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PRINCIPAL 2-BLOCKS OF THE SIMPLE GROUPS OF REE TYPE

BY

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ABSTRACT. The decomposition numbers in characteristic 2 of the groups of Ree type are determined, as well as the Loewy and socle series of the indecomposable projective modules. Moreover, we describe the Green correspondents of the simple modules. As an application of this and similar works on other simple groups with an abelian Sylow 2-subgroup, all of which have been classified apart from those considered in the present paper, we show that the Loewy length of an indecomposable projective module in the principal block of any finite group with an abelian Sylow 2-subgroup of order 2^n is bounded by $\max\{2n + 1, 2^n\}$. This bound is the best possible.

Introduction. The purpose of this paper is to determine the algebra structure of the principal 2-blocks B of the simple groups $R(q)$ of Ree type of order $|R(q)| = (q^3 + 1)q^3(q - 1)$, where $q = 3^{2n+1}$, and $n = 1, 2, \ldots$.

Let (F, R, S) be a splitting 2-modular system for $R(q)$, where F has characteristic 2, and S and R have characteristic zero. The character table of the groups $R(q)$ was determined by Ward [15] up to a few but very essential values missing because of the incomplete classification of these groups. Ward [15] also showed that B contains eight ordinary irreducible characters ξ_i, all of height zero, and five nonisomorphic simple $FR(q)$-modules φ_i, $i = 1, 2, \ldots, 5$, where $\varphi_1 = I$ denotes the trivial $FR(q)$-module. In [7], Fong determined the decomposition matrix up to three parameters a, b and c. He also showed that we may choose notation so that φ_2 and φ_3 are algebraically invariant and self-dual, while φ_4 and φ_5 are the duals and the algebraic conjugates of each other.

In §§2 and 3 we complete the decomposition matrix of B by showing that $a = 2$ and $b = c = 1$ for every $n = 1, 2, \ldots$ (Theorem 3.9).

If M is a finitely generated FG-module, G a finite group, then $\mathrm{soc}(M) = S_1(M)$ denotes the socle of M which is the sum of all simple FG-modules of M. Let $S_{i+1}(M)/S_i(M) = S_1(M/S_i(M))$. Then $0 < S_1(M) < S_2(M) < \cdots < S_{k-1}(M) < S_k(M) = M$ is called the socle series of M, and k is the socle length of M which coincides with the Loewy length $j(M)$ of M. If J denotes the Jacobson radical of FG, then $j(M)$ is the uniquely determined integer j with $MJ^{j-1} \neq 0 = MJ^j$. In order to describe the socle series of an FG-module M, we associate to M a matrix

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which in the ith row vector has the composition factors of $S_i(M)/S_{i-1}(M)$ with their multiplicities.

Let D be a Sylow 2-subgroup of $R(q)$. Then D is an elementary abelian group D of order $|D| = 8$. Its normalizer $N = N_{R(q)}(D)$ is a holomorph of D by a noncyclic Frobenius group F_{21} of order $|F_{21}| = 21$. Hence FN is a block algebra. Its structure was determined by the authors in **[12]**. It has five nonisomorphic simple FN-modules, I and four others denoted by 1, 1*, 3 and 3*, where our notation indicates the degree of the simple FN-module, and where m^* denotes the dual of m. By Knörr's theorem **[10]**, each simple B-module φ_i has vertex $\mathrm{vx}(\varphi_i) =_{R(q)} D$. Therefore we consider the Green correspondence f with respect to D between the indecomposable $FR(q)$-modules of B and the indecomposable FN-modules. The main results of §§2 and 5 are collected in Theorem 5.3 which asserts that the socle series of the Green correspondents $f(\varphi_i)$ are:

(a) $f(I) = I, f(\varphi_4) = 1^*$, and $f(\varphi_5) = 1$,

(b) $f(\varphi_2) = \begin{smallmatrix} 3^* \\ 3 \end{smallmatrix}$,

(c) $f(\varphi_3) = \begin{array}{ccc} & 3 & \\ 3 & & 3^* \\ & 3^* & \end{array}$.

These Green correspondents are used to compute the parameters a and b of the decomposition matrix of B in §§2 and 3. Furthermore, they enable us to find the structure of the indecomposable projective B-modules P_i, where P_i denotes the projective cover of φ_i, $i = 1, 2, \ldots, 5$.

THEOREM 4.1. *The indecomposable projective $FR(q)$-modules of the principal 2-block B of any simple group $R(q)$ of Ree type have Loewy and socle series*:

(a)

$$P_I:\quad \begin{array}{ccccccc} & & & I & & & \\ & & & \varphi_2 & & & \\ \varphi_3 & & \varphi_4 & & \varphi_5 & & I \\ & \varphi_2 & & & & \varphi_2 & \\ \varphi_3 & & \varphi_4 & & \varphi_5 & & I \\ & & & \varphi_2 & & & \\ & & & I & & & \end{array}$$

(b)

$$P_2:\quad \begin{array}{ccccccc} & & & \varphi_2 & & & \\ I & & \varphi_3 & & \varphi_4 & & \varphi_5 \\ & \varphi_2 & & \varphi_2 & & \varphi_2 & \\ I & \varphi_5 & \varphi_4 & \varphi_3 & \varphi_4 & \varphi_5 & I \\ & \varphi_2 & & \varphi_2 & & \varphi_2 & \\ I & & \varphi_3 & & \varphi_4 & & \varphi_5 \\ & & & \varphi_2 & & & \end{array}$$

(c)

$$P_3: \quad \begin{array}{ccc} & \varphi_3 & \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \\ & \varphi_3 & \end{array}$$

(d)

$$P_4 = P_5^*: \quad \begin{array}{ccccccc} & & \varphi_4 & & & & \\ & & & \varphi_2 & & & \\ \varphi_3 & & \varphi_4 & & \varphi_5 & & I \\ & \varphi_2 & & & & \varphi_2 & \\ \varphi_3 & & \varphi_4 & & \varphi_5 & & I \\ & & & \varphi_2 & & & \\ & & & \varphi_4 & & & \end{array}$$

Finally combining Theorem 4.1 together with recent work of J. Alperin [1], K. Erdmann [5] and the authors [12] we show in §6 that every indecomposable projective B_0-module P of the principal 2-block B_0 of an arbitrary finite group G with an abelian Sylow 2-subgroup D of order $|D| = 2^n$ has Loewy length $j(P) < \max\{2n + 1, 2^n\}$ (Theorem 6.1). This upper bound is sharp.

Concerning our terminology and notation we refer to Dornhoff [4], Feit [6], Gorenstein [8], and Green [9]. Discussions with L. Scott have been helpful.

1. Known results on the groups of Ree type. In this section we collect some known facts on the groups $R(q)$ of Ree type of order $|R(q)| = (q^3 + 1)q^3(q - 1)$, where $q = 3^{2n+1}$, $m = 3^n$, and $n = 0, 1, 2, \ldots$. These results either can be found in Ward's paper [15] or are due to Fong [7].

Throughout this section, (F, R, S) denotes a splitting 2-modular system for $R(q)$ and all its subgroups.

By E_n we denote an elementary abelian group of order n. Let E_8 be a fixed Sylow 2-subgroup of $R(q)$ and fix an involution $1 \neq u \in E_8$. All involutions of $R(q)$ are conjugate.

If U is a subgroup of $R(q)$, then $N(U)$ and $C(U)$ denote its normalizer and centralizer respectively in $R(q)$. By Ward [15], in each group $R(q)$ of Ree type the centralizer of the involution u has the form

$$C = C(u) = \langle u \rangle \times \mathrm{PSL}(2, q),$$

where $q \equiv 3 \bmod 8$, and E_4 is a Sylow 2-subgroup of $\mathrm{PSL}(2, q)$.

Let $N = N(E_8)$. Then N is a holomorph of E_8 by the Frobenius group F_{21} of order 21. Let E be the normal subgroup of N with index 3. Then $E = E_8 \cdot \mathbf{Z}_7$ is a Frobenius group, where $\mathbf{Z}_r$ denotes the cyclic group of order r. Furthermore,

$$K = N \cap C = \langle u \rangle \times \mathfrak{A}_4.$$

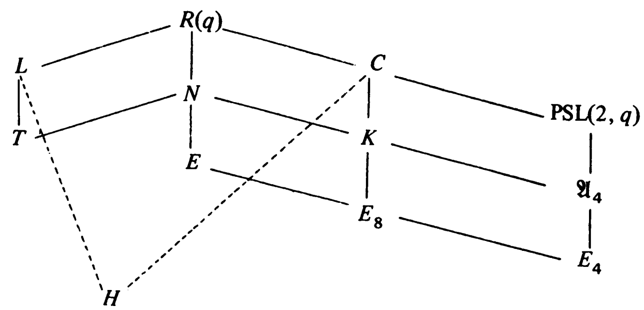
Let P be a fixed Sylow 3-subgroup of $R(q)$ such that its normalizer $L = N(P)$ contains the involution u. By [15], L is a holomorph of P by $\langle u \rangle \times \mathbf{Z}_{(q-1)/2}$, and

$$H = L \cap C = \langle u \rangle \times (E_q \cdot \mathbf{Z}_{(q-1)/2}),$$

where $E_q \cdot \mathbf{Z}_{(q-1)/2}$ is a Frobenius group. Furthermore, C, L and N are maximal subgroups, and

$$T = L \cap N = \langle u \rangle \times \mathbf{Z}_3.$$

Thus we have the following diagram of subgroups of $R(q)$:



I or I_G denotes the trivial (modular) representation of any group G, and P_I denotes the projective cover of I.

Throughout, $B = (B = eFG,\ \hat{B} = \hat{e}RG,\ B_S = \hat{B} \otimes_R S)$ denotes the principal 2-block of $R(q)$, where e and $\hat{e}$ denote the block idempotents of B and $\hat{B}$ respectively. As in [15], let ξ_i, $i = 1, 2, \ldots, 8$, $\xi_1 = I$, denote the ordinary irreducible characters of B. We also let φ_i, $i = 1, 2, \ldots, 5$, $\varphi_1 = I$, denote the Brauer characters of the corresponding simple $FR(q)$-modules.

Let P_i be the projective cover of φ_i, $i = 1, 2, \ldots, 5$, $P_1 = P_I$. In order to determine the socle series of P_i we have to compute the three parameters a, b and c in Fong's [7] decomposition matrix of B:

character	I	φ_2	φ_3	φ_4	φ_5	degree
I	1	0	0	0	0	1
ξ_2	1	1	0	0	0	$q^2 - q + 1$
ξ_3	1	a	1	b	b	q^3
ξ_4	1	$a - 1$	1	b	b	$q(q^2 - q - 1)$
ξ_5	0	c	0	1	0	$\frac{1}{2}(q - 1)m(q + 1 + 3m)$
ξ_7	0	c	0	0	1	$\frac{1}{2}(q - 1)m(q + 1 + 3m)$
ξ_6	0	$c - 1$	0	1	0	$\frac{1}{2}(q - 1)m(q + 1 - 3m)$
ξ_8	0	$c - 1$	0	0	1	$\frac{1}{2}(q - 1)m(q + 1 - 3m)$

Our computation of the integers a, b and c requires the following character tables and subsidiary results.

The principal 2-block of $C' \cong \mathrm{PSL}(2, q)$ is denoted by $b_0 = b_0(C')$. It has four ordinary characters I, χ_q, χ' and χ'', and three modular characters I, χ' and χ''. Furthermore, $\chi'' = (\chi')^*$. Let P_I', $P_{\chi'}'$ and $P_{\chi''}'$ be the projective covers of the simple b_0-modules of C'.

