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THE TRANSFER AND COMPACT LIE GROUPS

BY

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ABSTRACT. Let G be a compact Lie group with H and K arbitrary closed subgroups. Let BG, BH, BK be l-universal classifying spaces, with $\rho(H, G)$: $BH \to BG$ the natural projection. Then transfer homomorphisms $T(H, G)$: $h(BH) \to h(BG)$ are defined for h an arbitrary cohomology theory. One of the basic properties of the transfer for finite coverings is a double coset formula. This paper proves a double coset theorem in the above more general context, expressing $\rho^*(K, G) \circ T(H, G)$ as a sum of other compositions. The main theorems were announced in the Bulletin of the American Mathematical Society in May 1977.

I. Introduction. A map π: $X \to Y$ between two spaces induces a homomorphism π^*: $h(Y) \to h(X)$ between the cohomology groups of the spaces, where h is an arbitrary cohomology theory. In certain situations a transfer homomorphism τ^*: $h(X) \to h(Y)$ exists also. The properties of such transfers are of considerable importance. For example, in certain situations it is known that $\tau^* \circ \pi^*$: $h(Y) \to h(Y)$ is the identity, from which it follows that $h(X)$ contains $h(Y)$ as a direct summand. This paper develops a general theorem (Theorem II.11) which gives a formula for compositions similar to $\pi^* \circ \tau^*$: $h(X) \to h(X)$ which are much more complicated to calculate than the first type of composition mentioned. This double coset theorem generalizes the classical double coset theorem for finite groups (I.2). Theorem II.11 holds for all compact Lie groups.

The notion of an induced transfer has been around in various forms for many years. During the 1950's transfers were introduced into the cohomology of finite groups. In recent years Gottlieb and Becker have defined a transfer homomorphism τ^*: $h^*(E) \to h^*(B)$ for π: $E \to B$ a fibration with compact fibre. Dold has also defined a transfer homomorphism for arbitrary cohomology theories. It is closely related to that of Becker-Gottlieb.

These transfer maps have many nice properties, many of which are listed in §II. One of the most important deals with the composition $h(B) \xrightarrow{\pi^*} h(E) \xrightarrow{\tau^*} h(B)$. Let h be singular cohomology, H^*, and π: $E \to B$ a fibre bundle with B compact and with compact fibre F.

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Property I.1. In the above case, the composition $\tau^* \circ \pi^*: H^*(B) \to H^*(B)$ is multiplication by the Euler-Poincaré characteristic $\chi(F)$. If $\chi(F) = 1$ then $\tau^* \circ \pi^* = \text{id}$ and $H^*(B)$ is a direct summand of $H^*(E)$. See **[D0]** for a much more general version of this property.

The composition $\pi^* \circ \tau^*: h(E) \to h(E)$ $(h(E) \xrightarrow{\tau^*} h(B) \xrightarrow{\pi^*} h(E))$ is in general much more difficult to analyze than the other composition. In the case where G is a finite group with subgroup H and $\pi(H, G)$: $BH \to BG$ is the natural projection of classifying spaces, a result has been known since the fifties (see [CE]). In its more general form it is known as the *double coset theorem*. We state it here in terms of the cohomology of classifying spaces. It was originally stated and proven purely algebraically.

THEOREM I.2. *Let G be a finite group and let H, K be arbitrary subgroups of G. Let $\rho(H, G)$: $BH \to BG$ be the natural projection of classifying spaces. Associated to $\rho(H, G)$ is a transfer homomorphism $T(H, G)$: $H^*(BH) \to H^*(BG)$. (See* II.5.) *We have for $H^g = gHg^{-1}$, $g \in G$, and Cg: $H^*(BH) \to H^*(BH^g)$ the conjugation isomorphism* (II.3) *that*

$$\rho^*(K, G) \circ T(H, G) = \sum T(H^g \cap K, K) \circ \rho^*(H^g \cap K, H^g) \circ Cg$$

where the sum is over all points $\{KgH\}$ in the double coset space $K|G|H$.

These concepts are defined more clearly later. Since G is a finite group $K|G|H$ is finite also. Hence the sum is finite.

The main theorem of this paper generalizes the double coset theorem to the case where G is a compact Lie group with H and K arbitrary closed subgroups. The sum is once again finite and is indexed by the structure of the double coset space $K|G|H$.

II. The transfer and the double coset theorem. Let h be an arbitrary cohomology theory and let G be a compact Lie group with H and K arbitrary closed subgroups. Let E be a highly connected paracompact and locally compact space on which G acts freely on the right. Such a space may be easily constructed, e.g. by embedding G as a subgroup of $U(n)$ for suitable n and defining $E = U(n+m)/(I_n \times U(m))$ where I_n is the identity matrix in $U(n)$ and m is a very large integer.

DEFINITION II.1. Define a left action of G on $E \times (G/H)$ as follows. Let $e \in E$, $g, g' \in G$. Then $g(e, g'H) = (eg^{-1}, gg'H)$. Let $G(e, g'H)$ denote the G-orbit of $(e, g'H)$. Define $BH = E \times_G G/H$ to be the orbit space of this free G action.

DEFINITION II.2. Let $e \in E$, $g' \in G$. Define $\rho(H, G)$: $BH \to BG$ by $G(e, g'H) \mapsto G(e, G)$.

DEFINITION II.3. Let $e \in E$, $g, g' \in G$ and let $H^g = gHg^{-1}$ be the conjugate subgroup of H in G. Define $\tilde{C}g$: $BH^g \to BH$ by $G(e, g'gHg^{-1}) \mapsto G(e, g'gH)$ and let Cg: $h(BH) \to h(BH^g)$ denote the induced isomorphism in cohomology. This is called conjugation.

Associated to $\rho(H, G)$ is a transfer homomorphism $T(H, G)$: $h(BH) \to h(BG)$.

Note. At the present time it has not been constructed for maps of classifying spaces as a whole but only for the finite approximations of the classifying spaces constructed above. This technical difficulty should be overcome in the near future.

The double coset theorem (II.11) expresses the composition

$$\begin{array}{ccc} h(BK) & & h(BH) \\ {\scriptstyle \rho^*(K,G)} \nwarrow & & \swarrow {\scriptstyle T(H,G)} \\ & h(BG) & \end{array}$$

as a finite sum of compositions of the form

$$\begin{array}{ccccc} & h(BH^g \cap K) & & & \\ {\scriptstyle T(H^g \cap K,\, K)} \swarrow & & \nwarrow {\scriptstyle \rho^*(H^g \cap K,\, H^g)} & & \\ h(BK) & & h(BH^g) & \xleftarrow[Cg]{} & h(BH) \end{array}$$

which are often considerably easier to calculate.

The transfer $T(H, G)$ is defined via Dold's definition **[D1]**.

The definition of the transfer is subtle and somewhat cumbersome to use. However many useful properties of the transfer are easily derived from this definition. In practice it is the properties of the transfer that are used rather than the explicit definition given below.

Here is Dold's definition.

DEFINITION II.4. Let p: $D \to B$ be an ENR_B (a Euclidean Neighborhood Retract over B) where B is paracompact and locally compact. For example p could be a locally trivial fibration of finite type whose fibre is an ENR (Euclidean Neighborhood Retract). Let V be an arbitrary open set in D and let f: $V \to D$ be a *vertical* map (i.e., $pf = p|V$) and $\mathrm{Fix}(f) = \{u \in V | f(u) = u\}$. Assume that f is *compactly fixed*, i.e. $p|\mathrm{Fix}(f)$ is a proper map. Then for X an open set in V containing $\mathrm{Fix}(f)$, one has a transfer map t_f^X: $h(X) \to h(B)$.

t_f^X: $h(X) \to h(B)$ is defined as follows:

Case 1. $D = B \times \mathbf{R}^n$. t_f^X is the following composition

$$h^j(X) \overset{\sigma}{\simeq} h^{j+n}(X \times_B X, X \times_B X - \Delta) \overset{(i,f)^*}{\to} h^{j+n}(W, W - \mathrm{Fix}(f))$$
$$\overset{\mathrm{EXC}}{\simeq} h^{j+n}(D, D - \mathrm{Fix}(f)) \to h^{j+n}(D, D - D_q)$$
$$\simeq h^{j+n}(D, D - B \times \{0\}) \overset{\sigma}{\simeq} h^j(B)$$

where σ is the (iterated) suspension isomorphism, Δ is the diagonal of $X \times_B X = \{(x_1, x_2) | px_1 = px_2\}$, i is the inclusion of $W = X \cap f^{-1}(X)$ into X and q: $B \to (0, \infty)$ is a continuous function such that $\mathrm{Fix}(f) \subset D_q = \{(b, y) \in B \times \mathbf{R}^n | \, \|y\| < q(b)\}$. EXC is the excision isomorphism. t_f^X is independent of q.

Case 2. In general, by the definition of an ENR_B, any map f: $V \to D$ factors through some V' where V' is open in $D' = B \times \mathbf{R}^n$. Thus f is given by the composition $V \overset{\alpha}{\to} V' \overset{\beta}{\to} D$ $(f = \beta \circ \alpha)$. t_f^X is defined to equal the composition

$$h(X) \overset{\beta_X^*}{\to} h(\beta^{-1}(X)) \overset{t_{\alpha\beta}^{\beta^{-1}X}}{\to} h(B)$$

where β_X^* is defined via the restriction $\beta_X = \beta | \beta^{-1}(X)$ of β to $\beta^{-1}(X)$. $T_{\alpha\beta}^{\beta^{-1}(X)}$ is defined in Case 1.

DEFINITION II.5. Define $T(H, G)$ to be t_{id}^{BH} where id: $BH \to BH$ is the identity map over BG. id is compactly fixed since G/H is compact.

The double coset space $K|G|H$. A particular space, known as the double coset space $K|G|H$, plays an important role in this paper. This subsection is devoted to its definition and to a brief discussion of its general character. Much of the discussion is true in much more general form (cf. [**Br**]).

The group K acts on the left on the homogeneous space G/H by

$$k\colon G/H \to G/H \text{ given by } gH \mapsto kgH \text{ for } k \in K.$$

This is called the left action of K on G/H. It is, of course, a smooth action on a closed manifold since G is a compact Lie group and the action of K is the natural one.

The set $\{kgH | k \in K\} = KgH$ is called the orbit of gH under the left action of K. Hence we get the following

DEFINITION II.6. $K|G|H$ is the orbit space of G/H under the left action of K, with the natural topology induced from that of G/H via the projection π_K: $G/H \to K|G|H$. Let $K_{\bar{g}}$ be the isotropy group at $\bar{g} = gH$.

DEFINITION II.7. Two orbits are said to be of the same *orbit type* if the sets of isotropy groups of points in them are identical. This is equivalent to saying $K\bar{g}$ has the same type as $K\bar{g}_1$ iff $K_{\bar{g}}$ is conjugate to $K_{\bar{g}_1}$ in K.

There is a partial ordering on the set of orbit types. Type K/L_1 is less than type K/L_2 means L_2 is conjugate to a subgroup of L_1 (in K).

$K|G|H$ is not in general a manifold. However, it can be broken up into a disjoint union of manifolds of differing dimensions. Each of these manifolds consists of orbits of the same type.

(II.8). The closure of each orbit-type manifold consists of orbits which are of type less than or equal to it. The orbit-type manifolds may not be connected. However they consist of only finitely many components by the compactness of G/H. Also the number of orbit types is finite **[Br]**.

