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ABSTRACT. Let L be a finitely generated nonalgebraic extension of a field K of characteristic $p \neq 0$. A maximal separable extension D of K in L is distinguished if $L \subseteq K^{p^{-n}}(D)$ for some n. Let d be the transcendence degree of L over K. If every maximal separable extension of K in L is distinguished, then every set of d relatively p-independent elements is a separating transcendence basis for a distinguished subfield. Conversely, if $K(L^p)$ is separable over K, this condition is also sufficient. A number of properties of such fields are determined and examples are presented illustrating the results.

0. Introduction. Let L be a finitely generated extension of a field K of characteristic $p \neq 0$. If L is algebraic over K, then there is a unique intermediate field D such that D is separable over K and L is purely inseparable over D. If L is not algebraic over K, there will still be maximal separable extensions S of K in L and L will necessarily be purely inseparable finite dimensional over any such subfield S. However, in general S is far more being unique. If p^s is the minimum of the degrees $[L: S]$, s is called the order of inseparability of L/K (inor(L/K)). In [5], Dieudonne studied such maximal separable extensions and established that there must be an S such that $L \subseteq K^{p^{-\infty}}(S)$, that is, L can be obtained from S by adjoining p^nth roots of elements of K. Such a field S is called distinguished. In [7], Kraft established that the distinguished maximal separable intermediate fields are precisely those over which L is of minimal degree. In this paper we examine the question of when every maximal separable intermediate field is distinguished, a property which holds for algebraic extensions.

If n is the least nonnegative integer such that $K(L^{p^n})$ is separable over K, n is called the inseparability exponent, inex(L/K). Throughout this paper n will be used to denote inex(L/K) and d will denote the transcendence degree of L/K. If D is distinguished for L/K, then $K(L^{p^n}) = K(D^{p^n})$ and hence $L \subseteq K^{p^{-n}}(D)$. Thus, if Y is a relative p-basis of D over K, Y is relatively p-independent in L over K. Since D is separable over K, Y^{p^n} is a relative p-basis of $K(D^{p^n})$, i.e. $K(L^{p^n})$, over $K(D^{p^{n+1}})$. Thus D is of the form $K(L^{p^n})(Y)$. We also note that if S/K is separable and L/S is purely inseparable, S is a maximal separable subfield of L/K if and only if $L^p \cap S \subseteq K(S^p)$ [3, Lemma 1.2, p. 46]. L is modular over K if and only if L^{p^r} and K are linearly disjoint for all r. If L/K is finite dimensional purely

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inseparable, L/K is modular if and only if it is a tensor product of simple extensions [**11**].

Now suppose the order of inseparability of L/K is s. Any intermediate field L_1 of L/K which also has order of inseparability s will be called a form of L/K. In [**4**, Theorem 1.4, p. 657] it is shown that there exists a unique minimal intermediate field L^* of L/K which is a form of L/K. The field L^* is called the irreducible form of L/K and L^*/K is called the irreducible. For example, if P is a perfect field and $\{x, u, v\}$ is algebraically independent over P, let $K = P(u^p, v^p)$ and $L = K(x, ux^p + v)$. Then $K(x)$ is distinguished, $K(ux^p + v)$ is maximal separable but not distinguished and $K(x^p, ux^p + v)$ is a form of L/K (and in fact is the irreducible form).

In §I we develop necessary conditions for every maximal separable subfield of L/K to be distinguished. Theorem 4 establishes the condition that every set of d relatively p-independent elements must be a separating transcendence basis for a distinguished subfield. Extensions with this property are then shown to be separable algebraic extensions of irreducible extensions (Theorem 5). In particular, any set of d relatively p-independent elements must be algebraically independent over K, a property similar to the characterizing property of a separable extension that the elements of any relative p-basis must be algebraically independent [**8**, Theorem 11, p. 281].

§II deals exclusively with the inseparability exponent 1 case. It is shown that every maximal separable subfield is distinguished if and only if every set of d relatively p-independent elements is a separating transcendence basis for a distinguished subfield. Moreover, if L/K has the property, so does any intermediate field L_1/K. §III presents examples having the property and indicating why it is necessary to restrict the results of §II to exponent 1. §IV develops criteria which force those L/K having every maximal separable subfield distinguished to be of exponent less than or equal to 1. In fact, we conjecture that this must always be true (in the nonalgebraic case).