LEMMA 1.1. (a) *The indecomposable projective modules of the principal block* b_0 *of* $\mathrm{PSL}(2, q)$, $q \equiv 3 \bmod 8$, *have socle and Loewy series*

$$P_I'\colon \begin{matrix} & I & \\ \chi' & & \chi'' \\ & I & \end{matrix}, \qquad P_{\chi'}'\colon \begin{matrix} & \chi' & \\ I & & \chi'' \\ & \chi' & \end{matrix}, \qquad P_{\chi''}'\colon \begin{matrix} & \chi'' & \\ I & & \chi' \\ & \chi'' & \end{matrix}.$$

(b) *The simple modules of* b_0 *may also be considered as the simple modules of* $G = \mathbf{Z}_2 \times \mathrm{PSL}(2, q)$, $q \equiv 3 \bmod 8$, *in which group their projective covers have socle and Loewy series*

$$P_I\colon \begin{matrix} & I & \\ \chi' & I & \chi'' \\ \chi' & I & \chi'' \\ & I & \end{matrix}, \qquad P_{\chi'} = \begin{matrix} & \chi' & \\ I & \chi' & I \\ I & \chi' & I \\ & \chi' & \end{matrix}, \qquad P_{\chi''} = \begin{matrix} & \chi'' & \\ I & \chi'' & \chi' \\ I & \chi'' & \chi' \\ & \chi'' & \end{matrix}$$

(c) *Consider again* χ' *and* χ'' *as* $\mathrm{PSL}(2, q)$*-modules. Then*

$$\chi'_{|\mathfrak{A}_4} = 1 \oplus \textit{projectives}, \quad \chi''_{|\mathfrak{A}_4} = 1^* \oplus \textit{projectives}.$$

PROOF. (a) See K. Erdmann [5].[3]

(b) Since G contains a central involution, this follows from (a) and Green's theorem (see Dornhoff [4, p. 329]).

(c) By K. Erdmann [5], 1 is the Green correspondent of χ' in $\mathfrak{A}_4$, which is the normalizer of a Sylow 2-subgroup. Moreover, $\mathrm{PSL}(2, q)$ contains a dihedral subgroup of order $q - 1$. Let 1, 1^- be the irreducible characters of the principal block of this group. Any other block is of defect 0. We then compute

$$(\chi', 1) = 1, \qquad (\chi', 1^-) = 0,$$

which shows that $\chi'_{|\mathbf{Z}_2} = I \oplus$ projectives. Hence no component of $\chi'_{|\mathfrak{A}_4}$ has $\mathbf{Z}_2$ as a vertex. Since $\chi'' \cong (\chi')^*$ assertion (c) follows.

In order to restate the character table of C' we use Ward's notation [15] for representatives of conjugacy classes of $C' = \mathrm{PSL}(2, q)$. Thus the $R(q)$-conjugacy class of u is identified with the one of J.

[3]This result is due to J. L. Alperin who announced it without details in his paper *Minimal resolutions*, Finite Groups 1972, North-Holland, Amsterdam, 1973, pp. 1–2. MR **50** #1045.

Character table of the normalizer L of a Sylow 3-subgroup of $R(q)$

	centr.	$q^3(q-1)$	q^3	$3q$	$2q^2$	$2q^2$	$3q$	$3q$	$2q$	$2q$	$q-1$	$q-1$	$q(q-1)$
	order	1	3	9	3	3	9	9	6	6	$(q-1)/2$*	$q-1$*	2
defect	element	1	X	Y	T	T^{-1}	YT	YT^{-1}	JT	JT^{-1}	R^a	JR^a	J
1	I_q	1	1	1	1	1	1	1	1	1	1	1	1
	I_q^-	1	1	1	1	1	1	1	-1	-1	1	-1	-1
1	Φ^b	1	1	1	1	1	1	1	1	1	ϵ^{ab}	ϵ^{ab}	1
	Φ^{b-}	1	1	1	1	1	1	1	-1	-1	ϵ^{ab}	$-\epsilon^{ab}$	-1
0	Λ_1	$q-1$	$q-1$	-1	$q-1$	$q-1$	-1	-1	0	0	0	0	0
0	Λ_q	$q(q-1)$	$-q$	0	0	0	0	0	0	0	0	0	0
0	Λ_m	$m(q-1)$	$m(q-1)$	$-m$	$-m+im^2\sqrt{3}$	$-m-im^2\sqrt{3}$	$\frac{1}{2}(m+im\sqrt{3})$	$\frac{1}{2}(m-im\sqrt{3})$	0	0	0	0	0
0	$\bar{\Lambda}_m$	$m(q-1)$	$m(q-1)$	$-m$	$-m-im^2\sqrt{3}$	$-m+im^2\sqrt{3}$	$\frac{1}{2}(m-im\sqrt{3})$	$\frac{1}{2}(m+im\sqrt{3}$	0	0	0	0	0
1	$\Lambda_{m/2}$	$m(q-1)/2$	$m(q-1)/2$	m	$\frac{1}{2}(-m+im^2\sqrt{3})$	$\frac{1}{2}(-m-im^2\sqrt{3})$	$\frac{1}{2}(-m-im\sqrt{3})$	$\frac{1}{2}(-m+im\sqrt{3})$	$\frac{1}{2}(-1+im\sqrt{3})$	$\frac{1}{2}(-1-im\sqrt{3})$	0	0	$\frac{1}{2}(q-1)$
	$\Lambda_{m/2}^-$	$m(q-1)/2$	$m(q-1)/2$	m	$\frac{1}{2}(-m+im^2\sqrt{3})$	$\frac{1}{2}(-m-im^2\sqrt{3})$	$\frac{1}{2}(-m-im\sqrt{3})$	$\frac{1}{2}(-m+im\sqrt{3})$	$\frac{1}{2}(1-im\sqrt{3})$	$\frac{1}{2}(1+im\sqrt{3})$	0	0	$-\frac{1}{2}(q-1)$
1	$\bar{\Lambda}_{m/2}$	$m(q-1)/2$	$m(q-1)/2$	m	$\frac{1}{2}(-m-im^2\sqrt{3})$	$\frac{1}{2}(-m+im^2\sqrt{3})$	$\frac{1}{2}(-m+im\sqrt{3})$	$\frac{1}{2}(-m-im\sqrt{3})$	$\frac{1}{2}(-1-im\sqrt{3})$	$\frac{1}{2}(-1+im\sqrt{3})$	0	0	$\frac{1}{2}(q-1)$
	$\bar{\Lambda}_{m/2}^-$	$m(q-1)/2$	$m(q-1)/2$	m	$\frac{1}{2}(-m-im^2\sqrt{3})$	$\frac{1}{2}(-m+im^2\sqrt{3})$	$\frac{1}{2}(-m+im\sqrt{3})$	$\frac{1}{2}(-m-im\sqrt{3})$	$\frac{1}{2}(1+im\sqrt{3})$	$\frac{1}{2}(1-im\sqrt{3})$	0	0	$-\frac{1}{2}(q-1)$

where $a, b = 1, 2, \ldots, \frac{1}{2}(q-3)$.

*for $a=1$

Character table of $C' = \mathrm{PSL}(2, q)$

central index	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$	$q(q-1)$	$\frac{1}{2}(q(q-1))$
element	1	T	T^{-1}	R^h	S_0^j, j even	S_0^j, j odd	J
I	1	1	1	1	1	1	1
χ_q	q	0	0	1	-1	-1	-1
χ'	$\frac{1}{2}(q-1)$	$\frac{1}{2}(-1+\sqrt{-q})$	$-\frac{1}{2}(1+\sqrt{-q})$	0	-1	1	1
χ''	$\frac{1}{2}(q-1)$	$-\frac{1}{2}(1+\sqrt{-q})$	$\frac{1}{2}(-1+\sqrt{-q})$	0	-1	1	1
χ_k	$q-1$	-1	-1	0	$-(\varepsilon^{jk}+\varepsilon^{-jk})$	$-(\varepsilon^{jk}+\varepsilon^{-jk})$	$-2(-1)^k$
$\hat{\chi}_l$	$q+1$	1	1	$\omega^{hl}+\omega^{-hl}$	0	0	0

Where $1 < h, j, k, l < r = \frac{1}{4}(q-3)$,

$$\varepsilon = e^{\pi i/(r+1)}, \qquad \sum_{j=1}^{r} \varepsilon^{jk} + \varepsilon^{-jk} = -1 - (-1)^k,$$

$$\omega = e^{2\pi i/(2r+1)}, \qquad \sum_{l=1}^{r} (\omega^{hl} + \omega^{-hl}) = -1,$$

all characters χ_k belong to blocks of defect 1, and all characters $\hat{\chi}_l$ belong to blocks of defect zero of C'.

From this character table and Ward's character table of $R(q)$ ([**15**, pp. 87–88]) we deduce

LEMMA 1.2. *Let $\hat{e}$ be the block idempotent of the principal 2-block $\hat{B}$ of $R(q)$. Let I, χ', χ'' and χ_q be the ordinary irreducible characters of the principal block b_0 of $C' = \mathrm{PSL}(2, q)$, $q \equiv 3 \bmod 8$. Then*:

(a) $I^{R(q)}\hat{e} = I \oplus \xi_2 \oplus 3\xi_3 \oplus \xi_4$,

(b) $\chi'^{R(q)}\hat{e} = q\xi_3 \oplus (q-2)\xi_4 \oplus m\xi_7 \oplus m\xi_8$,

(c) $\chi''^{R(q)}\hat{e} = q\xi_3 \oplus (q-2)\xi_4 \oplus m\xi_5 \oplus m\xi_6$,

(d) $\chi_q^{R(q)}e = 2\xi_2 \oplus 2q\xi_3 \oplus 2q\xi_4 \oplus (m+1)\xi_5 \oplus (m-1)\xi_6 \oplus (m+1)\xi_7 \oplus (m-1)\xi_8$.

Using Fong's description [**7**, Lemma 2] of the Sylow 3-subgroup P of $R(q)$, it is not difficult to compute the character table of its normalizer $L = N(P)$. In fact we show that it is uniquely determined although the structure of P is not completely known. Again we use Ward's notation [**15**] of representatives of conjugacy classes. (See the Character table of the normalizer L of a Sylow 3-subgroup of $R(q)$.)

From this and Ward's character table [**15**, pp. 87–88], we obtain

LEMMA 1.3. (a) $\xi_{2|L} = I^- + \Lambda_q$.

(b) $\xi_{3|L} = I + q\Lambda_q + m\Lambda_m + m\overline{\Lambda}_m + \Lambda_1 + \frac{1}{2}(m+1)\Lambda_{m/2} + \frac{1}{2}(m-1)\Lambda_{m/2}^- + \frac{1}{2}(m+1)\overline{\Lambda}_{m/2} + \frac{1}{2}(m-1)\overline{\Lambda}_{m/2}^-$.

(c) $\xi_{6|L} = \frac{1}{2}(m-1)\Lambda_q + \Lambda_{m/2}$, $\xi_{5|L} = \frac{1}{2}(m+1)\Lambda_q + \Lambda_{m/2}^-$.

(d) $\xi_{8|L} = \frac{1}{2}(m-1)\Lambda_q + \overline{\Lambda}_{m/2}$, $\xi_{7|L} = \frac{1}{2}(m+1)\Lambda_q + \overline{\Lambda}_{m/2}^-$.

As in **[12]**, centralizers of involutions play an important role in the course of the proofs. If M is an FG-module or an RG-module and $u \neq 1$ is an involution of the finite group G, then

$$\operatorname{ann}_M(1-u) = \{m \in M \mid m(1-u) = 0\}$$

is called the centralizer of u in M.

If X is an R-form of the irreducible character ξ of the finite group G, and if $\overline{X} = X/X\pi$, where πR is the maximal ideal of R, then we write

$$z_X(\xi) = \dim_F(\operatorname{ann}_{\overline{X}}(1-u)).$$

For the irreducible characters ξ_2 and ξ_i, $i = 5, 6, 7, 8$, of the groups $R(q)$ of Ree type, this dimension $z_X(\xi)$ of a centralizer of the involution $u \neq 1$ is independent of the choice of the R-form X by the following result.

LEMMA 1.4. (a) $z_X(\xi_2) = \frac{1}{2}(\xi_2(1) + 1)$.

(b) $z_X(\xi_i) = \frac{1}{2}(\xi_i(1) + \frac{1}{2}(q-1))$, $i = 5, 6, 7, 8$.

