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A SPECTRAL SEQUENCE FOR GROUP PRESENTATIONS WITH APPLICATIONS TO LINKS

BY

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ABSTRACT. A spectral sequence is associated with any presentation of a group G. It turns out that this spectral sequence is independent of the chosen presentation. In particular if G is the fundamental group of a link L in R^3 the spectral sequence leads to invariants that compare to the Milnor invariants of L.

0. Introduction. Recently Stallings used the cobar construction of a resolution to associate to each group G a 2nd quadrant spectral sequence $E^r_{-s,t}$ which is 0 for $s > t$ and which satisfies $E^\infty_{-s,s} = I^sG/I^{s+1}G$ where IG is the fundamental ideal of G [**9**]. Here we present a different construction with all the properties mentioned above but with some advantages. First, it can be read off from any group presentation. Second, $E^r_{-s,t} = 0$ for $t \geqslant s + 2$. In Stallings' sequence one has no information on those terms (and they are definitely not zero). Third, and most important, the $E^r_{-s,s}$ and $E^r_{-s,s+1}$ terms are related to the Baer invariants of G [**1**]. This is better than the results in [**5**] which do depend upon the presentation while ours do not.

We describe our sequence in §1; in §2 we show that the sequence is intrinsically defined by using the results of [**4**]. In §3 we apply our results to the theory of links in R^3.

[The referee would like to thank J. Ratcliffe for pointing out to him the existence of [**4**]. Thanks to it, the referee was able to improve some results and prove a conjecture of the author's (the main theorem).]

1. The spectral sequence of a presentation.

(1.0) We shall consider complexes of algebras. A normal short complex is one

$$\cdots \to A_2 \to A_1 \xrightarrow{\partial} A_0, \qquad \mathbf{A}$$

for which $A_n = 0$, $n \geqslant 2$, and ∂ is a normal monomorphism (see [**4**, p. 225]). Then we have an exact sequence

$$0 \to A_1 \xrightarrow{\partial} A_0 \xrightarrow{\varepsilon} H_0(\mathbf{A}) \to 0.$$

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If A_0 is a projective (resp. free) algebra, then **A** is called a projective (resp. free) presentation of $H_0(\mathbf{A})$. We apply this to the case where our algebra is an integral group ring.

(1.1) Let G be a group; then G_n stands for the nth member of its lower central series [**6**, Chapter V, §9]. In particular G_2 is the commutator subgroup and $\overline{G} = G/G_2$ is the abelianization of G.

The ring ZG is the integral group ring of G with augmentation ε: $ZG \to Z$. Let $IG = \ker \varepsilon$; then I^nG stands for the nth power of IG.

(1.2) Let now

$$\langle x_i : r_j \rangle \qquad \text{(P)}$$

be the presentation [**6**, p. 205] for G. This means that we have a free group F in the x_i and that $G \cong F/R$, where R is the smallest normal subgroup of F generated by $\{r_j\} \subseteq F$. We write $R = \langle r_j \rangle^F$.

Consider the 2-sided ideal $N = (r_j - 1)$ of ZF generated by the $r_j - 1$. Then we have a free presentation

$$0 \to N \xrightarrow{\partial_1} ZF \to ZG \to 0$$

of ZG. Since $N \subseteq IF$ we may take the short complex [**4**, §2]

$$0 \to N \xrightarrow{\partial_1} IF \qquad \mathbf{J}$$

(here $J_q = 0$, $q \geqslant 2$, $J_1 = N$ and $J_0 = IF$), which is a free presentation of IG, via the isomorphism $H_0(J) \cong IG$, since IF is F-free [**6**, Chapter VI, Theorem 5.5]. By Lemma 5.2 of [**4**], IF is a projective algebra.

