ON THE LITTLEWOOD-PALEY THEORY FOR
MIXED NORM SPACES

BY

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ABSTRACT. An inequality of Littlewood-Paley type is proved for the mixed
norm spaces $L_p(l_r)$, $1 < p, r < \infty$, on the interval $[0, 1]$. This result makes
use of recent work by C. Fefferman and A. Cordoba on the boundedness of
singular integrals on these spaces. As an application of this inequality,
boundedness of the lacunary maximal partial sum operator for Walsh-Four-
ier series on $L_p(l_r)$ is established. This result can be viewed as an extension
of a similar result for the Hardy-Littlewood maximal function due to C.
Fefferman and E. M. Stein.

Introduction. Recently various authors (see [1], [3], [4], and [11], for exam-
ple) have established the boundedness of certain (sub-)linear operators on
mixed norm spaces. In this paper we prove similar results for a version of the
Littlewood-Paley function and for the lacunary maximal partial sum operator
for Walsh-Fourier series. A particular case of the latter result is the Feffer-
man-Stein inequality proved in [3]. A by-product of our proofs is an extension
of Khinchine's inequality to mixed norm spaces.

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for suggesting certain simplifications.

I. In this section we extend an inequality of Khinchine [12] to mixed norm
spaces.

LEMMA 1. *Let* $\{f_k\}_{k=1}^{\infty}$ *denote a sequence of functions on* $[0, 1]$, *each being
given by a Rademacher series* $\sum_{\nu=1}^{\infty} a_\nu^k r_\nu(t)$. *Let* $1 < p, r < \infty$. *Then there
exist constants* $C_{p,r}$ *and* $B_{p,r}$ *depending only on* p *and* r *such that*

$$\left(\int_0^1 \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{p/r} dx \right)^{1/p} \leqslant B_{p,r} \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=1}^{\infty} |a_\nu^k|^2 \right)^{r/2} \right)^{1/r}$$

and

$$\left(\int_0^1 \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{p/r} dx \right)^{1/p} \geqslant C_{p,r} \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=1}^{\infty} |a_\nu^k|^2 \right)^{r/2} \right)^{1/r}.$$

PROOF. For the first inequality, we consider the cases $p < r$ and $p > r$ separately. For $p < r$ it follows by Hölder's inequality and Khinchine's inequality for a single function that

$$\left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r\right)^{p/r} dx\right)^{1/p} < \left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r dx\right)\right)^{1/r}$$

$$= \left(\sum_{k=1}^\infty \int_0^1 |f_k(x)|^r dx\right)^{1/r} < C_r \left(\sum_{k=1}^\infty \left(\sum_{\nu=1}^\infty |a_\nu^k|^2\right)^{r/2}\right)^{1/r}.$$

For $p > r$, it follows by Minkowski's inequality and Khinchine's inequality for a single function that

$$\left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r\right)^{p/r} dx\right)^{1/p} < \left(\sum_{k=1}^\infty \left(\int_0^1 |f_k(x)|^p dx\right)^{r/p}\right)^{1/r}$$

$$< C_p \left(\sum_{k=1}^\infty \left(\sum_{\nu=1}^\infty |a_\nu^k|^2\right)^{r/2}\right)^{1/r}.$$

This proves the first inequality. To prove the second inequality, we consider the cases $p > r$ and $p < r$ separately. For $p > r$ it follows by Khinchine's inequality for a single function and Hölder's inequality that

$$\left(\sum_{k=1}^\infty \left(\sum_{\nu=1}^\infty |a_\nu^k|^2\right)^{r/2}\right)^{1/r} < C_r \left(\sum_{k=1}^\infty \int_0^1 |f_k(x)|^r\right)^{1/r}$$

$$= C_r \left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r\right)\right)^{1/r} < C_r \left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r\right)^{p/r}\right)^{1/p}.$$

Thus it remains to prove the second inequality only in the case $p < r$. To do this we use duality. Let $\{b_\nu^k\}$ denote an element of norm 1 in $l_r(l_2)$ where $1/r + 1/r' = 1$. Then let $\{a_\nu^k\}_{\nu=1}^N$, $1 < k < K$, be any NK numbers. Consider the corresponding b_k^ν and let $f_1, \ldots, f_K$, $g_1, \ldots, g_K$ be defined by $f_k = \sum_{\nu=1}^N a_\nu^k r_\nu(t)$ and $g_k = \sum_{\nu=1}^N b_\nu^k r_\nu(t)$. Then by Hölder's inequality with (p, r) (p', r'), and the first inequality of this lemma, we have

$$|\langle \{a_\nu^k\}, \{b_\nu^k\}\rangle| = \left|\int_0^1 \sum_{k=1}^K f_k(x) g_k(x)\, dx\right|$$

