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ERGODIC BEHAVIOUR OF NONSTATIONARY REGENERATIVE PROCESSES[1]

BY

DAVID MCDONALD

ABSTRACT. Let V_t be a regenerative process whose successive generations are not necessarily identically distributed and let A be a measurable set in the range of V_t. Let μ_n be the mean length of the nth generation and α_n be the mean time V_t is in A during the nth generation. We give conditions ensuring $\lim_{t\to\infty} \operatorname{Prob}\{V_t \in A\} = \alpha/\mu$ where $\lim_{n\to\infty}(1/n)\Sigma_{j=1}^n \mu_j = \mu$ and $\lim_{n\to\infty}(1/n)\Sigma_{j=1}^n \alpha_j = \alpha$.

Introduction and main results. A nonstationary regenerative process V_t may be viewed as a succession of independent generations or cycles whose cycles are not necessarily identically distributed (a good example is a classical regenerative process (see [8]) whose stochastic mechanism is nonuniformly perturbed). The successive cycles are labelled 1, 2, 3, . . . and we could start off the process at the nth cycle, say, in which case we denote the process by $V_t^{(n)}$ ($V_t^{(1)}$ is denoted by V_t). However the process is started, it is defined on a measure space $\{\Omega, \mathfrak{F}\}$. Starting off at the nth cycle merely induces a different probability measure $P^{(n)}$ on $\{\Omega, \mathfrak{F}\}$ ($P^{(1)}$ is denoted by P).

Let $(T_n)_{n=1}^\infty$ denote the lengths of the cycles starting at cycle 1. Hence $(T_n)_{n=1}^\infty$ is a sequence of independent random variables defined on $\{\Omega, \mathfrak{F}, P\}$. Also we assume $T_n > 0$ for all n. Define $S_n = \Sigma_{m=1}^n T_m$.

If the T_m all take values on a lattice (for simplicity the integers) we call it the lattice case; otherwise we call it the continuous case. As in [2] and [3] we maintain a dichotomy between the two cases. In the former case R $[R_+]$ represents the integers [nonnegative integers]; B_+ is the σ-field of subsets of R_+ and m is counting measure. In the latter case R $[R_+]$ represents $(-\infty, \infty)$ $[[0, \infty)]$; B_+ is the σ-field of Borel sets on R_+ and m is Lebesgue measure.

Let K_n be a partition of $\{1, 2, \dots\}$ of the form $K_n = \{i | i_n < i \leq i_{n+1}\}$. Given K_n define $Y_n = \Sigma_{i\in K_n} T_i$. For $d > 0$, and $\varepsilon = 1/2r$, r an integer, set

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$$B_k(\varepsilon) = \{x | -\varepsilon < x - 2k\varepsilon \leqslant \varepsilon\},$$

$$q_{nk}(\varepsilon, d) = \min[\operatorname{Prob}\{Y_n \in B_k(\varepsilon)\}, \operatorname{Prob}\{Y_n - d \in B_k(\varepsilon)\}],$$

$$q_n(\varepsilon, d) = \sum_{k=-\infty}^{\infty} q_{nk}(\varepsilon, d), \qquad Q_m = \sum_{n=1}^{m} q_n(\tfrac{1}{2}, 1).$$

Definition 1. The sequence $\{T_n\}_{n=1}^{\infty}$ is called strongly d-mixing if $\forall\varepsilon$ there exists a sequence K_n such that $\Sigma_{n=0}^{\infty} q_n(\varepsilon, d) = \infty$. Furthermore the sequence $\{T_n\}_{n=1}^{\infty}$ is called strongly mixing if the closure of the smallest subgroup containing $\{d | \{T_n\}_{n=1}^{\infty}$ is strongly d-mixing$\}$ is R.

Condition C(a). $(T_n)_{n=1}^{\infty}$ is a strongly mixing sequence (see also [2, Definition 3] and also Mineka [4]).

Let F^n be the distribution of T_n.

Condition C(b). There exists a distribution G with finite mean such that $F^n(s) \geqslant G(s)$ for all n and all $s \in R_+$. Let $\bar{\mu}$ be the mean of G.

Condition C(c). If $\mu_n = ET_n$ is the mean length of the nth cycle then $\inf \mu_n \geqslant \underline{\mu} > 0$. This condition is vacuous in the lattice case.

It is clear that V_t is a very special semiregenerative process (see [2] and [3]) with embedded semi-Markov chain $(n + 1, T_n)_{n=0}^{\infty}$ $(T_0 = 0)$. We may therefore apply the ergodic results ([2, Theorem 4] and [3, Theorem 4]) to our special case:

Theorem 1. *If A is a measurable set in the range of V_t, if Conditions* C(a)–C(c) *hold and if in the continuous case the functions $P^{(n)}\{V_s \in A, X_1 > s\}$ (X_1 denotes the length of the starting cycle) are uniformly directly Riemann integrable* (*see* [3, *Definition* 4]) *then*

$$\lim_{\substack{t\to\infty \\ t\in R_+}} \sum_{n=1}^{\infty} \left| P\{V_t \in A, S_{n-1} \leqslant t < S_n\} - \frac{\alpha_n}{\mu_n} P\{S_{n-1} \leqslant t < S_n\} \right| = 0 \quad (1)$$

where $\alpha_n = \int dP^{(n)} \int_0^{X_1} \chi_{\{V_s^{(n)} \in A\}}(s) \cdot m(ds)$; that is α_n is the mean time V_t is in A during the nth cycle.

The goal of this paper is to evaluate the weighting $P\{S_{n-1} \leqslant t < S_n\}$ as $t \to \infty$. We define the following supplementary conditions:

Let the variance of F^n be σ_n^2 and set $A_n = \Sigma_{j=1}^n \mu_j$, $B_n^2 = \Sigma_{j=1}^n \sigma_j^2$.

Condition S(a). There exist constants $\underline{\sigma}$ and $\bar{\sigma}$ such that $\sigma_n^2 \leqslant (\bar{\sigma})^2$ and $n(\underline{\sigma})^2 \leqslant B_n^2$.

