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NOTES ON SQUARE-INTEGRABLE COHOMOLOGY SPACES ON CERTAIN FOLIATED MANIFOLDS

BY

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ABSTRACT. We discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Applications to a contact manifold are given.

1. Introduction. On a compact foliated manifold with a bundle-like metric, B. L. Reinhart [9] proved that the cohomology of base-like differential forms is isomorphic to the harmonic space of a certain semidefinite Laplacian. In his paper [4], H. Kitahara discussed the square-integrable basic cohomology spaces on a foliated manifold with a complete bundle-like metric.

In this note, we discuss some square-integrable cohomology spaces on a foliated manifold with one-dimensional foliation whose leaves are compact and with a complete bundle-like metric. Moreover, we give applications to a contact manifold.

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2. Preliminaries. We assume that all objects and maps are of class C^∞. Let M be a connected, orientable, $(n+1)$-dimensional Riemannian foliated manifold with one-dimensional foliation $\mathfrak{F}$ whose leaves are compact. We assume that the Riemannian metric $(\ ,\ )$ on M is a bundle-like metric with respect to $\mathfrak{F}$ (cf. [8]).

Let $\{U; (x, y^1, \ldots, y^n)\}$ denote a flat coordinate neighborhood system with respect to $\mathfrak{F}$, that is, the integral manifolds of $\mathfrak{F}$ are given locally by $y^1 = c^1, \ldots, y^n = c^n$, for some constants $c^1, \ldots, c^n$ (cf. [8]). We may choose, in each flat coordinate neighborhood system $\{U; (x, y^1, \ldots, y^n)\}$, 1-form η and vectors $v_1, \ldots, v_n$ such that $\{\eta, dy^1, \ldots, dy^n\}$ and $\{\partial/\partial x, v_1, \ldots, v_n\}$ are dual bases for the cotangent and tangent spaces respectively at each point in U. Hence

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$$\eta = dx + \sum A_i dy^i,$$
$$v_i = \partial/\partial y^i - A_i \partial/\partial x, \qquad i = 1,2,\ldots,n \tag{1}$$

(cf. [8], [12]). Throughout this note, all local expressions for forms and vectors are taken with respect to these bases.

We may choose, in (1), $A_i = A_i(x,y)$ such that $(\partial/\partial x, v_i) = 0$, $i = 1,2,\ldots,n$, where $y = (y^1,\ldots,y^n)$. Then the metric has the local expression

$$ds^2 = g_{\Delta\Delta}(x,y)\eta\cdot\eta + \sum g_{ij}(y)\, dy^i\cdot dy^j, \tag{2}$$

where $g_{\Delta\Delta}(x,y) = (\partial/\partial x, \partial/\partial x)$ and $g_{ij}(y) = (v_i, v_j)$, since the metric $(\ ,\ )$ is a bundle-like metric with respect to $\mathfrak{F}$ (cf. [8], [12]).

DEFINITION (CF. [8], [12]). A form ϕ on M is called of type $(1,s)$ (resp. $(0,s)$) if it is expressed locally as

$$\phi = \frac{1}{1!s!}\sum \phi_{\Delta i_1\cdots i_s}(x,y)\eta\wedge dy^{i_1}\wedge\cdots\wedge dy^{i_s}$$

(resp. $\phi = (1/0!s!)\sum \phi_{i_1\cdots i_s}(x,y)dy^{i_1}\wedge\cdots\wedge dy^{i_s}$).

Let $\bigwedge^{1,s}(M)$ (resp. $\bigwedge^{0,s}(M)$) be the space of all forms on M of type $(1,s)$ (resp. $(0,s)$). The space $\bigwedge^s(M)$ of all s-forms on M is the direct sum of $\bigwedge^{1,s-1}(M)$ and $\bigwedge^{0,s}(M)$.

The operator d of exterior differentiation is decomposed as $d = d' + d'' + d'''$ where the components are of type $(1,0)$, $(0,1)$ and $(-1,2)$ respectively (cf. [9], [12]). Since the foliation $\mathfrak{F}$ is of one dimension, we notice the following: (i) If $\phi\in\bigwedge^{1,s}(M)$, then $d\phi = d''\phi + d'''\phi$, where $d''\phi\in\bigwedge^{1,s+1}(M)$ and $d'''\phi\in\bigwedge^{0,s+2}(M)$. (ii) If $\phi\in\bigwedge^{0,s}(M)$, then $d\phi = d'\phi + d''\phi$, where $d'\phi\in\bigwedge^{1,s}(M)$ and $d''\phi\in\bigwedge^{0,s+1}(M)$.

Among examples we show the following one for reference below:

EXAMPLE. Let R^3 be a Euclidean space with cartesian coordinates (x,y^1,y^2). We put $\eta = dx + (-y^2)dy^1$, then $\{\eta, dy^1, dy^2\}$ is a base for the cotangent space at each point in R^3. Let ξ be a dual of η. Then R^3 is considered a foliated manifold whose leaves are orbits of vector field ξ. We define a metric ds^2 on R^3 by

$$ds^2 = dx\cdot dx + 2(-y^2)dx\cdot dy^1 + \left(1 + (y^2)^2\right)dy^1\cdot dy^1 + dy^2\cdot dy^2$$

(cf. [10]). Then we have $ds^2 = \eta\cdot\eta + dy^1\cdot dy^1 + dy^2\cdot dy^2$. Thus the metric ds^2 is a complete bundle-like metric with respect to the foliation.