DEFINITION II.9. Let $\{M\}$ denote the set of orbit-type manifold components of $K|G|H$. Define

$$\chi^{\#}(M) = \chi(\overline{M}) - \chi(\overline{M} - M).$$

$\chi^{\#}(M)$ is called the internal euler characteristic of M. For example $\chi^{\#}(\text{point}) = 1$; $\chi^{\#}$ (interior of a line segment) $= -1$.

DEFINITION II.10. A map f: $K|G|H \to K|G|H$ is said to *preserve the orbit structure* if x and $f(x)$ are orbits of the same type for all $x \in K|G|H$. Similarly a homotopy h: $K|G|H \times I \to K|G|H$ *preserves the orbit structure* if x and $h(x, t)$ are orbits of the same type for all $x \in K|G|H$, and all $t \in I$.

THEOREM II.11 (DOUBLE COSET). *Let G be a compact Lie group and H and K arbitrary closed subgroups of G. Let $K|G|H$ be the double coset space considered as the orbit space of the left action of K on G/H* (II.6). *Let $\{M\}$ denote the set of orbit-type manifold components of $K|G|H$. Let $g \in G$ be such that $KgH \in M$. Let Cg: $h(BH) \to h(BH^g)$, H^g, $\rho(H, G)$, $T(H, G)$, $\chi^{\#}(M)$ be defined as in* II.3, II.3, II.2, II.5, *and* II.9, *respectively. Then*

$$\rho^*(K, G) \circ T(H, G) = \sum \chi^{\#}(M) T(K \cap H^g, K) \circ \rho^*(K \cap H^g, H^g) \circ Cg$$

where the sum is over all of the orbit-type manifold components $\{M\}$ of $K|G|H$.

Theorem II.11 appears very complicated. However in many instances it simplifies to only a few terms. A few examples of this are given in §VI. A larger number of applications will appear in a second paper.

Note that $H^g \cap K$ is the isotropy group of the left action of K on G/H at gH.

Here are several properties of the transfer. It should be noted that this is only a partial list consisting only of some of the more important properties. For a more complete list see **[D0]**. In addition to the fundamental property (I.1) (which is true in more generality than stated) we have:

Property II.12 (*Naturality in X*). Let f: $X \to D$ be a vertical map over B as above (II.4) and assume Y is an open neighborhood of $\mathrm{Fix}(f)$ in X. Let j: $Y \to X$ be the inclusion. Then

$$t_f^X = t_f^Y \circ j^*.$$

From this property we see that t_f^X depends only on the behavior of f near the fixed point set $\mathrm{Fix}(f)$. The transfer is thus essentially defined on the Čech-cohomology $\check{h}(\mathrm{Fix}(f))$ of $\mathrm{Fix}(f)$.

Property II.13 (*Naturality in B*). Let t_f^X: $h(X) \to h(B)$ be as above (II.4). Let B' be paracompact and locally compact. Let S: $B' \to B$ be given and let $S^!(D) = \{(d, b') | d \in D, b' \in B', p(d) = S(b')\}$, $S^!(X) = \{(x, b') | x \in X, b' \in B'$ and $p(x) = S(b')\}$ be the pullbacks of D and X respectively.

Then if F: $S^!(X) \to S^!(D)$ is the natural map defined by

$$(x, b') \mapsto (f(x), b')$$

we get that $t_F^{S^!(X)}$: $h(S^!(X)) \to h(B')$ is defined and the following diagram is commutative where $\tilde{S}$: $S^!(X) \to X$ is the projection $(x, b') \mapsto x$:

$$\begin{array}{ccc} h(S^!(X)) & \overset{\tilde{S}^*}{\leftarrow} & h(X) \\ t_F^{S^!(X)} \downarrow & & \downarrow t_f^X \\ h(B') & \underset{S^*}{\leftarrow} & h(B) \end{array}$$

Property II.14 (*Homotopy invariance*). Let F: $X \times I \to D \times I$ be a compactly fixed map over $B \times I$ where ρ: $D \times I \to B \times I$ is given by $(d, t) \mapsto (p(d), t)$ where p: $D \to B$, etc. is as above (II.4). Then if f_t is defined via $F(x, t) = (f_t(x), t)$, we have $t_{f_0}^X = t_{f_1}^X$.

It is important that the homotopy F be compactly fixed. Many examples exist where the conclusion is false without this hypothesis.

Property II.15 (*Additivity*). Suppose $Y = \amalg Y_i$ is a finite disjoint union of open sets $\{Y_i\}$ in X. Let j: $Y \to X$ and k_i: $Y_i \to Y$, j_i: $Y_i \to X$ be inclusions. As in the definition of t_f^X (II.4) let f: $X \to D$ be a map over B. Assume $\mathrm{Fix}(f) \subset Y$. Then $h(Y) = \bigoplus h(Y_i)$, and

$$t_f^Y = \sum t_f^{Y_i} \circ k_i^* \quad \text{and} \quad t_f^X = \sum t_f^{Y_i} \circ j_i^*$$

where the sum is over all Y_i.

Property II.16 (*Retraction*). Let D and Y be Euclidean neighborhood retracts over B (ENR_B) [D1]. Let X be an open set in D. Assume Y is included in X and hence in D, with inclusion maps i: $Y \to X$ and j: $Y \to D$. Assume also that there exists a retraction over B, r: $X \to Y$, and that f: $Y \to Y$ and $\tilde{f} = j \circ f \circ r$: $X \to D$ are compactly fixed maps over B (II.4). Hence $\tilde{f}$ is the composition $X \xrightarrow{r} Y \xrightarrow{f} Y \xrightarrow{j} D$ and $\mathrm{Fix}(\tilde{f}) = \mathrm{Fix}(f)$ (II.4). Then $t_{\tilde{f}}^X = t_f^Y \circ i^*$.

Note. The Retraction Property does not follow from Naturality in X (II.12). In II.12 t_f^Y and t_f^X are both defined with respect to the same ENR_B but in II.16 t_f^Y is defined with respect to the ENR_B, Y, whereas $t_{\tilde{f}}^X$ is defined with respect to D. Thus Y need not be open in D in II.16.

The following thoerem is often useful in simplifying specific double coset formulas.

THEOREM II.17. *Let $N_G(H) = \{g \in G \mid ghg^{-1} \in H \ \forall h \in H\}$ be the normalizer of H in G. Suppose $N_G(H)/H$ is not discrete. Then $T(H, G) = 0$.*

PROOF. The bundle $\rho(H, G)$: $BH \to BG$ has fibre G/H and structure group G.

Let $\mathring{N}$ be the identity component of $N_G(H)/H$. Then if $\{p_t H\}_{t \in I}$ is any path P: $I \to \mathring{N}$ beginning at H, $\bar{P}$: $G/H \times I \to G/H$, given by $\bar{P}(gH, t) = gp_t H$, is a G-equivariant homotopy which starts at the identity and ends at $\hat{P}$: $G/H \to G/H$ given by $\hat{P}(gH) = gp_1 H$. Hence if $P(1) \neq H$, $\bar{P}$ induces a compactly fixed homotopy of the identity on BH over $BG \times I$ which ends with no fixed points. Since there exists such a path when $\mathring{N}$ is not H, $T(H, G) = 0$ by homotopy invariance (II.14) and the fact that the transfer is dependent only on a neighborhood of the fixed point set (II.12). Since there is no fixed point set the transfer is 0.

Here is a general outline of the proof of the double coset theorem (II.11). Consider the following commutative diagram where Γ is the pullback of BH. $E \times_K (G/H)$ is defined analogously to $E \times_G G/H$ (II.1).

$$\begin{array}{ccccc} \Gamma = E \times_K (G/H) = \rho^!(K, G)(BH) & \xrightarrow{l} & BH & \\ \gamma\downarrow & & \downarrow\rho(H, G) & \\ BK & \xrightarrow{\rho(K, G)} & BG & \qquad \text{(II.18)} \end{array}$$

By naturality of the transfer (II.13) $\rho^*(K, G) \circ T(H, G) = T \circ l^*$ where T is the transfer associated to the identity map of Γ over BK. It is at this point that order of the transfer and the induced homomorphism of a map are switched. The detailed analysis of the proof concerns the transfer $T = t^{\Gamma}_{\mathrm{id}}$. This separate analysis is the subject of a more general theorem (V.14). Only at the end of the proof do we return to the original question. In subsequent analysis of this particular situation the conjugations (II.3) come into the formula.

In analyzing T we first note that γ: $\Gamma \to BK$ has fibre G/H and structure group K where K acts on the left. Thus a K-equivariant deformation of the identity map of G/H induces a compactly fixed homotopy of the identity map of Γ over BK. We construct such a homotopy on G/H whose induced resultant map $\hat{\beta}$ on Γ has considerably smaller fixed point set $\mathrm{Fix}(\hat{\beta})$. Furthermore $\mathrm{Fix}(\hat{\beta})$ will break up into a finite number of disjoint sets. By homotopy invariance (II.14) and additivity (II.15) T is equivalent to a sum of other transfers.

The construction of the K-equivariant homotopy of the identity map of G/H is done by carefully constructing a homotopy of the identity map of the double coset space $K|G|H$ which preserves the orbit structure (II.10). Then a variant of the covering homotopy theorem of Palais (IV.1) is used to lift this homotopy to G/H.

Various technical theorems are proven at this point so that one may identify the resulting sum with that in the double coset formula.

III. Hat constructions and the homotopy on $K|G|H$. A homotopy of the identity map of $K|G|H$, H': $K|G|H \times I \to K|G|H$, is constructed at the end of this section. The homotopy H' preserves orbit types (II.10). This allows the use of a variant of the covering homotopy theorem of Palais in §IV to construct a K-equivariant homotopy H: $G/H \times I \to G/H$.

(III.1). The homotopy H' is constructed by appealing to a theorem of C. T. Yang [Y] which states that the orbit space $K|G|H$ is triangulable. Furthermore, this triangulation agrees with the orbit structure of $K|G|H$, i.e. the interior of each simplex consists of orbits of the same type [I].

DEFINITION III.2. A map or a homotopy which sends the interior of each simplex into itself is said to *preserve the triangulation*.

A homotopy or map which preserves the triangulation of $K|G|H$ also preserves the orbit structure (II.10). The procedure therefore is to construct a homotopy of the identity of $K|G|H$, H', which preserves the triangulation. The construction of H' is inductive over the skeleta and is defined simplex by simplex.

We now define several basic constructions, called hat constructions. The first such construction is needed for the construction of H'. Others are closely related and will be needed for technical reasons later.

Hat constructions are general extensions of homotopies defined on certain parts of simplices.

DEFINITION III.3. Let b be the barycenter of the n-simplex Δ^n. Consider a homeomorphism of Δ^n with the standard n-dimensional unit disk D^n in $\mathbf{R}^n$ which sends b to the center of D^n. Let d: $\Delta^n \times \Delta^n \to [0, \infty)$ be the metric induced on Δ^n which corresponds to the standard metric on D^n.

DEFINITION III.4. For every point in $\Delta^n - \{b\}$ there is a unique closest point in $\partial\Delta^n$, the boundary of Δ^n. This defines a *radial projection* p: $\Delta^n - \{b\} \to \partial\Delta^n$.