Condition S(b). $\lim_{b\to\infty}(1/\sigma_k^2)\int_{|x|<b}(x - \mu_k)^2 F^k(dx) = 1$ uniformly in k.

Condition S(c). There exists a $\mu > 0$ such that $\lim_{n\to\infty} \sqrt{n}\,(A_n/n - \mu) = 0$.

The chief results of this paper (the proofs are given in the next section) are as follows.

THEOREM 2. *If Conditions* C(a)–C(c) *and* S(a)–S(c) *hold, then*

$$\lim_{\substack{t\to\infty \\ t\in R_+}} \sum_{n=1}^{\infty} \left| P\{S_{n-1} \leq t < S_n\} - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right| = 0. \quad (2)$$

THEOREM 3. *If Conditions* C(a)–C(c) *and* S(a)–S(c) *hold, if* (1) *holds and if there exists an* α *such that*

$$\lim_{n\to\infty} \sqrt{n} \left(\frac{1}{n} \sum_{j=1}^{n} \alpha_j - \alpha \right) = 0$$

then

$$\lim_{\substack{t\to\infty \\ t\in R_+}} P\{V_t \in A\} = \frac{\alpha}{\mu}.$$

This generalizes the classical ergodic result for regenerative processes (see [8]).

Lemmas and proofs. Conditions C(a)–C(c) are verified throughout.

LEMMA 1. *If Conditions* S(a) *are satisfied then for any* $\varepsilon > 0$ *there exists a* λ *such that*

$$\lim_{t\to\infty} \sum_{|(t-A_n)/\sqrt{t}|>\lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} < \varepsilon.$$

PROOF.

$$\sum_{(t-A_n)/\sqrt{t}>\lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\}$$

$$< \sum_{k=1}^{t/\gamma\underline{\mu}} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{(\underline{\sigma})^2((t-\lambda\sqrt{t})/\bar{\mu}-k)}} \exp\left\{ \frac{-(\lambda\sqrt{t}+k\underline{\mu})^2}{2(\bar{\sigma})^2((t-\lambda\sqrt{t})/\underline{\mu}-k)} \right\}$$

$$+ \sum_{k=t/\gamma\underline{\mu}+1}^{(t-\lambda\sqrt{t})/\underline{\mu}} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\underline{\sigma}} \exp\left\{ \frac{-(\lambda\sqrt{t}+k\underline{\mu})^2}{2(\bar{\sigma})^2((t-\lambda\sqrt{t})/\underline{\mu}-k)} \right\}$$

(where $0 < (1/\gamma)(\bar{\mu}/\underline{\mu}) < 1$ and $t > \lambda^2(1-1/\gamma)^{-2}$)

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{t/\gamma\underline{\mu}} \frac{1}{\sqrt{t-\lambda\sqrt{t}-x\bar{\mu}}} \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t}+x\underline{\mu})^2}{(t-\lambda\sqrt{t}-x\underline{\mu})} \right\} dx$$

$$+ \frac{t}{\underline{\mu}}\left(1-\frac{1}{\gamma}\right) \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\underline{\sigma}} \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(t/\gamma)^2}{t} \right\}.$$

We now choose $0 < \alpha < 1$ such that

$$t - \lambda\sqrt{t} - x\bar{\mu} > \alpha\left(t - \lambda\sqrt{t} - x\underline{\mu}\right)$$

for $0 < x < t/\gamma\underline{\mu}$. It suffices to find a suitable α for $x = t/\gamma\underline{\mu}$. For $x = t/\gamma\underline{\mu}$ the above inequality yields $t(1 - \alpha - \bar{\mu}/\gamma\underline{\mu} + \alpha/\gamma) > \lambda\sqrt{t}\,(1 - \alpha)$. Pick α so that $\beta = (1 - \alpha - \bar{\mu}/\gamma\underline{\mu} + \alpha/\gamma) > 0$; that is $0 < \alpha < (1 - 1/\gamma)^{-1}(1 - \bar{\mu}/\gamma\underline{\mu})$ and $\beta t > \lambda\sqrt{t}\,(1 - \alpha)$; that is $t > (\lambda(1 - \alpha)/\beta)^2$. Therefore for α, γ and β as above and $t > \max\{\lambda^2(1 - 1/\gamma)^{-2}, (\lambda(1 - \alpha)/\beta)^2\}$

$$\frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{t/\gamma\underline{\mu}} \frac{1}{\sqrt{t - \lambda\sqrt{t} - x\bar{\mu}}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\underline{\mu})^2}{(t - \lambda\sqrt{t} - x\underline{\mu})}\right\} dx$$

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_0^{t/\gamma\underline{\mu}} \frac{1}{\sqrt{t - \lambda\sqrt{t} - x\underline{\mu}}} \cdot \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\underline{\mu})^2}{(t - \lambda\sqrt{t} - x\underline{\mu})}\right\} dx$$

$$= \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\underline{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{t(1-1/\gamma)-\lambda\sqrt{t}}^{t-\lambda\sqrt{t}} \frac{1}{\sqrt{s}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(t - s)^2}{s}\right\} ds$$

(where $s = t - \lambda\sqrt{t} - x\underline{\mu}$)

$$= \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\underline{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{\lambda\sqrt{t}\,(t-\lambda\sqrt{t})^{-1/2}}^{(t/\gamma+\lambda\sqrt{t})(t(1-1/\gamma)-\lambda\sqrt{t})^{-1/2}} \left(1 - x(x^2 + 4t)^{-1/2}\right) \cdot \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} \cdot x^2\right\} dx$$

(where $x = (t - s)/\sqrt{s}$ or $\sqrt{s} = \left(\sqrt{x^2 + 4t} - x\right)/2$)

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\underline{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_\lambda^\infty \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} x^2\right\} dx.$$

Therefore,

$$\sum_{(t-A_n)/\sqrt{t} > \lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{\frac{-(t - A_n)^2}{2B_n^2}\right\}$$

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\underline{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_\lambda^\infty \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} x^2\right\} dx + \frac{t}{\underline{\mu}}\left(1 - \frac{1}{\gamma}\right)\frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\underline{\sigma}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2\gamma^2} t\right\} < \frac{\varepsilon}{2}$$

for λ sufficiently large and $t > \max\{\lambda^2(1 - 1/\gamma)^{-2}, (\lambda(1-\alpha)^2/\beta)\}$ where α, λ and β are defined as above.