Let $\varphi_m : R^3\to R^3$ be a map defined by

$$\varphi_m(x,y^1,y^2) = (x+m, (-1)^m y^1, (-1)^m y^2),$$

where m is an integer. Define an equivalence relation in R^3 by $(x,y^1,y^2)\sim(\tilde{x},\tilde{y}^1,\tilde{y}^2)$ if there exists an integer m such that $\varphi_m(x,y^1,y^2) = (\tilde{x},\tilde{y}^1,\tilde{y}^2)$. It is trivial that η and ds^2 are invariant by φ_m for any m. Thus the induced η on $M = R^3/\!\sim$ define a foliation on M whose leaves are of one dimension and compact, and the induced ds^2 on M is a complete bundle-like metric with respect to the foliation. We notice that the foliation is not regular (cf. [8]).

3. Spaces $\Delta^{1,*}(M)$ and $\Delta^{0,*}(M)$. Hereafter, we are interested in forms such that

$$\phi = \frac{1}{1!s!} \sum \phi_{\Delta i_1 \cdots i_s}(y) \eta \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_s} \tag{I}$$

and

$$\phi = \frac{1}{0!s!} \sum \phi_{i_1 \cdots i_s}(y)\, dy^{i_1} \wedge \cdots \wedge dy^{i_s}. \tag{II}$$

Let $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) be the subspace of $\bigwedge^{1,s}(M)$ (resp. $\bigwedge^{0,s}(M)$) satisfying (I) (resp. (II)) and $\Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) the subspace of $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) composed of forms with compact supports.

PROPOSITION 3.1 (CF. [9]). *For $\phi \in \bigwedge^{0,s}(M)$, $d'\phi = 0$ if and only if $\phi \in \Delta^{0,s}(M)$.*

A form in $\Delta^{0,s}(M)$ is called a basic or base-like form (cf. [4], [9]).

Restricted to $\Delta^{0,*}(M) = \sum \Delta^{0,s}(M)$, $d''^2 = d^2 = 0$ so we may consider the cohomology of $\Delta^{0,*}(M)$ and d'' (cf. [9], [12]). Next, if we have an assumption that

$$A_i = A_i(y), \qquad i = 1,2, \ldots, n, \tag{III}$$

in (1), then we have $d''^2 = d^2 = 0$ on $\Delta^{1,*}(M) = \sum \Delta^{1,s}(M)$ (cf. [12]). The example in §2 satisfies our assumption (III). Hereafter, we consider the spaces $\Delta^{1,s}(M)$ and $\Delta^{1,*}(M)$ under the assumption (III). Thus we may consider the cohomology of $\Delta^{1,*}(M)$ and d''.

4. Square-integrable basic cohomology spaces $\tilde{H}_2^{0,s}(M)$. This and the next sections are due to H. Kitahara [4], that is, they are the special cases of Kitahara's results. The methods are analogous to those of A. Andreotti and E. Vesentini [1] and K. Okamoto and H. Ozeki [7].

The $*''$-operation in $\Delta^{0,s}(M)$ is defined by

$$*''\phi = \frac{1}{(n-s)!s!} \sum g^{i_1 j_1} \cdots g^{i_s j_s} \delta^{1 \cdots n}_{j_1 \cdots j_s k_1 \cdots k_{n-s}} \times \sqrt{\det(g_{ij})}\, \phi_{i_1 \cdots i_s}\, dy^{k_1} \wedge \cdots \wedge dy^{k_{n-s}}$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) and $\delta^{1 \cdots n}_{j_1 \cdots j_s k_1 \cdots k_{n-s}}$ the Kronecker symbol (cf. [2], [4], [9], [12]). According to B. L. Reinhart [9], we define a pre-Hilbert metric $\langle\, ,\, \rangle_1$ on $\Delta_0^{0,s}(M)$ by

$$\langle \phi, \psi \rangle_1 = \int_M dx \wedge \phi \wedge *''\psi.$$

The differential operator d'' maps $\Delta^{0,s}(M)$ into $\Delta^{0,s+1}(M)$. We define $\tilde{\delta}''$: $\Delta^{0,s}(M) \to \Delta^{0,s-1}(M)$ by

$$\tilde{\delta}''\phi = (-1)^{ns+n+1} *'' d'' *''\phi.$$

Let $\tilde{L}_2^{0,s}(M)$ be the completion of $\Delta_0^{0,s}(M)$ with respect to the inner product $\langle\ ,\ \rangle_1$. We denote by $\tilde{\partial}_0$ the restriction of d'' to $\Delta_0^{0,s}(M)$ and by $\tilde{\theta}_0$ the restriction of $\tilde{\delta}''$ to $\Delta_0^{0,s}(M)$. We define $\tilde{\partial} = (\tilde{\theta}_0)^*$ and $\tilde{\theta} = (\tilde{\partial})^*$, where $(\ )^*$ denotes the adjoint operator of $(\ )$ with respect to the inner product $\langle\ ,\ \rangle_1$. Then $\tilde{\partial}$ (resp. $\tilde{\theta}$) is a closed, densely defined operator of $\tilde{L}_2^{0,s}(M)$ into $\tilde{L}_2^{0,s+1}(M)$ (resp. $\tilde{L}_2^{0,s-1}(M)$). Let $D_{\tilde{\partial}}^{0,s}$ (resp. $D_{\tilde{\theta}}^{0,s}$) be the domain of the operator $\tilde{\partial}$ (resp. $\tilde{\theta}$) in $\tilde{L}_2^{0,s}(M)$ and we put

$$Z_{\tilde{\partial}}^{0,s}(M) = \{\phi \in D_{\tilde{\partial}}^{0,s};\ \tilde{\partial}\phi = 0\},$$

$$Z_{\tilde{\theta}}^{0,s}(M) = \{\phi \in D_{\tilde{\theta}}^{0,s};\ \tilde{\theta}\phi = 0\}.$$

