_L$, in their respective $V_L$, deforming the underlying map $g|(M^n\text{-(discs)})$ each time as necessary. In the end, we have made $g$ a codimension-$k$ immersion of $M^n$-(discs) into $R^{n+k}$, with each $M_L^\circ$ being immersed in $V_L$.

Moreover, it is easy to arrange that for $M_L \cap M_J \neq \varnothing$ (i.e., $L$ a face of $J$ or vice versa) the immersion takes $M_L^\circ \cap M_J^\circ$ into the interior of $V_L \cap V_J$.

Now we have immersed $M_0^n$, i.e. a multiply-punctured $M^n$ but, since $M^n$ admitted a handle decomposition with no $n$-handle, $M_0^n$ contains a smaller copy of $M^n$ as a codimension-0 submanifold; thus this copy is immersed. Let $M'_L$ be the intersection of this smaller copy with $M_L^\circ$.

Now triangulate $R^{n+k}$ so that the immersion is a simplicial map on $M^n$ and all the $M'_L$. We claim that the Gauss map thus engendered, $M^n \to \mathcal{G}_{n,k}$, must have its image in $B$. For let $S$ be an arbitrary simplex $\sigma$ of $M^n$, and let $P = \{L|\sigma \subseteq M'_L\}$. This set is linearly ordered by the face relation, and thus, if $L_1$ is the maximal element, then a neighborhood of $\sigma$ must be embedded in $V_{L_1}$ since a neighborhood of $\sigma$ in $M'_L$ is embedded in $V_L \cap V_{L_1}$ for all $L$ in $P$.

Therefore the formal link $J$ of $\sigma$ (defined by the triangulation of $R^{n+k}$) must have the property mentioned in property (i) above, i.e., $V_J$ must coincide with $V_{J'}$ for some face $J'$ of $L$. Thus, since $B$ is geometric, $e_j \subseteq B$.

This completes the proof that if the normal bundle of $M^n$ is induced by a map to $B$, then $M^n$ immerses in $R^{n+k}$ with a Gauss map going into $B$.

The converse is, of course, trivial.

**4. Appendix.** (I) We briefly discuss here the smooth analogue of the problem addressed by Theorem 3.1. That is, given some condition on immersions that is "locally" defined, when may $M^n$ be immersed in $\mathbf{R}^{n+k}$ satisfying this condition. (It may be helpful to think of a typical condition of interest, e.g. positive (or negative) curvature.) We will briefly indicate why, for $n$-dimensional non-closed manifolds, this problem is, in some (not very useful) sense, a homotopy problem.

With some local condition for smooth immersions of $n$-manifolds in $\mathbf{R}^{n+k}$ in mind, consider the set consisting of all pairs $(V^n, k)$ where $V^n$ is some manifold and $h$: $V^n \to \mathbf{R}^{n+k}$ is some immersion of $V^n$ satisfying the condition. We consider the union $U$ of all such $V^n$ as a space and let $\gamma$ be the tangent bundle of this "nonparacompact" manifold. Clearly if $M^n$ is a nonclosed manifold and there is a bundle map $\tau(M^n) \to \gamma$, then there is a tangential map of $M^n$ to some $V^n$ where $V^n$ immerses via $h$, to satisfy the given condition. But then, by Hirsch's theorem $M^n$ immerses in $V^h$ via some map $g$, and thus $M^n$ immerses in $\mathbf{R}^{n+k}$, via $h \circ g$ so as to satisfy the condition. The foregoing falls under the heading of facts which are true, but not very interesting.

On the other hand the "geometric" subcomplexes of $\mathcal{G}_{n,k}$ studied in §3 are interesting precisely because they are given as C-W complexes determined purely by "local" data. That is, it seems entirely feasible that, for interesting examples $B$ of such subcomplexes ($B^+_{n,k}$ and $B^-_{n,k}$, for instance), the algebraic topological questions that naturally arise may be answerable. (That is, it might well be both possible and useful to compute the homotopy groups of the fiber of $B \to$ BPL and to determine relations on the characteristic classes of $\gamma_{n,k}(B)$.)

On the other hand, for smooth manifolds and immersions, I know of no such analogous possibilities. The $U$, $\gamma$ constructed above (or variants thereof) does not seem to hold out much promise of being understandable in terms of algebraic topology.

(II) Other versions of $\mathcal{G}_{n,k}$: Stability.

We first observe that there are natural mappings

$$\alpha\colon \mathcal{G}_{n,k} \to \mathcal{G}_{n+1,k}, \qquad \beta\colon \mathcal{G}_{n,k} \to \mathcal{G}_{n,k+1}.$$

The map $\alpha$ is completely obvious; a formal link $L$ of dimension $(n, k, j)$ is, by inspection of the definition, also a formal link of dimension $(n + 1, k, j)$. This induces the inclusion of complexes $\alpha$. As for $\beta$, consider a formal link $L = (U_L, T_L, \Sigma_L)$ having dimension $(n, k, j)$. To $L$ we associate the $(n, k + 1, j)$-dimensional link $J$ constructed as follows. If $\mathbf{R}^{n+k+1} = \mathbf{R}^{n+k} \oplus \mathbf{R}^1$ in a standard way let $U_J = U_L \oplus \mathbf{R}^1$; let $T_J$ be isomorphic with the suspension of $T_L$. That is, there is a natural homeomorphism of suspension $|T_L|$ to $S_{U_J}$ which is the identity on the "equator" $S_{U_L}$ and which takes the suspension points to the "north" and "south" poles of $S_{U_J}$. Let $\Sigma_J = \Sigma_L$. Again the assignment $L \to J$ defines a map of complexes $\mathcal{G}_{n,k} \to \mathcal{G}_{n,k+1}$.

