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CROSSED EXTENSIONS

BY

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ABSTRACT. We develop a natural five term exact sequence relating the second and third cohomology of groups. We show that this sequence is the proper framework for the problem of realizing an abstract kernel. As an application, we give an interpretation of the third cohomology of a group in terms of crossed sequences.

1. Introduction. If G is a group, then a *G-crossed module* C with boundary ∂ gives rise to a central group extension

$$0 \to A \to C \xrightarrow{\partial} N \to 1$$

where N is a normal subgroup of G and A is a Q-module with $Q = G/N$. We show that these extensions, suitably classified by a congruence relation, are the elements of an abelian group $\mathrm{Xext}_G(N, A)$. Our main result, Theorem 8.1, is that this group fits into the exact sequence

$$H^2(Q, A) \xrightarrow{\inf} H^2(G, A) \xrightarrow{\gamma} \mathrm{Xext}_G(N, A) \xrightarrow{\delta} H^3(Q, A) \xrightarrow{\inf} H^3(G, A). \qquad (1.1)$$

This sequence is part of a long exact sequence constructed by G. S. Rinehart in **[12]** as a special case of a long exact sequence constructed in a very general categorical setting. Our contribution is to identify the connecting homomorphism δ in (1.1) as the homomorphism which maps the class of a crossed extension to its Mac Lane-Whitehead obstruction **[10]**. This obstruction is important in topology because it gives the Eilenberg-Mac Lane k-invariant **[4]** determined by a space X in the group $H^3(\pi_1(X), \pi_2(X))$. Rinehart noted in **[12]** the connection of δ with the group extension obstruction theory of Eilenberg-Mac Lane **[3]**; however, because of his general cocycle-free methods, Rinehart did not identify the obstruction at the cocycle level.

We also show that the following diagram commutes:

$$\begin{array}{ccccc} & & H^1(Q, H^1(N, A)) & & \\ & & \downarrow \zeta & \searrow \text{cup} & \\ H^2(G, A) & \xrightarrow{\gamma} & \mathrm{Xext}_G(N, A) & \xrightarrow{\delta} & H^3(Q, A) \\ & \searrow \text{res} & \downarrow \varepsilon & & \\ & & H^2(N, A) & & \end{array}$$

where ζ maps $H^1(Q, H^1(N, A))$ isomorphically onto the subgroup of split crossed

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extensions, ε maps the class of a crossed extension to the class of its underlying central extension, and cup is the cup product homomorphism of Eilenberg and Mac Lane.

A G-crossed module C determines a 4-term exact sequence

$$0 \to A \to C \xrightarrow{\partial} G \to Q \to 1$$

called a *crossed sequence*. As an application of our constructions, we show that there is a natural isomorphism from $H^3(Q, A)$ to the group of congruence classes of crossed sequences from A to Q. This gives an interpretation of the third cohomology of a group in terms of extensions analogous to that of the second cohomology.

To summarize the contents of the paper: In §2 we define the functor Xext. In §3 we show that Baer sum induces an abelian group structure on $\text{Xext}_G(N, A)$. In §4 we generalize the results of §§2, 3. In §5 we define the homomorphism γ: $H^2(G, A) \to \text{Xext}_G(N, A)$ induced by restriction. In §6 we define the Mac Lane-Whitehead obstruction homomorphism δ: $\text{Xext}_G(N, A) \to H^3(Q, A)$. In §7 we define the monomorphism ζ: $H^1(Q, H^1(N, A)) \to \text{Xext}_G(N, A)$ and show that $\delta \circ \zeta$ is the cup product homomorphism. In §8 we prove the exactness of (1.1). In §9 we give an interpretation of the third cohomology of a group in terms of crossed sequences. In §10 we show that (1.1) provides a framework for the problem of realizing an abstract kernel.

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J. Huebschmann [5] has generalized the exact sequence (1.1) to the case where A is an arbitrary G-module. For another proof of the exactness of (1.1) see J.-L. Loday [7].

2. Basic definitions. Suppose G is a group, N is a normal subgroup of G, $Q = G/N$ and A is a Q-module. Regard A as a G-module with trivial action of N. We assume G acts on N by conjugation. Then a *G-crossed extension* of A by N is an extension

$$E\colon 0 \to A \xrightarrow{\iota} C \xrightarrow{\partial} N \to 1 \tag{2.1}$$

with operators in G such that C is a G-crossed module with boundary ∂ as in the sense of J. H. C. Whitehead [14]. This means simply that the action of G on C is compatible with conjugation in C, that is, $bcb^{-1} = \partial b \cdot c$ for each $b, c \in C$. This condition implies that a crossed extension is central. Such extensions always exist, since the trivial extension

$$E_0\colon 0 \to A \to A \times N \to N \to 1$$

is a G-crossed extension with G acting on $A \times N$ diagonally.

Two G-crossed extensions E and E' are said to be *G-congruent*, written $E \equiv_G E'$, if there is a G-homomorphism φ such that the following diagram commutes:

$$\begin{array}{lccccccccc} E: & 0 & \rightarrow & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & N & \rightarrow & 1 \\ & & & \| & & \downarrow\varphi & & \| & & \\ E': & 0 & \rightarrow & A & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & N & \rightarrow & 1 \end{array}$$

Clearly, G-congruence induces an equivalence relation on the set of all G-crossed extensions of A by N. Let $\mathrm{Xext}_G(N, A)$ denote the set of equivalence classes.

PROPOSITION 2.1. *If E is a G-crossed extension, then $E \equiv_G E_0$ if and only if E splits via a G-homomorphism.* □

Given a G-crossed extension (2.1) and a Q-module homomorphism $\zeta\colon A \to A'$, then the push out extension

$$\zeta E\colon 0 \to A' \xrightarrow{\iota'} C' \xrightarrow{\partial'} N \to 1$$

where $C' = (A' \times C)/T$ and $T = \{(-\zeta(a), \iota(a)) | a \in A\}$ is a G-crossed extension with action of G on C' induced by the diagonal action of G on $A' \times C$.

PROPOSITION 2.2. *If E is a G-crossed extension of A by N, $\zeta\colon A \to A'$ is a Q-module homomorphism, and E' is a G-crossed extension of A' by N, then $E' \equiv_G \zeta E$ if and only if there is a G-homomorphism η such that the following diagram commutes:*

$$\begin{array}{lccccccccc} E: & 0 & \rightarrow & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & N & \rightarrow & 1 \\ & & & \downarrow\zeta & & \downarrow\eta & & \| & & \\ E': & 0 & \rightarrow & A' & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & N & \rightarrow & 1 \end{array} \quad \square$$

It follows easily from Proposition 2.2 that $\zeta\colon A \to A'$ induces a function ζ_*: $\mathrm{Xext}_G(N, A) \to \mathrm{Xext}_G(N, A')$ given by $\zeta_*\{E\} = \{\zeta E\}$.