Since $\tilde{\partial}$ and $\tilde{\theta}$ are closed operators, $Z_{\tilde{\partial}}^{0,s}(M)$ and $Z_{\tilde{\theta}}^{0,s}(M)$ are closed in $\tilde{L}_2^{0,s}(M)$. Let $B_{\tilde{\partial}}^{0,s}(M)$ (resp. $B_{\tilde{\theta}}^{0,s}(M)$) be the closure of $\tilde{\partial}(D_{\tilde{\partial}}^{0,s-1})$ (resp. $\tilde{\theta}(D_{\tilde{\theta}}^{0,s+1})$).

Definition (cf. [4]). $\tilde{H}_2^{0,s}(M) = Z_{\tilde{\partial}}^{0,s}(M) \ominus B_{\tilde{\partial}}^{0,s}(M)$ is called the square-integrable basic cohomology space, where $\ominus$ denotes the orthogonal complement of $B_{\tilde{\theta}}^{0,s}(M)$.

Lemma 4.1 (cf. [4]). $\tilde{H}_2^{0,s}(M) = Z_{\tilde{\partial}}^{0,s}(M) \cap Z_{\tilde{\theta}}^{0,s}(M)$.

Since $Z_{\tilde{\partial}}^{0,s}(M)$ and $Z_{\tilde{\theta}}^{0,s}(M)$ are closed in $\tilde{L}_2^{0,s}(M)$, $\tilde{H}_2^{0,s}(M)$ has canonically the structure of a Hilbert space.

The following orthogonal decomposition theorem is proved analogously to L. Hörmander [3]. In fact, we have to notice that $B_{\tilde{\partial}}^{0,s}(M)$ and $B_{\tilde{\theta}}^{0,s}(M)$ are mutually orthogonal and that the intersection of the orthogonal complements of $B_{\tilde{\partial}}^{0,s}(M)$ and $B_{\tilde{\theta}}^{0,s}(M)$ is $\tilde{H}_2^{0,s}(M)$.

Theorem 4.2 (cf. [4]).

$$\tilde{L}_2^{0,s}(M) = \tilde{H}_2^{0,s}(M) \oplus B_{\tilde{\partial}}^{0,s}(M) \oplus B_{\tilde{\theta}}^{0,s}(M).$$

The following diagram is commutative:

$$\begin{array}{ccc}
\Delta_0^{0,s}(M) & \xrightarrow{\ *''\ } & \Delta_0^{0,n-s}(M) \\
\tilde{\theta}_0\downarrow\ \uparrow\tilde{\partial}_0 & & \tilde{\partial}_0\downarrow\ \uparrow\tilde{\theta}_0 \\
\Delta_0^{0,s-1}(M) & \xrightarrow[(-1)^s *'']{} & \Delta_0^{0,n-s+1}(M)
\end{array}$$

Then we have the Dolbeault-Serre type theorem.

Theorem 4.3 (cf. [4]). *If the bundle-like metric on M is complete,* $\tilde{H}_2^{0,s}(M) = \tilde{H}_2^{0,n-s}(M)$ (*isomorphic as Hilbert spaces*).

COROLLARY 4.4 (CF. [4]). *If the bundle-like metric on M is complete and* $\dim \tilde{H}_2^{0,s}(M)$ *is finite, then* $\dim \tilde{H}_2^{0,s}(M) = \dim \tilde{H}_2^{0,n-s}(M)$.

5. $\tilde{\square}$-harmonic forms. In this section, we assume that the bundle-like metric on M is complete.

We consider a function μ on $\mathbf{R}$ (the reals) satisfying

(i) $0 \leq \mu \leq 1$ on $\mathbf{R}$,

(ii) $\mu(t) = 1$ for $t \leq 1$,

(iii) $\mu(t) = 0$ for $t \geq 2$.

It is known that a geodesic orthogonal to a leaf is orthogonal to other leaves (cf. [8]). Let o be a point in M, and we fix the point o. For each point p in M, we denote by $\rho(p)$ the distance between leaves through o and p. Then we put $w_k(p) = \mu(\rho(p)/k)$, $k = 1, 2, 3, \ldots$. We remark that $d'w_k = 0$ and $w_k\phi$ has compact support for each $\phi \in \Delta^{0,s}(M)$. We have that $w_k\phi \in D_{\tilde{\partial}}^{0,s} \cap D_{\tilde{\theta}}^{0,s}$ for any $\phi \in \Delta^{0,s}(M)$ and

$$\tilde{\partial}(w_k\phi) = d''(w_k\phi), \qquad \tilde{\theta}(w_k\phi) = \tilde{\delta}''(w_k\phi). \tag{3}$$

LEMMA 5.1 (CF. [4]). *Under the above notations, there exists a positive number* A, *depending only on* μ, *such that*

$$\|d''w_k \wedge \phi\|^2 \leq (nA^2/k^2)\|\phi\|^2$$

and

$$\|d''w_k \wedge *''\phi\|^2 \leq (nA^2/k^2)\|\phi\|^2$$

for all $\phi \in \Delta^{0,s}(M)$, *where* $\|\phi\|^2 = \langle\phi, \phi\rangle_1$.