Similarly

$$\sum_{(t-A_n)/\sqrt{t} < -\lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\}$$

$$< \sum_{k=1}^{\infty} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{(\underline{\sigma})^2((t+\lambda\sqrt{t})/\bar{\mu}+k)}} \exp\left\{ \frac{-(\lambda\sqrt{t}+k\underline{\mu})^2}{2(\bar{\sigma})^2((t+\lambda\sqrt{t})/\underline{\mu}+k)} \right\}$$

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{t+\lambda\sqrt{t}+k\underline{\mu}}} \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t}+k\underline{\mu})^2}{(t+\lambda\sqrt{t}+k\underline{\mu})} \right\}$$

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{t+\lambda\sqrt{t}+x\underline{\mu}}} \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t}+x\underline{\mu})^2}{(t+\lambda\sqrt{t}+x\underline{\mu})} \right\}$$

$$= \frac{(\bar{\mu})^{3/2}}{\underline{\sigma\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{t+\lambda\sqrt{t}}^{\infty} \frac{1}{\sqrt{s}} \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(s-t)^2}{s} \right\} ds$$

(where $s = t + \lambda\sqrt{t} + x\underline{\mu}$)

$$= \frac{(\bar{\mu})^{3/2}}{\underline{\sigma\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\lambda\sqrt{t}/\sqrt{t+\lambda\sqrt{t}}}^{\infty} \left(1 + \frac{x}{\sqrt{x^2+4t}}\right) \exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot x^2 \right\} dx$$

(where $x = (t-s)/\sqrt{s}$ or $\sqrt{s} = (\sqrt{x^2+4t} + x)/2$)

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} 2\exp\left\{ \frac{-\underline{\mu}}{2(\bar{\sigma})^2} x^2 \right\} dx < \frac{\varepsilon}{2}$$

for λ sufficiently large.

The result follows. □

Lemma 2. *If Conditions* S(a) *are verified, then for any* $\varepsilon > 0$ *there exist* λ *and* $T(\lambda(\varepsilon))$ *such that*

$$\sum_{|(t-A_n)/\sqrt{t}| > \lambda} P\{S_{n-1} \leq t < S_n\} < \varepsilon \quad \text{for all } t > T.$$

Proof.

$$\sum_{|(t-A_n)/\sqrt{t}| > \lambda} P(S_{n-1} \leq t < S_n) \leq P(S_k > t) + P(S_l < t)$$

where

$$k = \sup\{n: A_n < t - \lambda\sqrt{t}\}$$

and

$$l = \inf\{n: A_n > t + \lambda\sqrt{t}\},$$

$$P(S_k > t) \leqslant P\left(S_k - A_k > \lambda\sqrt{t}\right) \leqslant \frac{B_k^2}{t\lambda^2} \leqslant \frac{k(\bar{\sigma})^2}{t\lambda^2}.$$

However $\underline{\mu}k \leqslant A_k < t - \lambda\sqrt{t}$. Hence

$$\frac{k}{t} \leqslant \frac{1}{\underline{\mu}}\left(1 - \frac{\lambda}{\sqrt{t}}\right) \leqslant \frac{1}{\underline{\mu}} \qquad \text{for } t > \lambda^2.$$

Therefore $P(S_k > t) \leqslant (1/\underline{\mu})(\bar{\sigma})^2/\lambda^2$.

$$P(S_l < t) \leqslant P\left(S_l - A_l < -\lambda\sqrt{t}\right) \leqslant \frac{B_l^2}{t\lambda^2} \leqslant \frac{l(\bar{\sigma})^2}{t\lambda^2}.$$

However $\underline{\mu}(l-1) \leqslant A_{l-1} < t + \lambda\sqrt{t}$. Hence

$$\frac{l}{t} \leqslant \frac{1}{\underline{\mu}}\left(1 + \frac{\lambda}{\sqrt{t}}\right) + \underline{\mu}.$$

Therefore,

$$P(S_k < t) \leqslant \left[\left(\frac{1}{\underline{\mu}} + \underline{\mu}\right) + \frac{1}{\underline{\mu}} \cdot \frac{\lambda}{\sqrt{t}}\right] \frac{(\bar{\sigma})^2}{\lambda^2}.$$

Hence,

$$\overline{\lim_{t \to \infty}}\, \left[P(S_k > t) + P(S_l < t)\right] \leqslant \frac{(\bar{\sigma})}{\lambda^2}\left(\frac{2}{\underline{\mu}} + \underline{\mu}\right).$$

This can be made arbitrarily small by taking λ large.

LEMMA 3. *If Conditions* S(a) *are satisfied then for* $L > 0$, $\varepsilon > 0$, $\lambda > 0$ *there exists a* $T > 0$ *such that for all* $t > T$ *and for all* x $(0 \leqslant x \leqslant L)$ *we have*

$$\left|\exp\left\{\frac{-(t - x - A_n)^2}{2B_n^2}\right\} - \exp\left\{\frac{-(t - A_{n+1})^2}{2B_{n+1}^2}\right\}\right| < \varepsilon$$

for all n *such that* $|(t - A_n)/\sqrt{t}\,| < \lambda$.

