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EXPANSIVE HOMEOMORPHISMS AND TOPOLOGICAL DIMENSION

BY

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ABSTRACT. Let K be a compact metric space. A homeomorphism $f\colon K \hookleftarrow$ is expansive if there exists $\varepsilon > 0$ such that if $x, y \in K$ satisfy $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \in \mathbf{Z}$ (where $d(\cdot, \cdot)$ denotes the metric on K) then $x = y$. We prove that a compact metric space that admits an expansive homeomorphism is finite dimensional and that every minimal set of an expansive homeomorphism is 0-dimensional.

A homeomorphism f of a compact metric space K is expansive if there exists $c > 0$ (called an expansivity constant for f) such that $d(f^n(x), f^n(y)) \leqslant c$ for all n implies $x = y$. This property has frequent applications in stability theory, symbolic dynamics and ergodic theory. Specially interesting are expansive homeomorphisms of 0-dimensional spaces because they can be embedded in shifts, or, in other words, they are equivalent to subshifts. In [1] Bowen proved that hyperbolic minimal sets of diffeomorphisms are 0-dimensional. Since the restriction of a diffeomorphism to a hyperbolic set is always expansive, it is natural to ask whether minimal sets of expansive homeomorphisms are 0-dimensional. The purpose of this paper is to prove this property.

THEOREM. *If $f\colon K \hookleftarrow$ is an expansive homeomorphism of the compact metric space K then* $\dim K < \infty$ *and every minimal set of f is* 0*-dimensional.*

Recall that a compact metric space has dimension $\leqslant n$ if for all $r > 0$ there exists a covering $\mathcal{U}$ of K by open sets with diameter $\leqslant r$ such that every point belongs to at most $n + 1$ sets of $\mathcal{U}$ [2]. Moreover it is known [2] that K is 0-dimensional if and only if it is totally disconnected i.e. if the connected component of every point x is $\{x\}$.

Let us see an application of the theorem to the symbolic dynamics of an expansive homeomorphism $f\colon K \hookleftarrow$. Let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a covering of K by open sets with diameter smaller than an expansivity constant c of f. Let $\Sigma(f, \mathcal{U})$ be the subshift associated to f and $\mathcal{U}$ i.e. the set of sequences θ: $\mathbf{Z} \to \mathcal{U}$ such that $\bigcap_{-\infty}^{+\infty} f^{-n}(\bar{\theta}_n) \neq \varnothing$. Endow $\Sigma(f, \mathcal{U})$ with the topology induced by the space $\mathcal{U}^{\mathbf{Z}}$. Then $\Sigma(f, \mathcal{U})$ is compact and since the diameter of the sets in $\mathcal{U}$ is smaller than c we have a continuous map $\pi\colon \Sigma(f, \mathcal{U}) \to K$

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defined by $\pi(\theta) = \bigcap_{-\infty}^{+\infty} f^{-n}(\bar{\theta}_n)$. Moreover if σ: $\Sigma(f, \mathfrak{U}) \hookleftarrow$ is the shift homeomorphism we have $f\pi = \pi\sigma$.

COROLLARY. *If $\Delta \subset K$ is a minimal set for f then $\pi^{-1}(\Lambda)$ contains a minimal set Λ_0 that is mapped homeomorphically onto Λ by π.*