(c) $\dim_F(\operatorname{ann}_{\varphi_i}(1-u)) = \frac{1}{4}(q-1) + \frac{1}{2}\dim_F \varphi_i$, $i = 4, 5$.

PROOF. (a) Let X be an R-form of ξ_2. Then Lemma 1.3(a) asserts that $\overline{X}_{|L} = \Lambda_q \oplus I$, because Λ_q is a projective FL-module by the character table of L. Therefore,

$$z_X(\xi_2) = \dim_F(\operatorname{ann}_{\overline{X}}(1-u)) = 1 + \tfrac{1}{2}\dim \Lambda_q$$
$$= 1 + \tfrac{1}{2}(q^2 - q) = \tfrac{1}{2}(\xi_2(1) + 1).$$

(b) Let X be an R-form of ξ_8. Then Lemma 1.3(d) asserts that $\overline{X}_{|L} = \frac{1}{2}(m-1)\Lambda_q \oplus \overline{\Lambda}_{m/2}$, because Λ_q is a projective FL-module by the character table of L. Furthermore, $\overline{\Lambda}_{m/2}$ belongs to a block of L with defect group conjugate to $\langle u \rangle$. Now $H = C \cap L = C_L(u) = \langle u \rangle \times (E_q \cdot \mathbf{Z}_{(q-1)/2})$, and the second direct factor is a Frobenius group. Let f be the Green correspondence between L and H with respect to $\langle u \rangle$. Then

$$\overline{\Lambda}_{(m/2)|H} = f\left(\overline{\Lambda}_{m/2}\right) \oplus \text{projective } FH\text{-modules.} \qquad (*)$$

Thus $f(\overline{\Lambda}_{m/2})$ is an irreducible FH-module of degree $\frac{1}{2}(q-1)$ with u in its kernel. Therefore (b) follows for ξ_8. By Lemma 1.3 the same argument applies to the other characters as well.

(c) Again let X be an R-form of ξ_8. Then by Fong's decomposition matrix there is an integer c such that $\overline{X}$ has composition factors $\varphi_5 + (c-1)\varphi_2$. Since $\varphi_{2|L} = \Lambda_q$ is a projective FL-module, it follows that

$$\overline{X}_{|L} = \varphi_{5|L} \oplus (c-1)\varphi_2.$$

Hence from Lemma 1.3(d) and the Krull-Remak-Schmidt theorem we obtain

$$\varphi_{5|L} = \tfrac{1}{2}(m + 1 - 2c)\Lambda_q \oplus \overline{\Lambda}^{-}_{m/2}.$$

As Λ_q is projective, equation $(*)$ now implies that

$$\dim_F(\operatorname{ann}\varphi_5(1-u)) = \tfrac{1}{2}(q-1) + \tfrac{1}{2}\left(\dim_F \varphi_5 - \tfrac{1}{2}(q-1)\right)$$
$$= \tfrac{1}{4}(q-1) + \tfrac{1}{2}\dim \varphi_5.$$

Hence (c) holds for φ_5. The same argument applies to φ_4, too.

For the sake of completeness we now restate the following inequality due to the first author [11]

LEMMA 1.5. *Let $\bar{X}$ be a liftable FG-module of the form $\bar{X} = X/X\pi$. Let χ be the character of $X \otimes_R S$. If $u \neq 1$ is any involution of the finite group G, then*

$$\dim_F(\operatorname{ann}_{\bar{X}}(1-u)) \geq \tfrac{1}{2}[\chi(1) + |\chi(u)|].$$

Another subsidiary result is

LEMMA 1.6. *Let A be a finite-dimensional algebra over a field F. Let $S, T_1, T_2, \ldots, T_n$ be simple A-modules such that $\dim_F \operatorname{Ext}_A^1(T_i, S) = 1$ for $i = 1, 2, \ldots, n$. If the A-module M has a submodule $M_0 \cong rS$ such that $M/M_0 \cong T_1 \oplus T_2 \oplus \cdots \oplus T_n$, where $r \geq 2$, then M is not indecomposable, and every direct summand has simple socle.*

PROOF. By induction we may assume that $r = 2$. Hence $M_0 = S_1 \oplus S_2 = \operatorname{soc} M$, where $S_i \cong S$ for $i = 1, 2$. Let $M_1 = M/S_1$. Then M_1 is a direct sum of a semisimple A-module U_1 and an A-module V_2 with $\operatorname{soc} V_2 = S_2$ and $V_2/S_2 \cong T_{i_1} \oplus \cdots \oplus T_{i_k}$. Let U be the preimage of U_1 and V the one of V_2. Then our hypothesis implies that U has simple socle $\operatorname{soc} U = S_1$ and that $V \cong V_2 \oplus S_1$. Hence V has a direct sum decomposition $V = V_1 \oplus S_1$, where $V_2 \cong V_1 + S_1/S_1 \cong V_1$. Thus $M = U \oplus V_1$.

We complete this section by restating some useful definitions and notations of [12].

Let G be an arbitrary finite group. For any pair of FG-modules X, Y and a subgroup U of G, denote $(X, Y)_U := \operatorname{Hom}_{FU}(X, Y)$. $X \circ Y$ denotes any extension of X by Y, so that there exists an exact sequence $0 \to Y \to X \circ Y \to X \to 0$. ΩX denotes the Heller module of X, so that there exists an exact sequence $0 \to \Omega X \to P \to X \to 0$, whenever P is a projective cover of X.

$(X, Y)_{1,G}$ consists of all FG-module homomorphisms from X into Y factorizing through a projective FG-module P, and

$$(X, Y)_G^1 = (X, Y)_G / (X, Y)_{1,G}.$$

DEFINITION. Let G be a finite group, $p \mid |G|$ a prime number, and F a splitting field for all subgroups of G of characteristic $p > 0$. Let H be a subgroup of G, and U an indecomposable FG-module which is H-projective. If a component $f^*(U)$ of U_H is the only direct summand E of U_H satisfying $U | E^G$, then $f^*(U)$ is called a generalized Green correspondent of U in H.

2. The Green correspondents of φ_2, φ_4 and φ_5. In this section we determine the Green correspondents of the simple $FR(q)$-modules φ_i, $i \in \{2, 4, 5\}$, of the principal 2-block B of the groups $R(q)$ of Ree type. We also show that $c = 1$.

In order to derive these assertions we study the structure of two permutation modules. Let $U = (I_L)^{R(q)}$ be the permutation module belonging to the 2-fold transitive permutation representation of $R(q)$, see Ward [15]. Then $\hat{U} \otimes_R S = I \oplus \xi_3$, and U is a self-dual indecomposable $FR(q)$-module belonging to B with the following properties, of which assertion (d) is due to L. Scott [14].

LEMMA 2.1. (a) $S(U) = I \cong U/UJ$.
(b) $\mathrm{Ext}^1_{FR(q)}(I, I) = 0 = \mathrm{Ext}^2_{FR(q)}(I, I)$.
(c) $S_2(U)/S(U) = \varphi_2$.
(d) *The projective cover* P_I *of* I *has a submodule* $\begin{matrix} I \\ \varphi_2 \\ I \end{matrix}$.
(e) U *has vertex* $\mathrm{vx}(U) =_{R(q)} \langle u \rangle = E_2$.
(f) U *has a minimal projective resolution* $0 \leftarrow U \leftarrow P_I \leftarrow \Omega U \leftarrow 0$, *and* $U \cong \Omega U$.

PROOF. As U is a permutation module, it is liftable, self-dual, I occurs in the head U/UJ and in the socle of U. Since $\hat{U} \otimes_R S = I \oplus \xi_3$, $\dim_F \mathrm{End}_{FG}(U) = 2$. If U were not indecomposable, then $\mathrm{End}_{FG}(U) = F \oplus F$, and I would be a direct summand of U. However, $E_2 = \langle u \rangle$ is a Sylow 2-subgroup of L, which implies that each component of U is E_2-projective. But $\mathrm{vx}(I) =_{R(q)} E_8$. Hence U is indecomposable, and its vertex $\mathrm{vx}(U) =_{R(q)} E_2$. Furthermore, $I = \mathrm{soc}(U) \cong U/UJ$. Thus (a) and (e) hold.

By [9], the Heller operator Ω commutes with induction from L to $R(q)$. Hence (f) follows.

As N is also the normalizer of the Sylow 2-subgroup of the smallest Janko group J_1, assertion (b) follows from [12], Lemma 1.1 and the proof of Lemma 6.3.

By Fong's decomposition matrix of B the character ξ_2 has an R-form X such that $\bar{X} = X/X\pi = \begin{matrix} \varphi_2 \\ I \end{matrix}$. As $\mathrm{Ext}^2_{FR(q)}(I, I) = 0$ by (b), the argument of L. Scott [14] yields the existence of the uniserial module

$$\begin{matrix} I \\ \varphi_2 \\ I \end{matrix}.$$

Thus (d) holds.

Furthermore, the same reasoning shows the existence of

$$T_i = \begin{matrix} I \\ \varphi_i \\ I \end{matrix},$$

whenever $\begin{matrix} \varphi_I \\ I \end{matrix}$ belongs to the second socle $S_2(U)$ of U. Since φ_3 has multiplicity 1 in U by (f) and Fong's decomposition matrix

$$U \ncong \begin{matrix} I \\ \varphi_2 \\ I \end{matrix}.$$

Therefore by (a) and (d), its Loewy length $j(U) > 3$. If T_i were a submodule of U, then it would therefore be a proper submodule of U. Hence the multiplicity of I in U would be at least 3 by (a). Thus by (f), the Cartan invariant $c_{11} \geq 2 \cdot 3 = 6$. But $c_{11} = 4$ by Fong's decomposition matrix. This contradiction proves (c) and completes the proof.

Besides the permutation module $U = (I_L)^{R(q)}$ we consider the component W in the principal block $B \leftrightarrow e$ of the permutation module $(I_C)^{R(q)}$, i.e. $W = (I_C)^{R(q)}e$.

Before we can determine its main properties, we have to study the restrictions of φ_2 to C and L.

LEMMA 2.2. (a) $\varphi_{2|L} = \Lambda_q$ *is a simple and projective FL-module.*

(b) *If* e_0 *denotes the block idempotent of the principal* 2-*block* b_0 *of* C, *then* $\varphi_{2|C}e_0$ *is indecomposable and has socle series*

$$\varphi_{2|C}e_0 = \begin{matrix} I & \chi' & \chi'' \\ I & \chi' & \chi'' \end{matrix},$$

where I, χ', *and* χ'' *are the simple FC-modules of* b_0. *Furthermore, its vertex is* E_8.

(c) $\varphi_{2|C}(1 - e_0)$ *is a projective FC-module.*

(d) *If* e_0 *also denotes the principal* 2-*block idempotent of* C', *then also* $\varphi_{2|C'}e_0$ *is indecomposable with* E_4 *as a vertex and has socle series*

$$\varphi_{2|C'}e_0 = \begin{matrix} I & \chi' & \chi'' \\ I & \chi' & \chi'' \end{matrix}.$$

(e) $\varphi_{2|C'}(1 - e_0)$ *is a projective FC'-module.*

PROOF. (a) holds by Lemma 1.3(a), Fong's decomposition matrix of B, and the character table of L.

(b) Since every nonprincipal 2-block of C' has defect at most 1, any nonprojective component V of $\varphi_{2|C'}(1 - e_0)$ has vertex $\mathrm{vx}(V) =_{C'} E_2$. However all involutions of $R(q)$ are conjugate. Hence V is projective by (a). Thus $V = 0$.

(c) is an immediate consequence of (e) by Nagao's lemma, see Dornhoff [**4**, p. 353].