$$< \left(\int_0^1 \left(\sum_{k=1}^K |f_k(x)|^r\right)^{p/r} dx\right)^{1/p} \left(\int_0^1 \left(\sum_{k=1}^K |g_k(x)|^r\right)^{p'/r} dx\right)^{1/r}$$

$$< C_{p,r} \left(\int_0^1 \left(\sum_{k=1}^K |f_k(x)|^r\right)^{p/r} dx\right)^{1/p} \left(\sum_{k=1}^K \left(\sum_{\nu=1}^N |b_\nu^k|^2\right)^r\right)^{1/r}$$

$$< C_{p,r} \left(\int_0^1 \left(\sum_{k=1}^K |f_k(x)|^r\right)^{p/r} dx\right)^{1/p}.$$

Taking the supremum over the unit ball of $l_{r'}(l_2)$ we obtain

$$\left(\sum_{k=1}^{K} \left(\sum_{\nu=1}^{N} |a_\nu^k|^2 \right)^{r/2} \right)^{1/r} \leqslant C_{p,r} \left(\int_0^1 \left(\sum_{k=1}^{K} |f_k(x)|^r \right)^{p/r} dx \right)^{1/p}.$$

Letting $N \to \infty$ and then $K \to \infty$ the lemma follows.

II. In this section we prove some basic estimates on certain multipliers. We note that partial results similar to these are proved in [11]. The proofs in [11] are for martingales and are based on a decomposition lemma of Gundy. We are able to give a short proof in view of recent work by A. Cordoba and C. Fefferman [2]. $S_{2^r} f$ will denote the 2^rth Walsh series partial sum.

LEMMA 2. *Let* $a = \{a_\nu\}_{\nu=1}^\infty$ *denote a sequence of real numbers with* $\|\{a_\nu\}\|_\infty \leqslant 1$. *Let* $T_a f = \sum_{\nu=1}^\infty a_\nu (S_{2^\nu} - S_{2^{\nu-1}})f$. *Let* $T_a\{f_k\} = \{T_a f_k\}_{k \geqslant 1}$. *Then for* $1 < p, r < \infty$, *there exists a constant* $C_{p,r}$ *such that*

$$\left(\int_0^1 \left(\sum_{k=1}^\infty |T_a f_k(x)|^r \right)^{p/r} dx \right)^{1/p} \leqslant C_{p,r} \left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r \right)^{p/r} dx \right)^{1/p}.$$

In particular $C_{p,r}$ *is independent of the sequence* a.

PROOF. For a locally integrable function f, let $f^*(x) = \sup_{x \in \omega} |\omega|^{-1} \int_\omega |f(t) - f_\omega(t)| \, dt$ where ω is a dyadic interval and $f_\omega(t) = |\omega|^{-1} \int_\omega f(x) \, dx$ for $t \in \omega$. In view of results in [2], it suffices to prove that, for $1 < p_1 < \infty$, $(T_a f)^*(x) \leqslant C_{p_1}[(|f|^{p_1})^*(x)]^{1/p_1}$ with C_{p_1} independent of f and a. We note that $\|T_a f\|_{p_1} \leqslant C_{p_1}\|f\|_{p_1}$ with C_{p_1} independent of f and a. This follows from the Littlewood-Paley theory for a single function. Let ω denote a dyadic interval and let ω^* denote the dyadic interval containing ω with $|\omega^*| = 2|\omega|$. (If $\omega = [0, 1]$, $\omega = \omega^*$.) Let $f = f_1 + f_2$ where $f_1(\cdot) = f(\cdot)\chi_{\omega^*}(\cdot)$. Then for $x \in \omega$ we have

$$|\omega|^{-1} \int_\omega |T_a f_1(t) - (T_a f_1)_\omega(t)| \, dt$$

$$\leqslant 2|\omega|^{-1} \int_\omega |T_a f_1(t)| \, dt \leqslant 2 \left(|\omega|^{-1} \int_\omega |T_a f_1(t)|^{p_1} \, dt \right)^{1/p_1}$$

$$\leqslant C_{p_1} \left(|\omega^*|^{-1} \int_{\omega^*} |f_1(t)|^{p_1} \, dt \right)^{1/p_1} \leqslant C_{p_1}[(|f|^{p_1})^*]^{1/p_1}(x).$$