PROPOSITION 2.3. *The assignment $A \mapsto \mathrm{Xext}_G(N, A)$ and $\zeta \mapsto \zeta_*$ is a covariant functor from the category of Q-modules to the category of sets.* □

3. Addition of crossed extensions. Suppose E_1 and E_2 are G-crossed extensions of A by N. Consider the extension

$$E_1 \oplus E_2\colon 0 \to A \oplus A \xrightarrow{\iota_1 \times \iota_2} C_1 \times C_2 \xrightarrow{\partial_1 \times \partial_2} N \times N \to 1.$$

The diagonal homomorphism $\Delta\colon N \to N \times N$ induces the pull back extension

$$(E_1 \oplus E_2)\Delta\colon 0 \to A \oplus A \xrightarrow{\iota} C \xrightarrow{\partial} N \to 1$$

where $C = \{(c_1, c_2, x) \in C_1 \times C_2 \times N | (\partial_1 c_1, \partial_2 c_2) = (x, x)\}$. Moreover, $(E_1 \oplus E_2)\Delta$ is a G-crossed extension with G acting on C diagonally. Let $\nabla\colon A \oplus A \to A$ be the codiagonal homomorphism. Since ∇ is a Q-module homomorphism, the push out extension $\nabla((E_1 \oplus E_2)\Delta)$ is a G-crossed extension of A by N. Define the *Baer sum* of E_1 and E_2 to be the extension $E_1 + E_2 = \nabla((E_1 \oplus E_2)\Delta)$.

Given a G-crossed extension (2.1), define $-E$ to be the extension

$$0 \to A \xrightarrow{-\iota} C \xrightarrow{\partial} N \to 1.$$

Clearly, $-E$ is also a G-crossed extension.

We want to show that Baer sum induces an addition in $\mathrm{Xext}_G(N, A)$. We do this indirectly by constructing an auxiliary abelian group $H^2_G(N, A)$ and a bijection from $\mathrm{Xext}_G(N, A)$ to $H^2_G(N, A)$.

Suppose we are given a G-crossed extension (2.1). For ease of notation, write the multiplication of C additively. For each $x \in N$, choose a representative $u(x) \in C$ such that $\partial(u(x)) = x$. In particular, choose $u(1) = 0$. We say that the function u: $N \to C$ *is inverse to* ∂.

For each $(g, x) \in G \times N$, we have $g \cdot u(x) - u(gxg^{-1}) \in \ker \partial$; hence, there is an $\alpha(g, x) \in A$ such that

$$g \cdot u(x) - u(gxg^{-1}) = \iota\alpha(g, x). \tag{3.1}$$

Moreover, $\alpha(g, 1) = 0 = \alpha(1, x)$ for each $g \in G$ and $x \in N$.

For each $(x, y) \in N \times N$, we have $u(x) + u(y) - u(xy) \in \ker \partial$; hence, there is a $\beta(x, y) \in A$ such that

$$u(x) + u(y) - u(xy) = \iota\beta(x, y). \tag{3.2}$$

Moreover, $\beta(x, 1) = 0 = \beta(1, y)$ for each $x, y \in N$.

The pair of functions (α, β) is called the *factor system* of E determined by u. The crossed module structure on C implies that (α, β) is a pair of normalized functions which satisfy the following four properties.

(i) $\alpha(gh, x) = g \cdot \alpha(h, x) + \alpha(g, hxh^{-1}), \qquad g, h \in G, x \in N,$ (3.3)

(ii) $\beta(x, y) + \beta(xy, z) = \beta(y, z) + \beta(x, yz), \qquad x, y, z \in N,$ (3.4)

(iii) $g \cdot \beta(x, y) + \alpha(g, xy)$
$= \alpha(g, x) + \alpha(g, y) + \beta(gxg^{-1}, gyg^{-1}), \qquad g \in G, x, y \in N,$ (3.5)

(iv) $\beta(x, y) - \beta(xyx^{-1}, x) = \alpha(x, y), \qquad x, y \in N.$ (3.6)

Formula (3.3) follows from the action associative law. Formula (3.4) follows from the multiplication associative laws. Formula (3.5) follows from the distributive law, and formula (3.6) follows from the conjugation property of a crossed module.

The set of all the pairs of normalized functions (α, β) which satisfy the properties (i)–(iv) forms an abelian group $Z_G(N, A)$ with component-wise addition. Let $B_G(N, A)$ be the subgroup of $Z_G(N, A)$ consisting of all pairs of the form $(\delta_G f, \delta_N f)$ where f: $N \to A$ is a normalized function and $\delta_G f$: $G \times N \to A$ and $\delta_N f$: $N \times N \to A$ are defined by

$$\delta_G f(g, x) = g \cdot f(x) - f(gxg^{-1}), \qquad g \in G, x \in N, \tag{3.7}$$

$$\delta_N f(x, y) = f(x) + f(y) - f(xy), \qquad x, y \in N. \tag{3.8}$$

Any two factor systems of E differ by an element of $B_G(N, A)$, so that E determines an element in $H^2_G(N, A) = Z_G(N, A)/B_G(N, A)$ independent of any choice. Moreover, if $E \equiv_G E'$, then E and E' determine the same element in $H^2_G(N, A)$. Hence, we may define a function ω: $\mathrm{Xext}_G(N, A) \to H^2_G(N, A)$ by $\omega\{E\} = \{(\alpha, \beta)\}$, where (α, β) is any factor system of E.

THEOREM 3.1. *The function* ω: $\mathrm{Xext}_G(N, A) \to H^2_G(N, A)$ *is a bijection; moreover,*
(i) $\omega\{E_0\} = 0$,
(ii) $\omega\{-E\} = -\omega\{E\}$,
(iii) $\omega\{E_1 + E_2\} = \omega\{E_1\} + \omega\{E_2\}$.

PROOF. To see that ω is injective, suppose E_1 and E_2 are G-crossed extensions of A by N, (α_i, β_i) is a factor system of E_i with inverse function u_i for $i = 1, 2$, and (α_1, β_1) and (α_2, β_2) determine the same element of $H^2_G(N, A)$. Then there is a normalized function f: $N \to A$ such that $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) + (\delta_G f, \delta_N f)$. Let $u_2' = \iota_2 \circ f + u_2$. Then u_2' is inverse to ∂_2 and determines the factor system (α_1, β_1). To show that $E_1 \equiv_G E_2$, define φ: $C_1 \to C_2$ by $\varphi(\iota_1(a) + u_1(x)) = \iota_2(a) + u_2'(x)$ for $a \in A$ and $x \in N$. It is straightforward to check that $E_1 \equiv_G E_2$ via φ. This shows that ω is injective.

To see that ω is surjective, suppose $(\alpha, \beta) \in Z_G(N, A)$. Let $A \times_{(\alpha,\beta)} N = A \times N$ as a set. Define an action of G on $A \times_{(\alpha,\beta)} N$ and a multiplication in $A \times_{(\alpha,\beta)} N$ by $g \cdot (a, x) = (g \cdot a + \alpha(g, x), gxg^{-1})$ and $(a, x) + (b, y) = (a + b + \beta(x, y), xy)$. Then formulas (3.3)–(3.6) imply that

$$E_{(\alpha,\beta)}\colon 0 \to A \to A \times_{(\alpha,\beta)} N \to N \to 1$$

is a G-crossed extension. Let $u(x) = (0, x)$ for each $x \in N$. Then u determines the factor system (α, β), so that $\omega\{E_{(\alpha,\beta)}\} = \{(\alpha, \beta)\}$. This shows that ω is surjective.

Property (i) follows since (0, 0) is a factor system for E_0. Property (ii) follows since if (α, β) is a factor system for E, then $(-\alpha, -\beta)$ is a factor system for $-E$. While (iii) follows since if (α_i, β_i) is a factor system for E_i, $i = 1, 2$, then $(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$ is a factor system for $E_1 + E_2$. $\square$

If (α, β) is a factor system of E and ζ: $A \to A'$ is a Q-module homomorphism, then $(\zeta \circ \alpha, \zeta \circ \beta)$ is a factor system of ζE. This implies that ζ_*: $\mathrm{Xext}_G(N, A) \to \mathrm{Xext}_G(N, A')$ is a homomorphism.