(d) By Lemma 1.2, $A' = \varphi_{2|C'}e_0$ has composition factors $2I + 2\chi' + 2\chi''$. Since all involutions of $R(q)$ are conjugate, it follows from (a) that $A'|_{\langle v\rangle}$ is projective for every involution v of C'. Therefore the sources of the components of A' are even-dimensional. Hence every component of A' has an even number of composition factors. As φ_2 is self-dual and algebraically invariant, Lemma 1.1 implies that A' is indecomposable and projective free. Therefore $\mathrm{vx}(A') = E_4$, and (d) follows.

(e) Certainly $A = \varphi_{2|C}e_0$ is indecomposable by (d). By Knörr's theorem [**10**], $\mathrm{vx}(\varphi_2) =_{R(q)} E_8$. Hence E_8 is a vertex of A. Furthermore, (a) implies

$$\dim_F \mathrm{ann}_A(1 - u) = \tfrac{1}{2} \dim A.$$

Again the self-duality and algebraic invariance of A imply that $\mathrm{ann}_A(1 - u)$ is semisimple. Hence (b) follows, because $\mathrm{soc}\, A = \mathrm{ann}_A(1 - u)$. This completes the proof of Lemma 2.2.

LEMMA 2.3. *The* $FR(q)$-*module* $W = (I_C)^{R(q)}e$ *of the principal* 2-*block* $B \leftrightarrow e$ *of* $R(q)$ *has the following properties*:

(a) $\hat{W} \otimes_R S = I \oplus 2\xi_3$,

(b) $\mathrm{End}_{FR(q)}(W) \cong \mathrm{End}_{RR(q)}(\hat{W})/\pi\mathrm{End}_{RR(q)}(\hat{W})$,

(c) $\dim_F \mathrm{End}_{FR(q)}(W) = 5$,

(d) $\dim_F \mathrm{Hom}_{FR(q)}(U, W) = 3 = \dim_F \mathrm{Hom}_{FR(q)}(W, U)$,

(e) $W = I \oplus V \oplus V^*$, *where the dual B-modules* V *and* V^* *have socles* $\mathrm{soc}\, V = \varphi_2$ *and* $\mathrm{soc}\, V^* = \varphi_3$.

PROOF. (a) follows immediately from Ward's character table [15] and the Frobenius relations.

(b) $Y = (I_C)^{R(q)}$ is a permutation module. Thus,

$$\operatorname{End}_{FR(q)}(Y) \cong \operatorname{End}_{RR(q)}(Y)/\pi\operatorname{End}_{RR(q)}(Y).$$

Since $\hat{W} = \hat{Y}\hat{e}$,

$$\operatorname{End}_{RR(q)}(\hat{W}) \cong \hat{e}\left[\operatorname{End}_{RR(q)}(\hat{Y})\right]\hat{e},$$

which implies (b).

(c) follows from (a) and (b).

(d) As $\hat{U} \otimes_R S = I \oplus \xi_3$ we obtain from (a) that

$$\dim_S \operatorname{Hom}_{SR(q)}(\hat{U} \otimes_R S, \hat{W} \otimes_R S) = 3 = \dim_S \operatorname{Hom}_{SR(q)}(\hat{W} \otimes_R S, \hat{U} \otimes_R S)$$

Since U and Y are permutation modules, it follows that

$$\operatorname{Hom}_{FR(q)}(U, W) \cong \operatorname{Hom}_{RR(q)}(\hat{U}, \hat{W})/\pi\operatorname{Hom}_{RR(q)}(\hat{U}, \hat{W}),$$

because $\hat{W} = \hat{Y}\hat{e}$. Interchanging U and W we obtain the other isomorphism. Thus,

$$\dim_F \operatorname{Hom}_{FR(q)}(U, W) = 3 = \dim_F \operatorname{Hom}_{FR(q)}(W, U).$$

(e) As $K = N \cap C$ we have a Green correspondence g_1 between K and C with respect to E_8. Since $g_1(I) = I$,

$$(I_K)^C \cong I_C \oplus Q,$$

where Q is a direct sum of indecomposable FC-modules with vertices contained in Klein four subgroups of C.

The index $|N : K| = 7$. Thus $(I_K)^N$ has composition factors I, 3 and 3^* by [12]. Clearly E_8 is normal in N and has odd index. Therefore $(I_{E_8})^N$ is a semisimple FN-module. As I_K has vertex E_8 it is a component of $(I_{E_8})^K$. Hence $(I_K)^N$ is a direct summand of the semisimple FN-module $(I_{E_8})^N$, which implies

$$(I_K)^N \cong I_N \oplus 3 \oplus 3^*.$$

Now let g be the Green correspondence between N and $R(q)$ with respect to E_8. Then

$$(I_K)^{R(q)} \cong I \oplus g(3) \oplus g(3^*) \oplus A,$$

where A is a direct sum of indecomposable $FR(q)$-modules with smaller vertices than E_8. So

$$(I_K)^{R(q)} = \left((I_K)^C\right)^{R(q)} \cong (I_C)^{R(q)} \oplus Q^{R(q)}$$

yields

$$W = (I_C^{R(q)})e \cong I \oplus g(3) \oplus g(3^*) \oplus W_0.$$

As

$$\dim_F(\operatorname{End}_{FR(q)}(I \oplus g(3) \oplus g(3^*)) > 5,$$

(c) implies $W_0 = 0$. Hence $W = I \oplus V \oplus V^*$ for some indecomposable $FR(q)$-module V.

By Lemma 2.2(b) φ_2 occurs once in the socle of W. Hence V and V^* are algebraically invariant. As $\dim_F \mathrm{End}_{FR(q)}(W) = 5$ it follows that neither φ_4 nor φ_5 occurs in the socle of W. Furthermore, we may assume without loss of generality that soc $V = \varphi_2$. Hence soc $V^* = \varphi_3$ which completes the proof.

PROPOSITION 2.4. *Let e_0 be the block idempotent of the principal 2-block b_0 of C. Then*:

(a) $c = 1$,

(b) $\varphi_{4|C}e_0 = \chi'' \oplus \frac{1}{2}(m-1)P_{\chi''}$,

(c) $\varphi_{5|C}e_0 = \chi' \oplus \frac{1}{2}(m-1)P_{\chi'}$.

PROOF. Let T be an R-form of ξ_8. Then by Fong's decomposition matrix, $\overline{T} = T/T\pi$ has composition factors $\varphi_5 + (c-1)\varphi_2$. By Lemma 2.2(b), $\varphi_{2|C}e_0$ has composition factors $2I + 2\chi' + 2\chi''$. Therefore Lemma 1.2 implies that $\varphi_{5|C}e_0$ has composition factors

$$(m+1-2c)I + (2m+1-2c)\chi' + (m+1-2c)\chi''. \qquad (*)$$

Since the involution $u \neq 1$ is in the center of $C = \langle u\rangle \times C'$, $Y = \mathrm{ann}_{\varphi_{5|C}}(1-u)$ is an FC-submodule of $\varphi_{5|C}$. As u acts trivially on Y, it follows that Y is an FC'-module. In particular, every nonprojective component of Y has Loewy length 2, because $C' \cong \mathrm{PSL}(2, q)$, $q \equiv 3 \bmod 8$. Since I_C does not occur in the socles of $\varphi_{5|C}$ and $\varphi_{4|C}$ by Lemma 2.3, it follows that I_C does not occur in the socle and the head of Y, because $\varphi_4^* = \varphi_5$. Hence by Donovan-Freislich [3] every indecomposable summand U of Ye_0 is isomorphic to one of the following indecomposable FC'-modules:

$$\chi',\ \chi'',\ \begin{matrix}\chi'\\ \chi''\end{matrix},\ \begin{matrix}\chi''\\ \chi'\end{matrix},\ P'_{\chi'} \text{ and } P'_{\chi''}.$$

Let r, s, t and u be the multiplicities of χ', χ'', $\begin{matrix}\chi'\\ \chi''\end{matrix}$ and $\begin{matrix}\chi''\\ \chi'\end{matrix}$ respectively in Ye_0, and let p be the number of indecomposable projective summands of Ye_0. Then

$$\dim_F Ye_0 = p\big(q + \tfrac{1}{2}(q-1)\big) + \tfrac{1}{2}(r+s)(q-1) + (u+t)(q-1). \qquad (**)$$

Hence by Lemma 1.1, the multiplicity of the composition factor I_C in Ye_0 is p.

Let $X = (1-u)\varphi_{5|C}$. Then $X \subseteq Y$, and by Lemma 1.4(c),

$$\dim_F(Y/X) = \tfrac{1}{2}(p-1).$$

As the multiplicity of I_C in $\varphi_{5|C}$ is

$$m + 1 - 2c = \tfrac{1}{3}\sqrt{3q} + 1 - 2c < \tfrac{1}{2}(q-1),$$

it follows from the character table of PSL(2, q) that Y/X is isomorphic to χ' or χ''. Thus,

$$Y/X \cong Ye_0/Xe_0. \qquad (***)$$

As $\varphi_5 e_0/Ye_0 \cong Xe_0$, it follows that I_C occurs $\frac{1}{2}(m+1-2c)$ times as a composition factor of Ye_0. Hence $p = \frac{1}{2}(m+1-2c)$.

By (*),

$$\dim_F \varphi_5 e_0 = (m+1)q + \frac{m}{2}(q-1) - 2cq.$$

Hence (***) implies

$$\dim_F Ye_0 = \frac{m+1}{2}q + \frac{m+1}{4}(q-1) - cq.$$

Inserting the value of p into (***) we also obtain

$$\dim_F Ye_0 = \tfrac{1}{2}(m+1-2c)\big(q+\tfrac{1}{2}(q-1)\big) + \tfrac{1}{2}(r+s)(q-1) + (u+t)(q-1).$$

Thus,

$$\tfrac{1}{2}(q-1)c = \tfrac{1}{2}(r+s)(q-1) + (u+t)(q-1),$$

and

$$c = r + s + 2(u+t).$$

Let x be the multiplicity of the direct summand $P'_{\chi'}$ in Ye_0 and let y be the one of $P'_{\chi''}$. Then

$$x + y = p = \tfrac{1}{2}(m+1-2c).$$

By Lemma 1.1, the multiplicity of χ'' in $xP'_{\chi'} \oplus yP'_{\chi''}$ is $x+2y$. Since (*) and the simplicity of Ye_0/Xe_0 imply that χ'' occurs $\frac{1}{2}(m+1-2c)$ times as a composition factor of Ye_0, it follows that

$$x + 2y < \tfrac{1}{2}(m+1-2c).$$

Therefore $y = 0$, and $x = \frac{1}{2}(m+1-2c)$. Hence Lemma 1.1 asserts that $t = u = s = 0$, and $c = r$. Thus

$$Ye_0 = c\chi' \oplus \tfrac{1}{2}(m+1-2c)P'_{\chi'}.$$

Since I_C is not contained in the head of Xe_0, it follows from (***) that

$$\varphi_5 e_0 / Ye_0 \cong Xe_0 \cong (c-1)\chi' \oplus \tfrac{1}{2}(m+1-2c)P'_{\chi'}.$$

Restricting φ_5 now to C' we obtain

$$A_1 = \varphi_{5|C'} e_0 = (2c-1)\chi' \oplus (m+1-2c)P'_{\chi'}.$$

Let $v \neq 1$ be an involution of C', and let $z_{A_1}(v) = \dim_F(\operatorname{ann}_{A_1}(1-v))$. Since by Lemma 1.1, the Green correspondent $f(\chi')$ of χ' in $\mathfrak{A}_4$ is one-dimensional and has a projective complement, we obtain

$$\begin{aligned} z_{A_1}(v) &= 2c - 1 + \tfrac{1}{2}(2c-1)(\dim_F \chi' - 1) + \tfrac{1}{2}(m+1-2c)\dim_F P_{\chi'} \\ &= \tfrac{1}{2}(2c - 1 + \dim A_1). \end{aligned}$$

We now choose the R-form T of ξ_8 so that $\varphi_5 = \operatorname{soc}(\overline{T})$, see [4, Lemma 68.10], due to J. Thompson. By Lemma 1.3, $\overline{T}$ has the following composition factors:

$$\begin{aligned} \overline{T}|_{C'} &= (m-1)I + (2m-1)\chi' + (m-1)\chi'' \\ &\quad + m\sum_{s=1}^{r/2} \chi_{2s-1} + (m-2)\sum_{s=1}^{r/2} \chi_{2s} + (m-1)\sum_{l=1}^{r} \hat{\chi}_l. \end{aligned}$$