We shall use the projection map in extending homotopies defined on parts of $\partial\Delta^n$ to other points in Δ^n. A point $x \in \Delta^n$ which projects to $y \in \partial\Delta^n$ will be sent to a point which projects to the given image of y in $\partial\Delta^n$. Exactly which such point x will be sent to will be determined by the distance of x to y, $d(x, y)$, and the particular construction.

DEFINITION III.5. (1) Let H: $\partial\Delta^n \times I \to \partial\Delta^n$ be a homotopy of the identity map which preserves the triangulation. We shall extend H to a homotopy of the identity map of Δ^n, $\widehat{H}_1$, which preserves the triangulation and which ends in a map which has only one additional fixed point, namely the barycenter b.

b will remain fixed throughout the homotopy. We will need to arrange the homotopy so that points near b move very little. This factor is k: $\Delta^n \to [0, 1]$.

For $\eta \in \Delta^n$ let

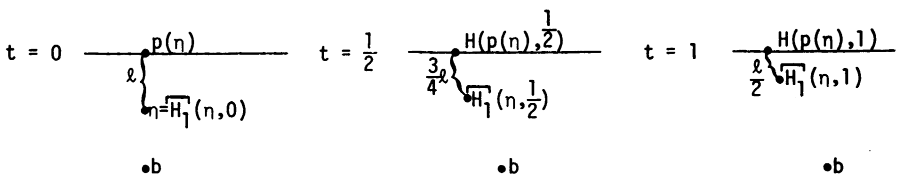
$$k(\eta) = \begin{cases} 2d(b, \eta) & \text{if } d(b, \eta) < \frac{1}{2}, \\ 1 & \text{if } d(b, \eta) \geq \frac{1}{2}. \end{cases}$$

We now define $\overline{H}_1$: $\Delta^n \times I \to \Delta^n$.

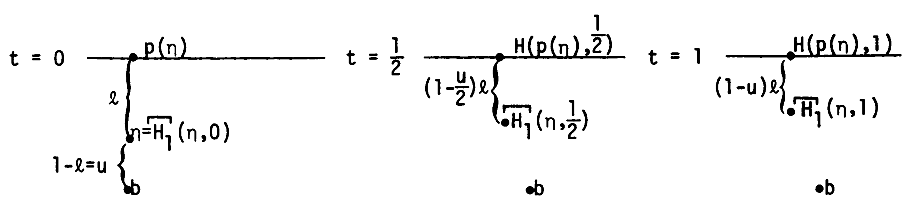
For $\eta \in \Delta^n - \{b\}$, $t \in I$, $\overline{H}_1(\eta, t)$ is that point which is $d(\eta, p(\eta))[1 - k(\eta)t/2]$ along the line from $H(p(\eta), t)$ to b.

$$\overline{H}_1(b, t) = b \quad \text{for all } t.$$

EXAMPLE. *Case* (1). $d(b, \eta) \geq \frac{1}{2}$. Hence $k(\eta) = 1$



Case (2). $d(b, \eta) < \frac{1}{2}$. Hence $k(\eta) = 2d(b, \eta)$

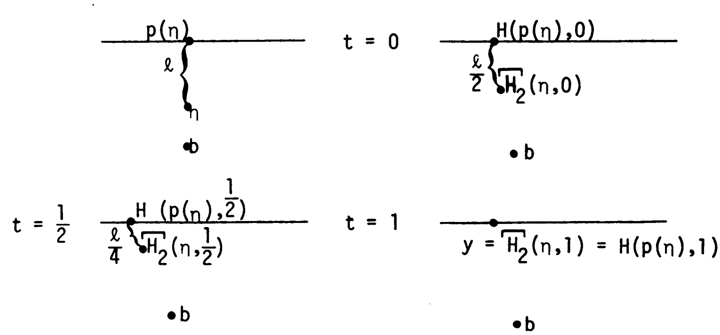


Note that for η near $\partial\Delta^n$, $\overline{H}_1(\eta, 1)$ equals that point which is $\frac{1}{2}d(\eta, p(\eta))$ along the line from $H(p(\eta), 1)$ to b. Points in general are pushed out from b.

DEFINITION III.5. (2) Let $A \subset \partial\Delta^n$ and let $N \subset \Delta^n - \{b\}$ project onto A. Let H: $A \times I \to \partial\Delta^n$ be a homotopy which preserves the triangulation for $t \neq 1$ and ends in the constant map which sends all points in A to some $y \in A$. Assume also that y is the only point which is fixed at any stage of H. Then we define $\overline{H}_2$: $N \times I \to \Delta^n$.

Let $\eta \in N$, $t \in I$. $\overline{H}_2(\eta, t)$ is that point in Δ^n which is $\frac{1}{2}(1 - t)d(\eta, p(\eta))$ along the line from $H(p(\eta), t)$ to b.

EXAMPLE.

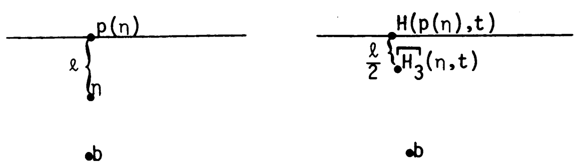


Hence one pushes η closer to $\partial\Delta^n$, finally ending at y.

$\overline{H_2}$ agrees with H on common points, preserves the triangulation for $t \neq 1$ and has no fixed points at any stage other than y which is the fixed point of the given homotopy. Hence $\overline{H_2}$ satisfies the assumption made for H with A replaced by N, $\partial\Delta^n$ replaced by Δ^n (which may be contained in some $\partial\Delta^{n+1}$).

DEFINITION III.5. (3) Let $A \subset \partial\Delta^n$ and let $N \subset \Delta^n - \{b\}$ project onto A. Let $H: A \times I \to \partial\Delta^n$ be a homotopy which preserves the triangulation. We define a homotopy $\overline{H_3}$: $N \times I \to \Delta^n$. For $\eta \in N$, $t \in I$, $\overline{H_3}(\eta, t)$ is that point which is $\frac{1}{2}d(\eta, p(\eta))$ along the line from $H(p(\eta), t)$ to b.

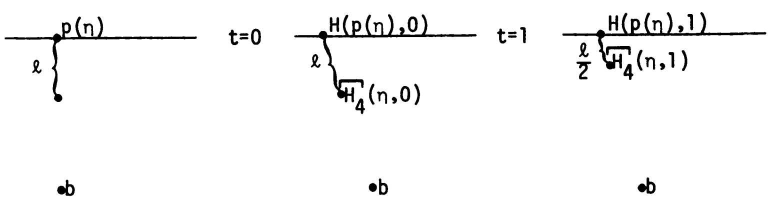
EXAMPLE.



$\overline{H_3}$ agrees with H on common points and does not introduce any new fixed points.

DEFINITION III.5. (4) Let $A \subset \partial\Delta^n$ and let $N \subset \Delta^n - \{b\}$ project onto A. Let H: $A \times I \to \partial\Delta^n$ be a homotopy which preserves the triangulation. We define a homotopy $\overline{H_4}$: $N \times I \to \Delta^n$. For $\eta \in N$, $t \in I$, $\overline{H_4}(\eta, r)$ is that point which is $(1 - t/2)d(\eta, p(\eta))$ along the line from $H(p(\eta), t)$ to b.

EXAMPLE.

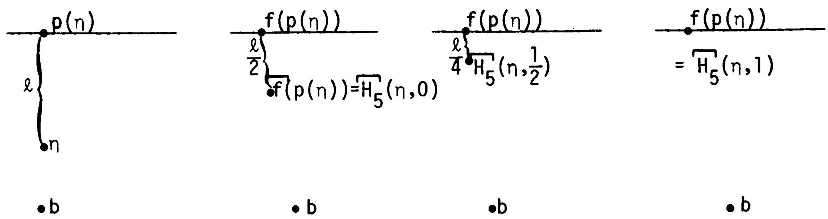


$\overline{H}_4$ agrees with H on common points and preserves the triangulation. Note that $\overline{H}_4(\eta, 1)$ equals that point which is $\frac{1}{2}d(\eta, p(\eta))$ along the line from $H(p(\eta), 1)$ to b.

DEFINITION III.5. (5) Let $A \subset \partial\Delta^n$ and let $N \subset \Delta^n - \{b\}$ project onto A. Let $\eta \in N$. Let $f\colon A \to \partial\Delta^n$ be given and let $\overline{f}\colon N \to \Delta^n$ be defined by letting $\overline{f}(\eta)$ = that point which is $\frac{1}{2}d(\eta, p(\eta))$ along the line from $f(p(\eta))$ to b. $\overline{H}_5$ is a homotopy of $\overline{f}$.

We define $\overline{H}_5\colon N \times I \to \Delta^n$. For $\eta \in N$, $t \in I$, $\overline{H}_5(\eta, t)$ is that point which is $(1 - t)d(\eta, p(\eta))/2$ along the line from $f(p(\eta))$ to b.

EXAMPLE.



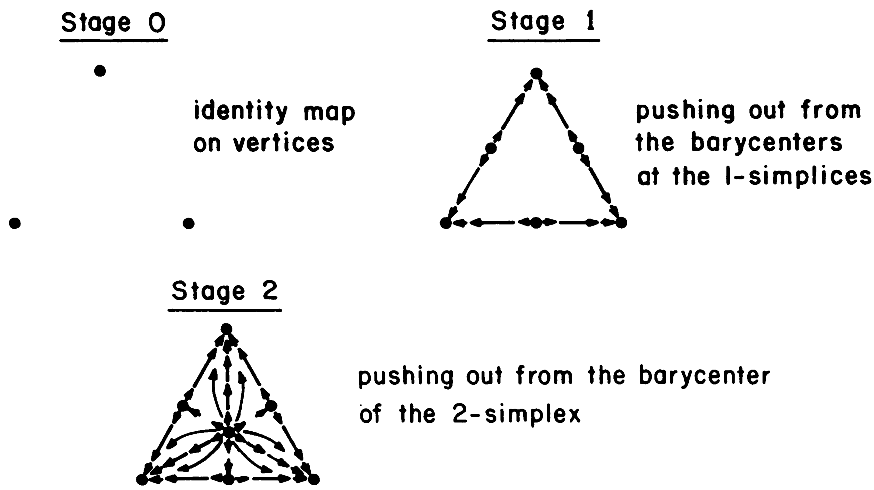
$\overline{f}$ and $\overline{H}_5$ are continuous and $\overline{f}$ agrees with f on A. If f agrees with the resultant map of a triangulation preserving homotopy $H\colon \partial\Delta^n \times I \to \partial\Delta^n$ on A and $d(\eta, b) > \frac{1}{2}$ then $\overline{f}$ agrees with $\overline{H}_1(\eta, 1)$ (see III.5(1)). The resultant map of $\overline{H}_5$ agrees with $f \circ p$ where p is the projection (III.4) and the total fixed point set of $\overline{H}_5$ equals that of f.

DEFINITION III.6. The definitions of the hat constructions allow one to define extensions of homotopies inductively over parts of skeleta. For example, consider the hat construction $\overline{H}_1$. Suppose we have a homotopy of the identity map of the $(n - 1)$st skeleton of $K|G|H$ which preserves the triangulation. Then using hat homotopy III.5(1) we can extend this homotopy n-simplex by n-simplex to a homotopy of the identity on the n-skeleton which preserves the triangulation and has additional fixed points at the barycenters of all the n-simplices. We can continue this process until we have a homotopy of the identity map of $K|G|H$ which preserves the triangulation. All such extensions are called *extensions by inductive use of the hat construction.*

We now define the homotopy of the identity map on $K|G|H$.