J can be considered to be the augmentation kernel of the complex

$$0 \to N \xrightarrow{\partial_1} ZF \qquad \mathbf{C}$$

and the powers $\mathbf{J}^p$ of **J** define a filtration $F_{-p}\mathbf{C} = \mathbf{J}^p$ on **C**. Notice that if we define

$$N(0) = N(1) = N \quad \text{and}$$

$$N(p) = N(1)I^{p-1}F + IFN(p-1) = N(p-1)IF + IFN(p-1) \qquad (1)$$

then $\mathbf{J}^p = N(p) \oplus I^pF$.

(1.3) The filtration F induces a spectral sequence in the usual manner [**6**, Chapter VIII, §2]. Since our filtration degree is negative, our sequence lies in the 2nd quadrant and since $\mathbf{C}_q = 0$, $q \geqslant 2$, then $E^r_{-s,s+k} = 0$ for $k \geqslant 2$, whereas

$$E^r_{-s,s} = I^sF/\left(I^{s+1}F + N(s-r+1) \cap I^sF\right), \qquad s \geqslant 0, \qquad (2)$$

and

$$E^r_{-s,s+1} = (N(s) \cap I^{s+r}F)/\left(N(s+1) \cap I^{s+r}F\right), \qquad s \geqslant 1. \qquad (3)$$

DEFINITION (1.4) The spectral sequence E is called the spectral sequence of G (associated to the presentation (P)).

2. The main theorem. Our main goal is to show that E depends only on G.

(2.1) Let then (P) be the presentation in (1.2) and let

$$\langle x'_k : r'_l \rangle \qquad \text{(Q)}$$

be another presentation. Put $F' = \langle x'_k : \rangle$ and $R' = \langle r'_l \rangle^{F'}$; let E' be the spectral sequence associated to (Q).

LEMMA (2.2) *If there exists an epimorphism* ϕ: $F \to F'$ *with* $\phi(R) = R'$ *then* ϕ *induces an isomorphism* Φ: $E \to E'$ *of spectral sequences.*

MAIN THEOREM (2.3) *If* (P) *and* (Q) *are any two presentations of the group* G *then the corresponding spectral sequences are isomorphic.*

This allows us to drop the parenthetical remark in Definition (1.4).

The theorem follows from (2.2) for there exists a presentation of G, (S): $\langle y_\alpha : s_\beta \rangle$ where $L = \langle y_\alpha : \rangle$ and $S = \langle s_\beta \rangle^L$ and epimorphisms ψ: $L \to F$ and ψ': $L \to F'$ with $\psi(S) = R$ and $\psi(S') = R'$.

(2.4) Now we proceed to prove (2.2). Let $(IG)^{(s)}$ be the s-fold tensor product of IG over G. By [4, Lemma 5.2], $(IG)^{(s)}$ has a structure of G-module. We contend that

$$E^1_{-s,s} \cong H_0(G, (IG)^{(s)}) \tag{4}$$

and

$$E^1_{-s,s+1} \cong H_1(G, (IG)^{(s)}). \tag{5}$$

To show this we employ [4, Theorem 7.1]: $\mathbf{J}$ is a normal short complex (cf. [4, §2]) and $H_0(\mathbf{J}) \cong IG$ is a projective presentation of IG. Notice that $\mathbf{J}_0 = IF$ and by formula (6.3) of [4], $V^s_1(\mathbf{J})$ is defined by (1) and so $V^s_1(\mathbf{J}) = N(s)$. Then by formulae (6.6) (loc. cit.),

$$\mathrm{Tor}^G_0((IG)^{(s)}, Z) = I^sF/\,(I^{s+1}F + N(s))$$

and

$$\mathrm{Tor}^G_1((IG)^{(s)}, Z) = (I^{s+1}F \cap N(s))/N(s+1)$$

which in view of (2) and (3) prove our claim. Now, if (P) and (Q) are presentations and ϕ: $F \to F'$ the epimorphism of the hypothesis, it induces an automorphism ϕ' of G and by [4, Lemma 5.2] an automorphism $\phi^{(s)}$ of $(IG)^{(s)}$. Then $\Phi'_{-s,t}$: $E^1_{-s,t} \to E'^1_{-s,t}$ is an isomorphism for all t: for $t = s$ and $s + 1$ by (4) and (5) and for $t \geqslant s + 2$ because both sides are trivial. The induced map is natural by definition and it commutes with the differentials. By construction $E^2 = H(E^1, d')$ so that Φ: $E^2 \to E'^2$ is an isomorphism as well. By induction $E^r = E'^r$ for all r. Q.E.D.