PROPOSITION 3.2. *The assignment* $A \mapsto \mathrm{Xext}_G(N, A)$ *and* $\zeta \mapsto \zeta_*$ *is a covariant functor from the category of Q-modules to the category of abelian groups.* $\square$

4. General crossed extensions. If C is a G-crossed module and C' is a G'-crossed module, then a *homomorphism* from C to C' is a pair of homomorphisms (φ, h): $(C, G) \to (C', G')$ such that $\partial' \circ \varphi = h \circ \partial$ and $\varphi(g \cdot c) = h(g) \cdot \varphi(c)$ for each $g \in G$ and $c \in C$. If C and C' are G-crossed modules, then a *G-morphism* from C to C' is a homomorphism φ: $C \to C'$ such that (φ, id_G) is a homomorphism of crossed modules.

Suppose C is a G-crossed module and A is a Q-module where $Q = G/N$ and N is the image of the boundary of C. A *G-crossed extension* of A by C is an extension

$$E\colon 0 \to A \xrightarrow{\iota} B \xrightarrow{\eta} C \to 1 \tag{4.1}$$

with operators in G such that B is a G-crossed module and η is a G-morphism.

Note that N is a G-crossed module with action of G on N given by conjugation and boundary the inclusion of N in G, so this definition generalizes the previous definition of a G-crossed extension.

Let $\mathrm{Xext}_G(C, A)$ denote the set of G-congruence classes of G-crossed extensions of A by C. The same proof as in Theorem 3.1 shows that $\mathrm{Xext}_G(C, A)$ is an abelian group with addition induced by Baer sum.

Given a G-crossed extension (4.1) and a homomorphism of crossed modules $(\varphi, h)\colon (C', G') \to (C, G)$, then the pull back extension

$$E(\varphi, h)\colon 0 \to A \xrightarrow{\iota'} B' \xrightarrow{\eta'} C' \to 1$$

where $B' = \{(b, c') \in B \times C' | \eta(b) = \varphi(c')\}$ is a G'-crossed extension with action of G' on B' given by $g' \cdot (b, c') = (h(g') \cdot b, g' \cdot c')$.

PROPOSITION 4.1. *If E is a G-crossed extension of A by C, $(\varphi, h)\colon (C', G') \to (C, G)$ is a homomorphism of crossed modules, and E' is a G'-crossed extension of A by C', then $E' \equiv_{G'} E(\varphi, h)$ if and only if there is a homomorphism ψ such that the following diagram commutes*:

$$\begin{array}{ccccccccccc} E'\colon & 0 & \to & A & \xrightarrow{\iota'} & B' & \xrightarrow{\eta'} & C' & \to & 1 \\ & & & \| & & \downarrow\psi & & \downarrow\varphi & & \\ E\colon & 0 & \to & A & \xrightarrow{\iota} & B & \xrightarrow{\eta} & C & \to & 1 \end{array}$$

and $\psi(g' \cdot b') = h(g') \cdot \psi(b')$ for each $g' \in G'$ and $b' \in B'$. □

It follows easily from Proposition 4.1 that a homomorphism of crossed modules $(\varphi, h)\colon (C', G') \to (C, G)$ induces a function $(\varphi, h)^*\colon \mathrm{Xext}_G(C, A) \to \mathrm{Xext}_{G'}(C', A)$ defined by $(\varphi, h)^*\{E\} = \{E(\varphi, h)\}$. If (α, β) is a factor system of E, then $(\alpha \circ h \times \varphi, \beta \circ \varphi \times \varphi)$ is a factor system of $E(\varphi, h)$. This implies that $(\varphi, h)^*$ is a homomorphism.

Call a pair (C, A) consisting of a G-crossed module C and a Q-module A *compatible* if $Q = G/N$ where N is the image of the boundary of C.

PROPOSITION 4.2. *The assignment $(C, A) \mapsto \mathrm{Xext}_G(C, A)$, $(\varphi, h) \mapsto (\varphi, h)^*$ and $\zeta \mapsto \zeta_*$ is a bifunctor from the category of compatible pairs to the category of abelian groups, which is contravariant in the first variable and covariant in the second.* □

Next, we define some special cases of the contravariant homomorphisms induced by the functor Xext. If $\varphi\colon C \to C'$ is a G-morphism, define $\varphi^*\colon \mathrm{Xext}_G(C', A) \to \mathrm{Xext}_G(C, A)$ by $\varphi^* = (\varphi, \mathrm{id}_G)^*$. Call a pair (G, N) consisting of a group G and a normal subgroup N a *normal group pair*. If $h\colon (G, N) \to (G', N')$ is a homomorphism of normal group pairs, define $h^*\colon \mathrm{Xext}_{G'}(N', A) \to \mathrm{Xext}_G(N, A)$ by $h^* = (h|N, h)^*$.

5. Extendable crossed extensions. Suppose A is a Q-module. Regard A as a G-module with trivial action of N. Suppose

$$E\colon 0 \to A \xrightarrow{\zeta} B \xrightarrow{\eta} G \to 1 \tag{5.1}$$

is an extension of A by G which induces the given G-module structure on A. Then the extension

$$E_N\colon 0 \to A \xrightarrow{\zeta} \eta^{-1}(N) \xrightarrow{\partial} N \to 1,$$

where ∂ is the restriction of η, is a G-crossed extension with action of G on $\eta^{-1}(N)$ induced by conjugation in B. In a sense E extends E_N. We say that a G-crossed extension of A by N is *extendable* if it is G-congruent to an extension of the form E_N.

Let $\mathrm{EXT}(G, A)$ denote the group of congruence classes of extensions of A by G which induce the given G-module structure of A. Define res: $\mathrm{EXT}(G, A) \to \mathrm{Xext}_G(N, A)$ by $\mathrm{res}\{E\} = \{E_N\}$. This corresponds to a homomorphism res: $H^2(G, A) \to H^2_G(N, A)$ given by $\mathrm{res}\{\mu\} = \{(\mu_G, \mu_N)\}$ where $\mu: G \times G \to A$ is a normalized cocycle and $\mu_G: G \times N \to A$ and $\mu_N: N \times N \to A$ are defined by

$$\mu_G(g, x) = \mu(g, x) - \mu(gxg^{-1}, g), \tag{5.2}$$

$$\mu_N(x, y) = \mu(x, y). \tag{5.3}$$

The following proposition is implicit in **[12]**.