PROOF.

$$\left|\exp\left\{\frac{-(t - x - A_n)^2}{2B_n^2}\right\} - \exp\left\{\frac{-(t - A_{n+1})^2}{2B_{n+1}^2}\right\}\right|$$
$$\leqslant \left|1 - \exp\left\{\frac{(t - A_{n+1})^2}{2B_{n+1}^2} - \frac{(t - x - A_n)^2}{2B_n^2}\right\}\right|.$$

Next

$$\frac{(t-A_{n+1})^2}{2B_{n+1}^2}-\frac{(t-x-A_n)^2}{2B_n^2}=\frac{B_n^2(t-A_{n+1})^2-B_{n+1}^2(t-x-A_n)^2}{2B_{n+1}^2B_n^2}.$$

Also

$$\begin{aligned}B_n^2(t-A_n-\mu_{n+1})^2-(B_n+\sigma_{n+1}^2)(t-x-A_n)^2\\ &=B_n^2(t-A_n)^2-2B_n^2\mu_{n+1}(t-A_n)+(\mu_{n+1})^2B_n^2\\ &\quad -B_n^2(t-A_n)^2-B_n^2x^2+2B_n^2x(t-A_n)\\ &\quad -\sigma_{n+1}^2(t-A_n)^2+2x\sigma_{n+1}^2(t-A_n)-\sigma_{n+1}^2x^2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{(t-A_{n+1})^2}{2B_{n+1}^2}-\frac{(t-x-A_n)^2}{2B_n^2}\\ &=\frac{-2\mu_{n+1}(t-A_n)+2x(t-A_n)}{2B_{n+1}^2}+\frac{(\mu_{n+1})^2-x^2}{2B_{n+1}^2}\\ &\quad+\frac{2x\sigma_{n+1}^2(t-A_n)-\sigma_{n+1}^2(t-A_n)^2}{2B_n^2B_{n+1}^2}-\frac{\sigma_{n+1}^2x^2}{2B_n^2B_{n+1}^2}\end{aligned}$$

as $t\to\infty$, $n\to\infty$. Moreover $B_n^2 \geqslant n(\underline{\sigma})^2$ hence

$$\{(\mu_{n+1})^2-x^2\}/2B_{n+1}^2\to 0,\quad t\to\infty,$$

and

$$-\sigma_{n+1}^2x^2/2B_n^2B_{n+1}^2\to 0,\quad t\to\infty,$$

for n such that $|(t-A_n)/\sqrt{t}\,|<\lambda$. Next $A_n-\lambda\sqrt{t}\leqslant t\leqslant A_n+\lambda\sqrt{t}$ implies that $t-\lambda\sqrt{t}\leqslant n\bar{\mu}$. Hence

$$\frac{|t-A_n|}{B_n^2}\leqslant\frac{\lambda\sqrt{t}}{B_n^2}\leqslant\frac{\lambda\sqrt{t}}{n(\underline{\sigma})^2}\leqslant\frac{\lambda\sqrt{t}\,\bar{\mu}}{\underline{\sigma}(t-\lambda\sqrt{t}\,)}.$$

Therefore $|t-A_n|/B_n^2\to 0$ and

$$\frac{(t-A_n)^2}{B_n^2B_{n+1}^2}=\frac{(t-A_n)}{B_n^2}\cdot\frac{(t-A_n)}{B_{n+1}^2}\to 0$$

as $t\to\infty$ by the same reasoning. Hence we have

$$\frac{(t-A_{n+1})^2}{2B_{n+1}^2}-\frac{(t-x-A_n)^2}{2B_n^2}\to 0$$

which gives the result. □

LEMMA 4. *If Conditions* S(a) *are verified then* $\forall \varepsilon > 0$, $\lambda > 0$ *there exists a* $T > 0$ *such that for* $t > T$

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} m(ds) \right. \\ \left. - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \varepsilon$$

for n *such that* $|(t - A_n)/\sqrt{t}\,| < \lambda$.

PROOF.

$$\int_L^\infty (1 - F^n(x)) m(dx) < \int_L^\infty (1 - G(x)) m(dx) \to 0$$

as $L \to \infty$. Let L be such that $\int_L^\infty (1 - G(x)) m(dx) < \varepsilon/4$. Next

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \right. \\ \left. \left. - \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right] m(ds) \right| \\ < \int_0^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \\ \left. - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds).$$

Moreover

$$\frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < 1;$$

hence

$$\int_0^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds) \\ < \int_{t-L}^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \\ \left. - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds) + \frac{\varepsilon}{4}.$$

By Lemma 3 there exists a $T > L$ such that for $t > T$ and s such that $t - L \leq s < t$ we have

$$\left|\exp\left\{\frac{-(s-A_n)^2}{2B_n^2}\right\} - \exp\left\{\frac{-(t-A_{n+1})^2}{2B_{n+1}^2}\right\}\right| < \frac{\varepsilon}{4\bar{\mu}}$$

for n such that $|(t - A_n)/\sqrt{t}\,| < \lambda$. Hence we have

$$\left|\int_0^t (1 - F^{n+1}(t-s))\left[\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(s-A_n)^2}{2B_n^2}\right\} - \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(t-A_{n+1})^2}{2B_{n+1}^2}\right\}\right] m(ds)\right|$$

$$< \bar{\mu}\cdot\frac{\varepsilon}{4\bar{\mu}} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

However

$$\left|\int_0^t (1 - F^{n+1}(t-s))m(ds) - \mu_{n+1}\right|\cdot\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(t-A_{n+1})^2}{2B_{n+1}^2}\right\}$$

$$< \left|\mu_{n+1} - \int_0^L (1 - F^{n+1}(s))m(ds)\right| < \frac{\varepsilon}{2}.$$

This gives the result. □

We now digress to establish certain useful results related to local limit theorems.