DEFINITION III.7. Start with the identity homotopy on the 0-skeleton of $K|G|H$. Then extend this by inductive use of hat homotopy Definition III.5(1) to $H'\colon K|G|H \times I \to K|G|H$. Let $\beta'\colon K|G|H \to K|G|H$ be the resultant map.

EXAMPLE. We show how this process works for a two-simplex.



Since H' preserves the triangulation of $K|G|H$ (III.2) it necessarily preserves the orbit structure (II.10). This will allow us to use a variant of the covering homotopy theorem to lift H' to a K-equivariant homotopy of the identity on G/H, whose resultant map lies above β'.

Note that the only fixed points of β' are the barycenters of the simplices.

IV. The covering homotopy thoerem and the homotopy on G/H. We construct the K-equivariant homotopy of the identity on G/H in this section.

First we need to state the covering homotopy theorem.

THEOREM IV.1 (COVERING HOMOTOPY **[Br]**, **[P]**). *Let K be a compact Lie group and let X and Y be left K-spaces* **[Br]**. *Assume that every open subspace of $K \setminus X$ is paracompact. Let $f: X \to Y$ be equivariant and let $f': K \setminus X \to K \setminus Y$ be the induced map. Let $F': K \setminus X \times I \to K \setminus Y$ be a homotopy that preserves orbit structure and starts at f'. Then there exists a K-equivariant homotopy F: $X \times I \to Y$ covering F' and starting at f.*

Moreover, any two such liftings of F' differ by composition with a self-equivalence of $X \times I$ covering the identity on $K \setminus X \times I$ and equal to the identity on $X \times \{0\}$. (A self-equivalence of $X \times I$ is a K-equivariant homeomorphism φ: $X \times I \to X \times I$ where K acts trivially on I.)

We need more precise control of the homotopy near the fixed points of β' (III.7) than that provided by this version of the covering homotopy theorem. In particular we will need the resultant map on G/H to agree with the following construction near the fixed points.

Slice construction. Let $S \subset G/H$ be a linear slice at $y \in G/H$ for the left action of K. Suppose $L = K_y$ is the isotropy group at y, with π_L: $S \to L \setminus S$

the canonical projection. Let $s \in S$. Then the isotropy group of L at s equals the isotropy group of K at s, i.e. $L_s = K_s$ [**Br**, II.3.4, 4.7]. If $K(S)$ is the linear tube in G/H associated to S (i.e. points in $K(S)$ are just points of the K-orbits of points in S), then there is a canonical homeomorphism of $L \setminus S$ with $K \setminus K(S)$ given by sending Ls to Ks. Consider the orbit structure of $L \setminus S$ and $K \setminus K(S)$ respectively. Since $L_s = K_s$ for $s \in S$, two points which are of the same orbit-type in $L \setminus S$ have images in $K \setminus K(S)$ which are of the same type. It is also true that if two points in $K \setminus K(S)$ are in the same orbit-type manifold component then the corresponding points in $L \setminus S$ are also. This follows since $L_s = K_s$ and also that points in the closure of an orbit-type manifold component consist of points of type less than or equal to it (II.8). Hence

(IV.2) The natural homeomorphism of $L \setminus S$ with $K \setminus K(S)$ also sends orbit-type manifold components to orbit-type manifold components.

DEFINITION IV.3. Suppose we have a K-orbit-type preserving homotopy of the identity map of some open subset, U, of $K \setminus K(S) \subset K|G|H$ into $K \setminus K(S)$ whose final map has one fixed point at the orbit Ky. By the above remark we can view this homotopy as a L-orbit-type preserving homotopy. Hence we can lift this homotopy by the covering homotopy theorem to an L-equivariant homotopy of the identity map of $\pi_L^{-1}(U)$ into S. Extend this homotopy to all points on K-orbits of points in $\pi_L^{-1}(U)$ in the unique K-equivariant manner. This procedure is called a *slice construction* at y. The resultant K-equivariant map is called a *slice construction map*.

An example of some slice constructions is given at the end of this section.

The following theorem and its corollaries essentially state that one may construct an equivariant homotopy covering so that the resulting map agrees with slice construction maps in neighborhoods of the fixed orbits.

Let $\overleftrightarrow{W}$ be a differentiable manifold on which K acts differentiably on the left. Let W be the orbit space of $\overleftrightarrow{W}$ and let $\pi_K\colon \overleftrightarrow{W} \to W$ be the projection. Let U be an open set in W, and let $U_1 \subset U$ be open also. Let $\overleftrightarrow{U_1} = \pi_K^{-1}(U_1)$, $\overleftrightarrow{U} = \pi_K^{-1}(U)$. Let $\rho\colon \overleftrightarrow{U} \to \overleftrightarrow{W}$ be a K-equivariant map which covers $\rho'\colon U \to W$. Assume there is a homotopy of ρ', $H'\colon U \times I \to W$, which preserves the orbit structure (II.10). Let $\beta'\colon U \to W$ be the resultant map defined by $\beta'(x) = H'(x, 1)$ for $x \in U$. Assume that there is only one fixed point, y', of β' in U_1 and that y' remains fixed throughout the homotopy. Also assume $U_1 \subset \pi_K(S)$ for some slice S at $y \in \pi_K^{-1}(y')$. Let H_1 be any lifting of $H'|(U_1 \times I)$: $U_1 \times I \to W$ to a K-equivariant homotopy of $\rho|\overleftrightarrow{U_1}$, given by a slice construction at $y \in S$. Let β_1 be the resultant map of H_1.

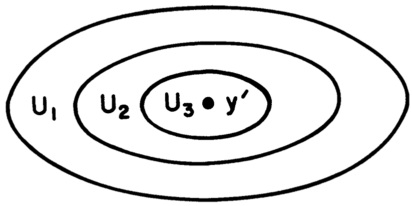
Note. We are assuming ρ is the identity map in $\pi_K(S)$.

THEOREM IV.4 (EXTENSION PROPERTY OF COVERING HOMOTOPY THEOREM). *Given the above notation and assumptions, there exist an open neighborhood U'*

of y' in U_1 (where $\overline{U'} = \pi_K^{-1}(U')$) and a K-equivariant homotopy of ρ, H: $\overline{U} \times I \to \overline{W}$, such that the resultant map β: $\overline{U} \to \overline{W}$ agrees with β_1 on $\overline{U'}$, that is, $\beta | \overline{U'} = \beta_1 | \overline{U'}$.

PROOF. Let S be the slice at $y \in \pi_K^{-1}(y')$ which is used to define the given slice construction map β_1. We shall define the homotopy in three successive stages. In the following $y' \in U_3 \subset U_2 \subset U_1$. Let U_2 be the image of the open disk of radius $\frac{1}{2}$ in $S \approx \mathbf{R}^n$ and U_3 the image of the open disk of radius $\frac{1}{3}$. Let $\overline{U_2} = \pi_K^{-1}(U_2)$, etc.

We have the following picture



(1) Let L be the isotropy group of K at y. L acts orthogonally on $\mathbf{R}^n$. Hence points in an L-orbit in $\mathbf{R}^n$ are the same distance from the origin. The homotopy of the identity map of the tube $K(S)$ used in the slice construction is induced in the unique K-equivariant manner from an L-equivariant homotopy of the identity map of S. Suppose h: $S \times I \to S$ is this L-equivariant homotopy. Let $\| \ \|$ be the standard norm in $\mathbf{R}^n$. Then define h': $S \times I \to S$ in the following manner: Let $s \in S$, $t \in I$. Then

$$h'(s, t) = \begin{cases} h(s, t) & \text{for } \|s\| \leq \frac{1}{3}, \\ h(s, t(3 - 6\|s\|)), & \frac{1}{3} < \|s\| \leq \frac{1}{2}, \\ h(s, 0), & \frac{1}{2} < \|s\|. \end{cases}$$

Since h' slows down the homotopy h the same amount on concentric spheres around the origin, h' is L-equivariant. h' induces a K-equivariant homotopy of the identity map of $K(S)$. This can be further extended to a K-equivariant homotopy of ρ which is constant outside of $K(S)$. This K-equivariant homotopy of ρ induces a homotopy of ρ' which is constant outside of U_2.

(2) Follow the homotopy of ρ' by one which is constant on U_3 and whose final map agrees with β'. The homotopy H' is used outside of U_2 and the linearity of the slice at y is used in a similar manner to its use in (1) to continue the homotopy inside U_2. This homotopy is constant on U_3. Lift the homotopy to a K-equivariant homotopy of the resultant equivariant map of (1) by the covering homotopy theorem.

(3) Restricting this second homotopy to $\overline{U_3}$ we see that the induced homotopy on U_3 is constant. Hence by the covering homotopy theorem

uniqueness condition it differs from the identity homotopy on $\overline{U_3}$ by a self-equivalence over the identity of $\overline{U_3} \times I$. This is used to correct (2) by continuing with a third K-equivariant homotopy so that the final map agrees with β_1 on $\overline{U'}$ where U' is some open neighborhood of y' contained in U_3. This final homotopy is scaled so as to be constant outside of $\overline{U_3}$.

COROLLARY IV.5. *In adition to the notation and assumptions of* IV.4 *let* ω': $U \to W$ *be the resultant map of the first half of the homotopy,* $H'|(U \times I)$, *i.e. let* $\omega'(x) = H'(x, \frac{1}{2})$ *for* $x \in U$. *Suppose* y' *is the only fixed point of* $\omega'|U_1$. *Let* ω_1: $\overline{U_1} \to \overline{W}$ *be the slice construction at* $y \in \overline{U_1}$ *defined by* $\omega_1(x) = H_1(x, \frac{1}{2})$ *for* $x \in \overline{U_1}$. *Suppose further that the second half of the homotopy,* $H'|(U \times [\frac{1}{2}, 1])$, *has compact total fixed point set. Then there exist an open neighborhood* U' *of* y' *in* U_1 (*where* $\overline{U'} = \pi_K^{-1}(U')$) *and a* K-*equivariant homotopy of* ρ, H: $\overline{U} \times I \to \overline{W}$, *with resultant map* β: $\overline{U} \to \overline{W}$ *which agrees with* β_1 *on* $\overline{U'}$. *Furthermore the map* ω: $\overline{U} \to \overline{W}$ *at some intermediate point of the homotopy agrees with* ω_1 *on* $\overline{U'}$ *and the homotopy from* ω *to* β *has total fixed point set compact.*

PROOF. We simply apply the proof of IV.4 twice. First we apply it to the homotopy $H'|(U \times [0, \frac{1}{2}])$. The resultant homotopy ends with some ω: $\overline{U} \to \overline{W}$ which agrees with ω_1 on some $\overline{U'}$. ω covers ω'. Apply the proof of IV.4 to the second half of the homotopy, $H'|(U \times [\frac{1}{2}, 1])$, where ω plays the role of ρ, and the second half of the homotopy H_1 plays the role of the slice construction homotopy in (1). The fixed point set of this second homotopy is compact since $H'|U \times [\frac{1}{2}, 1]$ has compact total fixed point set. The resultant map is β: $\overline{U} \to \overline{W}$, which agrees with β_1 on $\overline{U'}$.