We recall from the theory of Walsh-Fourier series (see [5]) that $S_{2^{\nu+1}}f - S_{2^\nu}f = f * \delta_\nu^*$ where δ_ν^* has support in $[0, 2^{-\nu}]$ and convolution is with respect to the dyadic structure on $[0, 1]$. Now suppose $|\omega| = 2^{-p}$. It then follows that, for $x \in \omega$, $S_{2^\nu}f_2(x) - S_{2^{\nu-1}}f_2(x) = 0$ for $\nu > p$. Consequently, for $x \in \omega$, $T_a f_2(x) = \sum_{\nu=1}^{p-1} a_\nu(S_{2^\nu} - S_{2^{\nu-1}})(f_2)(x)$. But $\sum_{\nu=1}^{p-1} a_\nu(S_{2^\nu} - S_{2^{\nu-1}})(f_2)$ is a polynomial of degree at most 2^p and therefore is constant on intervals of length 2^{-p}. It follows that

$$\left|\omega\right|^{-1}\int_{\omega}\left|T_{a}f_{2}(t)-(T_{a}f_{2})_{\omega}(t)\right|\,dt=0$$

and therefore

$$\left|\omega\right|^{-1}\int_{\omega}\left|T_{a}f(t)-(T_{a}f)_{\omega}(t)\right|\,dt=\left|\omega\right|^{-1}\int_{\omega}\left|T_{a}f_{1}(t)-(T_{a}f_{1})_{\omega}(t)\right|\,dt$$

$$<C_{p_{1}}\left[(\left|f\right|^{p_{1}})^{*}\right]^{1/p_{1}}(x).$$

Taking the supremum norm over ω such that $x\in\omega$, we obtain $(T_{a}f)^{*}(x)<C_{p_{1}}[(\left|f\right|^{p_{1}})^{*}]^{1/p_{1}}(x)$. This completes the proof of Lemma 2.

We mention a special case of this result. We shall see later that more is actually true. Let n be a fixed positive integer and let $S_{n}f(x)$ denote the nth partial sum of the Walsh-Fourier series of f. Then

$$\left(\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|S_{n}f_{k}(x)\right|^{r}\right)^{p/r}dx\right)^{1/p}<C_{p,r}\left(\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|f_{k}(x)\right|^{r}\right)^{p/r}\right)^{1/p}$$

where $C_{p,r}$ is independent of n and $\{f_{k}\}$. This follows by writing $S_{n}f_{k}=\omega_{n}S_{n}^{*}(f_{k}\omega_{n})$ where S_{n}^{*} denotes the modified partial sum (see [5]) and noting that $S_{n}^{*}f_{k}=T_{a}f_{k}$ where a is a sequence of 0's and 1's depending on n.

III. In this section we prove an inequality of Littlewood-Paley type for mixed norm spaces. For a function f in $L^{p}(0;1)$, $p>1$, we define the Littlewood-Paley function Gf as

$$Gf(x)=\left(\sum_{\nu=1}^{\infty}\left|S_{2^{\nu}}f(x)-S_{2^{\nu-1}}f(x)\right|^{2}\right)^{1/2}$$

where $S_{2^{\nu}}f(x)=\left|\omega\right|^{-1}\int_{\omega}f(x)\,dx$, $\left|\omega\right|=2^{-\nu}$, $\omega\ni x$. Then for $f\in L^{p}(0;1)$, $1<p<\infty$, $\left\|Gf\right\|_{p}<C_{p}\left\|f\right\|_{p}$ with C_{p} independent of f. This is the Littlewood-Paley inequality for L^{p} spaces [10].

THEOREM. *Let $1<p,r<\infty$ and let $\{f_{k}\}$ be a sequence of integrable functions on $[0,1]$ with $\int_{0}^{1}(\sum_{k=1}^{\infty}\left|f_{k}(x)\right|^{r})^{p/r}\,dx<\infty$. Then there exists a constant $C_{p,r}$ independent of $\{f_{k}\}$ such that*

$$\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|Gf_{k}(x)\right|^{r}\right)^{p/r}dx<C_{p,r}\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|f_{k}(x)\right|^{r}\right)^{p/r}dx.$$

PROOF. This result is a simple consequence of the results in the previous two sections. For each $t\in[0,1]$ consider the sequence $\{\sum_{\nu=1}^{\infty}r_{\nu}(t)(S_{2^{\nu}}f_{k}(x)-S_{2^{\nu-1}}f_{k}(x))\}_{k}$. For each t in $[0,1]$, Lemma 2 implies

$$\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|\sum_{\nu=1}^{\infty}r_{\nu}(t)(S_{2^{\nu}}f_{k}(x)-S_{2^{\nu-1}}f_{k}(x))\right|^{r}\right)^{p/r}dx$$

$$<C_{p,r}\int_{0}^{1}\left(\sum_{k=1}^{\infty}\left|f_{k}(x)\right|^{r}\right)^{p/r}dx.$$