I. Necessary conditions.

THEOREM 1. *If every maximal separable intermediate field of L/K is distinguished, then $K^{p^{-n}}(L^{p^i}) \cap L = K(L^{p^i})$ for $i = 0, 1, \ldots$.*

PROOF. The proof is by induction on i. The conclusion is immediate for $i = 0$. Assume the result for $0 \leqslant i < n$. Suppose $\theta \in K^{p^{-n}}(L^{p^{i+1}}) \cap L \setminus K(L^{p^{i+1}})$ and θ is transcendental over K. Now $\theta \in K^{p^{-n}}(L^{p^i}) \cap L = K(L^{p^i})$. Since $K(\theta)/K$ is separable, θ is in a maximal separable intermediate field of L/K, say S. We show θ is not in any distinguished intermediate field and hence S is not distinguished, a contradiction. Suppose $\theta \in D$, a distinguished intermediate field. Then

$$\theta \in D \cap K(L^{p^i}) \subseteq (D \otimes_K 1) \cap \left(K(D^{p^i}) \otimes_K K^{p^{-n}}\right) = K(D^{p^i}).$$

Now $D = K(L^{p^n})(Y)$ where Y is relatively p-independent in L/K and Y^{p^n} is a relative p-basis of $K(L^{p^n})/K$. Thus

$$\theta \in K(L^{p^n})(Y^{p^i}) \subseteq K(L^{p^{i+1}})(Y^{p^i}).$$

Now Y^{p^i} is relatively p-independent in $K(L^{p^i})/K$ and since $\theta \notin K(L^{p^{i+1}})$, there exists $y \in Y$ such that $\theta \notin K(L^{p^{i+1}})(Y^{p^i} \setminus \{y^{p^i}\})$. Thus

$$y^{p^i} \in K(L^{p^{i+1}})\left(Y^{p^i} \setminus \{y^{p^i}\}, \theta\right).$$

Note that

$$\theta \in K(L^{p^i}) \cap K^{p^{-n}}(L^{p^{i+1}}) \subseteq \left(K^{p^{-n+i}} \otimes_K K(D^{p^i})\right) \cap \left(K^{p^{-n}} \otimes_K K(D^{p^{i+1}})\right)$$
$$= K^{p^{-n+1}} \otimes_K K(D^{p^{i+1}})$$

and thus $\theta^{p^{n-i}} \in K(D^{p^{n+1}}) = K(L^{p^{n+1}})$. Hence

$$y^{p^n} \in K(L^{p^{n+1}})\left(Y^{p^n} \setminus \{y^{p^n}\}, \theta^{p^{n-i}}\right) = K(L^{p^{n+1}})\left(Y^{p^n} \setminus \{y^{p^n}\}\right),$$

which contradicts the relative p-independence of Y^{p^n} in $K(L^{p^n})/K$. Now suppose θ is algebraic over K. Let $t \in K(L^{p^{i+1}})$ be transcendental over K. Then $\theta + t \in K^{p^{-n}}(L^{p^{i+1}}) \cap L \setminus K(L^{p^{i+1}})$ and is transcendental over K. However, this case has been shown to be impossible.

Now assume $K^{p^{-n}}(L^{p^i}) \cap L = K(L^{p^i})$ for $i \geqslant n$ and let $\theta \in K^{p^{-n}}(L^{p^{i+1}}) \cap L$. Then $\theta \in K^{p^{-n}}(L^{p^i}) \cap L = K(L^{p^i})$. Thus

$$\theta \in K^{p^{-n}}(L^{p^{i+1}}) \cap K(L^{p^i}) = \left(K^{p^{-n}} \otimes_K K(L^{p^{i+1}})\right) \cap \left(1 \otimes_K K(L^{p^i})\right)$$
$$= K(L^{p^{i+1}})$$

since $K^{p^{-n}} \otimes_K K(L^{p^i})$ is a field. Thus $K^{p^{-n}}(L^{p^{i+1}}) \cap L = K(L^{p^{i+1}})$.