In order to prove this lemma, we have to notice that the function ρ is a locally Lipschitz function and, at points where the derivatives exist, it holds $\sum g^{ij} v_i(\rho) v_j(\rho) \leq n$. Then we have

$$|d''w_k|^2 = \sum g^{ij} v_i(w_k) v_j(w_k) \leq nA^2/k^2,$$

where A is a positive number depending only on $\sup|d\mu/dt|$.

Put

$$N_{d''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M);\ d''\phi = 0\},$$
$$N_{\delta''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M);\ \delta''\phi = 0\}.$$

Then we have

PROPOSITION 5.2 (CF. [4]). *If the bundle-like metric on M is complete, then*

$$N_{d''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M) \subset Z_{\tilde{\partial}}^{0,s}(M),$$
$$N_{\delta''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M) \subset Z_{\tilde{\theta}}^{0,s}(M).$$

PROOF. Let ϕ be in $N_{d''}^{0,s}(M) \cap \tilde{L}_2^{0,s}(M)$. By (3), we have

$$\tilde{\partial}(w_k\phi) = d''(w_k\phi) = d''w_k \wedge \phi + w_k d''\phi = d''w_k \wedge \phi.$$

Hence, from Lemma 5.1, we have $\|\tilde{\partial}(w_k\phi)\|^2 < (nA^2/k^2)\|\phi\|^2$. Putting $\phi_k = w_k\phi$, we have

$$\tilde{\partial}\phi_k \underset{\text{(strong)}}{\rightarrow} 0 \qquad (k \to \infty),$$

where "$\rightarrow_{\text{(strong)}}$" means "converges strongly to". On the other hand, $\phi_k \rightarrow_{\text{(strong)}} \phi$ $(k \to \infty)$. Since $\tilde{\partial}$ is a closed operator, ϕ is in $D_{\tilde{\partial}}^{0,s}$ and $\tilde{\partial}\phi = 0$. This proves $\phi \in Z_{\tilde{\partial}}^{0,s}(M)$. In the same way, we may prove the second part.

DEFINITION (CF. [4]). The Laplacian $\tilde{\square}$ acting on $\Delta^{0,*}(M)$ is defined by $\tilde{\square} = d''\tilde{\delta}'' + \tilde{\delta}''d''$.

Let $B(k)$ be an open tube of radius k of the leaf through the fixed point o in M and $\Delta_{B(k)}^{0,s}(M)$ the space of all forms of type $(0, s)$ with compact support contained in $B(k)$. For $\phi, \psi \in \Delta_{B(k)}^{0,s}(M)$, we put $\langle\phi, \psi\rangle_{B(k)} = \langle\phi, \psi\rangle_1$. For any $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$, we have

$$\langle d''\phi, d''\alpha\rangle_{B(k)} + \langle\tilde{\delta}''\phi, \tilde{\delta}''\alpha\rangle_{B(k)} = \langle\tilde{\square}\phi, \alpha\rangle_{B(k)} \tag{4}$$

for all $\alpha \in \Delta_{B(k)}^{0,s}(M)$. Putting $\alpha = w_k^2\phi$, we have

$$d''\alpha = w_k^2 d''\phi + 2w_k d''w_k \wedge \phi,$$

$$\tilde{\delta}''\alpha = w_k^2\tilde{\delta}''\phi + (-1)^{ns+n+1} *''(2w_k d''w_k \wedge *''\phi).$$

Substituting in (4), we have

$$\begin{aligned} \|w_k d''\phi\|_{B(k)}^2 + \|w_k\tilde{\delta}''\phi\|_{B(k)}^2 &< |\langle\tilde{\square}\phi, w_k^2\phi\rangle_{B(k)}| + |\langle d''\phi, 2w_k d''w_k \wedge \phi\rangle_{B(k)}| \\ &\quad + |\langle\tilde{\delta}''\phi, *''(2w_k d''w_k \wedge *''\phi)\rangle_{B(k)}|. \end{aligned} \tag{5}$$

On the other hand, the Schwarz inequality gives the following

$$|\langle d''\phi, 2w_k d''w_k \wedge \phi\rangle_{B(k)}| < \tfrac{1}{2}\left(\|w_k d''\phi\|_{B(k)}^2 + 4\|d''w_k \wedge \phi\|_{B(k)}^2\right),$$

$$|\langle\tilde{\delta}''\phi, *''(2w_k d''w_k \wedge *''\phi)\rangle_{B(k)}| < \tfrac{1}{2}\left(\|w_k\tilde{\delta}''\phi\|_{B(k)}^2 + 4\|d''w_k \wedge *''\phi\|_{B(k)}^2\right)$$

and

$$|\langle\tilde{\square}\phi, w_k^2\phi\rangle_{B(k)}| < \tfrac{1}{2}\left(\frac{1}{\sigma}\|w_k\phi\|_{B(k)}^2 + \sigma\|\tilde{\square}\phi\|_{B(k)}^2\right)$$

for every $\sigma > 0$.