Proposition 1. *In the continuous case suppose* $\{T_n\}_{n=1}^{\infty}$ *is strongly mixing* (*we need not assume that the* T_n *are positive valued*) *and suppose* T_1 *has a bounded density*; *then for all* s, $0 \leq s \leq l$, *where* $s \in R_+$ *and* $l \in R_+$,

$$\lim_{n\to\infty} \sum_{k=-\infty}^{\infty} P\{kl \leq S_n < kl + s\} = \frac{s}{l}.$$

Proof. First suppose $s = xp$, $l = xq$ where p and q are integers ($x \in R_+$). Consider the function $f(z) = \chi_{\cup_{k=-\infty}^{\infty}[kl, kl+x)}(z)$. Next let ${}^2S_n = \sum_{k=2}^{n} T_k$ so

$$Ef(S_n) = Ef({}^2S_n + T_1) = E\int_{-\infty}^{\infty} f({}^2S_n + y)F^1(dy) = Eg({}^2S_n)$$

where

$$g(x) = \int_{-\infty}^{\infty} f(x+y)F^1(dy).$$

Since F^1 has a bounded derivative, g is uniformly continuous and applying Theorem 2 in Orey [7] we have:

$$\lim_{n\to\infty}\left[Ef(S_n) - Ef(S_n + x)\right] = \lim_{n\to\infty}\left[Eg({}^2S_n) - Eg({}^2S_n + x)\right] = 0.$$

Similarly

$$\lim_{n\to\infty} \left[Ef(S_n) - Ef(S_n + kx) \right] = 0$$

so

$$\lim_{n\to\infty} \left[qEf(S_n) - E \sum_{k=0}^{q-1} f(S_n + kx) \right] = 0;$$

that is $\lim_{n\to\infty}[qEf(S_n) - 1] = 0$. Hence $\lim_{n\to\infty} E \sum_{k=0}^{p-1} f(S_n + kx) = p/q$ which gives:

$$\lim_{n\to\infty} \sum_{k=-\infty}^{\infty} P\{kl \leq S_n < kl + s\} = \frac{s}{l}$$

if s/l is rational. For arbitrary s, l we can find $S_L < s < S_U$ such that S_L/l and S_U/l are rational. Applying the above

$$\frac{S_L}{l} \leq \varliminf_{n\to\infty} \sum_{k=-\infty}^{\infty} P\{kl \leq S_n < kl + s\}$$

and

$$\varlimsup_{n\to\infty} \sum_{k=-\infty}^{\infty} P\{kl \leq S_n < kl + s\} \leq \frac{S_U}{l}.$$

The result follows by taking S_L and S_U arbitrarily close to s. □

The analogous result in the lattice case is obtained similarly. We also have:

PROPOSITION 2. *In the continuous case under the hypotheses of Proposition* 1 $\lim_{n\to\infty} \sup_{s\in[\varepsilon,\infty)}|\prod_{k=1}^{n} f_k(t)| = 0$ *where* $\varepsilon > 0$ *and* $f_k(t)$ *is the characteristic function of* T_k.

PROOF. Let $\bar{T}_1$ be a random variable independent of the T_k for $k \geq 2$ with characteristic function

$$\bar{f}_1(s) = \begin{cases} 1 - |s|/\varepsilon, & \text{for } |t| \leq \varepsilon, \\ 0 & \text{for } |t| > \varepsilon \end{cases}$$

(see Mineka [5]). Let $\bar{S}_n = \bar{T}_1 + T_2 + \cdots + T_n$. Now $(n + 1, T_n)_{n=0}^{\infty}$ $(T_0 = 0)$ is a semi-Markov chain with state space $\{1, 2, \ldots\}$ and transitions from n to $n + 1$ for all n. It is clear that all bounded, harmonic functions on the underlying chain are constants. Moreover $(n + 1, T_n)_{n=0}^{\infty}$ is strongly mixing. Therefore by Lemma 1(a) in [3], for any harmonic function h on $(n + 1, S_n)_{n=0}^{\infty}$, $h(n, x) = C_h$, a constant, a.e.-m for each n. We may consider $(n + 1, \bar{S}_n)_{n=0}^{\infty}$ to be the same chain with a different initial measure. By hypothesis both S_n and $\bar{S}_n$ are absolutely continuous w.r.t. m so, by Theorem 1 in [3],

$$\lim_{n\to\infty} \left\| P\{S_n \in dx\} - P\{\bar{S}_n \in dx\} \right\| = 0$$

($\| \ \|$ is the total variation on R).

Now $Ee^{is\bar{T}_1} = 0$ for $|s| > \varepsilon$ and hence $Ee^{is\bar{S}_n} = 0$ for $|s| > \varepsilon$. Hence

$$\lim_{n\to\infty} \sup_{s\in[\varepsilon,\infty)} \left| \prod_{k=1}^{n} f_k(s) \right| = \lim_{n\to\infty} \sup_{s\in[\varepsilon,\infty)} \left| Ee^{isS_n} - Ee^{is\bar{S}_n} \right|$$

$$\leqslant \lim_{n\to\infty} \left\| P\{S_n \in dx\} - P\{\bar{S}_n \in dx\} \right\| = 0$$

since $|e^{isx}| \leqslant 1$ for all s. □

Proposition 3. *In the lattice case if Condition* C(a) *is satisfied then*

$$\prod_{k=1}^{\infty} \left[\max_{0 \leqslant n < h} P\{T_k = n \ (\mathrm{mod}\ h)\} \right] = 0$$

for any integer $h \geqslant 2$.

Proof. Suppose there exists an integer $\bar{h}$ such that

$$\prod_{k=1}^{\infty} \left[\max_{0 \leqslant m < \bar{h}} P\{T_k = m \ (\mathrm{mod}\ \bar{h})\} \right] > 0.$$

Then there exists a sequence of integers $\{m_k\}_{k=1}^{\infty}$ such that

$$\prod_{k=1}^{\infty} \left[P\{T_k - m_k = 0 \ (\mathrm{mod}\ \bar{h})\} \right] \geqslant \varepsilon > 0.$$