Corollary 2. *If $K^{p^{-n}}(L^{p^i}) \cap L = K(L^{p^i})$, for $i = 0, 1, \ldots,$ then $K^{p^{-n}}(K^{p^i}(L^{p^{j+i}})) \cap K(L^{p^j}) = K(L^{p^{j+i}})$, for $i = 0, 1, \ldots,$ for any j, hence $K(L^{p^j})$ also has the necessary condition.*

Proof. $K^{p^{-n}}(K^{p^i}(L^{p^{j+i}})) \cap K(L^{p^j}) = K^{p^{-n}}(L^{p^{j+i}}) \cap K(L^{p^j}) \subseteq K^{p^{-n}}(L^{p^{j+i}}) \cap L = K(L^{p^{j+i}})$. Clearly $K(L^{p^{j+i}}) \subseteq K^{p^{-n}}(K^{p^i}(L^{p^{i+j}})) \cap K(L^{p^j})$. Although $K(L^{p^j})$ will be of inseparable exponent less than that of L/K, clearly $K(L^{p^j})$ has the necessary condition.

Proposition 3. *Let $\overline{K}$ be the algebraic closure of K in L. If every maximal separable intermediate field of L/K is distinguished, then $\overline{K}/K$ is separable.*

Proof. Let S be the maximal separable intermediate field of $\overline{K}/K$ and let D be a maximal separable intermediate field of L/S. Since any maximal separable intermediate field of L/K must contain S, D is maximal separable for L/K, whence distinguished for L/K and L/S. Thus by Theorem 1, $S^{p^{-1}} \cap L \subseteq S^{p^{-n}}(L^{p^n}) \cap L = S(L^{p^n})$, and since $S(L^{p^n})/S$ is separable, $S^{p^{-1}} \cap L = S$. Thus $S = \overline{K}$.

Theorem 4. *Assume every maximal separable intermediate field of L/K is distinguished. Then every set of d relatively p-independent elements is a separating transcendence basis for a distinguished subfield.*

Proof. We use induction on d. Assume $d = 1$ and let x be relatively p-independent. Since $\overline{K}/K$ is separable, x is transcendental over K. Let S be a maximal separable extension of K in L containing $K(x)$. Then S is distinguished. If B is a

p-basis of K, since $x \notin K(S^p) = S^p(B)$, $B \cup \{x\}$ is p-independent in S, i.e., S is separable over $K(x)$. Thus $S/K(x)$ is separable algebraic and hence x is a separating transcendence basis for a distinguished subfield.

Now assume $d > 1$ and let $\{x_1, \ldots, x_d\}$ be relatively p-independent in L/K. Then, as above, x_1 is transcendental over K. Since x_1 is relatively p-independent in L/K, any maximal separable extension of $K(x_1)$ in L will be a maximal separable extension of K in L, hence will be distinguished for L/K and hence for $L/K(x_1)$. Thus every maximal separable intermediate field of $L/K(x_1)$ is distinguished. By induction $\{x_2, \ldots, x_d\}$ is a separating transcendence basis for a distinguished subfield of $L/K(x_1)$, and hence $\{x_1, \ldots, x_d\}$ is one for L/K.

THEOREM 5. *If every set of d relatively p-independent elements form a separating transcendence basis for a distinguished subfield, then $L = L^*(\theta)$ where L^*/K is the irreducible form of L/K and θ is separable algebraic over L^*.*

PROOF. Let C^* be the unique intermediate field such that L/C^* is separable and C^*/K is reliable [2, Theorem 2.3, p. 141]. If $L \neq C^*(L^p)$, choose $L \supseteq L_1 \supseteq C^*(L^p)$ with $[L: L_1] = p$. Since $L_1 \supseteq C^*$, L_1 is a form of L/K [4, Theorem 1.2, p. 656]. Thus L_1 cannot contain a separating transcendence basis for a distinguished subfield of L/K, else the order of inseparability of L/K would be one more than that of L_1/K. But since $[L: L_1] = p$, and $[L: K(L^p)] \geqslant p^{d+1}$, at least d elements of L_1 which are relatively p-independent over K must remain p-independent in L. This contradicts the assumption of the theorem, hence $L = C^*(L^p)$, i.e. L/C^* is separable algebraic.

Now, if C^*/K is not irreducible, then since C^* is not separable over any intermediate field of L/K [9, Theorem 1, p. 523], it has a form L_0 over which C^* is purely inseparable and $[C^*: L_0] = p$. But now $L = C^* \otimes_{L_0} S$ where S/L_0 is separable. Since L_0 is a form of L/K, S is also a form of L/K and $[L: S] = p$. This leads to a contradiction as above.