To prove the Corollary take a covering of Λ by disjoint open sets in Λ, $\tilde{\mathfrak{U}} = \{V_1, \ldots, V_l\}$ with diameters so small that there exists a function φ: $\tilde{\mathfrak{U}} \to \mathfrak{U}$ with the property $\varphi(V_i) \supset V_i$ for all i. Consider the subshift $\Sigma_1 = \Sigma(f/\Lambda, \tilde{\mathfrak{U}})$ associated to f/Λ and $\tilde{\mathfrak{U}}$ and the maps $\tilde{\pi}$: $\Sigma_1 \to \Lambda$ defined by $\tilde{\pi}(\theta) = \bigcap_{-\infty}^{+\infty} f^{-n}(\bar{\theta}_n)$ and $\tilde{\varphi}$: $\Sigma_1 \to \Sigma(f, \mathfrak{U})$ defined by $\tilde{\varphi}(\theta) = \varphi \circ \theta$. Now $\tilde{\pi}$ is a homeomorphism because the sets in $\tilde{\mathfrak{U}}$ are disjoint and we have $\pi\tilde{\varphi} = \tilde{\pi}$, $\sigma\tilde{\varphi} = \tilde{\varphi}\sigma$, where σ also denotes the shift homeomorphism of Σ_1. Since $\tilde{\pi}$ is a homeomorphism then σ: $\Sigma_1 \hookleftarrow$ is a minimal homeomorphism (because $f\tilde{\pi} = \tilde{\pi}\sigma$). Hence $\tilde{\varphi}(\Sigma_1)$ is a minimal set for σ: $\Sigma(f, \mathfrak{U}) \hookleftarrow$ and $\pi\tilde{\varphi}(\Sigma_1) = \tilde{\pi}\Sigma_1 = \Lambda_1$. Finally $\pi/\tilde{\varphi}(\Sigma_1)$ is one-to-one because $\pi\tilde{\varphi}(\theta) = \pi\tilde{\varphi}(\theta')$ implies $\tilde{\pi}(\theta) = \tilde{\pi}(\theta')$ and then $\theta = \theta'$.

1. The dimension of minimal sets. Let K be a compact metric space, with metric $d(\cdot, \cdot)$ and f: $K \hookleftarrow$ an expansive homeomorphism with expansivity constant $c > 0$. In this section we shall assume that $\dim K > 0$ and we shall prove that f cannot be a minimal homeomorphism.

If $\varepsilon > 0$ and $x \in K$, let $W^s_\varepsilon(x)$, $W^u_\varepsilon(x)$ be the local stable and unstable sets defined by:

$$W^s_\varepsilon(x) = \{y \in K | d(f^n(x), f^n(y)) < \varepsilon \, \forall n > 0\},$$
$$W^u_\varepsilon(x) = \{y \in K | d(f^{-n}(x), f^{-n}(y)) < \varepsilon, \forall n > 0\}.$$

Fix $0 < \varepsilon < c/2$.

The idea of the proof is the following: using the expansiveness we show that for some $x \in K$ there exists a compact connected set $\Lambda_0 \subset W^s_\varepsilon(x)$ with $\text{diam}(\Lambda_0) = c > 0$. Then we prove that some power f^{-m} of f expands every compact connected set Λ with $\text{diam}(\Lambda) = c$ contained in a local stable set. More precisely $\text{diam}\, f^{-m}(\Lambda) > 3c$. Using this we show that $f^{-m}(\Lambda)$ contains two compact connected sets Λ', Λ'', contained in local stable sets, with $\text{diam}(\Lambda') = \text{diam}(\Lambda'') = c$ and satisfying $\inf\{d(x, y) | x \in \Lambda', y \in \Lambda''\} > c/2$. This property contradicts the minimality of f^m because if we take an open set U with $\text{diam}(U) < c/2$ then either Λ'_0 or Λ''_0 (where Λ'_0, Λ''_0 are related to Λ_0 as Λ', Λ'' to Λ in the previous explanation) does not intersect U. Suppose $\Lambda'_0 \cap U = \varnothing$. Define $\Lambda_1 = \Lambda'_0$. Again Λ'_1 or Λ''_1 does not intersect U. Suppose $\Lambda' \cap U = \varnothing$ and define $\Lambda_2 = \Lambda'_1$. Using this method we find $\Lambda_0, \Lambda_1, \ldots$ such that $f^{-m}(\Lambda_j) \supset \Lambda_{j+1}$ and $\Lambda_j \cap U = \varnothing$. Let $x \in \bigcap_{j>0} f^{jm}(\Lambda_j)$. Then the backwards orbit of x under f^m does not intersect U; hence K is not minimal for f^m. In order to prove that K is not minimal for f we shall follow the same idea being more careful in the choice of U in order to make possible the construction of the sequence Λ_j satisfying $f^{-i}(\Lambda_j) \cap U = \varnothing$ for all $0 < i < m$. The existence of the initial set Λ_0 is proved in Lemma III and the expanding property in Lemma IV.

Lemma I. *For all $r > 0$ there exists $N > 0$ such that*

$$f^n(W^s_\varepsilon(x)) \subset W^s_r(f^n(x)), \qquad f^{-n}(W^u_\varepsilon(x)) \subset W^u_r(f^{-n}(x))$$

for all $x \in K$, $n \geq N$.