Lemma 1.4(b) asserts

$$z_T(\xi_8) = \tfrac{1}{2}\big(\xi_8(1) + \tfrac{1}{2}(q-1)\big) = \tfrac{1}{4}\big[m(q-1)(q+1-3m) + q - 1\big].$$

Let $T_1 = \overline{T}|_{C'}e_0$ and $T_2 = \overline{T}|_{C'}(1-e_0)$, where e_0 denotes the block idempotent of the principal 2-block of C'. By Lemma 1.5,

$$z_{T_1}(v) \geqslant \tfrac{1}{2}\big(\tfrac{1}{2}m(q-1) + (m-1)q + 1\big) = c_1,$$

because T_1 has odd dimension. Likewise,

$$z_{T_2}(v) \geqslant \tfrac{1}{8}\big[(m-2)(q-1)(q-3) + (m-1)(q+1)(q-3) + (q+1)(q-3)\big] = c_2.$$

An easy computation shows

$$z_T(\xi_8) = z_{T_1}(v) + z_{T_2}(v) \geqslant c_1 + c_2 = z_T(\xi_8).$$

Hence $z_{T_1}(v) = c_1 = \frac{1}{2}(\dim_F T_1 + 1)$. Since $\overline{T}$ has composition factors $\varphi_5 + (c-1)\varphi_2$, and as $\varphi_{2|\langle v\rangle}$ is projective,

$$\begin{aligned} z_{T_1}(v) &= z_{A_1}(v) + (c-1)\dim_F \operatorname{ann}_{\varphi_{2|C'}e_0}(1-v) \\ &= z_{A_1}(v) + (c-1)q \\ &= \tfrac{1}{2}(2c - 1 + \dim A_1) + (c-1)q \\ &= \tfrac{1}{2}(2c - 1 + \dim_F T_1). \end{aligned}$$

Thus $2c - 1 = 1$, and $c = 1$, which proves (a).

Furthermore, $\varphi_{5|C'}e_0 = A_1 = \chi' \oplus (m-1)P'_{\chi'}$, and $Ye_0 = \chi' \oplus \frac{1}{2}(m-1)P'_{\chi'}$. As the involution u acts trivially on χ' and as $C = \langle u\rangle \times C'$, it follows from the decomposition of A_1 that χ' is a direct summand of $\varphi_{5|C}e_0 = \chi' \oplus U$, where U denotes a complement of χ' in $\varphi_{5|C}e_0$. Since $\varphi_{5|C}e_0$ and Ye_0 have the same socle, they also have the same injective hull, which is a direct sum of $\frac{1}{2}(m+1)$ copies of $P_{\chi'}$. By Lemma 1.1, U and $\frac{1}{2}(m-1)P_{\chi'}$ have the same composition factors including multiplicities. Thus $U = \frac{1}{2}(m-1)P_{\chi'}$, which proves (c). Now (b) follows by duality. This completes the proof of Proposition 2.4.

PROPOSITION 2.5. *The simple $FR(q)$-modules φ_2, φ_4 and φ_5 have Green correspondents with the following socle and Loewy series*:

(a) $f(\varphi_2) = \begin{smallmatrix} 3^* \\ 3 \end{smallmatrix}$, *and* $\varphi_{2|N} = f(\varphi_2) \oplus$ *projectives.*

(b) $f(\varphi_4) = 1^*$, *and* $\varphi_{4|N} = f(\varphi_4) \oplus E_2$*-projectives.*

(c) $f(\varphi_5) = 1$, *and* $\varphi_{5|N} = f(\varphi_5) \oplus E_2$*-projectives.*

PROOF. (a) By Lemma 2.2(b) and (c),

$$\varphi_{2|C} = \begin{matrix} I & \chi' & \chi'' \\ I & \chi' & \chi'' \end{matrix} \oplus \text{projectives},$$

where the nonprojective summand is indecomposable with vertex E_8. By Lemmas 1.1(c) and 2.2(d),

$$\varphi_{2|\mathfrak{A}_4} = \begin{matrix} I & 1 & 1^* \\ I & 1 & 1^* \end{matrix} \oplus \text{projectives},$$

where the nonprojective summand is indecomposable with vertex E_4. Using the argument of Lemma 2.2(b), it follows that

$$\varphi_{2|K} = \begin{matrix} I & 1 & 1^* \\ I & 1 & 1^* \end{matrix} \oplus \text{projectives,} \qquad (*)$$

where the nonprojective summand is indecomposable with vertex E_8. As $K = N \cap C$ and N is 2-closed,

$$f(\varphi_2)_{|K} = \begin{matrix} I & 1 & 1^* \\ I & 1 & 1^* \end{matrix}.$$

Hence self-duality of $f(\varphi_2)$ implies that its composition factors are $2I + 2 \cdot 1^* + 2 \cdot 1$ or $3 + 3^*$. If the first possibility occurs, an element of order 7 is in the kernel of $f(\varphi_2)$. Thus the kernel of $f(\varphi_2)$ contains E. However, E_8 is not in kernel, since $f(\varphi_2)_{|E_2}$ is projective. Therefore $f(\varphi_2)$ equals $\begin{matrix} 3 \\ 3^* \end{matrix}$ or $\begin{matrix} 3^* \\ 3 \end{matrix}$, as its Loewy length is 2. Now

$$\varphi_{2|N} = f(\varphi_2) \oplus \text{projective}$$

by (*). Hence Lemma 1.1 of [12] and Lemma 2.1(d) imply that

$$Y = f\begin{bmatrix} I \\ \varphi_2 \\ I \end{bmatrix}$$

is an indecomposable FN-module. Since Y is the only nonprojective summand of $\begin{matrix} I \\ \varphi_2 \\ I \end{matrix}$, it is not projective. Thus Proposition 2.3 of [12] implies $f(\varphi_2) = \begin{matrix} 3^* \\ 3 \end{matrix}$.

(b) By Proposition 2.4(b) and Lemma 1.1(c), $\varphi_{4|K}$ has only one summand with vertex E_8, namely 1. As in the proof of (a), we obtain $f(\varphi_4)_{|K} = 1$. Thus $f(\varphi_4) = 1$.

(c) follows now from (b) by duality.

3. The decomposition matrix of B. In this section we determine the parameters a and b of Fong's decomposition matrix of the principal 2-block B of the groups $R(q)$ of Ree type.

LEMMA 3.1. *B has Cartan matrix*

	I	φ_2	φ_3	φ_4	φ_5
I	4	$2a$	2	$2b$	$2b$
φ_2	$2a$	$2a^2 - 2a + 4$	$2a - 1$	$2ab - b + 1$	$2ab - b + 1$
φ_3	2	$2a - 1$	2	$2b$	$2b$
φ_4	$2b$	$2ab - b + 1$	$2b$	$2b^2 + 2$	$2b^2$
φ_5	$2b$	$2ab - b + 1$	$2b$	$2b^2$	$2b^2 + 2$

PROOF. By Proposition 2.4(a) we know that $c = 1$. Inserting this number into Fong's decomposition matrix, Lemma 3.1 follows from Theorem 48.8 of Dornhoff [4].

COROLLARY 3.2. $\mathrm{Ext}^1_{FR(q)}(\varphi_3, \varphi_3) = 0$.

LEMMA 3.3. $\dim_F \mathrm{Ext}^1_{FR(q)}(\varphi_2, \varphi_2) \leq 1$.

PROOF. By Propositionn 2.4(a), $\varphi_{2|N} = \begin{smallmatrix} 3^* \\ 3 \end{smallmatrix} \oplus$ projectives. Hence

$$\dim_F \operatorname{Ext}^1_{FR(q)}(\varphi_2, \varphi_2) \leq \dim_F \operatorname{Ext}^1_{FN}(f(\varphi_2), f(\varphi_2))$$

by Lemma 1.1 of [12]. Now Proposition 2.3 of [12] asserts that $\Omega f(\varphi_2)$ has Loewy series

$$\Omega f(\varphi_2) = \begin{matrix} I & 1 & 1^* & & 3^* \\ & 3^* & 3 & 3 & \\ & & 3^* & & \end{matrix} \qquad (*)$$

By Feit [6],

$$\begin{aligned} \dim_F \operatorname{Ext}^1_{FN}(f(\varphi_2), f(\varphi_2)) &= \dim_F(\Omega f(\varphi_2), f(\varphi_2))'_N \\ &\leq \dim_F(\Omega f(\varphi_2), f(\varphi_2)) = 1, \end{aligned}$$

where the last equality follows from $(*)$ and Proposition 2.3 of [12].

LEMMA 3.4. $\dim_F \operatorname{Ext}^1_{FR(q)}(\varphi_i, \varphi_2) = 1$ *for* $i = 4, 5$.

PROOF. By Fong's decomposition matrix of B,

$$\dim_F \operatorname{Ext}^1_{FR(q)}(\varphi_4, \varphi_2) \geq 1.$$

Let $\varphi_4 \circ \varphi_2$ be any nonsplit extension of φ_2 by φ_4. As $\dim_F(\varphi_4 \circ \varphi_2)$ is odd, it has vertex E_8.

By Lemma 2.2 and Proposition 2.4, φ_2 and φ_4 have generalized Green correspondents $f^*(\varphi_2)$ and $f^*(\varphi_4)$ in C. Furthermore,

$$(\varphi_4 \circ \varphi_2)_{|C} = f^*(\varphi_4) \circ f^*(\varphi_2) \oplus E_2\text{-projectives}.$$

By the definition of relative projectivity ([4, p. 322]), also $Y = f^*(\varphi_4) \circ f^*(\varphi_2)$ is a nonsplit indecomposable extension with vertex $\operatorname{vx}(Y) =_C E_8$. Since $f^*(\varphi_2)_{|\langle u \rangle}$ is projective,

$$\operatorname{ann}_{f^*(\varphi_2)}(1 - u) \cong I \oplus \chi' \oplus \chi'' \leq Y(1 - u).$$

As Y has seven composition factors, we obtain

$$Y(1 - u) = I \oplus \chi' \oplus \chi'' \quad \text{and} \quad \operatorname{ann}_Y(1 - u)/Y(1 - u) \cong \chi''.$$

Hence $Y/Y(1 - u)$ is a semisimple FC-module. As $Y(1 - u) \leq f^*(\varphi_2)$, this implies that Y has Loewy length $j(Y) = 2$. Since Y is indecomposable, Lemma 1.6 asserts that $\operatorname{ann}_Y(1 - u)$ is not a semisimple FC'-module. Hence $Y_{|C'}$ is indecomposable, because $f^*(\varphi_2)_{|C'}$ is indecomposable. Therefore $f^*(\varphi_2)_{|C'}$ has only one nonsplit extension by χ'' as Lemma 1.1(a) shows. This completes the proof.

LEMMA 3.5. $\operatorname{Ext}^1_{FR(q)}(\varphi_i, \varphi_i) = 0$ *for* $i = 4, 5$.

PROOF. Suppose $\varphi_4 \circ \varphi_4$ is a nonsplit extension. By Proposition 2.4(b),

$$\varphi_{4|C}e_0 = \chi'' \oplus \text{projectives}.$$

Therefore by the definition of relative projectivity,

$$(\varphi_4 \circ \varphi_4)_{|C}e_0 = \chi'' \circ \chi'' \oplus \text{projectives}, \qquad (*)$$

where $\chi'' \circ \chi''$ is a nonsplit extension. Hence $\chi'' \circ \chi''$ has vertex E_4 by Lemma 1.1 and Green's theorem. Moreover, $(*)$ and Lemma 1.1 imply

$$(\varphi_4 \circ \varphi_4)_{|C'}e_0 = \chi'' \oplus \chi'' \oplus \text{projectives}.$$

Since $\varphi_4 \circ \varphi_4$ is C'-projective, relative projectivity now asserts that $\varphi_4 \circ \varphi_4$ is a split extension. By duality the other part of the lemma follows.