PROPOSITION 5.1. *The following sequence is exact*:

$$\mathrm{EXT}(Q, A) \xrightarrow{\mathrm{inf}} \mathrm{EXT}(G, A) \xrightarrow{\mathrm{res}} \mathrm{Xext}_G(N, A).$$

PROOF. One checks easily that the composite is trivial. To see that ker res $\subset$ Im inf, suppose we are given an extension (5.1) which represents an element in $\mathrm{EXT}(G, A)$ which is in the kernel of res. By Proposition 2.1, the extension E_N splits via a G-morphism $\sigma: N \to \eta^{-1}(N)$. This implies that $\sigma(N)$ is a normal subgroup of B. Let $B_\sigma = B/\sigma(N)$ and let $(\tau, \pi): (B, G) \to (B_\sigma, Q)$ be the natural projections. Since $\eta(\sigma(N)) = N$, $\eta: B \to G$ induces a homomorphism $\eta_\sigma: B_\sigma \to Q$. Define $\zeta_\sigma: A \to B_\sigma$ by $\zeta_\sigma = \tau \circ \zeta$. One checks that

$$E_\sigma: 0 \to A \xrightarrow{\zeta_\sigma} B_\sigma \xrightarrow{\eta_\sigma} Q \to 1$$

is an extension of A by Q which induces the given Q-module structure on A. Observe that the following diagram commutes:

$$\begin{array}{ccccccccc} E: & 0 & \to & A & \xrightarrow{\zeta} & B & \xrightarrow{\eta} & G & \to & 1 \\ & & & \| & & \downarrow\tau & & \downarrow\pi & & \\ E_\sigma: & 0 & \to & A & \xrightarrow{\zeta_\sigma} & B_\sigma & \xrightarrow{\eta_\sigma} & Q & \to & 1 \end{array}$$

This implies $E \equiv (E_\sigma)\pi$. $\square$

Let $\gamma: H^2(G, A) \to \mathrm{Xext}_G(N, A)$ be the homomorphism corresponding to res: $H^2(G, A) \to H^2_G(N, A)$. Let $\varepsilon: \mathrm{Xext}_G(N, A) \to H^2(N, A)$ be the homomorphism corresponding to eval: $H^2_G(N, A) \to H^2(N, A)$ defined by $\mathrm{eval}\{(\alpha, \beta)\} = \{\beta\}$.

PROPOSITION 5.2. *The following diagram commutes*:

$$\begin{array}{ccc} H^2(G, A) & \xrightarrow{\gamma} & \mathrm{Xext}_G(N, A) \\ & \searrow \mathrm{res} & \downarrow\varepsilon \\ & & H^2(N, A) \end{array} \quad \square$$

6. The obstruction of a crossed extension. Suppose we are given a G-crossed extension (2.1). Let $v\colon Q \to G$ be inverse to the natural projection $\pi\colon G \to Q$. Choose a normalized function $\mu\colon Q \times Q \to C$ such that $\partial\mu(q, r) = v(q)v(r)v(qr)^{-1}$. According to S. Mac Lane and J. H. C. Whitehead **[10]**, the formula

$$\iota\kappa(q, r, s) = v(q) \cdot \mu(r, s) + \mu(q, rs) - \mu(qr, s) - \mu(q, r) \qquad (6.1)$$

defines a normalized cocycle $\kappa\colon Q \times Q \times Q \to A$ whose class in $H^3(Q, A)$ is independent of the choice of v and μ. We call κ the *obstruction cocycle* of E determined by v and μ, and $\{\kappa\}$ the *obstruction* of E. Moreover, if $E \equiv_G E'$, then E and E' determine the same obstruction in $H^3(Q, A)$. Hence, we may define a function $\delta\colon \mathrm{Xext}_G(N, A) \to H^3(Q, A)$ by $\delta\{E\} = \{\kappa\}$ where κ is an obstruction cocycle of E.

Let $u\colon N \to C$ be inverse to ∂. Define $\mu\colon Q \times Q \to C$ by $\mu(q, r) = u(v(q)v(r)v(qr)^{-1})$. Let κ be the obstruction cocycle of E determined by v and μ, and let (α, β) be the factor system determined by u. A calculation shows that

$$\begin{aligned}\kappa(q, r, s) = {}& \alpha\big(v(q), v(r)v(s)v(rs)^{-1}\big)\\ & - \beta\big(v(q)v(r)v(qr)^{-1}, v(qr)v(s)v(qrs)^{-1}\big)\\ & + \beta\big(v(q)v(r)v(s)(rs)^{-1}v(q)^{-1}, v(q)v(rs)v(qrs)^{-1}\big). \qquad (6.2)\end{aligned}$$

Hence, κ can be expressed solely in terms of (α, β) and v. This implies that δ corresponds to a homomorphism con: $H^2_G(N, A) \to H^3(Q, A)$ given by $\mathrm{con}\{(\alpha, \beta)\} = \{\kappa\}$ where κ is defined by (6.2).

7. Split crossed extensions. The evaluation homomorphism $\varepsilon\colon \mathrm{Xext}_G(N, A) \to H^2(N, A)$ corresponds to the homomorphism eval: $\mathrm{Xext}_G(N, A) \to \mathrm{EXT}(N, A)$ which forgets the crossed structure. Let $\mathrm{Sext}_G(N, A)$ denote the kernel. Clearly, $\mathrm{Sext}_G(N, A)$ consists of the classes of extensions which are split, but not necessarily G-split.

PROPOSITION 7.1. $\mathrm{Sext}_G(N, A)$ *corresponds to the subgroup of* $H^2_G(N, A)$ *consisting of all classes of the form* $\{(\alpha, 0)\}$. □

A normalized function $\alpha\colon G \times N \to A$ has the property that $(\alpha, 0)$ is a factor system if and only if α satisfies the following three properties:

(i) $\alpha(gh, x) = g \cdot \alpha(h, x) + \alpha(g, hxh^{-1}), \quad g, h \in G, x \in N,$ (7.1)

(ii) $\alpha(g, xy) = \alpha(g, x) + \alpha(g, y), \quad g \in G, x, y \in N,$ (7.2)

(iii) $\alpha(x, y) = 0, \quad x, y \in N.$ (7.3)

Let $\overline{N} = N/[N, N]$, and $(\rho, \pi)\colon (N, G) \to (\overline{N}, Q)$ be the natural projections. Note that $\mathrm{Hom}(\overline{N}, A)$ is a Q-module with action of Q given by $q \cdot \varphi = q_* \circ \varphi \circ q_*^{-1}$. If $\alpha\colon G \times N \to A$ is a normalized function which satisfies (i)–(iii), then $\bar{\alpha}\colon Q \to \mathrm{Hom}(\overline{N}, A)$ defined by $\bar{\alpha}(\pi(g))(\rho(x)) = \alpha(g, x)$ is a normalized two-sided cocycle [2, §5]. To arrive at a left cocycle, define $\hat{\alpha}\colon Q \to \mathrm{Hom}(\overline{N}, A)$ by $\hat{\alpha}(q) = \bar{\alpha}(q) \circ q_*^{-1}$.

Conversely, if $\eta\colon Q \to \mathrm{Hom}(\overline{N}, A)$ is a normalized cocycle, then $\tilde{\eta}\colon G \times N \to A$

defined by $\tilde{\eta}(g, x) = \eta(\pi(g))(\pi(g) \cdot \rho(x))$ is a normalized function which satisfies (i)–(iii). Moreover, if η is cohomologous to η', then $(\tilde{\eta}, 0)$ is cohomologous to $(\tilde{\eta}', 0)$. Hence, we may define a homomorphism inj: $H^1(Q, \mathrm{Hom}(\overline{N}, A)) \to H^2_G(N, A)$ b $\mathrm{inj}\{\eta\} = \{(\tilde{\eta}, 0)\}$. The mappings $\alpha \mapsto \hat{\alpha}$ and $\eta \mapsto \tilde{\eta}$ are inverse to each other. This implies the following theorem.

THEOREM 7.2. *The function* inj: $H^1(Q, \mathrm{Hom}(\overline{N}, A)) \to H^2_G(N, A)$ *is a monomorphism mapping onto the subgroup corresponding to the split crossed extensions.* □

Since N acts trivially on A, $H^1(N, A) \cong \mathrm{Hom}(\overline{N}, A)$ by the universal coefficient theorem. Let ζ: $H^1(Q, H^1(N, A)) \to \mathrm{Xext}_G(N, A)$ be the homomorphism corresponding to inj.