Integrating with respect to t and interchanging the order of integration, we obtain

$$\int_0^1 \int_0^1 \left(\sum_{k=1}^{\infty} \left| \sum_{\nu=1}^{\infty} r_\nu(t)(S_{2^\nu}f_k(x) - S_{2^{\nu-1}}f_k(x)) \right|^r \right)^{p/r} dt \, dx$$

$$\leq C_{p,r} \int_0^1 \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{p/r} dx.$$

By Lemma 1 we have

$$\int_0^1 \int_0^1 \left(\sum_{k=1}^{\infty} \left| \sum_{\nu=1}^{\infty} r_\nu(t)(S_{2^\nu}f_k(x) - S_{2^{\nu-1}}f_k(x)) \right|^r \right)^{p/r} dt \, dx$$

$$\geq \tilde{C}_{p,r} \int_0^1 \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=1}^{\infty} |S_{2^\nu}f_k(x) - S_{2^{\nu-1}}f_k(x)|^2 \right)^{r/2} \right)^{p/r} dx$$

$$= \tilde{C}_{p,r} \int_0^1 \left(\sum_{k=1}^{\infty} |Gf_k(x)|^r \right)^{p/r} dx.$$

Hence

$$\int_0^1 \left(\sum_{k=1}^{\infty} |Gf_k(x)|^r \right)^{p/r} dx \leq \tilde{\tilde{C}}_{p,r} \int_0^1 \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{p/r} dx$$

where $\tilde{\tilde{C}}_{p,r}$ is independent of $\{f_k\}$.

IV. For the basic properties of the Walsh-Paley system, we refer the reader to [5] or [7]. $S_n f(\cdot)$ will denote the nth partial sum of the Walsh-Paley-Fourier series of $f \in L^1[0, 1]$.

In this section we extend a result of Paley to mixed norm spaces. Let $\{n_\nu\}_{\nu=1}^{\infty}$ denote a sequence of integers such that $n_{\nu+1} > qn_\nu$, $q > 1$, $\nu = 1, 2, \ldots$. For f integrable on $[0, 1]$, let $Sf(x) = \sup_\nu |S_{n_\nu} f(x)|$. Paley [7] proved that for $1 < p < \infty$, there exists a constant $A_{p,q}$ independent of f such that $\|Sf\|_p \leq A_{pq}\|f\|_p$. Our result is the following

THEOREM. *Let* $\{n_\nu\}_{\nu=1}^{\infty}$ *denote a lacunary sequence as above. Let* $\{f_k\}$ *denote a sequence of integrable functions on* $[0, 1]$ *such that* $\int_0^1 (\sum_{k=1}^{\infty} |f_k(x)|^r)^{p/r} dx < \infty$. *Let* $1 < p, r < \infty$. *Then there exists a constant* $A_{p,q,r}$ *independent of* $\{f_k\}$ *such that*

$$\left(\int_0^1 \left(\sum_{k=1}^{\infty} |Sf_k(x)|^r \right)^{p/r} dx \right)^{1/p} \leq A_{p,q,r} \left(\int_0^1 \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{p/r} \right)^{1/p}.$$

REMARK. We note that in the special case $n_\nu = 2^\nu$ our inequality reduces to dyadic analogue of the Fefferman-Stein inequality since $S_{2^\nu}f(x)$ is just the average of f over the dyadic interval of length $2^{-\nu}$ which contains x.

PROOF. By splitting up the sequence $\{n_\nu\}_{\nu=1}^\infty$ into finitely many subsequences if necessary, we may assume $2^\nu < n_\nu < 2^{\nu+1}$. The number of such subsequences depends only on q. For any $k, \nu > 1$, let $d_\nu^k = S_{2^{\nu+1}} f_k - S_{2^\nu} f_k$. Then

$$S_{n_\nu} f_k = S_{n_\nu}(f_k - S_{2^\nu} f_k) + S_{2^\nu} f_k \equiv S_{n_\nu}(d_\nu^k) + S_{2^\nu} f_k.$$

Then

$$Sf(x) < \sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)| + \sup_{\nu > 1} |S_{2^\nu} f_k(x)|$$

and therefore

$$\left(\sum_{k=1}^\infty |Sf_k(x)|^r \right)^{1/r} < \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{1/r}$$
$$+ \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{2^\nu} f_k(x)| \right)^r \right)^{1/r}.$$

By the Fefferman-Stein inequality [3]

$$\left(\int_0^1 \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{2^\nu} f_k(x)|^r \right)^{p/r} \right)^{1/p} < C_{p,r} \left(\int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r \right)^{p/r} \right)^{1/p}.$$

Thus it suffices to prove

$$\int_0^1 \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{p/r} < C_{p,r} \int_0^1 \left(\sum_{k=1}^\infty |f_k(x)|^r \right)^{p/r} dx. \qquad (*)$$