Substituting in (5),

$$\|w_k d''\phi\|_{B(k)}^2 + \|w_k\tilde{\delta}''\phi\|_{B(k)}^2 < \sigma\|\tilde{\square}\phi\|_{B(k)}^2 + \left(\frac{1}{\sigma} + \frac{8nA^2}{k^2}\right)\|\phi\|_{B(k)}^2.$$

Letting $k \to \infty$, we have

$$\|d''\phi\|^2 + \|\tilde{\delta}''\phi\|^2 < \sigma\|\tilde{\square}\phi\|^2 + \frac{1}{\sigma}\|\phi\|^2$$

for every $\sigma > 0$. In particular, setting $\tilde{\Box}\phi = 0$ and letting $\sigma \to \infty$, we have

LEMMA 5.3 (CF. [4]). *Let the bundle-like metric on M be complete. If* $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ *such that* $\tilde{\Box}\phi = 0$, *then* $d''\phi = 0$ *and* $\tilde{\delta}''\phi = 0$, *i.e.* $\phi \in N_{d''}^{0,s}(M) \cap N_{\tilde{\delta}''}^{0,s}(M)$.

From Proposition 5.2 and Lemma 5.3, we have the following theorem.

THEOREM 5.4 (CF. [4]). *Let the bundle-like metric on M be complete. If* $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ *such that* $\tilde{\Box}\phi = 0$, *then* $\phi \in \tilde{H}_2^{0,s}(M)$.

6. Square-integrable cohomology spaces $H_2^{0,s}(M)$ and $H_2^{1,s}(M)$. In this section, we set situations under the assumptions

$$A_i = A_i(y) \quad \text{and} \quad g_{\Delta\Delta}(x, y) = 1 \tag{IV}$$

in (1) and (2). The manifold given in the example in §2 satisfies (IV).

We notice that the volume element of M is

$$\begin{aligned} dV_M &= \sqrt{\det\begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}}\ \eta \wedge dy^1 \wedge \cdots \wedge dy^n \\ &= \sqrt{\det(g_{ij})}\ \eta \wedge dy^1 \wedge \cdots \wedge dy^n \qquad \text{(from (IV))} \\ &= \sqrt{\det(g_{ij})}\ dx \wedge dy^1 \wedge \cdots \wedge dy^n \qquad \text{(from (1)).} \end{aligned}$$

The $*$-operation on $\Delta^{1,s}(M)$ or $\Delta^{0,s}(M)$ is defined as follows. For $\phi \in \Delta^{1,s}(M)$ and $\psi \in \Delta^{0,s}(M)$,

$$\begin{aligned} *\phi &= \frac{(-1)^{(1-1)s}}{(1-1)!\,(n-s)!1!s!} \sum g^{\Delta\Delta} g^{i_1 j_1} \cdots g^{i_s j_s} \\ &\qquad \times \delta^{\Delta 1 \cdots n}_{\Delta j_1 \cdots j_s k_1 \cdots k_{n-s}} \sqrt{\det\begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}}\ \phi_{\Delta i_1 \cdots i_s} dy^{k_1} \wedge \cdots \wedge dy^{k_{n-s}} \\ &= \frac{1}{(n-s)!s!} \sum g^{i_1 j_1} \cdots g^{i_s j_s} \delta^{1 \cdots n}_{j_1 \cdots j_s k_1 \cdots k_{n-s}} \\ &\qquad \times \sqrt{\det(g_{ij})}\ \phi_{\Delta i_1 \cdots i_s} dy^{k_1} \wedge \cdots \wedge dy^{k_{n-s}}, \end{aligned}$$

$$\begin{aligned} *\psi &= \frac{(-1)^{(1-0)s}}{(1-0)!\,(n-s)!0!s!} \sum g^{i_1 j_1} \cdots g^{i_s j_s} \delta^{1 \cdots n}_{j_1 \cdots j_s k_1 \cdots k_{n-s}} \\ &\qquad \times \sqrt{\det\begin{pmatrix} g_{\Delta\Delta} & 0 \\ 0 & g_{ij} \end{pmatrix}}\ \psi_{i_1 \cdots i_s} \eta \wedge dy^{k_1} \wedge \cdots \wedge dy^{k_{n-s}} \\ &= \frac{(-1)^s}{(n-s)!s!} \sum g^{i_1 j_1} \cdots g^{i_s j_s} \delta^{1 \cdots n}_{j_1 \cdots j_s k_1 \cdots k_{n-s}} \\ &\qquad \times \sqrt{\det(g_{ij})}\ \psi_{i_1 \cdots i_s} \eta \wedge dy^{k_1} \wedge \cdots \wedge dy^{k_{n-s}} \end{aligned}$$

(cf. [2], [12]). The operator $*$ maps $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) into $\Delta^{0,n-s}(M)$ (resp. $\Delta^{1,n-s}(M)$), since (IV) holds.

We define a pre-Hilbert metric on $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$) by $\langle \phi, \psi \rangle = \int_M \phi \wedge *\psi$. The differential operator d'' maps $\Delta^{1,s}(M)$ (resp. $\Delta^{0,s}(M)$) into $\Delta^{1,s+1}(M)$ (resp. $\Delta^{0,s+1}(M)$). We define δ'': $\Delta^{1,s}(M) \to \Delta^{1,s-1}(M)$ (or $\Delta^{0,s}(M) \to \Delta^{0,s-1}(M)$) by

$$\begin{aligned} \delta''\phi &= (-1)^{(n+1)(s+1)+(n+1)+1} * d'' * \phi \\ &= (-1)^{ns+s+1} * d'' * \phi, \\ \delta''\psi &= (-1)^{(n+1)s+(n+1)+1} * d'' * \psi \\ &= (-1)^{ns+s+n} * d'' * \psi \end{aligned}$$

for $\phi \in \Delta^{1,s}(M)$ and $\psi \in \Delta^{0,s}(M)$. Then we have $\langle d''\phi, \psi \rangle = \langle \phi, \delta''\psi \rangle$ for $\phi \in \Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) and $\psi \in \Delta_0^{1,s+1}(M)$ (resp. $\Delta_0^{0,s+1}(M)$).