Proof. If the lemma is false we can find sequences $x_n, y_n \in K$, $m_n > 0$ such that $y_n \in W^s_\varepsilon(x_n)$, $\lim m_n = +\infty$ and $d(f^{m_n}(x_n), f^{m_n}(y_n)) > r$. Since $y_n \in W^s_\varepsilon(x_n)$ we have $d(f^n(f^{m_n}(x_n)), f^n(f^{m_n}(y_n))) \leq \varepsilon$ for all $-m_n \leq n$. Then if $x_n \to x$, $y_n \to y$ when $n \to +\infty$ we obtain that $d(f^n(x), f^n(y)) \leq \varepsilon$ for all $n \in \mathbf{Z}$. Moreover $d(x,y) = \lim d(f^{m_n}(x_n), f^{m_n}(y_n)) \geq r$ thus contradicting the expansivity.

Now define the stable and unstable sets $W^s(x)$, $W^u(x)$ as

$$W^s(x) = \bigcup_{n>0} f^{-n}(W^s_\varepsilon(f^n(x))), \qquad W^u(x) = \bigcup_{n>0} f^n(W^u_\varepsilon(f^{-n}(x))).$$

By Lemma I we have:

$$W^s(x) = \left\{ y \in K \mid \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0 \right\},$$

$$W^u(x) = \left\{ y \in K \mid \lim_{n\to+\infty} d(f^{-n}(x), f^{-n}(y)) = 0 \right\}.$$

Lemma II. *If for some $x \in K$ and $m > 0$ we have $f^m(W^s(x)) \cap W^u(x) \neq \varnothing$ then K contains a periodic point.*

Proof. Suppose $f^m(W^s(x)) \cap W^s(x) \neq \varnothing$. Take $y \in W^s(x) \cap f^m(W^s(x))$. Let $z = f^{-m}(y)$. Then $f^m(z) \in W^s(x) = W^s(z)$. Therefore $\lim_{n\to\infty} d(f^n(f^m(z)), f^n(z)) = 0$. Suppose that for some subsequence m_n we have that $f^{m_n}(z)$ converges to some $w \in K$. Then

$$\begin{aligned} d(w, f^m(w)) &= \lim d(f^m(f^{m_n}(z)), f^{m_n}(z)) \\ &= \lim d(f^{m_n}(f^m(z)), f^{m_n}(z)) = 0. \end{aligned}$$

Define $\Sigma^s_\delta(x)$, $\Sigma^u_\delta(x)$ as the connected components of x in $W^s_\varepsilon(x) \cap B_\delta(x)$ and $W^u_\varepsilon(x) \cap B_\delta(x)$ respectively, where $B_\delta(x) = \{y/d(y,x) \leq \delta\}$. Let $S_\delta(x) = \{y/d(y,x) = \delta\}$.

Lemma III. *There exists $\varepsilon > r > 0$ such that if $0 < \delta < r$ there exists $a \in K$ such that $\Sigma^s_\delta(a) \cap S_\delta(a) \neq \varnothing$ or $\Sigma^u_\delta(a) \cap S_\delta(a) \neq \varnothing$.*

Proof. Let $\Sigma_r(x)$ be the connected component of x in $B_r(x)$. Since $\dim K > 0$ we can find $x \in K$ and $r > 0$ such that $\Sigma_r(x) \cap S_r(x) \neq \varnothing$. If $0 < \delta < r$ it follows that $\Sigma_\delta(x) \cap S_\delta(x) \neq \varnothing$. Suppose that for some $0 < \delta < r$ we have $\Sigma^u_\delta(y) \cap S_\delta(y) = \varnothing$ for all y. We shall prove that there exists

$a \in K$ such that $\Sigma_\delta^s(a) \cap S_\delta(a) \neq \varnothing$. To find a we shall construct a family of compact connected sets Λ_n, $n \geq 0$, and a sequence of points $x_n \in \Lambda_n$ such that for some sequence of integers $m_n > 0$ they satisfy the following conditions:

(1) $f^{-m_n}(x_n) = x_{n+1}$;

(2) $f^{-m_n}(\Lambda_n) \supset \Lambda_{n+1}$;

(3) $\Lambda_n \cap S_\delta(x_n) \neq \varnothing$;

(4) $f^n(\Lambda_n) \subset B_\varepsilon(f^n(x_n))$ if $0 \leq n \leq m_{n-1}$.