We will first show that

$$\left| \left\{ x \in [0,1] : \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{1/r} > \lambda \right\} \right|$$
$$< C_r \lambda^{-p} \int_0^1 \left(\sum_{k=1}^\infty |Gf_k(x)|^r \right)^{p/r} dx. \qquad (**)$$

Combining this with the result in §III gives

$$\left| \left\{ x \in [0,1] : \left(\sum_{k=1}^\infty \left(\sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{1/r} > \lambda \right\} \right|$$
$$< \tilde{C}_{p,r} \lambda^{-p} \int \left(\sum_{k=1}^\infty |f_k(x)|^r \right)^{p/r} dx.$$

An interpolation argument similar to the Marcinkiewicz theorem yields $(*)$. By the Calderón-Zygmund lemma there exists a sequence $\{\omega_j\}$ of mutually disjoint dyadic intervals of $[0,1]$ which satisfy

(i) $\lambda < (|\omega_j|^{-1} \int_{\omega_j} (\Sigma_{k=1}^\infty Gf_k(x)^r)^{p/r} dx)^{1/p} < 2\lambda$,

(ii) $\Sigma_{j=1}^\infty |\omega_j| < \lambda^{-p} \int_0^1 (\Sigma_{k=1}^\infty Gf_k(x)^r)^{p/r} dx$,

(iii) $(\sum_{k=1}^{\infty} Gf_k(x)^r)^{1/r} \leqslant \lambda$ for a.e. $x \notin \bigcup_j \omega_j$.

Fix a large positive integer M. We will prove the stated inequality with $\sup_{1 < \nu < M} |S_{n_\nu}(d_\nu^k)(x)|$ in place of $\sup_{\nu > 1} |S_{n_\nu}(d_\nu^k)(x)|$, and C_r will be independent of M. The desired inequality will then follow by letting M tend to ∞. Let $\Omega = \bigcup_j \omega_j$. We define, for $1 < \nu < M$,

$$g_\nu^k(x) = \begin{cases} \omega_{n_\nu}(x)\left(|\omega_j|^{-1} \int_{\omega_j} \omega_{n_\nu}(t) d_\nu^k(t)\, dt\right) & \text{if } x \in \omega_j, \\ d_\nu^k(x) & \text{if } x \notin \Omega. \end{cases}$$

Let $b_\nu^k = d_\nu^k - g_\nu^k$, $k > 1$, $1 < \nu < M$. As in [8] we note that for each $k > 1$ and $x \notin \Omega$, $S_{n_\nu}(d_\nu^k)(x) = S_{n_\nu}(g_\nu^k)(x)$, $1 < \nu < M$. Thus

$$\left| \left\{ x \in [0, 1]: \left(\sum_{k=1}^{\infty} \left(\sup_{1 < \nu < M} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{1/r} > \lambda \right\} \right|$$

$$\leqslant |\Omega| + \left| \left\{ x \in [0, 1] \setminus \Omega: \left(\sum_{k=1}^{\infty} \left(\sup_{1 < \nu < M} |S_{n_\nu}(g_\nu^k)(x)|^r \right) \right)^{1/r} > \lambda \right\} \right|$$

$$\leqslant |\Omega| + \left| \left\{ x \in [0, 1] \setminus \Omega: \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=1}^{M} |S_{n_\nu}(g_\nu^k)(x)|^2 \right)^{r/2} \right)^{1/r} > \lambda \right\} \right|$$

$$\leqslant |\Omega| + \lambda^{-r} \int_{[0, 1] \setminus \Omega} \left(\sum_{k=1}^{\infty} |S_{n_\nu}(g_\nu^k)(x)|^2 \right)^{r/2} dx$$

$$\leqslant |\Omega| + \lambda^{-r} \int_0^1 \sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{M} |S_{n_\nu} g_\nu^k(x)|^2 \right)^{r/2} dx$$

$$= |\Omega| + \lambda^{-r} \sum_{k=1}^{\infty} \int_0^1 \left(\sum_{\nu=0}^{M} |S_{n_\nu} g_\nu^k(x)|^2 \right)^{r/2} dx.$$