(2.5) We proceed to describe the terms $E^1_{-s,s}$ and $E^1_{-1,2}$: for $E^1_{-s,s}$ we employ (4)

$$\begin{aligned} H_0(G, (IG)^{(s)}) &= [IG \otimes_G \cdots \otimes_G IG] \otimes_G Z \\ &= [IG \otimes_G \cdots \otimes_G IG] \otimes_G (IG \otimes_G Z) \end{aligned}$$

where the first brackets enclose an s-fold product and the second enclose an $(s-1)$-fold product. By [6, Chapter VI, Lemma 4.1] $IG \otimes_G Z = \overline{G}$ which is a trivial G-module. Thus

$$\begin{aligned}[IG \otimes_G \cdots \otimes_G IG] \otimes_G \bar{G} &= [IG \otimes_G \cdots \otimes_G IG] \otimes_G (IG \otimes_G \bar{G}) \\ &= [IG \otimes_G \cdots \otimes_G IG] \otimes_G (IG \otimes_G (Z \otimes_Z \bar{G})) \\ &= [IG \otimes_G \cdots] \otimes_G ((IG \otimes_G Z) \otimes_Z \bar{G}) \\ &= [IG \otimes_G \cdots] \otimes_G (\bar{G} \otimes_Z \bar{G}).\end{aligned}$$

By succesive applications of this we get

LEMMA (2.6) *$E^1_{-s,s}$ is the s-fold tensor product of $\bar{G}$ over Z.*

REMARK. In the notation of [4, §5], $E^1_{-s,s} = \bar{G}^{(s)}$.

LEMMA (2.7) $E^1_{-1,2} = H_2(G; Z)$.

PROOF. $E^1_{-1,2} = H_1(G, IG) = H_2(G; Z)$ by [6, Chapter VI, Theorem 12.1].

(2.8) In our thesis we worked out an explicit isomorphism $E^1_{-s,s} \to \bar{G}^{(s)}$ as follows: $\bar{G}$ is naturally isomorphic to $IF/(N + I^2F)$. Consider

$$\gamma\colon (IF/(N + I^2F))^{(s)} \to I^sF/(N(s) + I^{s+1}F)$$

defined by

$$\gamma\left(\overline{(x_{i_1} - 1)} \otimes \cdots \otimes \overline{(x_{i_s} - 1)}\right) = \prod (x_{i_j} - 1) + (N(s) + I^{s+1}F).$$

If $\Phi'_{-1,1}\colon (IF/N + I^2F) \to (IF'/N' + I^2F')$ is the isomorphism defined by ϕ (and $N' = (r'_l - 1) \subseteq ZF'$) then

$$\Phi'_{-s,s}\gamma = (\Phi'_{-1,1})^{(s)}\gamma'.$$

Similarly, if $h\colon F \to ZF$ is a map $x \mapsto x - 1$ then h induces an isomorphism $(R \cap F_2)/[F, R] \to (N \cap I^2F)/N(2)$ and the former quotient is the well-known Hopf formula for $H_2(G; Z)$ [6, p. 204]. We omit the proofs.

PROPOSITION (2.9) $E^s_{-s,s} = E^\infty_{-s,s} = I^sG/I^{s+1}G$.