LEMMA 3.6. $\mathrm{Ext}^1_{FR(q)}(\varphi_i, \varphi_j) = 0$, $i, j \in \{4, 5\}$ *and* $i \neq j$.

PROOF. Suppose $\varphi_4 \circ \varphi_5$ is a nonsplit extension. As in the proof of the previous lemma, this implies

$$(\varphi_4 \circ \varphi_5)_{|C'}e_0 = \chi'' \circ \chi' \oplus \text{projectives},$$

where $\chi'' \circ \chi'$ again is a nonsplit extension. Moreover, u lies in the kernel of $\chi'' \circ \chi'$. Thus

$$(\chi'' \circ \chi')_{|K} = (\chi'' \circ \chi')_{|\mathfrak{A}_4}$$

considered as a K-module by the trivial action of u. From Lemma 1.1(c) follows

$$\begin{aligned}(\varphi_4 \circ \varphi_5)_{|K} &= (\chi' \circ \chi'')_{|K} \oplus E_2\text{-projectives}\\ &= 1^* \circ 1 \oplus E_2\text{-projectives}. \qquad (*)\end{aligned}$$

As $\varphi_4 \circ \varphi_5$ has vertex E_8, its Green correspondent $f(\varphi_4 \circ \varphi_5)$ is a component of $(1^* \circ 1)^N$. Since N is 2-closed, $f(\varphi_4 \circ \varphi_5)$ has Loewy length 2 by $(*)$. By Lemma 2.6 of [12] every E_2-projective FK-module has Loewy length at least 3. Hence $f(\varphi_4 \circ \varphi_5)_{|K} = 1^* \circ 1$ by $(*)$. Thus $\dim_F f(\varphi_4 \circ \varphi_5) = 2$, a contradiction to Proposition 2.3 of [12].

LEMMA 3.7. *For each* $\varphi \in \{I, \varphi_4, \varphi_5\}$ *there exists at most one uniserial $FR(q)$-module* $\begin{matrix}\varphi_2\\ \varphi\\ \varphi_2\end{matrix}$.

PROOF. Suppose there are two nonisomorphic $FR(q)$-modules with Loewy series $\begin{matrix}\varphi_2\\ I\\ \varphi_2\end{matrix}$. By Lemma 2.1(c), their largest isomorphic submodule is $\begin{matrix}I\\ \varphi_2\end{matrix}$. Amalgamating it, we obtain an $FR(q)$-module with socle series

$$\begin{array}{ccc}\varphi_2 & & \varphi_2\\ & I & \\ & \varphi_2 & \end{array},$$

which is a contradiction to Lemma 2.1(c).

Since φ_2 is self-dual, Lemma 3.4 asserts

$$\dim_F \mathrm{Ext}^1_{FR(q)}(\varphi_2, \varphi_4) = 1 = \dim_F \mathrm{Ext}^1_{FR(q)}(\varphi_4, \varphi_2).$$

Therefore the above argument shows that there exists at most one uniserial $FR(q)$-module

$$\begin{matrix}\varphi_2\\ \varphi_4\\ \varphi_2\end{matrix}.$$

As φ_4 and φ_5 are algebraic conjugate, the proof is complete.

After all these preparations we now can compute the remaining parameters a and b of Fong's decomposition matrix of the principal 2-block B of the groups $R(q)$ of Ree type.

PROPOSITION 3.8. (a) *Let V be the indecomposable $FR(q)$-module of Lemma* 2.3. *Then it has socle series*

$$V: \begin{array}{ccc} & \varphi_3 & \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \end{array}$$

(b) *The permutation module U of Lemma* 2.1 *has socle series*

$$U: \begin{array}{ccc} & I & \\ & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5 \\ & \varphi_2 & \\ & I & \end{array}$$

(c) $a = 2$ *and* $b = 1$.

PROOF. Let U be the permutation module of Lemma 2.1 and let W be the one of Lemma 2.3. By the latter,

$$W \simeq I \oplus V \oplus V^*,$$

where soc $V = \varphi_2$ and soc $V^* = \varphi_3$. Furthermore, it follows from Lemma 2.3(a) and (e) that modulo πR, any R-form of ξ_3 has the same composition factors (including multiplicities) as V. Hence I has multiplicity 1 in V by Fong's decomposition matrix.

Since

$$\begin{array}{c} I \\ \varphi_2 \\ I \end{array}$$

is not projective by Lemma 2.1, it follows from the proof of Proposition 2.5(a) that $\begin{smallmatrix} I \\ \varphi_2 \end{smallmatrix} |_C e_0$ has Loewy length 2. Therefore Lemma 2.2(b) implies that I occurs twice in the head of $\begin{smallmatrix} I \\ \varphi_2 \end{smallmatrix} |_C$. Hence

$$\dim_F\left(\begin{smallmatrix} I \\ \varphi_2 \end{smallmatrix}, W\right)_{R(q)} = \dim_F\left(\begin{smallmatrix} I \\ \varphi_2 \end{smallmatrix}|_C, I\right)_C = 2.$$

Therefore $\begin{smallmatrix} I \\ \varphi_2 \end{smallmatrix}$ is a submodule of V, and $\dim_F(U, V)_{R(q)} = 1$ by the multiplicity of I in V.

Now Lemma 2.3(d) asserts that $\dim_F(U, V)_{R(q)} = 1$. Hence there exists a submodule T of V^* with soc $T = \varphi_3$ and head$(T) = I$.

Since $\dim_F(V^*, V) = 1$ by Lemma 2.3(c) and (e), there is no submodule of V with socle series $\begin{smallmatrix} \varphi_2 \\ \varphi_2 \end{smallmatrix}$, because by Lemma 3.3, there exists at most one nonsplit extension of φ_2 by itself. Lemma 3.1 asserts that φ_3 occurs only once as a composition factor of V, namely as the head of V. Furthermore, the Loewy length $j(V) > 3$, because $a \geq 2$ by Lemma 2.1. Since V is algebraically invariant, it follows that either

$$\begin{array}{c}\varphi_2\\ I\\ \varphi_2\end{array} \quad \text{or} \quad \begin{array}{c}\varphi_4\\ \varphi_2\end{array} \quad \text{and} \quad \begin{array}{c}\varphi_5\\ \varphi_2\end{array}$$

exist as submodules of V. By Lemma 3.7 there is at most one uniserial $FR(q)$-module

$$\begin{array}{c}\varphi_2\\ I\\ \varphi_2\end{array}.$$

If it were a submodule of V, then Lemma 2.3 would imply that $2 \leq \dim_F(V^*, V)_{R(q)} = 1$, a contradiction. Hence the second socle of V is $S_2(V)/\varphi_2 \cong \varphi_4 \oplus I \oplus \varphi_5$ by Lemma 3.4.

By Lemma 2.1(c), $\mathrm{Ext}^1_{FR(q)}(\varphi_i, I) = 0$ for $i = 4, 5$. Therefore Lemma 3.6 implies that only φ_2 can occur as a composition factor of the third socle $S_3(V)$ of V.

If the multiplicity of φ_2 in the third socle of V is not 1, then V has a submodule Y with socle series

$$Y = \begin{array}{ccccccc} & \varphi_2 & & & \varphi_2 & & \\ I & & \varphi_4 & & & \varphi_5 & \\ & & \varphi_2 & & & & \end{array}$$

We claim that one of the $FR(q)$-modules

$$\begin{array}{c}\varphi_2\\ I\\ \varphi_2\end{array}, \quad \begin{array}{c}\varphi_2\\ \varphi_4\\ \varphi_2\end{array} \quad \text{and} \quad \begin{array}{c}\varphi_2\\ \varphi_5\\ \varphi_2\end{array}$$

is a submodule of Y.

By Lemma 1.6, the $FR(q)$-module Y/φ_2 is not indecomposable. As $\mathrm{soc}(Y/\varphi_2) = I \oplus \varphi_4 \oplus \varphi_5$, one of the direct summands is uniserial with φ_2 in its head. This proves our claim.

Now Lemma 3.7 and the claim yield $2 \leq \dim_F(V^*, V)_{R(q)} = 1$, a contradiction. Therefore $S_3(V)/S_2(V) \cong \varphi_2$.

As $\dim_F(V, U)_{R(q)} = 1$ it now follows that there is a short exact sequence

$$0 \to \begin{array}{ccc}\varphi_4 & & \varphi_5\\ & \varphi_2 & \end{array} \to V \to T^* \to 0.$$

Since φ_2 occurs a times in V, it has multiplicity $a - 1$ in T^*.

Now T^* is isomorphic to a submodule of U, and φ_3 is the head of T^*. Lemmas 2.1 and 3.1 assert that φ_3 occurs only once as a composition factor of U. Since φ_2 and U are self-dual $FR(q)$-modules, and since T is a factor module of U with $\mathrm{soc}(T) = \varphi_3$, we obtain

$$2(a - 1) \leq a,$$

because φ_2 has multiplicity a in U by Lemmas 2.1 and 3.1. Hence $a \leq 2$. Therefore $a = 2$.

Applying now Lemmas 2.1, 3.4, 3.5 and 3.6, it follows that U has one of the following socle series

$$\begin{array}{ccc} & I & \\ & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5 \\ & \varphi_2 & \\ & I & \end{array} \quad \text{or} \quad \begin{array}{ccc} & I & \\ & \varphi_2 & \\ \varphi_4 & & \varphi_5 \\ & \varphi_3 & \\ \varphi_4 & & \varphi_5 \\ & \varphi_2 & \\ & I & \end{array}.$$

If the second possibility were true, then T would have socle series

$$T = \begin{array}{ccc} & I & \\ & \varphi_2 & \\ \varphi_4 & & \varphi_5 \\ & \varphi_3 & \end{array}.$$

As $T < V^*$, it follows that

$$V = \begin{array}{ccc} & \varphi_3 & \\ \varphi_4 & & \varphi_5 \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \end{array}.$$

In particular, $b = 2$, and Lemma 2.1 implies that P_I has socle series

$$P_I\colon \begin{array}{ccccc} & & I & & \\ & & \varphi_2 & & \\ & \varphi_4 & I & \varphi_5 & \\ & \varphi_3 & & \varphi_2 & \\ \varphi_4 & \varphi_5 & & \varphi_4 & \varphi_5 \\ & \varphi_2 & & \varphi_3 & \\ & \varphi_4 & I & \varphi_5 & \\ & & \varphi_2 & & \\ & & I & & \end{array}.$$

Therefore P_I has one of the following factor modules

$$X_1 = \begin{array}{c} I \\ \varphi_2 \\ I \\ \varphi_2 \end{array} \quad \text{or} \quad X_2 = \begin{array}{ccc} & I & \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \end{array}.$$

In any case there is a self-dual indecomposable $FR(q)$-module X with $\mathrm{soc}(X) = \varphi_2 = \mathrm{head}(X)$ such that there is a nonsplit short exact sequence

$$0 \to I \to X_i^* \to X \to 0$$

for some $i \in \{1, 2\}$. Therefore by **[14]** we have

$$0 \to \mathrm{Ext}^1_{FR(q)}(I, X_i^*) \to \mathrm{Ext}^1_{FR(q)}(I, X) \to 0,$$

because $\mathrm{Ext}^1_{FR(q)}(I, I) = 0 = \mathrm{Ext}^2_{FR(q)}(I, I)$. By the structure of P_I,

$$\dim_F \mathrm{Ext}^1_{FR(q)}(I, X) = 1.$$

Hence either

$$\begin{matrix} I \\ \varphi_2 \\ I \\ \varphi_2 \\ I \end{matrix} \quad \text{or} \quad \varphi_4 \quad \begin{matrix} I \\ \varphi_2 \\ I \\ \varphi_2 \\ I \end{matrix} \quad \varphi_5$$

exists. However, I is not a composition factor of the fifth socle of P_I, a contradiction. Thus U has socle series

$$\begin{matrix} & I & \\ & \varphi_2 & \\ \varphi_4 & \varphi_3 & \varphi_5 \\ & \varphi_2 & \\ & I & \end{matrix}$$

Hence $b = 1$, and V has a socle series as asserted in (a). This completes the proof of Proposition 3.8.