PROPOSITION 6. *Let C be a subfield of L/K such that L is separable over C. If every maximal separable intermediate field of L/K is distinguished, then the same is true for C/K.*

PROOF. Let D be a maximal separable intermediate field of C/K. Since C/D is purely inseparable bounded exponent and L/C is separable, $L = F \otimes_D C$ for some intermediate field F of L/D such that F/D is separable [6, Proposition 1, p. 302]. Since L/C is separable, C is a form of L/K and hence if F is distinguished for L/K, D is for C/K by a degree argument. Hence it suffices to show F is maximal separable in L/K, i.e. $L^p \cap F \subseteq K(F^p)$.

But if $b^p \in F$, then $b \in (D^{p^{-1}} \cap C) \otimes_D F$ and hence $b^p \in (C^p \cap D) \otimes_{D^p} F^p \subseteq K(D^p)(F^p) = K(F^p)$, so F is maximal separable.

COROLLARY 7. *If every maximal separable intermediate field of L/K is distinguished, then L is a separable algebraic extension of an irreducible extension.*

II. Exponent one. Throughout this section we assume that the inseparability exponent of L/K is 1. With this restriction, the necessary condition of Theorem 4 is also sufficient.

THEOREM 8. *Every maximal separable intermediate field of L/K is distinguished if and only if every set of d relatively p-independent elements is a separating transcendence basis for a distinguished subfield.*

PROOF. We induct on the order of inseparability of L/K. Assume the order of inseparability is 1 and every set of d relatively p-independent elements is a separating transcendence basis for a distinguished subfield. By Corollary 7, $L = L^*(\theta)$ and L^*/K is irreducible. Let S be a maximal separable extension of K in L. Let $\alpha^p \in S$ and $\alpha \notin S$. Then $S(\alpha)$ has order of inseparability 1, and hence contains L^*. Thus $L/S(\alpha)$ is separable and purely inseparable and hence $L = S(\alpha)$. Thus $[L: S] = p$ and S is distinguished.

Now assume the order of inseparability is $r > 1$ and let S be a maximal separable intermediate field. Consider $S(L^p)(B \setminus b) \equiv L_0$ where B is a relative p-basis for L over S. (Note $r \geqslant 2$ so $|B| \geqslant 2$.) Then $[L: L_0] = p$. Thus L_0 contains at least d elements which remain p-independent in L. Hence L_0 contains a separating transcendence basis for a distinguished subfield of L/K and hence L_0/K is not a form of L/K. Since we are in exponent 1, L_0 must have one less element in a relative p-basis over K, and hence every relative p-basis for L_0/K remains relatively p-independent in L/K. Thus every set of d elements of a relative p-basis for L_0/K form a separating transcendence basis for a distinguished intermediate field of L_0 (since they do for L), and hence by induction L_0 has every maximal separable intermediate distinguished. Thus S is distinguished for L_0 and since the inseparability of L/K is one more than the inseparability of L_0/K, S is distinguished for L/K.

LEMMA 9. *Assume every maximal separable intermediate field of L/K is distinguished. If L_1 is an intermediate field of L/K and $[L: L_1] = p$, then every maximal separable intermediate field of L_1/K is distinguished.*

PROOF. Since $[L: L_1] = p$, L/L_1 is separable algebraic or purely inseparable. If L/L_1 is separable, Proposition 6 applies. Suppose L/L_1 is purely inseparable. By Corollary 7, L is separable algebraic over its irreducible form, and L_1 is not a form of L/K. Since L/K is of exponent 1, L_1 has one less element than L in a relative p-basis over K. Since $[L: L_1] = p$, the elements of any relative p-basis for L_1/K remain relatively p-independent in L/K. Thus if d is the transcendence degree of L/K, any set of d relatively p-independent elements of L_1/K remain p-independent in L, hence are a separating transcendence basis for a distinguished subfield of L (Theorem 4) which must also be distinguished for L_1. Thus every maximal separable intermediate field of L_1/K is distinguished by Theorem 8.