Let $\bar{T}_k = T_k - m_k$; $\bar{S}_n = \bar{T}_1 + \bar{T}_2 + \cdots + \bar{T}_n$. Hence $P\{\bar{T}_n = 0 \ (\mathrm{mod}\ \bar{h})$ for all $n\} \geqslant \varepsilon > 0$. Hence $P\{\bar{S}_n = 0 \ (\mathrm{mod}\ \bar{h})$ ult$\} \geqslant \varepsilon > 0$. However the tail field of $\{\bar{S}_n\}_{n=1}^{\infty}$ is clearly contained in the tail field of $\{S_n\}_{n=1}^{\infty}$ which is trivial by hypothesis C(a). Hence $P\{\bar{S}_n = 0 \ (\mathrm{mod}\ \bar{h})$ ult$\} = 1$. However

$$\sum_{i=-\infty}^{\infty} |P\{S_n = i\} - P\{S_n = i+1\}|$$

$$\geqslant \left| P\left\{ S_n = \sum_{k=1}^{n} m_k \ (\mathrm{mod}\ \bar{h}) \right\} - P\left\{ S_n = \sum_{k=1}^{n} m_k + 1 \ (\mathrm{mod}\ \bar{h}) \right\} \right|$$

$$= \left| P\{\bar{S}_n = 0 \ (\mathrm{mod}\ \bar{h})\} - P\{\bar{S}_n = 1 \ (\mathrm{mod}\ \bar{h})\} \right|$$

$$\to 1 \quad \text{as } n \to \infty$$

since $\bar{S}_n = 0 \ (\mathrm{mod}\ \bar{h})$ ultimately. But by the argument used in Proposition 2 (or by Theorem 2 in [7])

$$\lim_{n\to\infty} \|P\{S_n \in dx\} - P\{S_n + 1 \in dx\}\| = 0.$$

This gives a contradiction and hence completes the proof. □

We now state:

LEMMA 5. *If Conditions* C(a) *and* S(a)–S(b) *are verified and* T_1 *has a bounded density then*

$$\lim_{n\to\infty}\left|B_n p_n(s) - \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(s-A_n)^2}{2B_n^2}\right\}\right| = 0$$

uniformly in $s \in R$ *where* $p_n(s) = P\{S_n = s\}$ *in the lattice case and* $p_n(s)$ *is the density of* S_n *in the continuous case.*

PROOF. It suffices to check the conditions for Theorem 1 in Mineka [5] in the lattice case and Theorem 1 in Muhin [6] in the continuous case. This is easy using Propositions 1–3. □

In the discrete case Condition S(b) may be weakened to the Lindeberg condition: for each $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{B_n^2}\sum_{k=1}^{n}\int_{|x|>\varepsilon B_n}(x-\mu_k)^2F^k(dx) = 0$$

plus a stronger mixing condition: $\inf_{n\to\infty}\sqrt{Q_n}\,/B_n > 0$. See [1].

LEMMA 6. *If Conditions* S(a)–S(b) *are verified and if* $W_t(n) = P\{S_{n-1} < t < S_n\}$ *then, for all* $\lambda > 0$,

$$\lim_{t\to\infty}\sum_{|(t-A_n)/\sqrt{t}\,|<\lambda}\left|W_t(n) - \frac{\mu_n}{\sqrt{2\pi}}\cdot\frac{1}{B_n}\exp\left\{\frac{-(t-A_n)^2}{2B_n^2}\right\}\right| = 0.$$

PROOF. By Remark 1 in [3] if $\tilde{T}_1$ is a random variable independent of $\{T_n\}_{n=1}^{\infty}$ with bounded derivative then

$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\left|W_t(n) - P\{\tilde{S}_{n-1} < t < \tilde{S}_n\}\right| = 0$$

where $\tilde{S}_n = \tilde{T}_1 + T_2 + \cdots + T_n$. Hence we may assume that T_1 has a bounded derivative. Now take $\varepsilon > 0$. Next $B_n^2/n > (\underline{\sigma})^2$ for all n as $n \to \infty$, so there is a K_2 such that

$$\sqrt{t/B^2_{[t/\bar{\mu}-\lambda/\bar{\mu}\sqrt{t}\,]}} < k \quad \forall t > K_2.$$

By Lemma 5 there exists an N such that for all $n > N$

$$\left|B_n p_n(s) - \frac{1}{\sqrt{2\pi}}\exp\left\{\frac{-(s-A_n)^2}{2B_n^2}\right\}\right| < \frac{\varepsilon}{\lambda k\bar{\mu}} \tag{3}$$

for all s. Hence there is a $K_3 > K_2$ such that the inequality $|(t-A_n)/\sqrt{t}\,| < \lambda$ implies $n > N$ for $t > K_3$. By Lemma 4 there exists a $T > K_3$ such that, for $t > T$,

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} m(ds) \right.$$
$$\left. - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_n^2} \right\} \right| < \frac{\varepsilon}{\lambda k} \tag{4}$$

for n such that $|(t-A_n)/\sqrt{t}\,| < \lambda$. We remark that $W_t(n+1) = \int_0^t (1 - F^{n+1}(t-s)) p_n(s) m(ds)$, so, for $t > T$,

$$\sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \left| W_t(n+1) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_{n+1}}{B_n} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right|$$
$$= \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{B_n} \left| \int_0^t (1 - F^{n+1}(t-s)) B_n p_n(s) m(ds) - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right|$$
$$< \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{B_n} \int_0^t (1 - F^{n+1}(t-s)) \cdot \left| B_n p_n(s) - \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right| m(ds)$$
$$+ \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{B_n} \left| \int_0^t (1 - F^{n+1}(t-s)) \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} m(ds) - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right|$$
$$< \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{B_n} \bar{\mu} \cdot \frac{\varepsilon}{\lambda k \underline{\mu}} \quad \text{(by (3))}$$
$$+ \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{B_n} \frac{\varepsilon}{\lambda k} \quad \text{(by (4))}$$
$$< \frac{2\varepsilon}{\lambda k} \cdot \frac{1}{B_{[t/\bar{\mu} - \lambda\sqrt{t}/\bar{\mu}]}} \cdot \frac{2\lambda}{\underline{\mu}} \sqrt{t},$$

using $n\bar{\mu} \geqslant t - \lambda\sqrt{t}$. But $t > T > K_2$ so $\sqrt{t}\,/B_{[t/\bar{\mu}-\lambda\sqrt{t}/\bar{\mu}]} < k$. Therefore we have

$$\sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \left| W_t(n+1) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_{n+1}}{B_n} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \frac{4\varepsilon}{\underline{\mu}}.$$