PROOF. Since $ZG \cong ZF/N$, $IG = IF/N$. Consider

$$\begin{array}{ccccccccc} 0 & \to & N & \to & IF & \xrightarrow{f} & IG & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \ker h & \to & I^sF & \xrightarrow{h} & I^sG & \to & 0 \end{array}$$

where $h = f|I^sF$ then $\ker h = \ker f \cap I^sF = N \cap I^sF$. Hence

$$I^s(G) \simeq I^sF/(N \cap I^s) \simeq (N + I^sF)/N.$$

By the Noetherian isomorphism theorem,

$$\begin{aligned} I^{s+1}F \subset I^sF,\ E^s_{-s,s} &\cong I^sF/(I^sF \cap N + I^{s+1}F) \\ &= \frac{I^sF/(I^sF \cap N)}{(I^sF \cap N + I^{s+1}F)/(I^sF \cap N)} \cong \frac{I^sF/(I^sF \cap N)}{I^{s+1}F/(I^sF \cap N \cap I^{s+1}F)} \\ &= \frac{I^sF/(I^sF \cap N)}{I^{s+1}F/(I^{s+1}F \cap N)} = I^sG/I^{s+1}G. \end{aligned}$$

LEMMA (2.10) *If g is an element of G_n then $g - 1$ is an element in I^nG for all $n \geqslant 1$.*

THEOREM (2.11) *Let $\bar{E}$ be the spectral sequence of the group $\Gamma = G/G_{q+1}$; q is any integer $\geqslant 1$. Let E be the spectral sequence of G. Then*

$$E^r_{-r,r} \simeq \bar{E}^r_{-r,r} \quad \text{for } 1 \leqslant r \leqslant q, \tag{6}$$

$$E^r_{-s,s} \simeq \bar{E}^r_{-s,s} \quad \text{for } 1 \leqslant s \leqslant r \leqslant q. \tag{7}$$

PROOF. Statement (7) follows from (6) because

$$E^r_{-s,s} \simeq \cdots \simeq E^{s+1}_{-s,s} \simeq E^s_{-s,s} \simeq \bar{E}^s_{-s,s} \simeq \bar{E}^{s+1}_{-s,s} \simeq \cdots \simeq \bar{E}^r_{-s,s}.$$

To prove (6) it is enough to show that

$$IG/I^{r+1}G \simeq I\Gamma/I^{r+1}\Gamma \quad \text{for } r \leqslant q.$$

The canonical epimorphism $G \to G/G_{q+1}$ induces the ring epimorphism $ZG \to Z\Gamma$. Define

$$\phi\colon IG \to I\Gamma/I^{r+1}\Gamma \quad \text{by } g - 1 \to g' - 1 + I^{r+1}\Gamma,$$

where $g' = gG_{q+1}$. Since $\phi(I^{r+1}G) = I^{r+1}\Gamma$, ϕ induces the epimorphism

$$\Phi\colon IG/I^{r+1}G \to I\Gamma/I^{r+1}\Gamma$$

given by

$$g - 1 + I^{r+1}(G) \to g' - 1 + I^{r+1}\Gamma. \tag{8}$$

But $g - 1 + I^{r+1}G$ generates $IG/I^{r+1}G$. Finally, we define an inverse to Φ. Define

$$\psi\colon I\Gamma \to IG/I^{r+1}G, \qquad g' - 1 \to g - 1 + I^{r+1}G,$$

where $g' = gG_{q+1}$. The map ψ is well defined, for if $g' = h'$, then $h = gw$, where $w \in G_{q+1}$, but $gw - 1 = (g - 1)(w - 1) + (g - 1) + (w - 1)$ and by Lemma (2.10), $w - 1 \in I^{q+1}G \subset I^{r+1}G$, since $r \leqslant q$ and $(g - 1)(w - 1) \in I^{q+2}G \subset I^{r+1}G$. Therefore, $\psi(h' - 1) = (gw - 1) + I^{r+1}G = (g - 1) + I^{r+1}G$. Consider the composite map, $IG \to I\Gamma \to IG/I^{r+1}G$, this is a ring homomorphism, and it carries $I^{r+1}G \to I^{r+1}\Gamma \to 0$. Therefore ψ induces

$$\psi\colon I\Gamma/I^{r+1}\Gamma \to IG/I^{r+1}G.$$

But $\psi \circ \Phi = 1$ and $\Phi \circ \psi = 1$; hence the result.