THEOREM 10. *If L_1 is an intermediate field of L/K and every maximal separable intermediate field of L/K is distinguished, then the same is true for L_1/K.*

PROOF. L/L_1 is finitely generated so a finite number of applications of Proposition 6 and Lemma 9 yield the desired result.

III. Examples. We now present some examples to illustrate the results. It should be noted that there is a class of extensions which have every maximal separable intermediate field distinguished. For if L/K is any transcendental extension with order of inseparability 1, let L^* be the irreducible form of L/K. If S is a maximal separable extension if K in L^* and $\alpha \in L^* \setminus S$ with $\alpha^p \in S$, then $S(\alpha)$ has order of inseparability 1, and hence $S(\alpha) = L^*$ and S is distinguished. Since all examples in the literature have their irreducible forms of transcendence degree 1, notably those in **[9]** and **[10]**, we present the following example.

EXAMPLE 11. There exists a field extension of transcendence degree greater than one which has every maximal separable intermediate field distinguished.

Let P be a perfect field and let $\{v, x, y, z, w\}$ be algebraically independent indeterminates over P. Let $K = P(v, x, y)$ and $L = K(z, w, zx^{p^{-1}} + wy^{p^{-1}} + v^{p^{-1}})$. Then L/K has transcendence degree 2 and exponent 1. Let $\{b_1, b_2\}$ be relatively p-independent in L/K. We need to show $\{b_1, b_2\}$ is a separating transcendence basis for a distinguished subfield, i.e. $K(L^p) = K(L^{p^2})(b_1^p, b_2^p)$. By a degree argument, this is true if and only if $K(L^{p^2})(b_1^p) \not\subseteq K(L^{p^2})(b_2)$ and $K(L^{p^2})(b_2^p) \not\subseteq K(L^{p^2})(b_1^p)$. Thus suppose $K(L^{p^2})(b_1^p) \subseteq K(L^{p^2})(b_2^p)$. Then $L/K(L^{p^2})(b_2^p)$ is modular with a subbasis b_1, b_2 and some b_3 with exponents 1, 1, 2 respectively. We use the method of Sweedler [**11**, Example 1.1, p. 405] and prove this is impossible by showing the field of constants of all rank p higher derivatives on L/K is $K(L^p)$. Let $d = \{d_0, d_1, \dots, d_p\}$ be a rank p higher derivation on L/K. Then

$$\left[d_1\left(zx^{p^{-1}} + wy^{p^{-1}} + v^{p^{-1}}\right)\right]^p = d_p(z^px + w^py + v) = (d_1(z))^px + (d_1(w))^py.$$

Since $\{1, x, y\}$ is linearly independent over L^p, we have $d_p(z^p) = 0 = d_p(w^p)$. Clearly $d_i(z^p) = 0 = d_i(w^p)$, $i = 1, \dots, p-1$. Hence $K(L^p)$ is the field of constants as claimed. Thus any 2 relatively p-independent elements are a separating transcendence basis for a distinguished subfield, and hence by Theorem 8, L/K has every maximal separable intermediate field distinguished.

EXAMPLE 12. We show the converse of Theorem 5 is not true. Let P be a perfect field and let $\{w, x, y_1, y_2, z\}$ be algebraically independent indeterminates over P. Let $K = P(x, y_1, y_2)$ and $L = K(z, w, x^{p^{-1}}z + y_1^{p^{-1}}, x^{p^{-1}}w + y_2^{p^{-1}})$. Let L^* be the irreducible form of L/K. If $L = L^*(L^p)$, then L is separable algebraic over L^*, as desired.

If $L \neq L^*(L^p)$, then there is a subfield L_1 over which L is purely inseparable and $[L: L_1] = p$ and L_1 has order of inseparability 2. We show this is impossible. Such a field L_1 is of the form $K(L^p)(b_1, b_2, b_3)$ where $\{b_1, b_2, b_3\}$ is relatively p-independent in L/K. If $K(L^p) = K(L^{p^2})(b_1^p, b_2^p, b_3^p)$, then L_1/K would have order of inseparability 1, a contradiction. If $K(L^p) \neq K(L^{p^2})(b_1^p, b_2^p, b_3^p)$, then since $[K(L^p): K(L^{p^2})] = p^2$, $K(L^{p^2})(b_1^p, b_2^p, b_3^p) \subseteq K(L^{p^2})(b_3^p)$ say. But now since $[L: K(L^{p^2})] = p^6$ and is of exponent 2, $L/K(L^{p^2})(b_3)$ is modular with a subbasis b_1, b_2 and some b_4 with exponents 1, 1, 2 respectively. But as in Example 11, the field of constants of all rank p higher derivatives on L/K is $K(L^p)$, and we have a contradiction. Thus $L = L^*(\theta)$ where L^*/K is irreducible and θ is separable algebraic over L^*. However, it is clear that $\{z, x^{p^{-1}}z + y_1^{p^{-1}}\}$ is not a separating transcendence basis for a distinguished subfield.