Since the procedures of these proofs are essentially local in nature one can generalize them to where there is more than one fixed point of β'.

COROLLARY IV.6. *Assume the hypotheses preceding Theorem* IV.4 *except for replacing the assumption that there is only one fixed point of* β' *by the assumption that there are a finite number of fixed points* $\{y'\}$ *contained in disjoint open sets* $\{V\}$, *where each* V *is contained in* U. *We assume* $V \subset \pi_K(S)$ *for* S *a slice at* $y \in \pi_K^{-1}(y')$. *Then given slice constructions of* $H'|(V \times I)$ *at* $y \in S$ *for each* V *which end in* ν: $\overline{V} \to \overline{W}$ (*where* $\overline{V} = \pi_K^{-1}(V)$), *there are open sets* $V' \subset V$, *for each* V, *containing the fixed point* y' *and a* K-*equivariant homotopy of* ρ, H: $\overline{U} \times I \to \overline{W}$, *which ends in* β: $\overline{U} \to \overline{W}$ *where* $\beta(x) = \nu(x)$ *for each* $x \in \overline{V'}$ ($\overline{V'} = \pi_K^{-1}(V')$).

Thus, if ρ is the identity on G/H and H' is an orbit-type preserving homotopy of the identity one can assume that there is a K-equivariant homotopy of the identity on G/H so that the resulting map β agrees with slice constructions in neighborhoods of the isolated fixed point orbits.

We also have the analogue of Corollary IV.5.

COROLLARY IV.7. *Assume the notation and assumptions of* IV.6. *Let* ω': $U \to W$ *be the resultant map of the first half of the homotopy* H', *i.e. let* $\omega'(x) = H'(x, \frac{1}{2})$ *for* $x \in U$. *Suppose* y' *is the only fixed point of* $\omega' | \overline{V}$. *For each* V *let* ω_1: $\overline{V} \to \overline{W}$ *be the slice construction map at* $y \in \overline{V}$ *defined by* $\omega_1(x) = H_1(x, \frac{1}{2})$ *for each* $x \in \overline{V}$. *Suppose that the second half of the homotopy* H' *has compact total fixed point set. Then there exists a K-equivariant homotopy of* ρ, H, *with resultant map* β *and there exists an open neighborhood* V' *of* y' *in* V *for each* V (*where* $\overline{V'} = \pi_K^{-1}(V')$) *such that* $\beta | \overline{V'} = \nu | \overline{V'}$. *Furthermore at some intermediate stage of the homotopy* H *the resultant map* ω *agrees with each* ω_1 *on* $\overline{V'}$ *and the K-equivariant homotopy from* ω *to* β *has total fixed point set compact.*

We shall now define the homotopy on G/H.

The homotopy of the identity on $K|G|H$, H' (III.7), preserves the triangulation of $K|G|H$ (III.2) and hence preserves the orbit structure of $K|G|H$ (II.10). Recall that the resultant map β' has fixed points at the barycenters $\{b\}$ of all the simplices of $K|G|H$. Pick slices $\{S\}$ with isotropy groups $\{L\}$ at representatives in G/H of each of these orbits. For a small enough neighborhood of each barycenter the homotopy H' will stay within the projection of these slices. Since the homotopy preserves the K-orbit structure near b it also preserves the L-orbit structure since they agree locally (IV.2). Apply the slice construction (IV.3) near each barycenter. By Corollary IV.6 there exists a K-equivariant homotopy of the identity on G/H whose resultant map lies above β' (III.7) and agrees with each slice construction map ν in a tubular neighborhood $\overline{V}$ of each orbit $b \subset G/H$.

DEFINITION IV.8. Define H: $G/H \times I \to G/H$ to be this final homotopy and let β: $G/H \to G/H$ be the resultant K-equivariant map.

We end this section with a low dimensional example of the procedure up to this point.

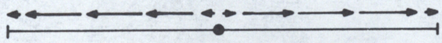
Let $U(n)$ be the complex nth unitary group viewed as a subgroup of $\mathrm{GL}(n, \mathbf{C})$. $U(1)$ is identified with the group of units of norm one in the complex numbers.

EXAMPLE IV.9. Let $G = U(2)$, $H = K = U(1) \times U(1) = T$, a maximal torus of $U(2)$. Then $G/H = U(2)/(U(1) \times U(1)) \approx S^2$. K acts on S^2 by rotation around the central axis. $(e^{i\varphi}, e^{i\psi})$ rotates S^2 by $(\varphi - \psi)$ radians. Hence $K|G|H$ is a line segment •——• with endpoints the north and south poles which are fixed points and correspond to the Weyl group. All other points are circle orbits corresponding to latitude circles.

There are three orbit-type manifold components, the endpoints and the interior of the line segment. There is an obvious triangulation of $K|G|H$ which preserves the orbit structure.

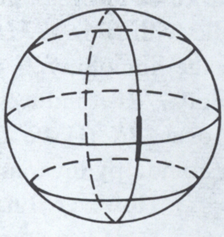
The homotopy on $K|G|H$ is given by pushing out from the barycenter of the 1-simplex as in the following diagram

a)



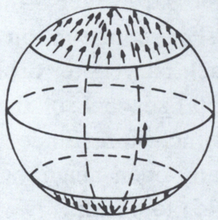
The slices consist of disks at the north and south poles and a portion of a longitude circle through the equator.

b)



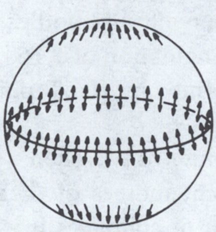
The slice procedure lifts the homotopy near the fixed points, first doing the homotopy in the slice (which is particularly simple to do in this case).

c)



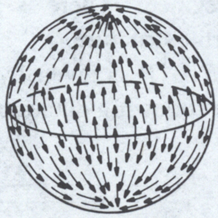
This is then extended K-equivariantly to the tubes around the fixed orbits.

d)



By the extension property of covering homotopy, this (essentially) can be extended to a K-equivariant homotopy of G/H.

e)



The fixed point set of the resulting map consists of three orbits, the north and south poles and the equator.

V. Proof of the double coset theorem and a generalization. Recall II.18. γ: $\Gamma \to BK$ is a fibre bundle with fibre G/H and structure group K acting on the left on G/H. Since the homotopy H (IV.8) is a K-equivariant deformation of the identity map of G/H it induces a compactly fixed deformation of the identity map of Γ over BK (II.4). This homotopy ends in $\hat{\beta}$: $\Gamma \to \Gamma$, induced by β (IV.8). By homotopy invariance (II.14), the transfers defined by $\hat{\beta}$ and id agree, i.e. $t_{\mathrm{id}}^{\Gamma} = t_{\hat{\beta}}^{\Gamma}$. However $\hat{\beta}$ has considerably smaller fixed point set than the identity map. Fix($\hat{\beta}$) has one component for each barycenter in the triangulation of $K|G|H$. Each of these barycenters b is a K-orbit and hence defines a subbundle of γ: $\Gamma \to BK$ with total space $\hat{b}$. We shall see later (V.13) that these subbundles are of the form $BL \to BK$ where L is an isotropy group for a point on the barycenter orbit b.

β agrees with a slice construction map ν in a K-equivariant tubular neighborhood $\overline{V}$ of each barycenter orbit b. Hence $\hat{\beta}$ agrees with the induced map of ν, $\hat{\nu}$, on the total space $\hat{V}$ of the subbundle of γ corresponding to V. Let i: $\hat{V} \to \Gamma$ be the inclusion. Since the transfer $t_{\hat{\beta}}^{\Gamma}$ is determined by the behavior of $\hat{\beta}$ inside any open neighborhood of the fixed point set (II.12) we can use additivity (II.15) to obtain:

$$t_{\mathrm{id}}^{\Gamma} = t_{\hat{\beta}}^{\Gamma} = \sum t_{\hat{\nu}}^{\hat{V}} \circ i^* \tag{V.1}$$

where the sum is over all subbundles $\{\hat{V}\}$ corresponding to all the barycenters $\{b\}$ in the triangulation of $K|G|H$.

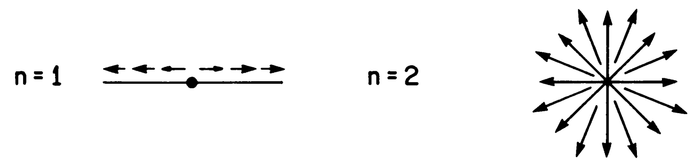
Dold **[D2]** has defined a fixed point index for ENR's which include triangulated spaces.

Let b be the barycenter of Δ^n. Then

(V.2) The fixed point indices of β' and $\beta'|\Delta^n$ at b equal $(-1)^n$. Furthermore, if W is a union of simplices in $K|G|H$ which contains Δ^n, then $\beta'|W$ has fixed point index $(-1)^n$ at b.

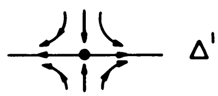
PROOF. $\beta'|\Delta^n$ pushes out in all directions from b and hence has fixed point index $(-1)^n$.

EXAMPLE. For $n = 1, 2$, $\beta'|\Delta^n$ near b looks like



Let W be a union of simplices in $K|G|H$ which contains Δ^n. By inductive use of hat homotopy Definition III.5(5) for a suitable open neighborhood U of b in W, $\beta'|U$ can be deformed without additional fixed points to the composition of a retraction of U into Δ^n followed by $\beta'|\Delta^n$. Hence it follows that the fixed point index of $\beta'|W$ at b is the same as that of $\beta'|\Delta^n$ at b and hence is $(-1)^n$.

EXAMPLE. When $n = 1$ the map near b may look like



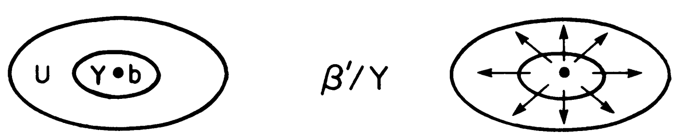
The only directions where β' is pointed away from b are in Δ^n. In the directions perpendicular to Δ^n the map points towards b. This assures the stability of the fixed point index in the inductive construction of β'.

Let $t_{\text{id}}^{\hat{b}}$ be the transfer associated to the fibre bundle $\gamma|\hat{b}: \hat{b} \to BK$. Let $j: \hat{b} \to \hat{V}$ be the inclusion. We shall see that

$$t_{\nu}^{\hat{V}} = (-1)^n t_{\text{id}}^{\hat{b}} \circ j^*. \tag{V.3}$$

(V.4) First suppose n is even so that the fixed point index of $\beta'|\Delta^n$ at b is plus one.

Let U be an open neighborhood of b in Δ^n which is homeomorphic to the standard open n-disk and which is contained in the image of the projection of the slice S used in the construction of ν. Let L be the isotropy group of S. Let $Y \subsetneq U$ be an open neighborhood of b in Δ^n which is homeomorphic to the standard open disk and such that $\beta'(Y) \subset U$. Then $\beta'|Y$ is homotopic in U to the constant map sending all points to b without any additional fixed points since both $\beta'|Y$ and the constant map have fixed point index plus one. For example when $n = 2$



A homotopy of $\beta'|Y$ exists which is induced by a small rotation of U about b so that after this homotopy a point in $Y - \{b\}$ and its image lie on different radii. This is then followed by a homotopy shrinking each radius to b.