We first note that $S_2 g_\nu^k(x) \equiv 0$, i.e., $\int_\omega g_\nu^k(x)\, dx = 0$ for every dyadic interval ω with $|\omega| = 2^{-k}$. If $\omega \cap \Omega = \varnothing$, this is immediate since $\int_\omega g_\nu^k(x)\, dx = \int_\omega d_\nu^k(x)\, dx = 0$ since each term in d_ν^k splits on ω. If $\omega = \omega_j \in \Omega$ for some j, $\int_\omega g_\nu^k(x)\, dx = C \int_\omega \omega_{n_\nu}(x)\, dx = 0$ (C is a constant). Finally suppose $\omega \cap \Omega \neq \varnothing$ but $\omega \neq \omega_j$ for any $\omega_j \in \Omega$. Let $\omega \cap \Omega = \bigcup_{j'} \omega_{j'}$. Note that $|\omega_{j'}| < 2^{-\nu-1}$ for each j'. Thus

$$\int_\omega g_\nu^k(x)\, dx = \sum_{j'} \int_{\omega_{j'}} g_\nu^k(x)\, dx + \int_{\omega \setminus \bigcup_{j'} \omega_{j'}} d_\nu^k(x)\, dx$$

$$= \sum_{j'} \int_{\omega_{j'}} \left(\frac{1}{|\omega_{j'}|} \omega_{n_\nu}(x) \int_{\omega_{j'}} \omega_{n_\nu}(t) d_\nu^k(t)\, dt \right) dx + \int_{\omega \setminus \bigcup_{j'} \omega_{j'}} d_\nu^k(x)\, dx$$

$$= \sum_{j'} \int_{\omega_{j'}} \frac{1}{|\omega_{j'}|} \int_{\omega_{j'}} d_\nu^k(t)\, dt\, dx + \int_{\omega \setminus \bigcup_{j'} \omega_{j'}} d_\nu^k(x)\, dx$$

(since $\omega_{n_r}(\cdot)$ is constant on each $\omega_{j'}$)

$$= \sum_{j'} \int_{\omega_{j'}} d_v^k(t) \, dt + \int_{\omega \setminus \bigcup_{j'} \omega_{j'}} d_v^k(x) \, dx = \int_\omega d_v^k(t) \, dt = 0.$$

Thus $S_{n_r} g_v^k(x) = S_{n_r} g_v^k(x) - S_{2^v} g_v^k(x)$. By a theorem of R. E. A. C. Paley on partial blocks [7, Lemma 13] there exists a constant C_r depending only on r such that

$$\int_0^1 \left(\sum_{v=0}^M |S_{n_r} g_v^k(x)|^2 \right)^{r/2} dx < C_r \int_0^1 \left(\sum_{v=0}^M |S_{2^{v+1}} g_v^k(x)|^2 \right)^{r/2} dx.$$

Returning to our inequalities, we now have

$$\left| \left\{ x : \left(\sum_{k=1}^\infty \left(\sup_{1 < v < M} |S_{n_r}(d_v^k)(x)| \right)^r \right) > \lambda \right\} \right|$$

$$< |\Omega| + C_r \lambda^{-r} \int_0^1 \sum_{k=1}^\infty \left(\sum_{v=0}^M |S_{2^{v+1}} g_v^k(x)|^2 \right)^{r/2} dx. \qquad (\dagger)$$

We now estimate $\sum_{k=1}^\infty (\sum_{v=1}^M |S_{2^{v+1}} g_v^k(x)|^2)^{r/2}$ separately for $x \notin \Omega$ and $x \in \Omega$. We first consider the case $x \notin \Omega$. Let ω denote the dyadic interval of length 2^{-v-1} which contains x. Then $\omega \notin \{\omega_j\}$ and the argument above shows that $\int_\omega g_v^k(x) \, dx = \int_\omega d_v^k(x) \, dx$. Thus

$$\sum_{k=1}^\infty \left(\sum_{v=0}^M |S_{2^{v+1}} g_v^k(x)|^2 \right)^{r/2} = \sum_{k=1}^\infty \left(\sum_{v=0}^M \left(\frac{1}{|\omega|} \left| \int_\omega g_v^k(x) \, dx \right| \right)^2 \right)^{r/2}$$

$$< \sum_{k=1}^\infty \left(\sum_{v=0}^M \left(\frac{1}{|\omega|} \int_\omega |d_v^k(x)| \right)^2 \right)^{r/2}$$

$$= \sum_{k=1}^\infty \left(\sum_{v=0}^M |d_v^k(x)|^2 \right)^{r/2}$$

(since $d_v^k(x)$ is constant on dyadic intervals of length 2^{-v-1})

$$< \sum_{k=1}^\infty \left(\sum_{v=0}^\infty |d_v^k(x)|^2 \right)^{r/2} = \sum_{k=1}^\infty Gf_k(x)^r.$$