Once the sets Λ_n are constructed the lemma follows easily taking $a = \lim x_n$ and defining Λ as the set of points $y \in K$ such that $y = \lim y_n$ for some sequence $y_n \in \Lambda_n$. Then Λ is connected and from (4) follows that $\Lambda \subset W_\varepsilon^s(a)$. By (3) $\Lambda \cap S_\delta(a) \neq \varnothing$. Hence the connected component of a in Λ intersects $S_\delta(a)$. Therefore the same thing is true for $\Sigma_\delta^s(a)$ because $\Lambda \subset W_\varepsilon^s(a)$. To construct the sets Λ_n start taking $\Lambda_0 = \Sigma_\delta(x)$. Suppose $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$ is constructed. If $\tilde{\Lambda}_{n-1}$ is the connected component of x_{n-1} in $B_\delta(x_{n-1}) \cap \Lambda_{n-1}$ then

$$\tilde{\Lambda}_{n-1} \cap S_\delta(x_{n-1}) \neq \varnothing$$

(because Λ_{n-1} is connected) and $\tilde{\Lambda}_{n-1}$ cannot be contained in $W_\varepsilon^u(x_{n-1})$. Hence there exists $y \in \tilde{\Lambda}_{n-1} \setminus W_\varepsilon^u(x_{n-1})$. Then for some $m > 0$ we have

$$\sup\{d(f^{-m}(z), f^{-m}(x_{n-1})) | z \in \tilde{\Lambda}_{n-1}\} \geq d(f^{-m}(y), f^{-m}(x_{n-1})) > \varepsilon$$

and we can suppose that:

$$\sup\{d(f^{-j}(z), f^{-j}(x_{n-1})) | z \in \tilde{\Lambda}_{n-1}, 0 \leq j < m\} \leq \varepsilon. \tag{0}$$

Let Λ_n be the connected component of $x_n = f^{-m}(x_{n-1})$ in $B_\delta(x_n) \cap f^{-m}(\tilde{\Lambda}_{n-1})$. Since $f^{-m}(\tilde{\Lambda}_{n-1})$ is connected we obtain that $\Lambda_n \cap S_\delta(x_n) \neq \varnothing$ thus proving (3). From (0) follows that Λ_n satisfies (4).

Lemma IV. *For all $0 < \delta < \varepsilon$ there exists $N > 0$ such that for all $x \in K$ and $y \in W_\varepsilon^s(x)$ with $d(y, x) = \delta$ there exists $0 < n \leq N$ satisfying*

$$d(f^{-n}(y), f^{-n}(x)) > \varepsilon.$$

Proof. If the lemma were false there would exist sequences $x_n \in K$, $y_n \in W_\varepsilon^s(x_n)$ such that $d(x_n, y_n) = \delta$ and if $j \leq n$, $d(f^{-j}(x_n), f^{-j}(y_n)) \leq \varepsilon$. Then if $x_n \to x$, $y_n \to y$ it is easy to check that $x \neq y$ and $d(f^n(x), f^n(y)) \leq \varepsilon$ for all $n \in \mathbf{Z}$.

Lemma V. *There exists $\delta_0 > 0$ such that $W_\varepsilon^s(x) \cap B_\delta(x) = W_{2\varepsilon}^s(x) \cap B_\delta(x)$ for all $x \in K$, $0 < \delta < \delta_0$.*

Proof. If the lemma is false there exist sequences $x_n, y_n \in K$ such that $d(x_n, y_n) \to 0$, and $y_n \in W_{2\varepsilon}^s(x_n)$. Hence for some $m_n > 0$ we must have $d(f^{m_n}(x_n), f^{m_n}(y_n)) > \varepsilon$ and $m_n \to +\infty$. We also have $d(f^n(f^{m_n}(x_n)),$

$f^n(f^{m_n}(y_n))) \leqslant 2\varepsilon$ for all $-m_n \leqslant m$, because $y_n \in W^s_{2\varepsilon}(x_n)$. Then if $f^{m_n}(x_n) \to x$ and $f^{m_n}(y_n) \to y$ when $n \to +\infty$ we conclude that $d(x, y) \geqslant \varepsilon$ and $d(f^m(x), f^m(y)) \leqslant 2\varepsilon$ for all $m \in \mathbf{Z}$ thus contradicting the expansivity of f.