Thus it is natural to ask whether the limit of the double sequence

$$\begin{array}{ccccccc}
\cdots \to & \mathcal{G}_{n,k} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k} & \to \\
 & \downarrow\beta & & \downarrow\beta & \\
\cdots \to & \mathcal{G}_{n,k+1} & \xrightarrow{\alpha} & \mathcal{G}_{n+1,k+1} & \\
 & \downarrow & & \downarrow &
\end{array}$$

is $\mathrm{B}\widetilde{\mathrm{PL}} = \mathrm{BPL}$.

The answer does not seem to be an unqualified "yes".

There are still other complexes which may be thought of as PL Grassmannians for certain purposes. For instance, we might consider what happens, if we were to define a "formal link" of dimension $(n, k, j)$ by a pair $(U_L, \Sigma_L)$ where $U_L$ is as before, a $j + k$-plane of $\mathbf{R}^{n+k}$ and where $\Sigma_L$ is, as before, a curvilinear triangulation of some $(j - 1)$-dimensional subsphere of $S_{U_L}$. That is, consider two formal links (in the original sense) $L = (U_L, T_L, \Sigma_L)$, $L' = (U_{L'}, T_{L'}, \Sigma_{L'})$ to be identical if $U_L = U_{L'}$, $\Sigma_L = \Sigma_{L'}$ (i.e. discount $T_L$ on the complement of $\Sigma_L$). Let a formal link (in the new sense) be an equivalence class of formal links (in the old sense). We then may construct $\mathcal{H}_{n,k}$ from these new formal links, with one $j$-cell for each $(h, k, j)$-dimensional link. There is a natural map $\mathcal{G}_{n,k} \to \mathcal{H}_{n,k}$ coming from the equivalence relation on old-style formal links. However, it is no longer natural to consider block bundles over $\mathcal{H}_{n,k}$. On the other hand we claim that there is a natural $n$-dimensional PL microbundle $\mathcal{O}_{n,k}$ over $\mathcal{H}_{n,k}$ and that the Gauss map $M^n \to \mathcal{H}_{n,n}$ is naturally covered by a map from the tangent microbundle of $M^n$ to $\mathcal{O}_{n,k}$. Moreover, a version of Theorem 3.1 may be proved for subcomplexes of $\mathcal{H}_{n,k}$.

Finally, we may want to consider what happens when we retopologize $\mathcal{H}_{n,k}$ in the following way: Consider two points $x, y$ as being $\varepsilon$-close if $x$ is the image of $\bar{x}$ in $c\Sigma_L$, $y$ is the image of $\bar{y}$ in $c\Sigma_J$ where $L$, $J$ are of the same dimension, and where $\bar{x}$ is $\varepsilon$-close to $\bar{y}$ in $\mathbf{R}^{n+k}$, and $\Sigma_L$ is $\varepsilon$-close to $\Sigma_J$, as closed subsets of $\mathbf{R}^{n+k}$, as are $S_{U_L}$ and $S_{U_J}$. Another way of putting this is as follows. Consider the first barycentric subdivision of $\mathcal{H}_{n,k}$ (which is a simplicial complex). Consider the set of $j$-simplices.

Each such $\sigma$ is the image of $\bar{\sigma}$, a simplex of the cone on the first derived subdivision of $\Sigma_L$ for some $L$. Put a metric on this set which makes $\sigma$ close to $\tau$ when $\bar{\sigma}$ is close to $\bar{\tau}$ in $\mathbf{R}^{n+k}$, and the corresponding $(\Sigma_L, \Sigma_J)$, $(S_{U_L}, S_{U_J})$ are also close. The metric converts the abstract simplicial complex to a simplicial space, and its geometric realization $\mathcal{I}_{n,k}$ is a new topology on $\mathcal{H}_{n,k}$.

$\mathcal{I}_{n,k}$ is a Grassmannian for certain kinds of piecewise-differentiable immersions. A theorem analogous to 3.1 holds for it as well. Moreover, there are equivariant versions of 3.1 for finite orthogonal actions on $\mathbf{R}^{n+k}$, (which induce actions on $\mathcal{G}_{n,k}$ (resp., $\mathcal{H}_{n,k}$, $\mathcal{I}_{n,k}$) in an obvious way). These results concern equivariant immersions of manifolds in $\mathbf{R}^{n+k}$. Finally, there are analogues to the notion of the $G_{n,k}$ bundle associated to the tangent bundle of a smooth manifold. In the case of a triangulated manifold $W^{n+k}$, it is possible to construct a complex $\mathcal{H}_{n,k}(W)$ which receives the Gauss map of a PL immersion $M^n \to W^{n+k}$ having certain properties. Furthermore, for a smooth Riemannian manifold $W^{n+k}$, one may speak of an associated $\mathcal{I}_{n,k}$-bundle which receives the Gauss map from a manifold $M^n$ piecewisedifferentiably immersed in $W^{n+k}$. These generalizations will be dealt with in future papers.

With these other "Grassmannians" in mind, I shall finally note that the $\mathcal{G}_{n,k}$ construction was emphasized in §1–3 since it is the simplest, most elegant, and most directly geometric.

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