Combining Proposition 2.4 with Proposition 3.8, we obtain

THEOREM 3.9. (a) *The principal 2-block B of a group $R(q)$ of Ree type has decomposition matrix*

character	I	φ_2	φ_3	φ_4	φ_5	*degree*
I	1	0	0	0	0	1
ξ_2	1	1	0	0	0	$q^2 - q + 1$
ξ_3	1	2	1	1	1	q^3
ξ_4	1	1	1	1	1	$q(q^2 - q + 1)$
ξ_5	0	1	0	1	0	$\frac{1}{2}(q - 1)m(q + 1 + 3m)$
ξ_7	0	1	0	0	1	$\frac{1}{2}(q - 1)m(q + 1 + 3m)$
ξ_6	0	0	0	1	0	$\frac{1}{2}(q - 1)m(q + 1 - 3m)$
ξ_8	0	0	0	0	1	$\frac{1}{2}(q - 1)m(q + 1 - 3m)$

(b) *B has Cartan matrix*

	I	φ_2	φ_3	φ_4	φ_5
I	4	4	2	2	2
φ_2	4	8	3	4	4
φ_3	2	3	2	2	2
φ_4	2	4	2	4	2
φ_5	2	4	2	2	4

4. Socle series of the indecomposable projective B-modules. In this section we determine the structure of the indecomposable projective $FR(q)$-modules P_n, $n \in \{I, 2, \ldots, 5\}$ belonging to the principal 2-block B of a group $R(q)$ of Ree type.

THEOREM 4.1. *Let B be the principal 2-block of a group $R(q)$ of Ree type. Then the nonisomorphic indecomposable projective $FR(q)$-modules $P_I, P_2, \ldots, P_5$ of B have socle and Loewy series as stated in the introduction.*

PROOF. By Lemma 2.1 and Proposition 3.8(b), the indecomposable projective B-module P_I has the asserted socle and Loewy series.

The structure of P_3 follows immediately from Proposition 3.8(a), Theorem 3.9 and the self-duality of P_3. Hence the socle series of P_I/I and P_3/φ_3 are

$$P_I/I\colon \begin{array}{cccccc} & & I & & & \\ & & \varphi_2 & & & \\ I & \varphi_4 & & \varphi_5 & \varphi_3 & \\ & \varphi_2 & & \varphi_2 & & \\ I & \varphi_4 & & \varphi_5 & \varphi_3 & \\ & & \varphi_2 & & & \end{array} \quad \text{and} \quad P_3/\varphi_3\colon \begin{array}{ccc} & \varphi_3 & \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \end{array}.$$

Furthermore, in both modules the socle and Loewy series coincide. By Lemma 2.3, P_2 contains a submodule Y with socle and Loewy series

$$Y = \begin{array}{ccc} & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \end{array}$$

which is not self-dual.

Let $X = P_I/I$. Then Lemma 1.6 asserts that

$$S_3(X)/S_1(X) \cong \begin{pmatrix} & \varphi_2 & \\ I & \varphi_4 & \varphi_5 \end{pmatrix} \oplus \begin{pmatrix} \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

Hence

$$\begin{array}{c} \varphi_2 \\ \varphi_3 \\ \varphi_2 \end{array}$$

exists. Thus $Y \not\cong Y^*$ implies now that $S_3(P_2)/S_2(P_2) > \varphi_2 \oplus \varphi_2 \oplus \varphi_2$.

Since P_2 is self-dual, and since φ_2 has multiplicity 8 in P_2, by Theorem 3.9 and Lemma 3.1, it follows that $\mathrm{Ext}^1_{FR(q)}(\varphi_2, \varphi_2) = 0$, and

$$S_3(P_2) = I \quad \begin{array}{c} \varphi_2 \\ \varphi_4 \end{array} \quad \begin{array}{c} \varphi_2 \\ \varphi_5 \\ \varphi_2 \end{array} \quad \begin{array}{c} \varphi_2 \\ \varphi_3 \end{array}.$$

Using the self-duality of P_2, the structure of P_I/I and P_3/φ_3, it is now easy to see that P_2 has the asserted socle and Loewy series.

As $P_4 \cong (P_5)^*$, it suffices to determine the structure of P_4. The socle series of P_3 implies

$$\mathrm{Ext}^1_{FR(q)}(\varphi_4, \varphi_3) = 0 = \mathrm{Ext}^1_{FR(q)}(\varphi_3, \varphi_4).$$

Furthermore, $\mathrm{Ext}^1_{FR(q)}(I, \varphi_3) = 0$. Therefore Corollary 3.2 and Lemma 3.4 assert that $S_2(P_4)/\varphi_4 = \varphi_2$.

Considering now the homomorphisms from P_3 into P_4 we find two submodules of P_4 with socle series

$$\begin{matrix} & \varphi_3 & \\ & \varphi_2 & \\ \varphi_4 & I & \varphi_5 \\ & \varphi_2 & \\ & \varphi_4 & \end{matrix} \quad \text{and} \quad \begin{matrix} \varphi_3 \\ \varphi_2 \\ \varphi_4 \end{matrix}.$$

Hence $S_3(P_4)/S_2(P_4) = \varphi_4 \oplus I \oplus \varphi_5 \oplus \varphi_3$ by the socle series of P_2. By Lemma 3.1 and Theorem 3.9, φ_3 has multiplicity 2 in P_4. Therefore the existence of the two above submodules of P_4 implies that P_4 has Loewy length $j(P_4) = 7$. Since φ_2 is the second socle of P_I, P_4 and P_5, it follows from Lemma 3.1 and Theorem 3.9 that $S_4(P_4)/S_3(P_4) = \varphi_2 \oplus \varphi_2$. Using the duality between P_4 and P_5, it is now easy to complete the proof of Theorem 4.1.

5. The Green correspondents of all simple B-modules. Let f be the Green correspondence between $R(q)$ and $N = N_{R(q)}(E_8)$ with respect to a Sylow 2-subgroup E_8 of a group $R(q)$ of Ree type. By Proposition 2.5 it suffices to determine the structure of $f(\varphi_3)$. This is done in this section.

LEMMA 5.1. *$\varphi_{3|L}$ is a projective FL-module.*

PROOF. Let X be an R-form of ξ_4 such that $\overline{X} = X/X\pi$ has socle φ_2. Then Theorem 3.9 and Theorem 4.1 imply that $\overline{X}$ has socle series

$$\overline{X} = \begin{matrix} I & \varphi_3 & & \varphi_4 & \varphi_5 \\ & & \varphi_2 & & \end{matrix}. \qquad (*)$$

By Lemma 2.2, $\varphi_{2|L} = \Lambda_q$ is projective. Lemma 1.3 asserts

$$\varphi_{4|L} = \tfrac{1}{2}(m-1)\Lambda_q \oplus \Lambda_{m/2}, \qquad \varphi_{5|L} = \tfrac{1}{2}(m-1)\Lambda_q \oplus \overline{\Lambda}_{m/2}.$$

Therefore there are two indecomposable B-modules Y and Y' with $\mathrm{soc}(Y) = \varphi_4$ and $\mathrm{soc}(Y') = \varphi_5$ such that

$$(\Lambda_{m/2})^{R(q)}e = Y \oplus \tfrac{1}{2}(m-1)P_3, \qquad \left(\overline{\Lambda}_{m/2}\right)^{R(q)}e = Y' \oplus \tfrac{1}{2}(m-1)P_3,$$

and E_2 is a vertex of Y and Y'. By means of Nakayama's relations it follows that $\Lambda_{m/2}$ and $\overline{\Lambda}_{m/2}$ occur both $\frac{1}{2}(m-1)$ times in the socle of $\varphi_{3|L}$. Hence,

$$\xi_{4|L} = I^- + \Lambda_1 + (q-1)\Lambda_q + m\left(\Lambda_m + \overline{\Lambda}_m\right) + \tfrac{1}{2}(m+1)\Lambda_{m/2} + \tfrac{1}{2}(m-1)\Lambda^-_{m/2} + \tfrac{1}{2}(m+1)\overline{\Lambda}_{m/2} + \tfrac{1}{2}(m-1)\overline{\Lambda}^-_{m/2}$$

implies

$$\overline{X}_{|L} = I \oplus \Lambda_{m/2} \oplus \overline{\Lambda}_{m/2} \oplus \Lambda_1 \oplus (q-1)\Lambda_q \oplus m\left(\Lambda_m \oplus \overline{\Lambda}_m\right) \oplus \tfrac{1}{2}(m-1)P_{m/2} \oplus \tfrac{1}{2}(m-1)\overline{P}_{m/2},$$

where $P_{m/2}$ and $\bar{P}_{m/2}$ denote the projective covers of $\Lambda_{m/2}$ and $\bar{\Lambda}_{m/2}$ respectively. On the other side, (*) implies

$$\bar{X}_{|L} = I \oplus \Lambda_{m/2} \oplus \bar{\Lambda}_{m/2} \oplus \varphi_{3|L} \oplus m\Lambda_q.$$

Therefore $\varphi_{3|L}$ is a projective FL-module by the character table of L and the Krull-Remak-Schmidt theorem.

PROPOSITION 5.2. (a) *The Green correspondent $f(\varphi_3)$ of the simple $FR(q)$-module φ_3 in FN has socle and Loewy series*

$$f(\varphi_3) = 3 \quad \begin{matrix} 3 \\ \\ 3^* \end{matrix} \quad 3^*.$$

(b) $\varphi_{3|N} = f(\varphi_3) \oplus$ *projectives.*

(c) φ_3 *has a periodic projective resolution, and* $\Omega^7(\varphi_3) \cong \varphi_3$.

PROOF. Let e_0 be the block idempotent of the principal 2-block b of C, and let $Y = (\varphi_{3|C})e_0$. By Lemma 1.2, Y has composition factors

$$2(q-m)I + [3(q-m)-2]\chi' + [3(q-m)-2]\chi''. \qquad (*)$$

Let $X = \operatorname{ann}_Y(1-u)$. Then $X = Y(1-u)$ by Lemma 5.1, because all involutions of $R(q)$ are conjugate. Furthermore, X is a self-dual, algebraically invariant FC'-module.

By Lemma 2.3(e), I has multiplicity 1 in $\operatorname{soc}(X) = \operatorname{soc}(Y)$. Let X_I be the indecomposable component of X with I in the socle. Then X_I has one of the following socle series

$$I, \begin{pmatrix} \chi' & & \chi'' \\ & I & \end{pmatrix}, \begin{pmatrix} & \chi' & & \chi'' & \\ I & & \chi' & & \chi'' \end{pmatrix}, \begin{pmatrix} I & \chi' & \chi'' \\ I & \chi' & \chi'' \end{pmatrix}, P_I'.$$

Moreover, any other nonprojective component of X belongs to $\left\{\chi', \chi'', \begin{matrix}\chi' \\ \chi''\end{matrix}, \begin{matrix}\chi'' \\ \chi'\end{matrix}\right\}$ by Donovan-Freislich [3]. Since $P'_{\chi'}$ and $P'_{\chi''}$ occur with the same multiplicity as a direct summand of X, and since the even number $(q-m)$ is the multiplicity of I in X, Lemma 1.1 implies that $X_I \neq I$. Let $X_1 = X_I$ if X_I is self-dual, and let $X_1 = X_I \oplus X_I^*$ otherwise. Then there are integers $r, s, t \in \mathbf{N}$ such that

$$X = X_1 \oplus t(\chi' \oplus \chi'') \oplus r\begin{pmatrix} \chi' & \oplus & \chi'' \\ \chi'' & & \chi' \end{pmatrix} + s(P'_{\chi'} \oplus P'_{\chi''}). \qquad (**)$$

If x_1 is the multiplicity of χ' in X_1 and x_2 is the one of I, then (*) and (**) imply

$$x_1 + t + 2r + 3s = \tfrac{1}{2}[3(q-m)-2], \qquad x_2 + 2s = q - m.$$

Hence $2x_1 < 2x_1 + 4r + 2t = 3x_2 - 2$, which shows that

$$X_I \neq \begin{pmatrix} & \chi' & & \chi'' & \\ I & & \chi' & & \chi'' \end{pmatrix}.$$

If $X_I = P_I'$, then $t = 1$ and $r = 0$. From (**) follows

$$X = P_I' \oplus \chi' \oplus \chi'' \oplus s(P'_{\chi'} \oplus P'_{\chi''}).$$

As $\operatorname{soc}(Y) = \operatorname{soc}(X)$, we obtain

$$Y = P_I \oplus \begin{matrix} \chi' \\ \chi' \end{matrix} \oplus \begin{matrix} \chi'' \\ \chi'' \end{matrix} \oplus s(P_{\chi'} \oplus P_{\chi''}).$$

Hence Y is E_4-projective, which is a contradiction, because φ_3 has vertex $\operatorname{vx}(\varphi_3) = E_8$ by Knörr's theorem **[10]**. Thus $X_I \neq P_{I'}$, and in any of the remaining two cases (**) implies

$$X = X_1 \oplus s(P'_{\chi'} \oplus P'_{\chi''}),$$

where $s = \frac{1}{2}(q - m - 2)$. Furthermore, $\operatorname{soc}(X_1) = I \oplus \chi' \oplus \chi''$.