Now define the constant $\bar{\varepsilon} = \inf\{d(x, y) | d(f^{-1}(x), f^{-1}(y)) > \varepsilon\}$.

Lemma VI. *For all $0 < \delta < \min(\delta_0, \bar{\varepsilon}/3)$ (where δ_0 is given by Lemma V) there exists $N = N(\delta) > 0$ such that if $x \in K$ and $\Lambda \subset W^s_\varepsilon(x)$ is a compact connected set containing x and intersecting $S_\delta(x)$ then there exist $0 < m < N$, points $\alpha, \beta \in f^{-m}(\Lambda)$ and compact connected sets Λ_α, Λ_β satisfying*:

(a) $\alpha \in \Lambda_\alpha$, $\beta \in \Lambda_\beta$, $\alpha \in W^s_{2\varepsilon}(\beta)$;

(b) $\Lambda_\alpha \cap S_\delta(\alpha) \neq \varnothing$, $\Lambda_\beta \cap S_\delta(\beta) \neq \varnothing$;

(c) $\inf\{d(z, w) | z \in B_\delta(\alpha), w \in B_\delta(\beta)\} > \delta$;

(d) $\Lambda_\alpha \subset W^s_\varepsilon(\alpha) \cap B_\delta(\alpha)$, $\Lambda_\beta \subset W^s_\varepsilon(\beta) \cap B_\delta(\beta)$.

Proof. Take $N = N(\delta)$ given by Lemma IV. Since $S_\delta(x) \cap \Lambda \neq \varnothing$, by Lemma IV there exists $0 \leqslant m < N$ such that

$$\sup\{d(f^{-(m+1)}(z), f^{-(m+1)}(x)) | z \in \Lambda\} > \varepsilon$$

and we can suppose:

$$\sup\{d(f^{-j}(z), f^{-j}(x)) | x \in \Lambda, 0 \leqslant j \leqslant m\} \leqslant \varepsilon.$$

Hence $f^{-m}(\Lambda) \subset W^s_\varepsilon(f^{-m}(x))$ and then:

$$f^{-m}(\Lambda) \subset W^s_{2\varepsilon}(w) \tag{1}$$

for all $w \in f^{-m}(\Lambda)$. Moreover by the definition of $\bar{\varepsilon}$ we have $\operatorname{diam} f^{-m}(\Lambda) < \bar{\varepsilon}$. Then we can find points $\alpha, \beta \in f^{-m}(\Lambda)$ such that $B_\delta(\alpha)$, $B_\delta(\beta)$ satisfy (c) (here is used the property $3\delta < \bar{\varepsilon}$). Let Λ_α, Λ_β be the connected components of α and β in $f^{-m}(\Lambda) \cap B_\delta(\alpha)$, $f^{-m}(\Lambda) \cap B_\delta(\beta)$ respectively. Since $f^{-m}(\Lambda)$ is connected, contains α and β, and $\alpha \notin B_\delta(\beta)$, $\beta \notin B_\delta(\alpha)$, it follows that Λ_α, Λ_β satisfy (b). By (1) α and β satisfy (a) and $\Lambda_\alpha \subset B_\delta(\alpha) \cap W^s_{2\varepsilon}(\alpha)$, $\Lambda_\beta \subset B_\delta(\beta) \cap W^s_{2\varepsilon}(\beta)$. Hence, by Lemma V, Λ_α, Λ_β satisfy (d).

Now we are ready to prove that f cannot be a minimal homeomorphism. Take $0 < \delta < \min(\delta_0, \bar{\varepsilon}/3, r)$ (r given by Lemma III) and define:

$$r_1 = \inf\{d(f^j(x), f^i(y)) | x \in W^s_{4\varepsilon}(y), d(x, y) \geqslant \delta, \quad 0 \leqslant j \leqslant N(\delta), 0 \leqslant i \leqslant N(\delta)\}$$