Let $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) be the completion of $\Delta_0^{1,s}(M)$ (resp. $\Delta_0^{0,s}(M)$) with respect to the inner product $\langle\ ,\ \rangle$. We denote by ∂_0 the restriction of d'' to $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$) and by θ_0 the restriction of δ'' to $\Delta_0^{1,s}(M)$ (or $\Delta_0^{0,s}(M)$). Define $\partial = (\theta_0)^*$ and $\theta = (\partial)^*$ where $(\ )^*$ denotes the adjoint operator of $(\ )$ with respect to the inner product $\langle\ ,\ \rangle$. Then ∂ is a closed, densely defined operator of $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) into $L_2^{1,s+1}(M)$ (resp. $L_2^{0,s+1}(M)$), and θ is a closed, densely defined operator of $L_2^{1,s}(M)$ (resp. $L_2^{0,s}(M)$) into $L_2^{1,s-1}(M)$ (resp. $L_2^{0,s-1}(M)$).

The following objects are defined by the same ways as in §4:

$$\begin{array}{llll} D_\partial^{1,s}, & D_\partial^{0,s}, & D_\theta^{1,s}, & D_\theta^{0,s}, \\ Z_\partial^{1,s}(M), & Z_\partial^{0,s}(M), & Z_\theta^{1,s}(M), & Z_\theta^{0,s}(M), \\ B_\partial^{1,s}(M), & B_\partial^{0,s}(M), & B_\theta^{1,s}(M), & B_\theta^{0,s}(M). \end{array}$$

Then

DEFINITION. $H_2^{1,s}(M) = Z_\partial^{1,s}(M) \ominus B_\partial^{1,s}(M)$ and $H_2^{0,s}(M) = Z_\partial^{0,s}(M) \ominus B_\partial^{0,s}(M)$.

By the same ways as in §4, we have

LEMMA 6.1. *Under the assumption* (IV),

$$H_2^{1,s}(M) = Z_\partial^{1,s}(M) \cap Z_\theta^{1,s}(M)$$

and

$$H_2^{0,s}(M) = Z_\partial^{0,s}(M) \cap Z_\theta^{0,s}(M).$$

THEOREM 6.2. *Under the assumption* (IV),

$$L_2^{1,s}(M) = H_2^{1,s}(M) \oplus B_\partial^{1,s}(M) \oplus B_\theta^{1,s}(M)$$

and

$$L_2^{0,s}(M) = H_2^{0,s}(M) \oplus B_\partial^{0,s}(M) \oplus B_\theta^{0,s}(M).$$

THEOREM 6.3. *Under the assumption* (IV), *if the bundle-like metric on M is complete, then $H_2^{0,s}(M) = H_2^{1,n-s}(M)$ (isomorphic as Hilbert spaces).*

In order to prove Theorem 6.3, we have to notice that $\langle \phi, \psi \rangle = \langle {*\phi}, {*\psi} \rangle$ for $\phi,\psi \in \Delta_0^{0,s}(M)$.

COROLLARY 6.4. *Under the assumption* (IV), *if the bundle-like metric on M is complete and* $\dim H_2^{0,s}(M)$ *is finite, then* $\dim H_2^{0,s}(M) = \dim H_2^{1,n-s}(M)$.

Now, we have $\langle \phi, \psi \rangle = \langle \eta \wedge \phi, \eta \wedge \psi \rangle$ for $\phi,\psi \in \Delta_0^{0,s}(M)$. Let ξ denote the dual to η and i_ξ the interior product by ξ operator. Then we have $i_\xi \phi \in \Delta^{0,s}(M)$ and $\eta \wedge i_\xi \phi = \phi$ for $\phi \in \Delta^{1,s}(M)$. The following diagram is commutative.

$$\begin{array}{ccc} \Delta_0^{0,s}(M) & \xrightarrow{e(\eta)} & \Delta_0^{1,s}(M) \\ \vartheta_0 \downarrow \uparrow \partial_0 & & \partial_0 \uparrow \downarrow \vartheta_0 \\ \Delta_0^{0,s-1}(M) & \xrightarrow{(-1)e(\eta)} & \Delta_0^{1,s-1}(M) \end{array}$$

where $e(\eta)$ denotes the exterior product by η operator. Thus we have

THEOREM 6.5. *Under the assumption* (IV), *if the bundle-like metric on M is complete, then $H_2^{0,s}(M) = H_2^{1,s}(M)$ (isomorphic as Hilbert spaces).*

From Theorems 6.3 and 6.5, we have

THEOREM 6.6. *Under the assumption* (IV), *if the bundle-like metric on M is complete, then $H_2^{1,s}(M) = H_2^{1,n-s}(M)$ and $H_2^{0,s}(M) = H_2^{0,n-s}(M)$ (isomorphic as Hilbert spaces).*

Next, from (IV), it is easy to see that $\langle \phi, \psi \rangle_1 = \langle \phi, \psi \rangle$ for $\phi,\psi \in \Delta_0^{0,s}(M)$. The following diagram is commutative.

$$\begin{array}{ccc} \Delta_0^{0,s}(M) & \xrightarrow{I} & \Delta_0^{0,s}(M) \\ \tilde{\vartheta}_0 \downarrow \uparrow \tilde{\partial}_0 & & \partial_0 \uparrow \downarrow \vartheta_0 \\ \Delta_0^{0,s-1}(M) & \xrightarrow[I]{} & \Delta_0^{0,s-1}(M) \end{array}$$

where I denotes the identity map. Thus we have the following theorem.