Extend the homotopy of $\beta'|Y$ to a homotopy of $\beta'|U'$, where U' is a neighborhood of b in $K|G|H$, by inductive use of hat homotopy Definition III.5(2). This homotopy starts with $\beta'|U'$ and ends with the constant map sending all points in U' to b. Until the final map it preserves the triangulation and hence the orbit structure. By the covering homotopy theorem this homotopy, except for the final map, can be lifted to an L-equivariant deformation of the lifting of β' to S used in the slice construction map, ν. This deformation can be continued to a final map since there is a unique point in S lying above b. Extend this homotopy in the unique K-equivariant manner to a deformation of the slice construction map ν to a retraction onto the orbit b. This K-equivariant deformation induces a deformation of $\hat{\nu}$ to a retraction onto $\hat{b}$ without the addition of any new fixed points. By invariance of the transfer under compactly fixed homotopies (II.14) and the retraction property (II.16) we have shown

$$t_{\nu}^{\hat{V}} = t_{\mathrm{id}}^{\hat{b}} \circ j^{*} \quad \text{if } n \text{ is even.} \tag{V.5}$$

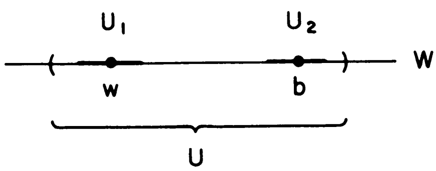
We now consider the case where n is odd so that the fixed point index of $\beta'|\Delta^n$ at b is minus one.

This case is more complicated than the case where n is even.

We shall use the following procedure several times.

Procedure V.6. U_1, U_2, U, W are open subsets of the interior of Δ^n which are homeomorphic to open disks in $\mathbf{R}^n$ such that U_1 and U_2 are disjoint and are contained in U and U is contained in W. Let $b \in U_2$ and $w \in U_1$.

For example ($n = 1$)



We are given a homotopy of the identity map of W into itself. α, the resultant map restricted to U, has isolated fixed points in $\{b, w\}$. We are also given a homotopy of α into W with compact fixed point set whose resultant map, δ, has fixed points in the set $\{b, w\}$.

We do the following. First we extend the homotopy of the identity map of W and U to neighborhoods W', U', U_1', U_2' of W, U, U_1, U_2 in $K|G|H$. This is done by inductive use of hat homotopy Definition III.5(4). Let α': $U' \to W'$ be the resultant map of this first homotopy restricted to U'. α' has fixed point set in $\{b, w\}$. Now we extend the given homotopy of α to a homotopy of α' by inductive use of hat homotopy Definition III.5(3). This homotopy of

α' has compact fixed point set and ends in a map δ': $U' \to W'$ which has fixed point set in $\{b, w\}$. Both homotopies preserve the triangulation and hence the orbit structure. Let $\overline{W}$, $\overline{U}$, $\overline{U}_1$, $\overline{U}_2$ be the set of points in G/H which project down to W', U', U_1', U_2' under the orbit map. Let $\hat{U}$, $\hat{U}_1$, $\hat{U}_2$, $\hat{w}$, $\hat{b}$ be the total spaces of the subbundles of γ (II.18) corresponding to $\overline{U}$, $\overline{U}_1$, $\overline{U}_2$, w, b. Let i_1: $\hat{U} \to \hat{U}$, i_2: $\hat{U}_2 \to \hat{U}$ be the inclusion maps. The identity map of $\overline{U}$ is a K-equivariant map which covers the identity map of U'. If w or b remains fixed under the first homotopy we can apply the slice construction to a neighborhood of it in G/H. We can assume the neighborhood is $\overline{U}_1$ or $\overline{U}_2$. Similarly if w or b remains fixed under both homotopies we can apply the slice construction to the composite homotopy in $\overline{U}_1$ or $\overline{U}_2$.

We now apply the covering homotopy theorem extension property Corollary IV.7 to the composition of the homotopies, where the identity map of $\overline{U}$ covers that of U'. We obtain a k-equivariant homotopy of the identity map of $\overline{U}$ which agrees with any given slice construction maps consistent with the homotopy of U' at the fixed points. Let δ: $\overline{U} \to \overline{W}$ be the resultant map. The homotopy also agrees with slice construction maps at the end of the first homotopy at the fixed points. Let α: $\overline{U} \to \overline{W}$ be the resultant map of this first part of the homotopy. We have shown that there is a K-equivariant homotopy of α which ends in δ and has compact fixed point set.

Let $\hat{\alpha}$: $\hat{U} \to \hat{W}$, $\hat{\delta}$: $\hat{U} \to \hat{W}$ be the maps induced by α, δ. The K-equivariant homotopy of α induces a compactly fixed homotopy of $\hat{\alpha}$ which ends in $\hat{\delta}$. Thus by homotopy invariance of the transfer (II.14) we have $t_{\hat{\alpha}}^{\hat{U}} = t_{\hat{\delta}}^{\hat{U}}$.

$t_{\hat{\alpha}}^{\hat{U}}$ and $t_{\hat{\delta}}^{\hat{U}}$ are only dependent on the behavior of $\hat{\alpha}$ and $\hat{\delta}$ near the fixed point set (II.12), and this agrees with slice constructions there. Hence if $\{b, w\}$ is the fixed point set of δ and α we have by (II.15) that

$$t_{\hat{\alpha}}^{\hat{U}} = t_{\hat{\alpha}}^{\hat{U}_1} \circ i_1^* + t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\delta}}^{\hat{U}_1} \circ i_1^* + t_{\hat{\delta}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\delta}}^{\hat{U}},$$

since the fixed point set equals $\hat{b} \cup \hat{w}$.

If b or w is not a fixed point for α' or δ', then the corresponding terms will not appear. Thus we have shown that if α: $U \to W$ and δ: $U \to W$ are the initial and resultant maps of a homotopy with compact fixed point set, as above, then the above equation holds, i.e. the transfer associated to the map $\hat{\alpha}$ equals that associated to $\hat{\delta}$, and furthermore that these equal the sum of transfers induced by slice constructions.

We shall apply this procedure to the following situation.

Start with a homotopy of the identity map of W which satisfies the following conditions.

(1) w and b remain fixed throughout the homotopy.

(2) The homotopy restricted to U_2 agrees with $H'|U_2$ (H': $K|G|H \times I \to K|G|H$ is our deformation of the identity map) (III.7).

(3) The fixed point set of the final map $\alpha\colon U \to W$ equals $\{b, w\}$, and the local fixed point indices of α at b, w are -1, $+1$ respectively.

In the above procedure we have $\hat{\alpha}|\hat{U}_2 = \hat{\nu}$, the given map induced by a slice construction near b. There exists a homotopy of α into W which ends in a map δ with no fixed points and which has compact total fixed point set. This follows since α has total fixed point index zero and U is homeomorphic to $\mathbf{R}^n$.

Hence by the above procedure the transfer

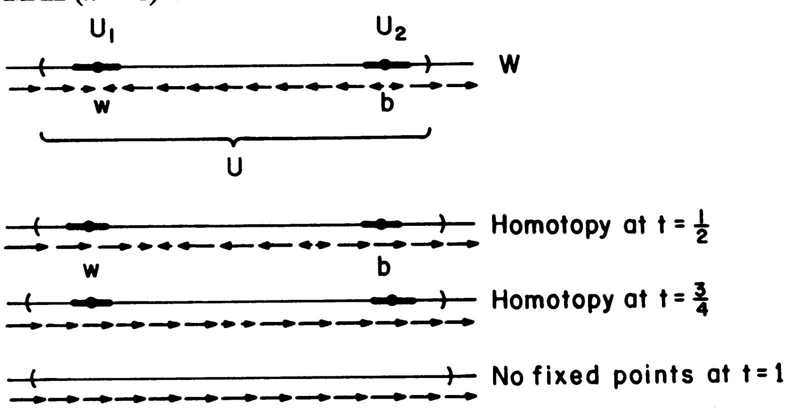
$$t_{\hat{\alpha}}^{\hat{U}} = t_{\hat{\alpha}}^{\hat{U}_1} \circ i_1^* + t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\delta}}^{\hat{U}} = 0$$

since $\hat{\delta}$ has no fixed points. Hence

$$-t_{\hat{\alpha}}^{\hat{U}_1} \circ i_1^* = t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\nu}}^{\hat{U}_2} \circ i_2^* \tag{V.7}$$

since $\hat{\alpha}$ and $\hat{\nu}$ agree on $\hat{U}_2$.

Example ($n = 1$). α looks like



We now apply the procedure again to a new situation. To distinguish these new maps from the previous situation we shall use the subscript 1. U, W, etc. will be the same however.

We start with a homotopy of the identity map of W which satisfies the following conditions.

(1) w and b remain fixed throughout the homotopy.

(2) The homotopy restricted to U_1 agrees with the homotopy of the previous application of the procedure.

(3) The fixed point set of the final map α_1 equals $\{b, w\}$ and has fixed point indices 0, $+1$ respectively.

We have, of course, $\alpha_1|U_1 = \alpha|U_1$. Hence $t_{\hat{\alpha}_1}^{\hat{U}_1} = t_{\hat{\alpha}}^{\hat{U}_1}$.

There is a homotopy of α_1 into W such that (1) b remains fixed under the homotopy, (2) the total fixed point set is compact and (3) the final map δ_1 has one fixed point at b of index $+1$. Such a homotopy exists since the total fixed point index of α_1 is $+1$.

By the above procedure (V.6)

$$t_{\hat\alpha_1}^{\hat U} = t_{\hat\alpha_1}^{\hat U_1}\circ i_1^* + t_{\hat\alpha_1}^{\hat U_2}\circ i_2^* = t_{\hat\delta_1}^{\hat U_2}\circ i_2^* = t_{\hat\delta_1}^{\hat U}.$$

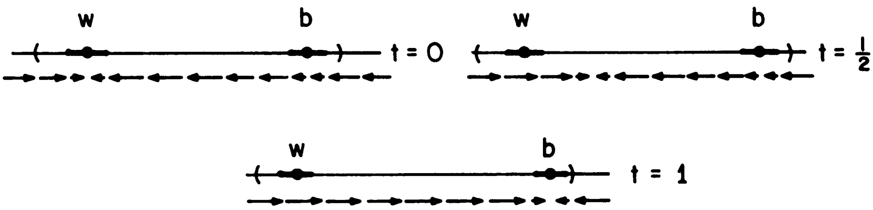
On the other hand if we use a homotopy of α_1 which remains constant inside U_1 but removes the fixed point at b (which can be done since the fixed point index at b is 0), we get by the above procedure that $t_{\hat\alpha_1}^{\hat U_1} = t_{\hat\alpha_1}^{\hat U_1}\circ i_1^*$.

Hence by (V.7)

$$t_{\hat\nu}^{\hat U_2}\circ i_2^* = -t_{\hat\alpha_1}^{\hat U_1}\circ i_1^* = -t_{\hat\delta_1}^{\hat U_2}\circ i_2^*.$$

δ_1: $U_2 \to W$ induces a slice construction map $\overline{\delta_1}$ at b (which induces $\hat\delta_1$). Since δ_1 has fixed point index $+1$ at b, we can use a procedure entirely analogous to the situation where the fixed point index of b was $+1$, where δ_1 plays the role of $\beta'|Y$, to get $t_{\hat\delta_1}^{\hat U_2}\circ i_2^* = t_{\mathrm{id}}^{\hat b}\circ j_1^*$ where j_1: $\hat b \to \hat U_2$ is the inclusion. (See (V.4).) Thus $t_{\hat\nu}^{\hat U_2}\circ i_2^* = -t_{\mathrm{id}}^{\hat b}\circ j_1^*$. Hence $t_{\hat\nu}^{\hat V} = -t_{\mathrm{id}}^{\hat b}\circ j^*$ since $\hat U_2$ can be chosen to be open in $\hat V$.