REMARKS. (1) In the course of the proof of Theorem (2.11) we have shown that

$$\Phi\colon IG/I^nG \xrightarrow{\sim} I\Gamma/I^n\Gamma$$

where $\Gamma = G/G_n$ (see (8)).

(2) Let E be the sequence of G associated to the presentation (P) as defined in (1.4), and let K be an Eilenberg-Mac Lane space of type $(G, 1)$. If Λ^pK denotes the p-fold smash product **[9]** of K with itself, then the formula $\bar{E}^1_{-p,q} = H_q(\Lambda^pK)$ describes a spectral sequence $\bar{E}$ whose 1-skeleton is described in **[5**, §1] and **[9**, §3]. Since $\bar{E}^1_{-p,p}$ is isomorphic to $E^1_{-p,p}$ and since $\bar{E}^\infty$ is isomorphic to E^∞, there is a natural map $\bar{E}^1 \to E^1$. This map, however, is not monic because the terms $\bar{E}^1_{-p,p+k}$ $(k \geqslant 2)$ are not zero while the corresponding terms in E are. The map, on the other hand, is onto.

3. Applications to links. Let $S^{(n)}$ be the space consisting of n-disjoint circles $S_1, \cdots, S_n$. Assume that fixed orientations have been chosen for $S^{(n)}$ and R^3. By an oriented *n-link* l in R^3 is meant a homeomorphic image of $S^{(n)}$ in R^3. Thus l can be thought of as an ordered collection $(l_1, l_2, \ldots, l_n)$ of homeomorphisms l_i: $S_i \to R^3$; where the images $L_1, L_2, \ldots, L_n$ of the l_i's are to be disjoint. Denote $L_1 \cup L_2 \cup \cdots \cup L_n$ by L, and the fundamental group of the complement of L in R^3, $\pi(R^3 - L, x_0)$ by $G(L)$, where $x_0 \in R^3 - L$ is a fixed chosen base point. Let $N_1, N_2, \ldots, N_n$ be solid tori chosen to be regular, disjoint neighborhoods of $L_1, L_2, \ldots, L_n$ respectively. Let $p_i(t)$ $(0 \leqslant t \leqslant 1)$ be a path from the point x_0 to a point on the boundary of N_i. A *meridian-longitude* pair (α_i, β_i) for L is a pair of elements of $G(L)$ where:

(i) α_i is represented by a closed loop in $R^3 - L$ described as follows: traverse p_i, then traverse a closed loop on the boundary of $N_i - L_i$ which has linking number $+1$ with l_i and finally return to x_0 along p_i;

(ii) β_i is represented by a closed loop in $R^3 - L$ described as follows: traverse p_i, then traverse a simple closed curve on the boundary of N_i which has linking number 0 with l_i and which is nullhomologous in $R^3 - L_i$, and finally return to x_0 along p_i.

The elements α_i, β_i of $G(L)$ are well defined in $G(L)$ up to the choice of p_i and the orientations chosen for $S^{(n)}$ and R^3. Any other ith meridian-longitude pair (α_i', β_i') for L is obtained from (α_i, β_i) by simultaneous conjugation, that is, $\alpha_i' = g\alpha_i g^{-1}$ and $\beta_i' = g\beta_i g^{-1}$ for some $g \in G(L)$.

Two links l and l' are said to be *isotopic* if there exists a continuous family h_t: $S^{(n)} \to R^3$ of homeomorphisms, for $0 \leqslant t \leqslant 1$, with $h_0 = l$ and $h_1 = l'$. The fundamental group $G(L)$ of the complement of L in R^3 is not invariant under isotopy of the link. In 1952, K. T. Chen proved [2] *that $G(L)/G_q(L)$, where $G_q(L)$ is the qth lower central subgroup of $G(L)$, is invariant under isotopy of the link for any arbitrary posititive integer q.* In 1957, Milnor gave [7] a presentation describing the group $G(L)/G_q(L)$ and defined the so-called Milnor invariants for a link.