THEOREM 6.7. *Under the assumption* (IV), *if the bundle-like metric on M is complete, then $\tilde{H}_2^{0,s}(M) = H_2^{0,s}(M)$ (isomorphic as Hilbert spaces).*

7. $\Box$-harmonic forms. In this section, we assume that the assumption (IV) holds and that the bundle-like metric on M is complete. We put

$$N_{d''}^{1,s}(M) = \{\phi \in \Delta^{1,s}(M);\ d''\phi = 0\},$$

$$N_{\delta''}^{1,s}(M) = \{\phi \in \Delta^{1,s}(M);\ \delta''\phi = 0\},$$

$$N_{d''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M);\ d''\phi = 0\},$$

$$N_{\delta''}^{0,s}(M) = \{\phi \in \Delta^{0,s}(M);\ \delta''\phi = 0\}.$$

Then, by the same ways as in §5, we have

PROPOSITION 7.1. *Let the assumption* (IV) *hold and the bundle-like metric on* M *be complete. Then*

$$N_{d''}^{1,s}(M) \cap L_2^{1,s}(M) \subset Z_{\partial}^{1,s}(M), \qquad N_{\delta''}^{1,s}(M) \cap L_2^{1,s}(M) \subset Z_{\theta}^{1,s}(M),$$
$$N_{d''}^{0,s}(M) \cap L_2^{0,s}(M) \subset Z_{\partial}^{0,s}(M), \qquad N_{\delta''}^{0,s}(M) \cap L_2^{0,s}(M) \subset Z_{\theta}^{0,s}(M).$$

DEFINITION. The Laplacian acting on $\Delta^{1,*}(M)$ (or $\Delta^{0,*}(M)$) is defined by $\square = d''\delta'' + \delta''d''$.

By the same ways as in §5, we have

LEMMA 7.2. *Let the assumption* (IV) *hold and the bundle-like metric on* M *be complete. If* $\phi \in L_2^{1,s}(M) \cap \Delta^{1,s}(M)$ (*resp.* $L_2^{0,s}(M) \cap \Delta^{0,s}(M)$) *such that* $\square\phi = 0$, *then* $d''\phi = 0$ *and* $\delta''\phi = 0$, *i.e.* $\phi \in N_{d''}^{0,s}(M) \cap N_{\delta''}^{1,s}(M)$ (*resp.* $N_{d''}^{0,s}(M) \cap N_{\delta''}^{0,s}(M)$).

From Proposition 7.1 and Lemma 7.2, we have

THEOREM 7.3. *Let the assumption* (IV) *hold and the bundle-like metric on* M *be complete. If* $\phi \in L_2^{1,s}(M) \cap \Delta^{1,s}(M)$ (*resp.* $L_2^{0,s}(M) \cap \Delta^{0,s}(M)$) *such that* $\square\phi = 0$, *then* $\phi \in H_2^{1,s}(M)$ (*resp.* $H_2^{0,s}(M)$).

From Theorems 5.4 and 6.7, we have

THEOREM 7.4. *Let the assumption* (IV) *hold and the bundle-like metric on* M *be complete. If* $\phi \in \tilde{L}_2^{0,s}(M) \cap \Delta^{0,s}(M)$ *such that* $\tilde{\square}\phi = 0$, *then* $\phi \in H_2^{0,s}(M)$.

From Theorems 6.7 and 7.3, we have

THEOREM 7.5. *Let the assumption* (IV) *hold and the bundle-like metric on* M *be complete. If* $\phi \in L_2^{0,s}(M) \cap \Delta^{0,s}(M)$ *such that* $\square\phi = 0$, *then* $\phi \in \tilde{H}_2^{0,s}(M)$.

REMARK. I. Vaisman **[12]**, **[13]** already noticed that on a compact orientable Riemannian foliated manifold M, the space $\mathcal{H}^{r,s}(M)$ of foliated harmonic forms is a subspace of the de Rham cohomology space $H^{r,s}(M)$.

REMARK. For the relations between certain cohomology spaces and the existence of bundle-like metrics, see H. Kitahara and S. Yorozu **[5]**.

8. Applications to a contact manifold. First, we cite the definition of the contact manifold. A 1-form η on a connected $(2n + 1)$-dimensional manifold is called a contact form if $\eta \wedge (d\eta)^n \neq 0$ at each point in the manifold (cf. **[10]**). A connected $(2n + 1)$-dimensional manifold with a contact form is called a contact manifold. On a contact manifold with a contact form η, there exists a global vector field ξ such that $\eta(\xi) = 1$ and $i_\xi d\eta = 0$ (cf. **[10]**). A connected paracompact contact manifold with a contact form η has a Riemannian metric $(\, , \,)$ such that

$$\eta(X) = (X, \xi) \tag{6}$$

for any vector field X (cf. **[10]**). In fact, let $(\, , \,)'$ be an arbitrary Riemannian metric, and we define

$$(X, Y) = (X - \eta(X)\xi, Y - \eta(Y)\xi)' + \eta(X)\cdot\eta(Y)$$

for any vector fields X and Y. Such a metric $(\, , \,)$ satisfies (6).

Now, let N be a connected $(2n + 1)$-dimensional contact manifold with a contact form η and a Riemannian metric $(\, , \,)$ satisfying (6). We assume that ξ is a Killing vector field on N with respect to the metric $(\, , \,)$ and that the orbits of ξ are compact. An example of such a manifold N is the manifold given in the example in §2.