where $N(\delta)$ is as in Lemma VI. Using Lemma II it is easy to see that this number is positive, otherwise we should have sequences $x_n, y_n \in K$, $n \geqslant 0$, $y_n \in W^s_{4\varepsilon}(x_n)$, $d(x_n, y_n) \geqslant \delta$, $0 \leqslant j_n \leqslant i_n \leqslant N(\delta)$ such that $d(f^{j_n}(x_n), f^{i_n}(y_n)) \to 0$ when $n \to +\infty$. We can suppose that $x_n \to x$, $y_n \to y$ when $n \to +\infty$ and that $j_n = j$, $i_n = i$ for all $n \geqslant 0$. Then $y \in W^s_{4\varepsilon}(x)$, $d(y, x) \geqslant \delta$ and $f^j(x) = f^i(y)$. Hence $j \neq i$ and $f^{j-i}(W^s(x)) \cap W^s(x) \neq \varnothing$. This, by Lemma II, implies that K contains a periodic point. So we can assume $r_1 > 0$. We shall show that $r_1 > 0$ also contradicts the minimality of K by showing the existence of an open set U and a point p such that $f^n(p) \notin U$ for all $n \geqslant 0$.

First we construct a family of compact connected sets Λ_n, $n \geqslant 0$, a sequence of points $x_n \in \Lambda_n$ and an open set $U \subset K$ such that:

(a) $\Lambda_n \cap S_\delta(x_n) \neq \varnothing$;

(b) $\Lambda_n \subset W^s_\varepsilon(x_n)$;

(c) For some $0 < m_n < N(\delta)$, $f^{-m_n}(\Lambda_n) = \Lambda_{n+1}$;

(d) $f^{-j}(\Lambda_n) \cap U = \varnothing$ for all $0 \leqslant j \leqslant m_n$.

By Lemma III we can suppose that $\Sigma^s_\delta(a) \cap S_\delta(a)$ for some $a \in K$ (because $\delta < r$). Define $x_0 = a$, $\Lambda_0 = \Sigma^s_\delta(a)$ and take an open set U with diameter $\leqslant r/2$, and $U \cap \Lambda_0 = \varnothing$. Suppose that we constructed $\Lambda_0, \Lambda_1, \ldots, \Lambda_{n-1}$. To find Λ_n we apply Lemma VI obtaining two compact connected sets Λ_α, Λ_β such that defining Λ_n as one of them, properties (a), (b), (c) would be satisfied. In order to satisfy condition (d) observe that by the definition of r_1 and the fact that the diameter of U is $\leqslant r_1/2$ then if $U \cap (\bigcup_{j=0}^m f^{-j}(\Lambda_\alpha)) \neq \varnothing$ (m being the number given in Lemma VI) then $U \cap (\bigcup_{j=0}^m f^{-j}(\Lambda_\beta)) = \varnothing$ and we chose $\Lambda_n = \Lambda_\beta$. Finally define $p = \bigcap_{n \geqslant 0} f^{-N_n}(\Lambda_n)$ where $N_n = \sum_{j=0}^n m_j$. Clearly $f^n(p) \notin U$ for all $n \geqslant 0$, thus proving that K is not minimal.

2. The dimension of K. Let f, K be as in §1. Here we shall prove that $\dim K < \infty$. Let, as in §1, $c > 0$ be an expansivity constant for f. Fix $0 < \varepsilon < c/2$.

LEMMA. *There exists $\delta > 0$ such that if $x, y \in K$, $d(x, y) \leqslant \delta$, and for some $n > 0$ satisfy $\varepsilon \leqslant \sup\{d(f^j(x), f^j(y)) | 0 \leqslant j \leqslant n\} \leqslant 2\varepsilon$, then $d(f^n(x), f^n(y)) \geqslant \delta$.*

PROOF. If this property is false we can find sequences $x_n, y_n \in K$, $m_n > l_n > 0$ such that $d(x_n, y_n) \to 0$, $d(f^{m_n}(x_n), f^{m_n}(y_n)) \to 0$, $d(f^{l_n}(x_n), f^{l_n}(y_n)) \geqslant \varepsilon$ and $\sup\{d(f^m(x_n), f^m(y_n)) | 0 \leqslant m \leqslant m_n\} \leqslant 2\varepsilon$. Suppose that $f^{l_n}(x_n) \to x$ and $f^{l_n}(y_n) \to y$. Then $d(x, y) \geqslant \varepsilon$ and $d(f^n(x), f^n(y)) \leqslant 2\varepsilon$ for all $n \in \mathbf{Z}$.