Let T be the nonprojective part of Y. Then

$$\operatorname{soc}(T) = \operatorname{soc}(X_1) = I \oplus \chi' \oplus \chi'' = \operatorname{head}(T),$$
$$S_2(T)/S_1(T) = 2I \oplus 2\chi' \oplus 2\chi'',$$

and T has Loewy length $j(T) = 3$. If T were decomposable, then

$$T \cap X = \begin{matrix} \chi' & & \chi'' \\ & I & \end{matrix} \oplus \begin{matrix} & I & \\ \chi' & & \chi'' \end{matrix}.$$

Since $T(1 - u) = T \cap X$ and T have the same socle, it follows that $T = T_1 \oplus T_1^*$ for some indecomposable FC-module, T_1 satisfying $T_1 \cap X = \begin{matrix} \chi' & & \chi'' \\ & I & \end{matrix}$. Hence,

$$T_1 = \begin{matrix} \chi' & & \chi'' \\ \chi' & I & \chi'' \\ & I & \end{matrix} \cong \Omega\begin{pmatrix} I \\ I \end{pmatrix}$$

by Lemma 1.1. Therefore T_1 and hence φ_3 are E_4-projective. Again by Knörr's theorem **[10]**, this is a contradiction. Thus T is an indecomposable FC-module, and

$$\varphi_{3|C} = T \oplus \text{projective } FC\text{-modules} \qquad (***)$$

by Nagao's lemma **[4**, p. 353]. Furthermore,

$$T \cap X = \operatorname{ann}_T(1 - u) = T(1 - u) = \begin{matrix} I \\ I \end{matrix} \quad \begin{matrix} \chi' \\ \chi' \end{matrix} \quad \begin{matrix} \chi'' \\ \chi'' \end{matrix}$$

or

$$\begin{matrix} \chi' & & \chi'' \\ & I & \end{matrix} \oplus \begin{matrix} & I & \\ \chi' & & \chi'' \end{matrix}.$$

In particular, $T \cap X$ is an FC'-module. By Lemma 1.1, its restriction to K decomposes as follows.

$$(T \cap X)_{|K} = \begin{matrix} I \\ I \end{matrix} \quad \begin{matrix} 1 \\ 1 \end{matrix} \quad \begin{matrix} 1^* \\ 1^* \end{matrix} \oplus \text{projective } F\mathfrak{A}_4\text{-modules}$$

or

$$(T \cap X)_{|K} = \begin{matrix} 1 & & 1^* \\ & I & \end{matrix} \oplus \begin{matrix} & I & \\ 1 & & 1^* \end{matrix} \oplus \text{projective } F\mathfrak{A}_4\text{-modules}.$$

As T restricted to any involution is projective, the socles of $T_{|K}$ and $(T \cap X)_{|K} = T_{|K}(1 - u)$ coincide. Since there is a Green correspondence between C and K with respect to E_8, it follows that

$$T_{|K} = I \quad 1 \quad \begin{matrix} I \\ 1^* \\ I \end{matrix} \quad \begin{matrix} 1 \\ I \\ 1 \end{matrix} \quad \begin{matrix} 1^* \\ 1 \\ 1^* \end{matrix} \quad 1^* \oplus \text{projective } FK\text{-modules},$$

where the nonprojective part of $T_{|K}$ is indecomposable.

Now let $f(\varphi_3)$ be the Green correspondent of φ_3 in N. Since N is 2-closed, $f(\varphi_3)_{|K}$ is indecomposable and coincides with the nonprojective part of $T_{|K}$. Therefore, $\dim_F f(\varphi_3) = 12$. As φ_3 is self-dual and algebraically invariant, Proposition 2.3 of **[12]** implies that $f(\varphi_3)$ has composition factors $2 \cdot 3 + 2 \cdot 3$. Since $f(\varphi_3)_{|K}$ has Loewy length 3, also $f(\varphi_3)$ has Loewy length 3. Thus

$$\text{either} \quad f(\varphi_3) = \begin{matrix} & 3 & \\ 3 & & 3^* \\ & 3^* & \end{matrix} \quad \text{or} \quad f(\varphi_3) = \begin{matrix} & 3^* & \\ 3 & & 3^* \\ & 3 & \end{matrix}.$$

In either case, $(***)$ implies

$$\varphi_{3|N} = f(\varphi_3) \oplus \text{projective } FN\text{-modules},$$

which proves assertion (b).

By Theorem 4.1, $\dim_F \operatorname{Ext}^1_{FR(q)}(\varphi_3, \varphi_2) = 1$. Hence,

$$\dim_F \operatorname{Ext}^1_{FN}(f(\varphi_3), f(\varphi_2)) = \dim_F(\Omega(f(\varphi_3)), f(\varphi_2))^1_N \neq 0$$

by (b), Proposition 2.5(a) and Lemma 1.1 of **[12]**. If

$$f(\varphi_3) = \begin{matrix} & 3^* & \\ 3 & & 3^* \\ & 3 & \end{matrix},$$

then

$$\Omega(f(\varphi_3)) = \begin{matrix} I & & 1 & & 1^* \\ & 3 & & 3^* & \\ & & 3^* & & \end{matrix},$$

and

$$\operatorname{Hom}_{FN}(\Omega(f(\varphi_3)), f(\varphi_2)) = (\Omega(f(\varphi_3)), f(\varphi_2))^1_N = 0$$

by Proposition 2.5(a). This contradiction proves assertion (a).

In order to prove (c) by Green **[9**, p. 151], it suffices to show that $\Omega^7 f(\varphi_3) \cong f(\varphi_3)$. From (a) and Proposition 2.3 of **[12]**, it follows that

$$0 \leftarrow f(\varphi_3) \leftarrow P_3 \leftarrow P_{3^*} \leftarrow P_{3^*} \leftarrow (P_I \oplus P_1 \oplus P_{1^*}) \leftarrow P_3 \leftarrow P_3 \leftarrow P_{3^*} \leftarrow f(\varphi_3) \leftarrow 0$$

is a minimal projective resolution of $f(\varphi_3)$. Hence $\Omega^7(\varphi_3) \cong \varphi_3$, which completes the proof of Proposition 5.2.

Combining the results of Propositions 2.5 and 5.2, we have shown

THEOREM 5.3. *The simple $FR(q)$-modules belonging to the principal 2-block B of a group $R(q)$ of Ree type have Green correspondents in FN with the following socle and Loewy series*:

(a) $f(I) = I$, $f(\varphi_4) = 1^*$, *and* $f(\varphi_5) = 1$,

(b) $f(\varphi_2) = \begin{matrix} 3^* \\ 3 \end{matrix}$,

(c) $f(\varphi_3) = \begin{matrix} & 3 & \\ 3 & & 3^* \\ & 3^* & \end{matrix}$.

6. An application. By means of Theorem 4.1 and recent work of J. Alperin [**1**], K. Erdmann [**5**] and the authors [**12**] it is now easy to give a sharp upper bound for the Loewy lengths $j(P)$ of the indecomposable projective FG-modules P belonging to the principal 2-block B_0 of an *arbitrary* finite group G with an abelian Sylow 2-subgroup. Since the index of nilpotency of the Jacobson radical $J = J(FG)$ is preserved by field extensions, the following result holds for arbitrary 2-modular systems (F, R, S) for G.

THEOREM 6.1. *Every indecomposable projective FG-module P of the principal* 2-*block* B_0 *of a finite group G with an abelian Sylow* 2-*subgroup D of order* $|D| = 2^n$ *has Loewy length* $j(P) \leq \max\{2n + 1, 2^n\}$.

PROOF. Let $O(G)$ be the maximal normal subgroup of G with odd order. Then by J. Walter's theorem (see Gorenstein [**8**, p. 485]) G contains a normal subgroup $T \geq O(G)$ with odd index $|G : T|$ such that $T/O(G)$ is a direct product of a 2-group and simple groups of the following types:

(a) $G \cong \mathrm{PSL}(2, q)$, $q > 3$, $q \equiv 3$ or 5 mod 8.

(b) $G \cong \mathrm{PSL}(2, 2^a)$, $a > 1$.

(c) $G \cong J_1$, the smallest Janko group of order $|J_1| = 175, 560$.

(d) $G \cong R(q)$, a simple group of Ree type.

Let $j(A)$ be the index of nilpotency of the Jacobson radical of an algebra A. Then for every indecomposable projective B_0-module P we have $j(P) \leq j(B_0) = j$.

Since B_0 is the principal 2-block of G, the elements of $O(G)$ act trivially on B_0. Hence we may assume that $O(G) = 1$. As $|G : T|$ is odd, Villamayor's theorem (see [**13**, p. 524]) asserts that $j(FG) = j(FT)$. Thus we may also assume that $G = T$.

If $G = G_1 \times G_2$, then $FG = FG_1 \otimes_F FG_2$. From Theorem 71.10 of Curtis-Reiner [**2**, p. 485], it follows easily that Jacobson radical

$$J(FG) = J(FG_1) \otimes FG_2 + FG_1 \otimes J(FG_2).$$

Therefore $j(B_0(G)) = j(B_0(G_1)) + j(B_0(G_2)) - 1$. Now suppose that the assertion of Theorem 6.1 holds in G_1 and G_2. Let p^{d_i} be the order of a Sylow 2-subgroup S_i of G_i, $i = 1, 2$. Then $S_1 \times S_2$ is a Sylow 2-subgroup of $G = G_1 \times G_2$ with order $p^{(d_1 + d_2)}$. Hence

$$\begin{aligned} j(B_0(G)) &\leq \max\{2d_1 + 1, 2^{d_1}\} + \max\{2d_2 + 1, 2^{d_2}\} - 1 \\ &\leq \max\{2(d_1 + d_2) + 1, 2^{(d_1 + d_2)}\}. \end{aligned}$$

Therefore we may assume that G is either an abelian 2-group or one of the simple groups of Walter's list (a)–(d). If G is abelian, then $j(B_0) \leq |D|$. In case (a), $j(B_0) \leq \max\{2n + 1, 2^n\}$ (see K. Erdmann [**5**][4]). In case (b), $j(B_0) = 2n + 1$ by Theorem 4 of J. Alperin [**1**]. In case (c), $j(B_0) = 7 = 2n - 1$ by Theorem 6.7 of [**12**]. Finally, Theorem 4.1 asserts that also $j(B_0) = 7 = 2n - 1$, if G is a group $R(q)$ of Ree type. Thus Theorem 6.1 holds.

[4]See Footnote 3.

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