EXAMPLE. $n = 1$ homotopy moving fixed point of index $+1$ to b



Hence we have shown (V.3) ((V.5) established the case n = even). It follows from (V.1) that

$$t_{\mathrm{id}}^{\Gamma} = \sum (-1)^n t_{\mathrm{id}}^{\hat b}\circ j^* \tag{V.8}$$

where the sum is over all barycenters $\{b\}$ in the triangulation of $K|G|H$, where n is the dimension of the simplex with barycenter b.

We will show that the composition $t_{\mathrm{id}}^{\hat b}\circ j^*$ is the same for all barycenters in the same orbit-type manifold component in $K|G|H$. We actually demonstrate a somewhat stronger fact: Let w and q be arbitrary points in an orbit-type manifold component. Then there is a path lying in that orbit-type manifold component starting at w and ending at q. Let $\hat w$, $\hat q$ be the total spaces of the subbundles of γ corresponding to the K-orbits w and q. Let j_1: $\hat w \to \Gamma$ and j_2: $\hat q \to \Gamma$ be the inclusion. Then we show

$$t_{\mathrm{id}}^{\hat w}\circ j_1^* = t_{\mathrm{id}}^{\hat q}\circ j_2^*. \tag{V.9}$$

We have two cases.

Case 1. First suppose q and w lie in the interior of Δ^n. Let q play the role of b in Procedure V.6 outlined above. We can find U_1, U_2, U, W as in the above procedure. Start with a homotopy of the identity map of W which satisfies

(1) q, w are fixed under the homotopy.

(2) The fixed point set of α, the resultant map restricted to U, is $\{q, w\}$.

(3) The fixed point index of α is $+1$ at q and 0 at w.

There is a homotopy of α such that

(1) w is fixed under the homotopy.

(2) The total fixed point set is compact.

(3) The fixed point set of the resultant map δ is $\{w\}$ which has index plus 1.

Hence

$$t_{\hat{\alpha}}^{\hat{U}} = t_{\hat{\alpha}}^{\hat{U}_1} \circ i_1^* + t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\delta}}^{\hat{U}_1} \circ i_1^*.$$

If instead of taking the second homotopy we took one which removed the fixed point at w, remaining constant inside U_2, we would get

$$t_{\hat{\alpha}}^{\hat{U}} = t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^*.$$

Hence $t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\hat{\delta}}^{\hat{U}_1} \circ i_1^*$. Since $\hat{\alpha}$, $\hat{\delta}$ agree with slice procedure maps in $\hat{U}_2$, $\hat{U}_1$ and α, δ have fixed point index $+1$ at q, w, we can use a procedure entirely similar to the procedure used in (V.4) to get

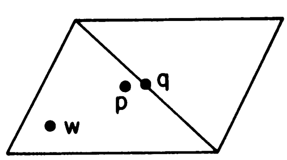
$$t_{\hat{\alpha}}^{\hat{U}_2} \circ i_2^* = t_{\mathrm{id}}^{\hat{q}} \circ l_2^* = t_{\mathrm{id}}^{\hat{w}} \circ l_1^*$$

where $l_1 \colon \hat{w} \to \hat{U}$, $l_2 \colon \hat{q} \to \hat{U}$ are the inclusion maps. Hence $t_{\mathrm{id}}^{\hat{q}} \circ j_2^* = t_{\mathrm{id}}^{\hat{w}} \circ j_1^*$.

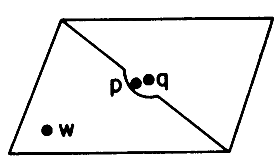
Case 2. Suppose q and w are not in the interior of the same simplex. Clearly it suffices to prove (V.9) if q is in the boundary of Δ^n, where w is contained in the interior of Δ^n. This follows since any path can be thought of as the union of paths which remain in the same simplex and those which end or begin in the boundary of a simplex and then stay within the interior of that simplex.

We shall reduce this case to Case 1 by noting that we can perturb the triangulation of $K|G|H$ slightly near q so that it still preserves the orbit structure but so that q is now in the interior of the same simplex as some point p which was formerly in the interior of the same simplex as w.

EXAMPLE.



Original Triangulation



New Triangulation

The following argument shows we can perturb the triangulation near q in the way described.

Let S be a linear slice at some point y in the K-orbit q. Let L be the isotropy group at y. Let 0_n, 0_m be the origins in $\mathbf{R}^n$, $\mathbf{R}^m$. Let S be homeomorphic to $\mathbf{R}^m \times \mathbf{R}^n$, where $\mathbf{R}^m \times 0_n$ is the fixed point set of L and L acts orthogonally on $0_m \times \mathbf{R}^n$. Then $\mathbf{R}^m \times 0_n$ projects down homeomorphically onto its image in $K|G|H$. Hence that part of the triangulation in the image of S which lies in the orbit-type manifold component corresponding to q can be viewed in $\mathbf{R}^m \times 0_n \subset S$. Similarly other parts of the triangulation lift to subspaces of S, namely their preimages. Label the preimages in S of these parts of simplices by their names in the triangulation below them in $K|G|H$. Call such a collection of subspaces of S a *pre-triangulation*. Let D^n, D^m be the closed unit disks in $\mathbf{R}^n$, $\mathbf{R}^m$. We shall alter the pre-triangulation inside $D^m \times D^n$ but leave it fixed outside $D^m \times D^n$. This will be accomplished by translating pre-simplices (i.e. preimages of simplices) by elements of $\mathbf{R}^m \times 0_n$.

Specifically let $u \in \mathbf{R}^m$, $v \in \mathbf{R}^n$ and let $\| \ \|$ be the metric in $\mathbf{R}^n$. Let h: $D^m \times I \to D^m$ be any deformation of the identity map of D^m which fixes the boundary of D^m and which is a homeomorphism at all stages.

Suppose (u, v) is in the original pre-simplex in S lying above part of some Δ^n. Then the following point corresponds to (u, v) in the corresponding pre-simplex of the new pre-triangulation

$$\begin{array}{ll} (h(u, 1 - \|v\|), v) & \text{if } \|v\| < 1 \text{ and } u \in D^m, \\ (u, v) & \text{if } \|v\| > 1 \text{ or } u \notin D^m. \end{array}$$

That is we translate the original point in the pre-simplex, $(u, v) \in D^m \times D^n$, by $h(u, 1 - \|v\|) - u \in \mathbf{R}^m$ to obtain the corresponding point in the new pre-simplex. We must show that this is in fact a pre-triangulation. First, this correspondence is continuous. Since L acts trivially on $\mathbf{R}^m$ and orthogonally on $\mathbf{R}^n$, translation by an element in $\mathbf{R}^m$ which depends only on u and the magnitude of v sends L-orbits to L-orbits. Hence under our correspondence L-orbits are sent to L-orbits. Furthermore the correspondence is one-to-one. Also if $(u, v) \notin D^m \times D^n$ it is fixed under the correspondence. Hence this is indeed a pre-triangulation, and the new pre-triangulation agrees with the old one outside $D^m \times D^n$. Hence when we project down to $K|G|H$ we have altered the triangulation near q. It remains to show that this new triangulation preserves the orbit structure of $K|G|H$. This follows by the following observation. Translation by an element of $\mathbf{R}^m$ in S sends a point of one orbit-type to a point of the same orbit-type since L acts trivially on $\mathbf{R}^m$ and orthogonally on $\mathbf{R}^n$. Hence since the orbit structure of S/L agrees with that of its image in $K|G|H$ (as pointed out in IV.2), the new triangulation preserves the orbit structure of $K|G|H$.

If we pick our deformation h: $D^m \times I \to D^m$ so that $h(p, 1) = q$ for some p in the interior of the simplex containing w, we will have altered the triangulation in the desired manner. Thus we have established (V.9).

We thus have $t_{\mathrm{id}}^{\hat{b}} \circ j^*$ equals $t_{\mathrm{id}}^{\hat{q}} \circ k^*$ where k: $\hat{q} \to \Gamma$ is the inclusion and q

is any point in the same orbit-type manifold component as b. By (V.2), the coefficient of $t_{\mathrm{id}}^{\hat{b}} \circ j^*$ in the sum (V.8) equals the fixed point index of β' at b. Group the terms in (V.8) by the orbit-type manifold components $\{M\}$. All terms corresponding to barycenters in M add to some number times $t_{\mathrm{id}}^{\hat{q}} \circ k^*$ where q is any point in M. The coefficient of this term is the sum of the local fixed point indices of β' at all the barycenters in M. This number equals $\chi^{\#}(M)$ because of the following argument.

The homotopy of the identity (III.7), H': $K|G|H \to K|G|H$, preserves the triangulation and ends in β'. Thus the total fixed point indices of $\beta'|M$, $\beta'|(\overline{M} - M)$ equal $\chi(\overline{M})$, $\chi(\overline{M} - M)$ respectively. Since the total fixed point index is the sum of the local fixed point indices, $\chi(\overline{M})$ equals the sum of all the fixed point indices at the barycenters in $\overline{M}$; $\chi(\overline{M} - M)$ equals the sum of all the fixed indices of the barycenters which are in $\overline{M}$ but not in M. Hence $\chi^{\#}(M) = \chi(\overline{M}) - \chi(\overline{M} - M)$ equals the sum of all the fixed point indices at the barycenters in M.

Note. We are using the fact that the local fixed point indices are the same for $\beta'|\overline{M}$ and $\beta'|(\overline{M} - M)$ (V.2).

Thus we have shown:

$$t_{\mathrm{id}}^{\Gamma} = \sum \chi^{\#}(M) t_{\mathrm{id}}^{\hat{q}} \circ k^* \qquad \text{(V.10)}$$

where the sum is over all orbit-type manifold components $\{M\}$ of $K|G|H$, q is any point in M and $\hat{q}$ is the total space of the subbundle of γ induced by the K-orbit q. k: $\hat{q} \to \Gamma$ is the inclusion $\chi^{\#}(M) = \chi(\overline{M}) - \chi(\overline{M} - M)$.

Note. (V.10) is really a theorem about the fibre bundle $\Gamma \to BK$. We shall return to this point in (V.14).

We now finish the proof of the double coset theorem, showing how the conjugations come into the formula.

Let $q = KgH$ for some $g \in G$. Then

$$\hat{q} = E \times_K (KgH) = \{K(e, kgH) | k \in K, e \in E\}$$

(II.18). Let f: $E \times_G (G/(K \cap H^g)) = B(K \cap H^g) \to \hat{q}$ be given by

$$G(e, g'(K \cap H^g)) \mapsto K(eg', gH).$$

Then f is a homeomorphism with inverse given by

$$K(e, kgH) \mapsto G(e, k(K \cap H^g)).$$

There is a natural projection $\rho(H^g \cap K, K)$ of $B(H^g \cap K)$ onto BK given by

$$G(e, g'(K \cap H^g)) \mapsto G(e, g'K).$$

This projection agrees with the natural projection of $\hat{q}$ onto BK given by $\gamma|\hat{q}$ (II.18) which sends $K(e, kgH)$ to $G(e, K)$, i.e. $\rho(H^g \cap K, K) = \gamma \circ f$.