By property (iii) of the decomposition, it follows that for a.e. $x \notin \Omega$ we have

$$\sum_{k=1}^{\infty}\left(\sum_{\nu=0}^{M}|S_{2^{\nu+1}}g_{\nu}^{k}(x)|^{2}\right)^{r/2} < \sum_{k=1}^{\infty}Gf_{k}(x)^{r}$$

$$< \left(\sum_{k=1}^{\infty}Gf_{k}(x)^{r}\right)^{p/r}\left(\sum_{k=1}^{\infty}Gf_{k}(x)^{r}\right)^{(r-p)/r}$$

$$< \lambda^{r-p}\left(\sum_{k=1}^{\infty}Gf_{k}(x)^{r}\right)^{p/r}.$$

We now consider the case $x \in \Omega$. Let $\bar{\omega}$ denote the element of Ω which contains x and let $|\bar{\omega}| = 2^{-S}$. Then letting ω_{ν} denote the dyadic interval of length $2^{-\nu}$ containing x, we have

$$\left(\sum_{\nu=0}^{S-2}|S_{2^{\nu+1}}g_{\nu}^{k}(x)|^{2}\right)^{r/2} = \left(\sum_{\nu=0}^{S-2}\left(\frac{1}{|\omega_{\nu+1}|}\left|\int_{\omega_{\nu+1}}g_{\nu}^{k}(x)\,dx\right|\right)^{2}\right)^{r/2}$$

$$= \left(\sum_{\nu=0}^{S-2}\left(\frac{1}{|\omega_{\nu+1}|}\left|\int_{\omega_{\nu+1}}d_{\nu}^{k}(x)\,dx\right|\right)^{2}\right)^{r/2}$$

(since each $\omega_{\nu+1} \neq \omega_{j}$ and the earlier argument applies)

$$< \left(\sum_{\nu=0}^{S-2}|d_{\nu}^{k}(x)|^{2}\right)^{r/2} = \left(\frac{1}{|\bar{\omega}|}\int_{\bar{\omega}}\left(\sum_{\nu=0}^{S-2}|d_{\nu}^{k}(x)|^{2}\right)^{1/2}dx\right)^{r}$$

(since each $d_{\nu}^{k}(\cdot)$ is constant on $\bar{\omega}$). Also we have

$$\left(\sum_{\nu=S-1}^{M}|S_{2^{\nu+1}}g_{\nu}^{k}(x)|^{2}\right)^{r/2} = \left(\sum_{\nu=S-1}^{M}\frac{1}{|\omega_{\nu+1}|}\left|\int_{\omega_{\nu+1}}g_{\nu}^{k}(x)\,dx\right|^{2}\right)^{r/2}$$

$$= \left(\sum_{\nu=S-1}^{M}\frac{1}{|\omega_{\nu+1}|}\left|\int_{\omega_{\nu+1}}\omega_{n_{\nu}}(x)\frac{1}{|\bar{\omega}|}\int_{\bar{\omega}}\omega_{n_{\nu}}(t)d_{\nu}^{k}(t)\,dt\,dx\right|^{2}\right)^{r/2}$$

$$< \left(\sum_{\nu=S-1}^{M}\left(\frac{1}{|\omega_{\nu+1}|}\int_{\omega_{\nu+1}}\frac{1}{|\bar{\omega}|}\left|\int_{\bar{\omega}}\omega_{n_{\nu}}(t)d_{\nu}^{k}(t)\,dt\right|dx\right)^{2}\right)^{r/2}$$

$$= \left(\sum_{\nu=S-1}^{M}\left(\frac{1}{|\bar{\omega}|}\left|\int_{\bar{\omega}}\omega_{n_{\nu}}(t)d_{\nu}^{k}(t)\,dt\right|\right)^{2}\right)^{r/2}$$

$$< \left(\sum_{\nu=S-1}^{M}\left(\frac{1}{|\bar{\omega}|}\int_{\bar{\omega}}|d_{\nu}^{k}(t)|\,dt\right)^{2}\right)^{r/2}$$

$$< \left(\frac{1}{|\bar{\omega}|}\left(\int_{\bar{\omega}}\sum_{\nu=S-1}^{M}|d_{\nu}^{k}(t)|^{2}\right)^{1/2}\right)^{r}.$$

Returning to our inequality we have for $x \in \Omega$, say $x \in \bar{\omega}$,

$$\sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{r/2}$$

$$< \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{S-2} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{1/2} + \left(\sum_{\nu=S-1}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{1/2} \right)^r$$

$$< 2^r \left\{ \sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{S-2} |S_{2^{\nu+1}} g_\nu^k(x)| \right)^{r/2} + \sum_{k=1}^{\infty} \left(\sum_{\nu=S-1}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{r/2} \right\}$$

$$< 2^r \left\{ \sum_{k=1}^{\infty} \left(\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{\nu=0}^{S-2} |d_\nu^k(x)|^2 \right)^{1/2} dx \right)^r \right.$$

$$\left. + \sum_{k=1}^{\infty} \left(\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{\nu=S-1}^{M} |d_\nu^k(x)|^2 \right)^{1/2} dx \right)^r \right\}$$