EXAMPLE 13. We show that the exponent 1 restriction is essential to Theorem 8. Let P be a perfect field and let $\{x, y, z\}$ be algebraically independent indeterminates over P. Let $K = P(x, y)$ and $L = K(z, zx^{p^{-2}} + y^{p^{-2}})$. It is straightforward that $L/K(L^{p^3})$ is not modular. Thus if b is p-independent in L/K, $b^{p^2} \notin K(L^{p^3})$, i.e. b is a separating transcendence basis for a distinguished subfield. Thus every set of 1 relatively p-independent element is a separating transcendence basis for a distinguished subfield. However, let $S = K((z^p + x(z^p x^{p^{-1}} + y^{p^{-1}})))$. Then $[L\colon S] = p^3$ and L/S is not modular. Thus $[S^{p^{-1}} \cap L\colon S] = p$. Clearly S is distinguished in $K(L^p)/K$ and since $[S^{p^{-1}} \cap K(L^p)\colon S] = p$, $L^p \cap S = (K(L^p))^p \cap S \subseteq K(S^p)$, i.e. S is a maximal separable extension of K in L. Since $[L\colon S] = p^3$ and the order of inseparability of L/K is p^2, S is not distinguished.

IV. Restrictions for exponent one. In this section we develop results which force an extension L/K which has every maximal separable subfield distinguished to have inseparability exponent 1.

LEMMA 14. *Suppose the transcendence degree of L over K is one. If* $K^{p^{-n}}(L^p) \cap L = K(L^p)$, *then every maximal separable intermediate field is either distinguished or contained in* $K(L^p)$.

PROOF. Let D be maximal separable and let x be a separating transcendence basis for D/K. If $x \in K(L^p)$, then since D is separable over $K(x)$, $D \subseteq K(L^p)$. If $x \notin K(L^p)$, then $x \notin K^{p^{-n}}(L^p)$. Thus $x^{p^n} \notin K(L^{p^{n+1}})$ and the separable algebraic closure of $K(x)$, i.e. D, is a distinguished subfield.

PROPOSITION 15. *Suppose the transcendence degree of L over K is* 1 *and every maximal separable intermediate field is distinguished. Then*:

(1) *every maximal separable intermediate field of* $K(L^{p^i})/K$ *is distinguished or contained in* $K(L^{p^{i+1}})$;

(2) *every maximal separable intermediate field of* $K(L^{p^{n-1}})$ *is distinguished.*

PROOF. (1) follows from Theorem 1 and Lemma 14. For (2), if D is distinguished for $K(L^{p^{n-1}})/K$, $K(D^p) = K(L^{p^n})$, and hence a maximal separable extension of K in $K(L^{p^{n-1}})$ cannot be contained in $K(L^{p^n})$.

L is a finite dimensional purely inseparable extension of $K(L^{p^n})$. We let L_m denote the unique minimal purely inseparable extension of L such that $L_m/K(L^{p^n})$ is modular [**11**, Theorem 6, p. 408]. Note that $L_m/K(L^{p^n})$ is also of exponent n.

THEOREM 16. *Suppose every maximal separable intermediate field of* L/K *is distinguished. Then* $K^{p^{-1}} \not\subseteq L_m$ *if and only if* $\operatorname{inex}(L/K) = 1$.

PROOF. Suppose $K^{p^{-1}} \not\subseteq L_m$ and $\operatorname{inex}(L/K) = n \geqslant 2$ Let X be a set of $d - 1$ relatively p-independent elements of L/K. Then X is part of a separating transcendence basis for a distinguished subfield by Theorem 4. Also, since X is relatively

p-independent, any maximal separable subfield for $L/K(X)$ is one for L/K and hence every maximal separable subfield of $L/K(X)$ is distinguished. Now X is a subbasis of $K(L^{p^n})(X)/K(L^{p^n})$ and since every element of X has maximal exponent, X is part of a subbasis of $L_m/K(L^{p^n})$. Thus $L_m/K(L^{p^n})(X)$ is modular.