COROLLARY 7.3. *The following sequence is exact*:

$$0 \to H^1(Q, H^1(N, A)) \xrightarrow{\zeta} \mathrm{Xext}_G(N, A) \xrightarrow{\varepsilon} H^2(N, A). \quad \square$$

Consider the extension

$$E: 1 \to N \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1.$$

The *fundamental class* of E is the class in $H^2(Q, \overline{N})$ corresponding to the extension

$$\overline{E}: 0 \to \overline{N} \xrightarrow{\bar{\iota}} G/[N, N] \xrightarrow{\bar{\pi}} Q \to 1.$$

More precisely, let v: $Q \to G$ be inverse to π. Define $\bar{\lambda}$: $Q \times Q \to \overline{N}$ by $\bar{\lambda}(q, r) = \rho(v(q)v(r)v(qr)^{-1})$. Then $\bar{\lambda}$ represents the fundamental class of E. We call $\bar{\lambda}$ the *fundamental cocycle* of E determined by v.

The cup product homomorphism cup: $H^1(Q, \mathrm{Hom}(\overline{N}, A)) \to H^3(Q, A)$ is defined by $\mathrm{cup}\{\eta\} = \{\eta \cup \bar{\lambda}\}$ where $\eta \cup \bar{\lambda}(q, r, s) = \eta(q)(q \cdot \bar{\lambda}(r, s))$, see [2, §4], [9, p. 248].

PROPOSITION 7.4. *The following diagram commutes*:

$$\begin{array}{ccc} H^1(Q, H^1(N, A)) & & \\ \downarrow \zeta & \searrow \mathrm{cup} & \\ \mathrm{Xext}_G(N, A) & \xrightarrow{\delta} & H^3(Q, A) \end{array}$$

PROOF. Observe that $\delta \circ \zeta\{\eta\} = \{\kappa\}$ where κ is the obstruction cocycle determined by $(\tilde{\eta}, 0)$ and v. Formula (6.2) implies that $\kappa = \eta \cup \bar{\lambda}$ where $\bar{\lambda}$ is the fundamental cocycle determined by v. □

8. The main theorem.

THEOREM 8.1. *If G is a group, N is a normal subgroup of G, and A is a Q-module where $Q = G/N$, then there is a natural exact sequence*

$$H^2(Q, A) \xrightarrow{\mathrm{inf}} H^2(G, A) \xrightarrow{\gamma} \mathrm{Xext}_G(N, A) \xrightarrow{\delta} H^3(Q, A) \xrightarrow{\mathrm{inf}} H^3(G, A)$$

where γ is induced by restriction and δ maps the class of a crossed extension to its Mac Lane-Whitehead obstruction.

PROOF. Suppose $1 \to R \overset{i}{\to} F \overset{p}{\to} G \to 1$ is a presentation for G as a quotient of a free group F. Then there is induced presentations for Q and N

$$\begin{array}{ccccccccc} 1 & \to & S & \overset{j}{\to} & F & \overset{\pi \circ p}{\to} & Q & \to & 1 \\ 1 & \to & R & \overset{k}{\to} & S & \overset{t}{\to} & N & \to & 1 \end{array}$$

where i, j, k are inclusions and t is the restriction of p. Since p: $(F, S) \to (G, N)$ is a homomorphism of normal group pairs, p induces a homomorphism p^*: $\mathrm{Xext}_G(N, A) \to \mathrm{Xext}_F(S, A)$. Since k is an F-morphism, it induces a homomorphism k^*: $\mathrm{Xext}_F(S, A) \to \mathrm{Xext}_F(R, A)$. The idea of the proof is to show that the following diagram commutes:

$$\begin{array}{ccccccccc} \mathrm{EXT}(Q, A) & \overset{\mathrm{inf}}{\to} & \mathrm{EXT}(G, A) & \overset{\mathrm{res}}{\to} & \mathrm{Xext}_G(N, A) & \overset{p^*}{\to} & \mathrm{Xext}_F(S, A) & \overset{k^*}{\to} & \mathrm{Xext}_F(R, A) \\ \downarrow\omega & & \downarrow\omega & & \downarrow\omega & & \downarrow\delta & & \downarrow\delta \\ H^2(Q, A) & \overset{\mathrm{inf}}{\to} & H^2(G, A) & \overset{\mathrm{res}}{\to} & H^2_G(N, A) & \overset{\mathrm{con}}{\to} & H^3(Q, A) & \overset{\mathrm{inf}}{\to} & H^3(G, A) \end{array}$$

and the vertical homomorphisms are isomorphisms and the top row is exact.

The proof proceeds as follows. (i) We show that the vertical maps are isomorphisms. (ii) We show that the diagram commutes. (iii) We prove exactness of the top row at $\mathrm{Xext}_G(N, A)$. (iv) We prove exactness at $\mathrm{Xext}_F(S, A)$. (v) We show that sequence (1.1) is natural.

(i) The first three vertical maps are natural equivalences. Since S is a free group, $\mathrm{Xext}_F(S, A) = \mathrm{Sext}_F(S, A)$. By Corollary 7.3, ζ: $H^1(Q, H^1(S, A)) \to \mathrm{Xext}_F(S, A)$ is an isomorphism. By the cup product reduction theorem [2], cup: $H^1(Q, H^1(S, A)) \to H^3(Q, A)$ is an isomorphism. Hence, by Proposition 7.4, δ: $\mathrm{Xext}_F(S, A) \to H^3(Q, A)$ is an isomorphism. Similarly, δ: $\mathrm{Xext}_F(R, A) \to H^3(G, A)$ is an isomorphism.

(ii) Clearly the first two squares of the diagram commute. To see that the third square commutes, consider the sequence of crossed extensions

$$\begin{array}{ccccccccccc} E\colon & 0 & \to & A & \to & C & \overset{\partial}{\to} & N & \to & 1 \\ E(t, p)\colon & 0 & \to & A & \to & C' & \overset{\partial'}{\to} & S & \to & 1 \end{array}$$

where $C' = \{(c, s) \in C \times S \,|\, \partial(c) = t(s)\}$. Let w: $Q \to F$ be inverse to $\pi \circ p$: $F \to Q$, then $v = p \circ w$ is inverse to π: $G \to Q$. Choose a normalized function μ: $Q \times Q \to C$ such that $\partial\mu(q, r) = v(q)v(r)v(qr)^{-1}$. Define μ': $Q \times Q \to C'$ by $\mu'(q, r) = (\mu(q, r), w(q)w(r)w(qr)^{-1})$. Let κ, κ' be the obstruction cocycle of E, $E(t, p)$ determined by v and μ, and v and μ' respectively. One checks easily that $\kappa = \kappa'$. Hence $\delta \circ p^* = \delta = \mathrm{con} \circ \omega$ and the third square commutes. The fourth square commutes since the homomorphism ζ and the cup product homomorphism are natural.