Since f is a homeomorphism over BK we have by naturality (II.13)

$$t_{\mathrm{id}}^{\hat{q}} = T(H^g \cap K, K) \circ f^*$$

(see (II.5)) for definition of $T(H^g \cap K, K)$. Hence for $l: \Gamma \to BH$ as in (II.18) and $k: \hat{q} \to \Gamma$ the inclusion, we have

$$t_{\mathrm{id}}^{\hat{q}} \circ k^* \circ l^* = T(H^g \cap K, K) \circ f^* \circ k^* \circ l^*$$

$$= T(H^g \cap K, K) \circ (l \circ k \circ f)^* \qquad \text{(V.11)}$$

(V.12) *Claim.*

$$l \circ k \circ f = \tilde{C}g \circ \rho(H^g \cap K, H^g) \colon B(H^g \cap K) \to BH.$$

Let $g, g' \in G$, $e \in E$ where $\tilde{C}g$ is as in (II.3). $l \circ k \circ f$ is given by

$$G(e, g'(K \cap H^g)) \overset{k \circ f}{\mapsto} K(eg', gH) \overset{l}{\mapsto} G(eg', gH) = G(e, g'gH).$$

On the other hand $\tilde{C}g \circ \rho(H^g \cap K, H^g)$ is given by

$$G(e, g'(K \cap H^g)) \overset{\rho(H^g \cap K, H^g)}{\to} G(e, g'H^g) \overset{\tilde{C}g}{\mapsto} G(e, g'gH).$$

Hence we have by naturality (II.13), (II.18) and (V.10)

$$\rho^*(K, G) \circ T(H, G) = t_{\mathrm{id}}^{\Gamma} \circ l^* = \sum \chi^{\#}(M) t_{\mathrm{id}}^{\hat{q}} \circ k^* \circ l^*$$

$$= \sum \chi^{\#}(M) T(H^g \cap K, K) \circ f^* \circ k^* \circ l^*$$

$$= \sum \chi^{\#}(M) T(H^g \cap K, K) \circ \rho^*(H^g \cap K, H^g) \circ Cg$$

by (V.11), (V.12) and the definition of Cg (II.3).

The sums are over all orbit-type manifold components $\{M\}$ in the space $K|G|H$ viewed as the orbit space of G/H under the left action of K. This proves the double coset theorem.

(V.13) *Note.* $H^g \cap K$ is the isotropy group of the left action of K on G/H at gH.

As noted, the proof of (V.10) applies almost verbatim to the following thoerem which should be thought of as a generalization of the double coset theorem.

THEOREM V.14. *Let B be a paracompact and locally compact space. Let γ: $\Gamma \to B$ be a fibre bundle with fibre F a compact differentiable manifold and with structure group a compact Lie group K acting differentiably on F on the left. Let $\{M\}$ be the set of orbit-type manifold components of the orbit space $K \setminus F$, and let q be any K-orbit in M. Let $\hat{q}$ be the total space of the subbundle of γ corresponding to q. Let k: $\hat{q} \to \Gamma$ be the inclusion and $\chi^{\#}(M) = \chi(\overline{M}) - \chi(\overline{M} - M)$. Then*

$$t_{\mathrm{id}}^{\Gamma} = \sum \chi^{\#}(M) t_{\mathrm{id}}^{\hat{q}} \circ k^*$$

where the sum is over all orbit-type manifold components $\{M\}$ of $K \setminus F$.

VI. Examples. A few examples of specific double coset formulas are given here. Many more examples and applications of the double coset theorem will be given in subsequent papers.

EXAMPLE VI.1. First we continue Example IV.9 where $G = U(2)$, $H = K = U(1) \times U(1)$. $K|G|H$ is a line segment.

The local fixed point indices of the resulting map of the homotopy on $K|G|H$ are $+1$ at the endpoints and -1 at the barycenter of the 1-simplex, which corresponds to the equator of G/H.

Since K is abelian the isotropy groups of points on a K-orbit are all equal. Hence we can talk about the isotropy group of an orbit. The isotropy groups of the fixed point orbits are $U(1) \times U(1)$. The isotropy group of the equator of G/H is the diagonal representation of $U(1)$ in $U(1) \times U(1)$ which equals $\{(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}) | c \in U(1)\}$ and is denoted by $\Delta U(1)$.

The double coset formula has three terms in this case, corresponding to each orbit-type manifold component of $K|G|H$. Consider the term corresponding to the interior of the line segment. Let $g \in G$ be any representative of this orbit-type manifold component. Then $H^g \cap K = \Delta U(1)$ is the isotropy group at $gH \in G/H$ under the left action of K. The term in the double coset formula corresponding to this involves $T(H^g \cap K, K) = T(\Delta U(1), U(1) \times U(1))$. Let $N_K(\Delta U(1))$ be the normalizer in K of $\Delta U(1)$, which equals K. Then $N_K(\Delta U(1))/\Delta U(1)$ is homeomorphic to $U(1)$ and hence is nondiscrete. By Theorem II.17, $T(\Delta U(1), U(1) \times U(1)) = 0$. Hence this term can be removed from the double coset formula.

We thus have only two terms, corresponding to the fixed point orbits, the north and south poles, which correspond to elements of the Weyl group of G. Since $\chi^{\#}(\mathrm{pt}) = 1$, and $T(K, K) = \mathrm{id}$, we have:

$$\rho^*(U(1) \times U(1), U(2)) \circ T(U(1) \times U(1), U(2)) = 1 + Cq$$

where $q = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \in G$ is a representative of the south pole orbit. (We picked $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ as a representative of the north pole orbit.)

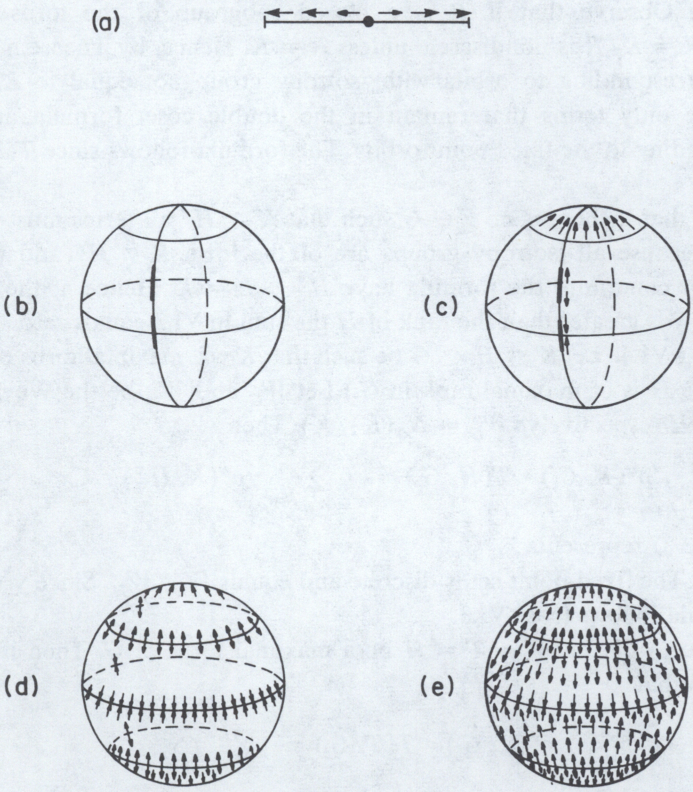
The sum is thus over the elements of the Weyl group.

Let Σ_n be the symmetric group on n letters. Let $\Sigma_n \sim U(1)$ be the wreath product.

EXAMPLE VI.2. A slightly more complicated example is where $G = U(2)$, $H = U(1) \times U(1)$ again, but $K = N_G(H) = \Sigma_2 \sim U(1)$. Then $G/H \approx S^2$ again and K acts on S^2 by rotation and a kind of flipping action, so that the orbit space, $K|G|H$, is again a line segment.

One endpoint corresponds to an orbit consisting of the north and south poles. The other endpoint corresponds to the equator. The open interval corresponds to orbits consisting of two circles.

The diagrams that correspond to (a), (b), (c), (d) and (e) of Example IV.9 are



Once again there are three terms that appear, corresponding to the poles, the equator, and the principal orbit consisting of two circles. The first two have coefficient $+1$, the last coefficient -1, since these are the corresponding internal euler characteristics of the orbit-type manifold components of $K|G|H$. The two-circle orbit has isotropy group $\Delta U(1)$. $N_K(\Delta U(1))/\Delta U(1) = K/\Delta U(1)$ is nondiscrete. Hence, this term is 0 by Theorem II.17. The isotropy groups of the other two orbits do not have this property. Hence, they remain in the formula. Let $g = (1/\sqrt{2})\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right)$ and $L = H^g \cap K$. We have:

$$\rho^*(K, G) \circ T(H, G) = T(H, K) + T(L, K) \circ \rho^*(L, H^g) \circ Cg.$$

We now derive some more general examples.

EXAMPLE VI.3. Let G be a compact Lie group with H any closed subgroup of G. Let K be an arbitrary torus in G. Then the double coset formula simplifies to:

$$\rho^*(K, G) \circ T(H, G) = \sum \chi^{\#}(M)\rho^*(K, H^g) \circ Cg$$

where the sum is over the manifold components $\{M\}$ of the *fixed point* orbits of K in $K|G|H$, where $g \in G$ is a representative of M.

PROOF. Observe that if R is a closed subgroup of the torus K then $N_K(R)/R = K/R$ is nondiscrete unless $R = K$. Hence, by Theorem II.17 all terms corresponding to orbits with isotropy group not equal to K vanish. Thus, the only terms that remain in the double coset formula are those corresponding to the fixed point orbits. The formula follows since $T(K, K) =$ id.

Notice that if there is no $g \in G$ such that $K \subset H^g$ all terms must be zero. This is because all isotropy groups are of the form $K \cap H^g$ and the only terms that come into the formula have $H^g \cap K = K$. Hence, if the rank of the torus K is greater than the rank of H the sum in VI.3 equals zero.

EXAMPLE VI.4. Let $K \subset H \subset G$ be such that K is a maximal torus of H and G. (Hence H is of maximal rank in G.) Let W_G and W_H be the Weyl groups of G and H respectively ($W_G = N_G(K)/K$). Then

$$\rho^*(K, G) \circ T(H, G) = \sum_{x \in W_G/W_H} \rho^*(K, H^x) \circ Cx$$

where $x \in G$ represents $\bar{x}$.

PROOF. The fixed point set is discrete and equals W_G/W_H. Since $\chi^{\#}(\mathrm{pt}) = 1$ the formula follows from VI.3.

EXAMPLE VI.5. Let $K = T = H$ be a maximal torus of G. Then it follows from VI.4 that

$$\rho^*(T, G) \circ T(T, G) = \sum_{x \in W_G} Cx,$$

where $x \in G$ represents $\bar{x}$.

The results of the two preceding examples are due to Brumfiel and Madsen **[BM]** who arrived at them by considerably different methods. Their theorems are stated slightly differently. VI. 1 is a special case of this result.

Note that all these formulas are true for arbitrary cohomology theories.

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