$$< 2^r \left\{ \left[\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{S-2} |d_\nu^k(x)|^2 \right)^{r/2} \right)^{1/r} \right)^r \right.$$

$$\left. + \left[\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{k=1}^{\infty} \left(\sum_{\nu=S-1}^{M} |d_\nu^k(x)|^2 \right)^{r/2} \right)^{1/r} \right)^r \right] \right\}$$

$$< 2^r \left\{ \left[\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{k=1}^{\infty} Gf_k^r \right)^{1/r} \right]^r + \left[\frac{1}{|\bar{\omega}|} \int_{\bar{\omega}} \left(\sum_{k=1}^{\infty} Gf_k^r \right)^{1/r} \right]^r \right\}$$

$$< 2^{r+1} [2\lambda]^r = 2^{2r+1} \lambda^r$$

by property (i) of the decomposition. This is the estimate for $x \notin \Omega$. Combining these results with inequality (†) we have

$$\left| \left\{ x : \left(\sum_{k=1}^{\infty} \left(\sup_{1 < \nu < M} |S_{n_\nu}(d_\nu^k)(x)| \right)^r \right)^{1/r} > \lambda \right\} \right|$$

$$< |\Omega| + C_r \lambda^{-r} \int_0^1 \sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{r/2} dx$$

$$= |\Omega| + C_r \lambda^{-r} \int_\Omega \sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{r/2} dx$$

$$+ C_r \lambda^{-r} \int_{[0,1]\setminus\Omega} \sum_{k=1}^{\infty} \left(\sum_{\nu=0}^{M} |S_{2^{\nu+1}} g_\nu^k(x)|^2 \right)^{r/2} dx$$

$$< |\Omega|\{1 + C_r\lambda^{-r}(2^{2r+1}\lambda^r)\} + C_r\lambda^{-r}\int_{[0,1]\setminus\Omega} \sum_{k=1}^{\infty} Gf_k(x)^r \, dx$$

$$< |\Omega|\{1 + C_r 2^{2r+1}\} + C_r\lambda^{-r}\lambda^{(r-p)}\int_{[0,1]\setminus\Omega} \left(\sum_{k=1}^{\infty} Gf_k(x)^r\right)^{p/r} dx$$

$$< |\Omega|\{1 + C_r 2^{2r+1}\} + C_r\lambda^{-p}\int_0^1 \left(\sum_{k=1}^{\infty} Gf_k(x)^r\right)^{p/r} dx$$

$$< \lambda^{-p}\{1 + C_r 2^{2r+1} + C_r\}\int_0^1 \left(\sum_{k=1}^{\infty} Gf_k(x)^r\right)^{p/r} dx$$

by property (ii) of the decomposition. Letting M tend to ∞, we obtain

$$\left|\left\{ x \in [0,1]: \left(\sum_{k=1}^{\infty} \left(\sup_\nu |S_{n_\nu}(d_\nu^k)(x)|\right)^r\right)^{1/r} > \lambda\right\}\right|$$

$$< \tilde{C}_r\lambda^{-p}\int_0^1 \left(\sum_{k=1}^{\infty} Gf_k(x)^r\right)^{p/r} dx$$

which is the desired inequality.

REMARKS. (i) In the case $f_k = 0$ for $k > 2$, our proof reduces to a direct proof of the strong type (p, p) inequality for the lacunary maximal partial sum operator for Walsh-Fourier series. The only proof in the literature known to the author is that of R. E. A. C. Paley which is not based on the Calderón-Zygmund theory.

(ii) The above proof can be easily modified to yield a partial analogue of the result in [8]:

$$\left|\left\{ x \in [0,1]: \left(\sum_{k=1}^{\infty} \left(\sup_k |S_{n_k}(d_k^\nu)(x)|\right)^r\right)^{1/r} > \lambda\right\}\right|$$

$$< C_r\lambda^{-1}\int_0^1 \left(\sum_{k=1}^{\infty} Gf_k(x)^r\right)^{1/r} dx.$$

Whether the full analogue is true depends upon integrability conditions between $(\sum_{k=1}^{\infty} Hf_k(x)^r)^{1/r}$ and $(\sum_{k=1}^{\infty} Gf_k(x)^r)^{1/r}$ where H denotes the Hardy-Littlewood maximal function. Such results are unknown to the author.

(iii) A natural question which arises from this work is whether analogous inequalities can be proved for the full maximal partial sum operator.

ADDENDUM. A positive answer to the question raised in (iii) was recently announced by Jose L. Rubio de Francia, Universidad de Zaragoza (Spain) in the Notices Amer. Math. Soc. **26** (1979), p. A-216.

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