It is known that: if *G is the fundamental group of the complement of an n-link l in R^3 then G/G_2 is free abelian of rank n.*

In Theorem (2.11) we found that if $\bar{E}$ is the spectral sequence of G/G_{s+1}, $s > 1$, and E is the spectral sequence of G that then $E^s_{-s,s} \simeq \bar{E}^s_{-s,s}$. In the light of the above stated result of Chen we can conclude:

THEOREM (3.1) *Let G be the group of a certain n-link l. Let E be the spectral sequence of G/G_{s+1}. Then $E^s_{-s,s}$ is an isotopy invariant of the link l.*

Let $\langle a_{ij} : r_{ij} \rangle$ $(i = 1, 2, \ldots, n;\ j = 1, \ldots, k_i)$ be a *Wirtinger presentation* for $G(L)$ (henceforth we shall write G for $G(L)$) where to each crossing point Q_{ij} of the projection corresponds a relation $r_{ij} = 1$, $r_{ij} = [b_{ij}, a_{ij}]a_{ij}a_{ij+1}^{-1}$ with $b_{ij} = a^{\varepsilon_{ij}}_{\lambda(ij)\mu(ij)}$, $(\lambda(ij), \mu(ij))$ are given by the segment of L which crosses over at Q_{ij}, and $\varepsilon_{ij} = \pm 1$ is the signature of the crossing. Let $v_{ij} = [b_{ij}, a_{ij}]$ and $a_{i1} = a_i$. Define

$$u_{i1} = 1 \quad \text{and} \quad u_{ij} = v_{ij-1}v_{ij-2} \cdots v_{i1} \quad (j = 2, 3, \ldots, k_i) \tag{9}$$

and

$$w_{ik_i} = b_{i1}^{-1}b_{i2}^{-1} \cdots b_{ik_i}^{-1}. \tag{10}$$

Then G may be presented by

$$\langle a_{ij} : h_{ij}, s_i \rangle \quad (i = 1, \ldots, n; j = 1, 2, \ldots, k_i),$$
$$h_{i1} = 1, \qquad h_{ij} = u_{ij}a_i a_{ij}^{-1} \qquad (j = 2, \ldots, k_i),$$
$$s_i = [a_i, w_{ik_i}]. \tag{11}$$

Note, w_{ik_i} is an ith longitude of L in G. Thus $ZG \simeq ZF/N$ where F is the free group on the a_{ij}'s and N is the ideal of ZF generated by $h_{ij} - 1$, $s_i - 1$ ($i = 1, \ldots, n$; $j = 2, \ldots, k_i$). Since $h_{ij} - 1 = (u_{ij} - a_{ij}a_i^{-1})a_i a_{ij}^{-1}$ and $a_i a_{ij}^{-1}$ is a unit of ZF, N is generated as an ideal of ZF by

$$\{u_{ij} - a_{ij}a_i^{-1}, s_i - 1\} \qquad (i = 1, \ldots, n; \quad j = 2, \ldots, k_i). \tag{12}$$

Lemma (3.2) *Let N_1 be the ideal of ZF generated by $\{u_{ij} - a_{ij}a_i^{-1}\}$ ($i = 1, \ldots, n$; $j = 2, \ldots, k_i$). Then*

$$N_1 \cap I^sF = N_1(s), \tag{13}$$

where $IF = \ker(ZF \to Z)$, $N_1(1) = N_1$, and $N_1(s) = IFN_1(s-1) + N_1(s-1)IF$ ($s > 1$).