LEMMA II. *For all $\rho > 0$ there exists $N = N(\rho)$ such that $d(x, y) \geqslant \rho$ implies that* $\sup\{d(f^n(x), f^n(y)) | \, |n| \leqslant N\} > \varepsilon$.

PROOF. If the property is false there exist sequences $x_n, y_n \in K$ with $d(x_n, y_n) \geqslant \rho$ and such that $\sup\{d(f^j(x), f^j(y)) | \, |j| \leqslant n\} \leqslant \varepsilon$. Then if $x_n \to x$, $y_n \to y$, we obtain $d(x, y) \geqslant \rho$ and $d(f^n(x), f^n(y)) \leqslant \varepsilon$ for all $n \in \mathbf{Z}$.

To prove that $\dim K < \infty$ take a covering $\{U_i | 1 \leqslant i \leqslant l\}$ of K by open sets with diameter $\leqslant \delta$, δ as in Lemma I. We claim that $\dim(K) \leqslant l^2 - 1$. To prove this for each $n \geqslant 0$ choose $\delta_n > 0$ such that $d(x, y) \leqslant \delta_n$ implies $d(f^j(x), f^j(y)) < \varepsilon$ for all $|j| \leqslant n$. Let $U^n_{i,j} = f^n(U_i) \cap f^{-n}(U_j)$ and let $U^{n,k}_{i,j}$, $1 \leqslant k \leqslant k(i, j, n)$, be the δ_n-components of $U^n_{i,j}$, i.e. the equivalence classes of $U^n_{i,j}$ under the relation $x \sim y$ if there exists a sequence $x = x_0, x_1, \ldots, x_p = y$

such that $d(x_r, x_{r+1}) < \delta_n$ for all $0 < r < p - 1$ and $x_r \in U^n_{i,j}$ for all $0 < r < p$. Observe that the sets $U^{n,k}_{i,j}$ are open and cover K. We have that:

$$\lim_{n\to+\infty}\left(\sup_{k,i,j} \operatorname{diam} U^{n,k}_{i,j}\right) = 0 \tag{2}$$

because otherwise we could find $\rho > 0$ and large values of n, say $n > 2N(\rho)$, $N(\rho)$ given by Lemma II, such that in some $U^{n,k}_{i,j}$ there exist x, y with $d(x,y) > \rho$. Let $x = x_0, \ x_1, \ldots, x_p = y$ a sequence in $U^n_{i,j}$ such that $d(x_r, x_{r+1}) < \delta_n$ for all $0 < r < \rho$. Define $S_r = \sup\{d(f^m(x_r), f^m(x_0))| \ |m| < n\}$. By Lemma II $S_p > \varepsilon$ and by the choice of δ_n $S_1 < \varepsilon$ and $|S_{r+1} - S_r| < \varepsilon$ for $1 < r < p$. Take r such that $s_{r'} < \varepsilon$ if $r' < r$ and $s_r > \varepsilon$. Then $s_r < 2\varepsilon$. Therefore the points $x = x_0$ and x_r satisfy $d(f^{-n}(x), f^{-n}(x_r)) < \delta$ because $f^{-n}(x), f^{-n}(x_r)$ belong to U_i, and $d(f^n(x), f^n(x_r)) < \delta$ because $f^n(x), f^n(x_r)$ belong to U_j, and $\varepsilon < s_r = \sup\{d(f^m(x), f^m(x_r))| \ |m| < n\} < 2\varepsilon$ thus contradicting Lemma I. Then (2) is proved and it remains to show only that for each n, every point of K belongs at most to l^2 sets of the covering $\{U^{n,k}_{i,j}|1 < i < l, \ 1 < j < l, \ 1 < k < k(i,j)\}$. Suppose that $\bigcap\{U^{n,k_m}_{i_m j_m}|1 < m < s\} \neq \emptyset$. Then $(i_m, j_m) = (i_{\bar{m}}, j_{\bar{m}})$ implies that $U^{n,k_m}_{i_m j_m} = U^{n,k_{\bar{m}}}_{i_m j_m}$ because they are both δ_n-components of $U^n_{i_m j_m}$ and have nonempty intersection.

This means that to different values of m correspond different values of the couple (i_m, j_m). Therefore $s < m^2$.

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