(iii) Proposition 5.1 says that the top row is exact at $\mathrm{EXT}(G, A)$. To see exactness at $\mathrm{Xext}_G(N, A)$, consider the sequence of extensions

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \to & B & \xrightarrow{\eta} & G & \to & 1 \\ E_N: & 0 & \to & A & \to & \eta^{-1}(N) & \xrightarrow{\partial} & N & \to & 1 \\ E_N(t,p): & 0 & \to & A & \to & C & \xrightarrow{\partial'} & S & \to & 1 \end{array}$$

where $C = \{(b, s) \in \eta^{-1}(N) \times S | \partial(b) = t(s)\}$. Since F is free, the homomorphism p: $F \to G$ lifts to a homomorphism φ: $F \to B$ such that $\eta \circ \varphi = p$. Define σ: $S \to C$ by $\sigma(s) = (\varphi(s), s)$. Then σ is a F-morphism which splits $E_N(t, p)$. By Proposition 2.1, $E_N(t, p) \equiv_F E_0$. This shows that Im res $\subset \ker p^*$.

To see the reverse inclusion, consider a sequence of crossed extensions

$$\begin{array}{lccccccccc} E: & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & N & \to & 1 \\ E(t,p): & 0 & \to & A & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & S & \to & 1 \end{array}$$

such that $E(t, p) \equiv_F E_0$. By Proposition 2.1, there is an F-morphism σ': $S \to C'$ which splits $E(t, p)$. Since $\partial' \circ \sigma' = \mathrm{id}_S$, there is a homomorphism σ: $S \to C$ such that $\sigma'(s) = (\sigma(s), s)$. Since $\partial \circ \sigma(R) = t(R) = 1$, σ induces a homomorphism $\bar{\sigma}$: $R \to A$.

Observe that p: $F \to G$ induces an F-module structure on A which allows us to form the semidirect product $A \rtimes F$. Since S acts trivially on A, the direct product $A \times S$ sits naturally in $A \rtimes F$. Let $T = \{(-\bar{\sigma}(r), r) \in A \times R | r \in R\}$. Note that T is a normal subgroup of $A \rtimes F$, since $\bar{\sigma}(frf^{-1}) = p(f) \cdot \bar{\sigma}(r)$ for $r \in R$ and $f \in F$. Let $B = (A \rtimes F)/T$ and let $[a, t]$ denote the image of an element (a, t). Define $\bar{i}$: $A \to B$ by $\bar{i}(a) = [a, 1]$, $\bar{p}$: $B \to G$ by $\bar{p}[a, f] = p(f)$ and $\bar{\rho}$: $F \to B$ by $\bar{\rho}(f) = [0, f]$. We have following push out diagram:

$$\begin{array}{lccccccccc} P: & 1 & \to & R & \xrightarrow{i} & F & \xrightarrow{p} & G & \to & 1 \\ & & & \bar{\sigma}\downarrow & & \bar{\rho}\downarrow & & \| & & \\ \bar{\sigma}P: & 0 & \to & A & \xrightarrow{\bar{i}} & B & \xrightarrow{\bar{p}} & G & \to & 1 \end{array}$$

We claim that $(\bar{\sigma}P)_N \equiv_G E$. Consider the crossed extension

$$(\bar{\sigma}P)_N: 0 \to A \to \bar{p}^{-1}(N) \xrightarrow{\bar{\partial}} N \to 1.$$

Observe that $\bar{p}^{-1}(N) = (A \times S)/T$. Define φ: $A \times S \to C$ by $\varphi(a, s) = \iota(a)\sigma(s)$. Since $\iota(A)$ is in the center of C, φ is a homomorphism. Since $T \subset \ker \varphi$, φ induces a homomorphism $\bar{\varphi}$: $\bar{p}^{-1}(N) \to C$ given by $\bar{\varphi}[a, s] = \iota(a)\sigma(s)$. We have the following congruence diagram:

$$\begin{array}{lccccccccc} (\bar{\sigma}P)_N: & 0 & \to & A & \xrightarrow{\bar{i}} & \bar{p}^{-1}(N) & \xrightarrow{\bar{\partial}} & N & \to & 1 \\ & & & \| & & \downarrow\bar{\varphi} & & \| & & \\ E: & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & N & \to & 1 \end{array}$$

Moreover, $\bar{\varphi}$ is a G-homomorphism, so that $(\bar{\sigma}P)_N \equiv_G E$ via $\bar{\varphi}$. This shows that $\ker p^* \subset$ Im res.

(iv) To see the exactness of the top row at $\mathrm{Xext}_F(S, A)$, observe that $k^* \circ p^* = (k, \mathrm{id}_F)^* \circ (t, p)^* = (t \circ k, p)^*$ which is trivial, since $t \circ k$ is trivial. To see that ker $k^* \subset \operatorname{Im} p^*$, suppose

$$E\colon 0 \to A \xrightarrow{\iota} C \xrightarrow{\partial} S \to 1$$

represents an element in the kernel of k^*. The image of E under k^* is represented by

$$E_R\colon 0 \to A \xrightarrow{\iota} \partial^{-1}(R) \xrightarrow{\eta} R \to 1$$

where η is the restriction of ∂. By Proposition 2.1, the extension E_R splits via an F-morphism $\sigma\colon R \to \partial^{-1}(R)$. Since σ is an F-morphism, $\sigma(R)$ is a normal subgroup of C. Let $C_\sigma = B/\sigma(R)$ and let $\tau\colon C \to C_\sigma$ be the natural projection. Since $\partial(\sigma(R)) = R$, ∂ induces a homomorphism $\partial_\sigma\colon C_\sigma \to N$. Let $\iota_\sigma = \tau \circ \iota$. One checks easily that

$$E_\sigma\colon 0 \to A \xrightarrow{\iota_\sigma} C_\sigma \xrightarrow{\partial_\sigma} N \to 1$$

is a G-crossed extension with action of G on C_σ given by $p(f) \cdot \tau(c) = \tau(f \cdot c)$. We have the following commutative diagram:

$$\begin{array}{cccccccccc}
E\colon & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & S & \to & 1 \\
 & & & \| & & \downarrow\tau & & \downarrow t & & \\
E_\sigma\colon & 0 & \to & A & \xrightarrow{\iota_\sigma} & C_\sigma & \xrightarrow{\partial_\sigma} & N & \to & 1
\end{array}$$

By Proposition 4.1, $E \equiv_F (E_\sigma)(t, p)$. Hence $\{E\}$ is in the image of p^*.

(v) The bottom cohomology exact sequence is easily seen to be natural in A, while the top extension group sequence is easily seen to be natural in the extension $1 \to N \to G \to Q \to 1$. Moreover, the vertical isomorphisms are also natural. It follows that the exact sequence (1.1) is completely natural. □

9. An interpretation of the third cohomology group. A G-crossed module C determines a 4-term exact sequence

$$S\colon 0 \to A \xrightarrow{\iota} C \xrightarrow{\partial} G \xrightarrow{\pi} Q \to 1 \tag{9.1}$$

called a *crossed sequence* by S. Mac Lane and J. H. C. Whitehead in **[10]**. Suppose S and S' are two crossed sequences from A to Q. We say that S is *simply congruent* to S' if there is a homomorphism of crossed modules $(\varphi, h)\colon C \to C'$ such that the following diagram commutes:

$$\begin{array}{cccccccccccc}
S\colon & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & G & \xrightarrow{\pi} & Q & \to & 1 \\
 & & & \| & & \downarrow\varphi & & \downarrow h & & \| & & \\
S'\colon & 0 & \to & A & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & G' & \xrightarrow{\pi'} & Q & \to & 1
\end{array}$$

This relation of simple congruence generates an equivalence relation on the set of all crossed sequences from A to Q which we call *congruence*. Let $\operatorname{Xseq}(Q, A)$ denote the set of congruence classes.