Proof. The elements $\{u_{ij} - a_{ij}a_i^{-1} + N_1(2)\}$ generate the Z-module $N_1/N_1(2)$. Moreover we shall show that $\{u_{ij} - a_{ij}a_i^{-1} + N_1(2)\}$ forms a basis for $N_1/N_1(2)$. Indeed, if for some integers n_{ij}, $\sum n_{ij}(u_{ij} - a_{ij}a_i^{-1}) = 0 + N_1(2)$, where the summation is over $i = 1, \ldots, n$ and $j = 2, \ldots, k_i$. Then $\sum n_{ij}(u_{ij} - a_{ij}a_i^{-1}) \in N_1(2)$, hence $\sum n_{ij}(u_{ij} - a_{ij}a_i^{-1}) \in I^2F$. Thus

$$(\partial/\partial a_{st})\sum n_{ij}\big(u_{ij} - a_{ij}a_i^{-1}\big)(1) = 0 \quad (\text{cf. } [3]). \tag{14}$$

But $u_{ij} \in F_2$, (9), hence $u_{ij} - 1 \in I^2$, so, $(\partial/\partial a_{st})(u_{ij} - 1)(1) = 0$ and $\partial a_{ij}/\partial a_{st} = 0$ if $(i,j) \neq (s,t)$ and $\partial a_{st}/\partial a_{st} = 1$. Therefore,

$$\begin{aligned}(\partial/\partial a_{st})\sum n_{ij}\big(u_{ij} - a_{ij}a_i^{-1}\big)(1) &= (\partial/\partial a_{st})\sum n_{ij}\big(u_{ij} - 1 - a_{ij}a_i^{-1} + 1\big)(1)\\ &= \sum -n_{ij}(\partial/\partial a_{st})\big(a_{ij}a_i^{-1}\big)(1)\\ &= \sum -n_{ij}\big((\partial/\partial a_{st})a_{ij}(1) + a_{ij}(1) + a_{ij}(\partial/\partial a_{st})a_i^{-1}(1)\big)\\ &= -n_{st}.\end{aligned}$$

Hence $n_{st} = 0$ (see (14)).

Thus the sequence of Z-modules

$$0 \to N_1(2) \to N_1 \to N_1/N_1(2) \to 0,$$

is split exact. Let M be the Z-submodule of N_1 generated by $\{u_{ij} - a_{ij}a_i^{-1}\}$ ($i = 1, \ldots, n; j = 2, \ldots, k_i$). Then $N_1 = M + N_1(2)$. Since

$$(\partial/\partial a_{ij})\big(u_{ij} - a_{ij}a_i^{-1}\big)(1) = -1, \qquad u_{ij} - a_{ij}a_i^{-1} \in IF,$$

but not in I^2F. So $M \cap I^2F = \{0\}$, and

$$N_1 \cap I^2F = N_1(2). \tag{15}$$

But

$$N_1(s+1) = \sum_{i=1}^{s-1} I^i F N_1(2) I^{s-i-1} F$$

and

$$N_1(s) \cap I^{s+1}F = \sum_{i=0}^{s-1} I^i F(N_1 \cap IF) I^{s-i-1} F.$$

Therefore

$$N_1(s+1) = \sum_{i=0}^{s-1} I^i F(N_1 \cap I^2F) I^{s-i-1} F = N_1(s) \cap I^{s+1}F. \tag{16}$$

The proof of (13) follows from (15) by induction on s.

Let N_2 be the ideal of ZF generated by $\{s_i - 1\}$ $(i = 1, \ldots, n)$, then one can write $N = N_1 + N_2$, N_1 is the ZF-ideal generated by $\{u_{ij} - a_{ij}a_i^{-1}\}$ $(i = 1, \ldots, n;$ $j = 2, \ldots, k_i)$ (see Lemma (3.2)).

Lemma (3.3) *If $s_i - 1$ is in I^sF for $i = 1, \ldots, n$, then*

$$E^{s-1}_{-s,s} \simeq E^{s-2}_{-s,s} \simeq \cdots \simeq E^1_{-s,s} \simeq \otimes^s IF/(N + I^2F).$$