Finally

$$\sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \frac{1}{\sqrt{2\pi}} \mu_{n+1} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \left| \frac{1}{B_n} - \frac{1}{B_{n+1}} \right|$$

$$\leqslant 2\frac{\lambda}{\mu}\sqrt{t}\,\frac{\sigma_G}{B_{n-1}B_n}$$

since $\sqrt{B_{n+1}^2 - B_n^2} \geqslant B_{n+1} - B_n$. This tends to 0 as $t \to \infty$. We remark that

$$W_t([\,t/\bar{\mu} + \lambda\sqrt{t}\,/\bar{\mu}]) \to 0 \qquad \text{as } t \to \infty$$

which implies

$$\sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \left| W_t(n) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_n}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right| < 4\varepsilon$$

which yields the result. □

PROPOSITION 1. *If Conditions* S(a)–S(b) *are verified then*

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \left| W_t(n) - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right| = 0.$$

PROOF. This is immediate from Lemmas 1, 2 and 6.

REMARKS. Proposition 1 gives

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} = \lim_{t\to\infty} \sum_{n=1}^{\infty} W_t(n) = 1.$$

LEMMA 7. *If Conditions* S(a)–S(c) *are satisfied,*

$$\lim_{t\to\infty} \sum_{|(t-A_n)/\sqrt{t}\,|<\lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right| = 0.$$

PROOF.

$$\sum_{|(t-A_n)/\sqrt{t}|<\lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right|$$

$$< \sum_{|(t-A_n)/\sqrt{t}|<\lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \cdot \left| 1 - \exp\left\{ \frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} \right\} \right|.$$

However

$$\frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} = -\frac{[A_n - n\mu]}{2B_n} \cdot \left[\frac{2(t-A_n)}{B_n} + \frac{A_n - n\mu}{B_n} \right].$$

But

$$\frac{A_n - n\mu}{B_n} = \frac{A_n - n\mu}{\sqrt{n}} \cdot \frac{\sqrt{n}}{B_n} \to 0 \qquad \text{as } n \to \infty.$$

Also $|(t - A_n)/\sqrt{t}\,| < \lambda$ implies $\sqrt{n} \geqslant ((t - \lambda\sqrt{t}\,)/\bar{\mu})^{1/2}$, hence

$$\left| \frac{t - A_n}{\sqrt{n}} \right| < \frac{\lambda\sqrt{t}}{\sqrt{n}} \leqslant \lambda\sqrt{t} \left(\frac{t - \lambda\sqrt{t}}{\bar{\mu}} \right)^{-1/2} \quad \text{if} \left| \frac{t - A_n}{\sqrt{t}} \right| < \lambda$$

$$< k \quad \text{for all } t \text{, for some constant } k.$$

Therefore $(t - n\mu)/B_n = O(1)$. Hence for any $\varepsilon > 0$, we may choose a $T > 0$ such that for $t > T$

$$\left| \exp\left\{ \frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} \right\} - 1 \right| < \varepsilon$$

for all n such that $|(t - A_n)/\sqrt{t}\,| < \lambda$. Therefore

$$\overline{\lim} \sum_{|(t-A_n)/\sqrt{t}|<\lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right|$$

$$< \varepsilon \varlimsup_{t\to\infty} \sum_{|(t-A_n)/\sqrt{t}|<\lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\}.$$

Now

$$\varlimsup_{t\to\infty} \sum_{|(t-A_n)/\sqrt{t}|<\lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} < 1.$$

The result follows. □

LEMMA 8. *If Conditions* S(a)–S(c) *are satisfied then for any* $\varepsilon > 0$ *there exists a* λ_2 *such that*

$$\lim_{t\to\infty} \sum_{|(t-n\mu)/\sqrt{t}|>\lambda_2} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} < \varepsilon.$$

PROOF. This follows from Lemma 1.

Note that

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(\mu^2/2\sigma^2)s^2} ds = \frac{1}{\mu}.$$

Hence there is a $\bar{\lambda} > 0$ such that

$$\left| \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\sqrt{\mu}} e^{-(\mu^2/2\sigma^2)s^2} ds - \frac{1}{\mu} \right| < \tilde{\varepsilon}.$$

If we break $[-\bar{\lambda}/\sqrt{\mu}, \bar{\lambda}/\sqrt{\mu}]$ at the following points:

$$\left\{ \dots, \frac{-n}{\sqrt{\bar{n}-n}}, \dots, \frac{-1}{\sqrt{\bar{n}-1}}, 0, \frac{1}{\sqrt{\bar{n}+1}}, \dots, \frac{n}{\sqrt{\bar{n}+n}}, \dots \right\},$$

we have

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\sqrt{\mu}} e^{-(\mu^2/2\sigma^2)s^2} ds = \lim_{\bar{n}\to\infty} \sum_{|n|<\bar{\lambda}\sqrt{\bar{n}/\mu}} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \frac{1}{\sqrt{\bar{n}+n}} \exp\left\{ -\frac{1}{2}\frac{\mu^2}{\sigma^2}\left(\frac{n}{\sqrt{\bar{n}+n}}\right)^2 \right\}$$

since $(n+1)/\sqrt{\bar{n}+n+1} - n/\sqrt{\bar{n}+n} \approx 1/\sqrt{\bar{n}+n}$. Let $\mu\bar{n} = t$:

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\mu} e^{-(\mu^2/2\sigma^2)s^2} ds = \lim_{t\to\infty} \sum_{|(t-n\mu)/\sqrt{t}|<\bar{\lambda}} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{n}} \exp\left\{ \frac{-(t-n\mu)^2}{2\sigma^2 n} \right\}.$$

This gives

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{n}} \exp\left\{ \frac{-(t-n\mu)^2}{2\sigma^2 n} \right\} = \frac{1}{\mu}.$$

PROOF OF THEOREM 2.