We define the operators δ, $e(\eta)$, i_ξ, L and Λ on $\bigwedge^s(N)$ as follows:

$$\delta\phi = (-1)^s * d * \phi, \qquad e(\eta)\phi = \eta \wedge \phi,$$
$$i_\xi\phi = (-1)^{s-1} * e(\eta) * \phi,$$
$$L\phi = d\eta \wedge \phi, \qquad \Lambda\phi = * L * \phi$$

(cf. **[2]**, **[6]**, **[11]**).

DEFINITION. A form ϕ in $\bigwedge^s(N)$ is called a C-harmonic form (resp. C^*-harmonic form) if $i_\xi\phi = 0$, $d\phi = 0$ and $\delta\phi = e(\eta)\Lambda\phi$ (resp. $e(\eta)\phi = 0$, $d\phi = i_\xi L\phi$ and $\delta\phi = 0$).

REMARK. The notion of C-harmonic forms was introduced by S. Tachibana **[11]**, and Y. Ogawa **[6]** gave the definition of C^*-harmonic forms. They discussed it on compact normal contact metric manifolds. A normal contact metric manifold is a so-called Sasakian manifold (for the definition, see **[6]**, **[10]**, **[11]**).

For each point in N, there exists a local coordinate neighborhood system $\{U; (x, y^1, \ldots, y^n, y^{n+1}, \ldots, y^{2n})\}$ such that

$$\eta = dx + \sum(-y^{n+i})\, dy^i \qquad (i = 1, 2, \ldots, n)$$

and the orbits of ξ are given locally by

$$y^1 = c^1, \ldots, y^n = c^n, y^{n+1} = c^{n+1}, \ldots, y^{2n} = c^{2n}$$

for the same constants $c^1, \ldots, c^n, c^{n+1}, \ldots, c^{2n}$ (cf. **[10]**). $\{\eta, dy^1, \ldots, dy^n, dy^{n+1}, \ldots, dy^{2n}\}$ and $\{\partial/\partial x, v_1, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ are dual bases for the cotangent and tangent spaces respectively at each point in U, where

$$v_i = \partial/\partial y^i + (y^{n+i})\partial/\partial x \quad \text{and} \quad v_{n+i} = \partial/\partial y^{n+i}.$$

We may consider N as a foliated manifold whose leaves are orbits of ξ. From (6), we have

$$(v_i, \xi) = \eta(v_i) = 0,$$
$$(v_{n+i}, \xi) = \eta(v_{n+i}) = 0.$$

Then, since ξ is a Killing vector field on N, the metric (,) on N is a bundle-like metric with respect to the foliation, that is, the local expression of the metric (,) in U is

$$ds^2 = \eta \cdot \eta + \sum g_{AB}\, dy^A \cdot dy^B$$

where $A, B = 1, 2, \ldots, 2n$. Thus the contact manifold N is a Riemannian foliated manifold with one-dimensional foliation $\mathfrak{F}$ whose leaves are compact and the Riemannian metric (,) on N is a bundle-like metric with respect to $\mathfrak{F}$. Moreover, the assumption (IV) in §6 is satisfied. Therefore, we may apply the discussions of above sections to the contact manifold N.

In order to obtain the applications to N, we have to prepare the decomposition of the operator δ. We have the decomposition of the operator d:

$$d = d' + d'' + d'''$$

(cf. §2). Then, according to I. Vaisman [12], we define the operators δ', δ'' and δ''' as follows:

$$\delta'\phi = (-1)^{r+s} * d' * \phi,$$
$$\delta''\phi = (-1)^{r+s} * d'' * \phi,$$
$$\delta'''\phi = (-1)^{r+s} * d''' * \phi,$$

where $\phi \in \bigwedge^{r,s}(N)$, $r = 0$ or 1. Then we have the decomposition of the operator δ:

$$\delta = \delta' + \delta'' + \delta'''.$$

We notice the following: (i) If $\phi \in \bigwedge^{1,s}(N)$, then $\delta\phi = \delta'\phi + \delta''\phi$, where $\delta'\phi \in \bigwedge^{0,s}(N)$ and $\delta''\phi \in \bigwedge^{1,s-1}(N)$. (ii) If $\phi \in \bigwedge^{0,s}(N)$, then $\delta\phi = \delta''\phi + \delta'''\phi$, where $\delta''\phi \in \bigwedge^{0,s-1}(N)$ and $\delta'''\phi \in \bigwedge^{1,s-2}(N)$.

We have easily the following lemma.

LEMMA 8.1. *For $\phi \in \Delta^{1,s}(N)$ and $\psi \in \Delta^{0,s}(N)$,*

$$d'''\phi = i_\xi L\phi, \qquad \delta'''\psi = e(\eta)\Lambda\psi.$$

From Theorem 7.4 and Lemma 8.1, we have

THEOREM 8.2. *Let the metric (,) on N be complete. If $\phi \in \tilde{L}_2^{0,s}(N) \cap \Delta^{0,s}(N)$ such that $\tilde{\square}\phi = 0$, then ϕ is a C-harmonic form.*

From Theorem 7.3 and Lemma 8.1, we have

THEOREM 8.3. *Let the metric (,) on N be complete. If $\phi \in L_2^{1,s}(N) \cap \Delta^{1,s}(N)$ such that $\square\phi = 0$, then ϕ is a C^*-harmonic form.*

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