Let $k \in K \setminus L_m^p$, $z \in L \setminus K(L^p)(X)$, and $w \in L \setminus K(L^p)(X, z)$ such that w has exponent n over $K(L^{p^n})(X, z)$. Such a w exists because $K(L^{p^n})(X, z)$ is distinguished and $\text{inex}(L/K) = n$. Set $D = K(L^{p^n})(X)(z^p + kw^p)$. Since $D \subseteq K(L^p)(X)$, D is not distinguished in $L/K(X)$. We show D is maximal separable in $L/K(X)$ and hence have a contradiction to $n \geqslant 2$.

Clearly $D/K(X)$ is separable and L/D is purely inseparable. Thus it suffices to show $L^p \cap D \subseteq K(X)(D^p)$ [3, Lemma 1.2, p. 46]. We first calculate $K(X)(D^p)$. $D/K(L^{p^n})(X)$ has exponent $n-1$ since $X \cup \{z, w\}$ is part of a subbasis of $L_m/K(L^{p^n})$ and each element is of exponent n. Thus $z^{p^{n-1}} + k^{p^{n-2}}w^{p^{n-1}} \notin K(L^{p^n})(X)$, so $z^{p^{n-1}} + k^{p^{n-2}}w^{p^{n-1}}$ is in a relative p-basis of $K(L^{p^{n-1}})(X)/K(X)$. Since every maximal separable subfield of $L/K(X)$ is distinguished, the same is true for $K(L^{p^{n-1}})(X)/K(X)$ by Lemma 14. Thus $z^{p^{n-1}} + k^{p^{n-2}}w^{p^{n-1}}$ is a separating transcendence basis for a distinguished subfield of $K(L^{p^{n-1}})(X)/K(X)$. Therefore $z^{p^n} + k^{p^{n-1}}w^{p^n} \notin K(L^{p^{n+1}})(X)$ and hence $K(X)(D^p) = K(L^{p^n})(X)(z^{p^2} + k^p w^{p^2})$.

Now suppose $L^p \cap D \not\subseteq K(X)(D^p)$, i.e. there exists $c \in L$ such that $c^p \in D \setminus K(X)(D^p)$. Then $D = K(L^{p^n})(X)(c^p)$ and c must have exponent n over $K(L^{p^n})(X)$. Thus $X \cup \{c\}$ is part of a subbasis of $L_m/K(L^{p^n})$ and hence L_m/D is modular. But, using the same methods as in Example 11, if L_m/D is modular, z^p and w^p are in D and $[D\colon K(L^{p^n})(X)] > p^{n-1}$, a contradiction.

Conversely, if $\text{inex}(L/K) = 1$, $L_m = L$ and Proposition 3 shows $K^{p^{-1}} \not\subseteq L$.

COROLLARY 17. *Suppose every maximal separable intermediate field of L/K is distinguished. If any of the conditions below hold, then* $\text{inex}(L/K) \leqslant 1$;

(a) $L/K(L^{p^n})$ *is modular*;

(b) $[L\colon K(L^p)] = p^{d+1}$;

(c) $[K\colon K^p] = p^e$ *where* $e = 0, 1, 2,$ *or* ∞.

PROOF. If L/K is separable, the result is trivial. Thus assume L/K is inseparable. By Proposition 3, $K^{p^{-1}} \cap L = K$. If $L/K(L^{p^n})$ is modular, $L = L_m$ and $K^{p^{-1}} \not\subseteq L_m$. By Theorem 16, $\text{inex}(L/K) = 1$. If $[L\colon K(L^p)] = p^{d+1}$, then it follows that $L/K(L^{p^n})$ is modular [1, Theorem 22, p. 1308]. If $[K\colon K^p] \leqslant 2$, $L/K(L^{p^n})$ is modular [1, Corollary 2.3, p. 1308]. If $[K\colon K^p] = \infty$, since $[L_m\colon L] < \infty$, $K^{p^{-1}} \not\subseteq L_m$.

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