$$\lim_{t\to\infty}\sum_{n=1}^{\infty}\left|W_t(n)-\frac{\mu_n}{\sqrt{2\pi}}\cdot\frac{1}{B_n}\exp\left\{\frac{-(t-n\mu)^2}{2B_n^2}\right\}\right|$$

$$< \lim_{t\to\infty}\sum_{n=1}^{\infty}\left|W_t(n)-\frac{\mu_n}{\sqrt{2\pi}}\cdot\frac{1}{B_n}\exp\left\{\frac{-(t-A_n)^2}{2B_n^2}\right\}\right|$$

$$+\lim_{t\to\infty}\sum_{n=1}^{\infty}\left|\frac{\mu_n}{\sqrt{2\pi}}\cdot\frac{1}{B_n}\exp\left\{\frac{-(t-A_n)^2}{2B_n^2}\right\}-\frac{\mu_n}{\sqrt{2\pi}}\cdot\frac{1}{B_n}\exp\left\{\frac{-(t-n\mu)}{2B_n^2}\right\}\right|.$$

The first expression goes to 0 as $n\to\infty$ by Proposition 1. The second goes to 0 as $n\to\infty$ using Lemma 7 and Lemmas 1 and 8.

PROOF OF THEOREM 3.

$$\lim_{\substack{t\to\infty\\ t\in R_+}}\left|P\{V_t\in A\}-\sum_{m=1}^{\infty}\frac{\alpha_m}{\mu_m}W_t(m)\right|=0$$

by Theorem 1. By Theorem 2,

$$\lim_{\substack{t\to\infty\\ t\in R_+}}\left|\sum_{m=1}^{\infty}\frac{\alpha_m}{\mu_m}W_t(m)-\sum_{m=1}^{\infty}\frac{\alpha_m}{\sqrt{2\pi}}\cdot\frac{1}{B_m}\exp\left\{\frac{-(t-m\mu)^2}{2B_m^2}\right\}\right|=0.$$

Let $\bar{\mu}_m=(\mu/(\alpha+1))(\alpha_m+1)$; hence $\mu/(\bar{\mu}+1)<\bar{\mu}_m<\mu(\bar{\mu}+1)$. Let $\bar{A}_n=\Sigma_{k=1}^n\bar{\mu}_k$ and $\gamma_n=\Sigma_{k=1}^n\alpha_k$; therefore

$$\frac{\bar{A}_n-n\mu}{\sqrt{n}}=\frac{1}{\sqrt{n}}\left[\frac{\mu}{(\alpha+1)}\gamma_n+\frac{n\mu}{\alpha+1}-n\mu\right]=\frac{\mu}{\alpha+1}\cdot\frac{(\gamma_n-\alpha n)}{\sqrt{n}}\to 0$$

as $n\to\infty$. Next since $\bar{\sigma}^2n>B_n^2\geqslant(\underline{\sigma})^2n$ for all n there is a subsequence n_k such that $\sigma_{n_k}^2\geqslant(\underline{\sigma})^2$ for all k. We may construct an i.i.d. sequence $\{L_{n_k}\}_{k=1}^{\infty}$ of uniformly distributed random variables with mean 0 and variance v where $\sqrt{3v}<\frac{1}{2}\mu/(\bar{\mu}+1)$ (i.e., uniform on $[-\sqrt{3v},\sqrt{3v}]$. Set $L_n=0$ off the subsequence. We may construct a sequence $\{Y_n\}_{n=1}^{\infty}$ of independent random variables, also independent of $\{L_{n_k}\}_{k=1}^{\infty}$, such that Y_n has mean $\bar{\mu}_n$, has support on $(+\frac{1}{2}\mu/(\bar{\mu}+1),\infty)$ and such that the variance of $\bar{T}_n=Y_n+L_n$ is σ_n^2. It is quite easy to check that $\{\bar{T}_n\}_{n=1}^{\infty}$ satisfies Condition C(a) since $\bar{T}_{n_k}$ has a bounded density. By construction $\bar{T}_n$ is positive.

Applying Theorem 2 to $\{\bar{T}_k\}_{k=1}^{\infty}$ we have

$$\lim_{t\to\infty}\sum_{m=1}^{\infty}\frac{\bar{\mu}_m}{\sqrt{2\pi}}\cdot\frac{1}{B_m}\exp\left\{\frac{-(t-m\mu)^2}{2B_m^2}\right\}=1.$$

Hence

$$\lim_{t\to\infty}\sum_{m=1}^{\infty}\frac{\alpha_m+1}{\sqrt{2\pi}}\cdot\frac{1}{B_m}\exp\left\{\frac{-(t-m\mu)^2}{2B_m^2}\right\}=\frac{\alpha+1}{\mu}.$$

But setting $0=\alpha=\alpha_1=\alpha_2=\ldots$ gives

$$\lim_{t\to\infty}\sum_{m=1}^{\infty}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{B_m}\exp\left\{\frac{-(t-m\mu)^2}{2B_m^2}\right\}=\frac{1}{\mu},$$

so that

$$\lim_{t\to\infty}\sum_{m=1}^{\infty}\frac{\alpha_m}{\sqrt{2\pi}}\cdot\frac{1}{B_m}\exp\left\{\frac{-(t-m\mu)^2}{2B_m^2}\right\}=\frac{\alpha}{\mu}.$$

This gives the result. □

COROLLARY 1. *In the lattice case if Conditions* S(a)–S(c) *are verified, then*

$$\lim_{\substack{t\to\infty\\ t\in R_+}} P\{\textit{renewal at } t\}=\frac{1}{\mu}.$$

PROOF. Define $N_t=\inf\{n-1 \mid S_{n-1}\leqslant t<S_n\}$.

The process $V_t=t-S_{N_t}$, $t\in R_+$, is a nonstationary regenerative process which regenerates itself when $V_t=0$. Also $V_t=0$ if and only if $S_{N_t}=t$. The mean time $V_t=0$ per cycle is 1, so applying Theorem 3

$$\lim_{\substack{t\to\infty\\ t\in R_+}} P\{\text{renewal at } t\}=\frac{1}{\mu}. \quad \square$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OTTAWA, ONTARIO, CANADA