The set $\operatorname{Xseq}(Q, A)$ is nonempty since it contains the class of the trivial crossed sequence

$$S_0\colon 0 \to A = A \to Q = Q \to 1.$$

The negative of the crossed sequence (9.1) is the crossed sequence

$$-S\colon 0 \to A \xrightarrow{-\iota} C \xrightarrow{\partial} G \xrightarrow{\pi} Q \to 1.$$

If S_1 and S_2 are two crossed sequences from A to Q, then the Baer sum $S_1 + S_2 = \nabla((S_1 \times S_2)\Delta)$ is also a crossed sequence from A to Q.

Suppose that S is simply congruent to S' via (φ, h): $C \to C'$. The naturality of the connecting homomorphism δ: $\mathrm{Xext}_G(N, A) \to H^3(Q, A)$ implies that S and S' determine the same obstruction in $H^3(Q, A)$. Thus we may define a function η: $\mathrm{Xseq}(Q, A) \to H^3(Q, A)$ which maps the class of a crossed sequence to its obstruction.

THEOREM 9.1. *The function* η: $\mathrm{Xseq}(Q, A) \to H^3(Q, A)$ *is a bijection; moreover,*
(i) $\eta\{S_0\} = 0$,
(ii) $\eta\{-S\} = -\eta\{S\}$,
(iii) $\eta\{S_1 + S_2\} = \eta\{S_1\} + \eta\{S_2\}$.

PROOF. Let P: $1 \to R \to F \xrightarrow{p} Q \to 1$ be a presentation for Q as the quotient of a free group. Define σ: $\mathrm{Xext}_F(R, A) \to \mathrm{Xseq}(Q, A)$ by $\sigma\{X\} = \{X \circ P\}$. Clearly $\eta \circ \sigma = \delta$. Since δ is an isomorphism, σ is injective.

To see that σ is surjective, suppose we are given a crossed sequence (9.1). Since F is free, p: $F \to Q$ lifts to a homomorphism h: $F \to G$ such that $\pi \circ h = p$. Observe that h restricts to a homomorphism ψ: $R \to N$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \to & R & \to & F & \xrightarrow{p} & Q & \to & 1 \\ & & \downarrow\psi & & \downarrow h & & \| & & \\ 1 & \to & N & \to & G & \xrightarrow{\pi} & Q & \to & 1 \end{array}$$

Consider the pull-back diagram:

$$\begin{array}{cccccccccc} X(\psi, h)\colon & 0 & \to & A & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & R & \to & 1 \\ & & & \| & & \downarrow\varphi & & \downarrow\psi & & \\ X\colon & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & N & \to & 1 \end{array}$$

By composing the two diagrams, we see that $X(\psi, h) \circ P$ is simply congruent to S via (φ, h). This shows that σ is surjective. Hence σ and η are bijections. The verifications of (i)–(iii) is straightforward. □

The equivalence of $H^3(Q, A)$ with $\mathrm{Xseq}(Q, A)$ gives a nice interpretation of the connecting homomorphism δ: $\mathrm{Xext}_G(N, A) \to H^3(Q, A)$ in terms of the composition of extensions. Consider the extension

$$E\colon 1 \to N \to G \xrightarrow{\pi} Q \to 1.$$

Define σ: $\mathrm{Xext}_G(N, A) \to \mathrm{Xseq}(Q, A)$ by $\sigma\{X\} = \{X \circ E\}$. Observe that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Xext}_G(N, A) & \xrightarrow{\sigma} & \mathrm{Xseq}(Q, A) \\ & \searrow\delta & \downarrow\eta \\ & & H^3(Q, A) \end{array}$$

This explains the compatibility of δ with cup products.

Note that the crossed sequence (9.1) is the composite of the extensions

$$X\colon\ 0 \to A \xrightarrow{\iota} C \xrightarrow{\partial} N \to 1$$

$$E\colon\ 0 \to N \to G \xrightarrow{\pi} Q \to 1$$

COROLLARY 9.2. *A crossed sequence $S = X \circ E$ from A to Q represents the zero element of* $\operatorname{Xseq}(Q, A)$ *if and only if the crossed extension X is extendable.*

PROOF. If X is extendable, then $\{X\}$ is in the image of $\gamma\colon H^2(G, A) \to \operatorname{Xext}_G(N, A)$. Hence by Theorem 8.1, $\delta\{X\} = 0$, so $\sigma\{X\} = \{S\}$ is zero.

Conversely, if $\{S\} = 0$, then $\delta\{X\} = 0$. Hence, X is extendable by Theorem 8.1. □

If $\zeta\colon A \to A'$ is a Q-module homomorphism, define the push out crossed sequence induced from S by ζ by $\zeta S = \zeta X \circ E$. If $f\colon Q' \to Q$ is a homomorphism, define the pull-back crossed sequence induced from S by f by $Sf = X \circ Ef$. Here $G' = \{(g, q') \in G \times Q' | \pi(g) = f(q')\}$ acts on C by $(g, q') \cdot c = g \cdot c$.

PROPOSITION 9.3. *If S is a crossed sequence from A to Q, $\zeta\colon A \to A'$ is a Q-module homomorphism, $f\colon Q \to Q'$ is a homomorphism, and S' is a crossed sequence from A' to Q', then ζS is simply congruent to $S'f$ if and only if there is a homomorphism of crossed modules* (φ, h) *such that the following diagram commutes*:

$$\begin{array}{ccccccccccc} S\colon & 0 & \to & A & \xrightarrow{\iota} & C & \xrightarrow{\partial} & G & \xrightarrow{\pi} & Q & \to & 1 \\ & & & \downarrow\zeta & & \downarrow\varphi & & \downarrow h & & \downarrow f & & \\ S'\colon & 0 & \to & A' & \xrightarrow{\iota'} & C' & \xrightarrow{\partial'} & G' & \xrightarrow{\pi'} & Q' & \to & 1 \end{array}$$ □

Proposition 9.3 implies that if S_1 is simply congruent to S_2, then ζS_1 is simply congruent to ζS_2. Thus we may define a function $\zeta_*\colon \operatorname{Xseq}(Q, A) \to \operatorname{Xseq}(Q, A')$ by $\zeta_*\{S\} = \{\zeta S\}$. If κ is an obstruction cocycle for S, then $\zeta \circ \kappa$ is an obstruction cocycle for ζS. Hence ζ_* corresponds to $\zeta_*\colon H^3(Q, A) \to H^3(Q, A')$.

Similarly, Proposition 9.3 implies that if S_1 is simply congruent to S_2, then $S_1 f$ is simply congruent to $S_2 f$. Thus we may define a function $f^*\colon \operatorname{Xseq}(Q', A) \to \operatorname{Xseq}(Q, A)$ by $f^*\{S\} = \{Sf\}$. If κ is an obstruction cocycle for S, then $\kappa \circ f \times f \times f$ is an obstruction cocycle for Sf. Hence f^* corresponds to $f^*\colon H^3(Q', A) \to H^3(Q, A)$. Summarizing, we have the following theorem.

THEOREM 9.4. *The natural transformation* $\eta\colon \operatorname{Xseq} \to H^3$ *defined by mapping the class of a crossed sequence to its Mac Lane-Whitehead obstruction is an equivalence of functors.* □

One should compare this section with Mac Lane's treatment of crossed sequences in [8]. There he proved that the third cohomology group was equivalent to the equivalence classes of crossed sequences with a fixed free group of operators (in our terminology, that $\delta\colon \operatorname{Xext}_F(R, A) \to H^3(Q, A)$ in the proof of Theorem 9.1 is an isomorphism). Mac Lane just missed proving Theorem 9.1. What he lacked at this point was the definition of congruence of 3-fold exact sequences, in particular for crossed sequences.

The definition of congruence for n-fold exact sequences was given by Yoneda in [16]. Yoneda's theory gave the interpretation of $\mathrm{EXT}^n_Q(\mathbf{Z}, A) \cong H^n(Q, A)$ as congruences classes of n-fold exact sequences of Q-modules from A to $\mathbf{Z}$. Y.-C. Wu [15], using only extension theory, proved the equivalence $\mathrm{EXT}^3_Q(\mathbf{Z}, A) \cong \mathrm{Xseq}(Q, A)$ modulo the correction that his definition of a special 2-fold exact sequence must be strengthened to that of a crossed sequence. Leedham-Green and McKay [6] first proved the equivalence $\mathrm{Xseq}(Q, A) \cong H^3(Q, A)$ using category theory and varietal cohomology.

10. Application to abstract kernels. If E: $1 \to K \to B \to Q \to 1$ is an extension of groups, then E determines a homomorphism ψ: $Q \to \mathrm{Out}(K)$. Conversely, call a pair (K, ψ) consisting of a group K and a homomorphism ψ: $Q \to \mathrm{Out}(K)$ an *abstract kernel.* A classical problem of group extension theory is that of determining whether or not an abstract kernel is realized by an extension.

R. Baer [1] approached the problem as follows: Consider the pull-back diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{In}(K) & \xrightarrow{i} & G & \xrightarrow{\pi} & Q & \to & 1 \\ & & \| & & \downarrow\theta & & \downarrow\psi & & \\ 1 & \to & \mathrm{In}(K) & \xrightarrow{j} & \mathrm{Aut}(K) & \xrightarrow{\tau} & \mathrm{Out}(K) & \to & 1 \end{array}$$

where $G = \{(\varphi, q) \in \mathrm{Aut}(K) \times Q \,|\, \tau(\varphi) = \psi(q)\}$. The group G is called the *graph* of (K, ψ). One sees easily that the extension

$$X: 0 \to A \xrightarrow{\iota} K \xrightarrow{\partial} N \to 1$$

where $A = Z(K)$ and $N = \mathrm{In}(K)$ is a G-crossed extension with action of G on K given by θ.

Baer showed that the kernel (K, ψ) is realizable if and only if X is extendable as in the sense of §5 to an extension X' of A by G. To see this, suppose X' extends X. Then there is a homomorphism ζ: $K \to B$ such that the following diagram commutes:

$$\begin{array}{cccccccccc} X: & 0 & \to & A & \xrightarrow{\iota} & K & \xrightarrow{\partial} & N & \to & 1 \\ & & & \| & & \downarrow\zeta & & \downarrow i & & \\ X': & 0 & \to & A & \xrightarrow{\iota'} & B & \xrightarrow{\partial'} & G & \to & 1 \end{array}$$

Define η: $B \to Q$ by $\eta = \pi \circ \partial'$. One sees easily that

$$E: 1 \to K \xrightarrow{\zeta} B \xrightarrow{\eta} Q \to 1$$

is an extension which realizes (K, ψ).

Conversely, suppose E realizes (K, ψ). Define ι': $A \to B$ by $\iota' = \zeta \circ \iota$, and ∂': $B \to G$ by $\partial'(b) = (b_*, \eta(b))$ where b_* is the automorphism of K induced by conjugation by b. One sees easily that the following sequence extends X:

$$X': 0 \to A \xrightarrow{\iota'} B \xrightarrow{\partial'} G \to 1.$$

Later, Eilenberg and Mac Lane [3] showed that an abstract kernel (K, ψ) determines an obstruction in $H^3(Q, A)$ which vanishes if and only if the kernel is realizable. Next, Robert Taylor [13] tied Baer's and Eilenberg and Mac Lane's work together when he showed that a crossed extension is extendable if and only if its obstruction vanishes, and then showed that the obstruction of the crossed extension determined by an abstract kernel is the Eilenberg-Mac Lane obstruction of the kernel. Summarizing, we obtain the following theorem.

Theorem 10.1. *If (K, ψ) is an abstract kernel and G is its graph, then (K, ψ) determines a G-crossed extension*

$$X: 0 \to A \xrightarrow{\iota} K \xrightarrow{\partial} N \to 1$$

where $A = Z(K)$ and $N = \mathrm{In}(K)$, and the following are equivalent:

(i) *(K, ψ) is realizable;*

(ii) *X is extendable;*

(iii) *$\{X\}$ is in the image of γ: $H^2(G, A) \to \mathrm{Xext}_G(N, A)$;*

(iv) *$\{X\}$ is in the kernel of δ: $\mathrm{Xext}_G(N, A) \to H^3(Q, A)$;*

(v) *the Mac Lane-Whitehead obstruction of X vanishes;*

(vi) *the Eilenberg-Mac Lane obstruction of (K, ψ) vanishes.*

Proof. The equivalence of (i) and (ii) is due to Baer. The equivalence of (ii) and (iii), and (iv) and (v) is by definition. The equivalence of (iii) and (iv) is by exactness at $\mathrm{Xext}_G(N, A)$ in the sequence (1.1). The equivalence of (v) and (vi) is by Taylor. The equivalence of (vi) and (i) is by Eilenberg and Mac Lane. □

We see that the problem of realizing an abstract kernel can be translated into the problem of extending a crossed extension. To understand this problem better, consider the following commutative exact diagram:

$$\begin{array}{ccccc} & & H^1(Q, H^1(N, A)) & & \\ & & \downarrow \zeta & \searrow \text{cup} & \\ H^2(G, A) & \xrightarrow{\gamma} & \mathrm{Xext}_G(N, A) & \xrightarrow{\delta} & H^3(Q, A) \\ & \searrow \text{res} & \downarrow \varepsilon & & \\ & & H^2(N, A) & & \end{array} \tag{10.1}$$

Observe that if E is a G-crossed extension of A by N, then $\{E\} \in \mathrm{Im}\,\gamma$ only if $\varepsilon\{E\} \in \mathrm{Im}$ res. In other words, a necessary condition for E to be extendable is that E, regarded as a central extension of A by N, represents an element in the image of res: $H^2(G, A) \to H^2(N, A)$. By chasing the diagram (10.1), one sees that $\varepsilon\{E\} \in$ Im res if and only if $\delta\{E\} \in$ Im cup. A more general relation is that Im ε/Im res $\cong$ Im δ/Im cup.

If $\varepsilon\{E\} \in$ Im res, then there is an extension E' of A by G such that $E'_N \equiv E$, but is not necessarily G-congruent. Observe that $E - E'_N$ determines an element κ of $H^1(Q, H^1(N, A))$ and E is extendable if and only if $\kappa \in$ ker cup. Thus we are led to the problem of determining the kernel of the cup product homomorphism cup: $H^1(Q, H^1(N, A)) \to H^3(Q, A)$. One sees from the diagram (10.1) that ζ(ker cup) = γ(ker res). This implies that ker cup $\cong$ ker res/ker γ.

In [11], we elaborate on these ideas and show the relationship between the diagram (10.1) and the transgression τ: $H^2(N, A) \to H^3(Q, A)$ of the Lyndon-Hochschild-Serre spectral sequence.

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