Proof. By (2) and (3),

$$E^r_{-s,s} = I^sF/\big(N(s-r+1) \cap I^sF + I^{s+1}F\big).$$

Let $t = s - r + 1$, then $2 \leqslant t \leqslant s$. Now $N(t) = N_1(t) + N_2(t)$, where N_2 is the ideal of ZF generated by $s_i - 1$; hence $N_2 \subset I^sF$. So, $N(t) \cap I^sF = N_2(t) + N_1(t) \cap I^sF$. But $N_1(t) = N_1 \cap I^tF$ (see (13)). Therefore $N_1(t) \cap I^sF = N_1 \cap I^sF$. Since $N_2 \subset I^sF$, $N_2(t) \subset I^{s+1}F$. Hence for $1 \leqslant r \leqslant s - 1$, $N(s - r + 1) \cap I^sF + I^{s+1}F = N_1 \cap I^sF + I^{s+1}F = N_1(s) + I^{s+1}F$; the last equality follows from (13). Therefore

$$I^sF/\big(N_1(s) + I^{s+1}F\big) \simeq E^{s-1}_{-s,s} \simeq E^{s-2}_{-s,s} \simeq \cdots \simeq E^1_{-s,s} \simeq \otimes^s IF/(N + 1^2F).$$

Corollary (3.4) *If $s_i - 1$ is in I^sF for $(i = 1, \ldots, n)$ then $E^r_{-s,s}$ $(1 \leqslant r \leqslant s - 1)$ is free abelian of rank n^s.*

Proof. This follows from the fact that G/G_2 is free abelian of rank n, Lemma (3.3) and the isomorphism $I/N + I^2 \simeq G/G_2$.

Next we shall describe a basis for $E^r_{-s,s} = I^sF/(N_1(s) + I^{s+1}F)$ $(1 \leqslant r \leqslant s - 1)$. Here again we assume that $s_i - 1 \in I^sF$, $i = 1, \ldots, n$.

Recall that N_1 is the ideal of ZF generated by $\{u_{ij} - a_{ij}a_i^{-1}\}$ $(i = 1, 2, \ldots, n;$ $j = 2, 3, \ldots, k_i)$. Let $\eta_{ij} - a_{ij}a_i^{-1}$ and $\chi_i = a_i - 1$. Then

$$\begin{aligned}\eta_{ij} &= u_{ij} - 1 - a_{ij}a_i^{-1} + 1\\ &= (a_i - 1) - (a_{ij} - 1) + (u_{ij} - 1) - (a_{ij} - 1)(a_i^{-1} - 1) + (a_i^{-1} - 1).\end{aligned}$$

Let $W_{ij} = (u_{ij} - 1) - (a_{ij} - 1)(a_i^{-1} - 1) + (a_i - 1)(a_i^{-1} - 1)$. Then $W_{ij} \in I^2F$. Hence

$$\begin{cases} a_{ij} = 1 + \chi_i + W_{ij} + \eta_{ij}, \\ a_{ij}^{-1} = 1 - \chi_i - W'_{ij} - \eta_{ij}, \end{cases} \tag{17}$$

where $W'_{ij} = W_{ij} + (a_{ij} - 1)(a_{ij}^{-1} - 1) \in I^2F$ and $\eta_{ij} \in N_1$.

Since $G \simeq F/R$, where F is the free group on $\{a_{ij}: i = 1, \ldots, n; j = 1, \ldots, k_i\}$, the set $\{(a_{i_1j_1} - 1)(a_{i_2j_2} - 1) \cdots (a_{i_sj_s} - 1) + N_1(s) + I^{s+1}F\}$ $(i_1, i_2, \ldots, i_s = 1, \ldots, n$ and $j_1, j_2, \ldots, j_s = 1, 2, \ldots, k_i)$ generates $I^sF/(N_1(s) + I^{s+1}F)$. Using the equalities (17) one can write

$$\prod_{t=1}^{s} (a_{i_tj_t} - 1) + N_1(s) + I^{s+1}F = \prod_{t=1}^{s} \chi_{i_t} + N_1(s) + I^{s+1}F.$$

Thus,

$$\left\{(a_{i_1} - 1)(a_{i_2} - 1), \ldots, (a_{i_s} - 1) + N_1(s) + I^{s+1}F\right\} \qquad (i_1, \ldots, i_s = 1, \ldots, n) \tag{18}$$

forms a generating set of $I^s/(N_1(s) + I^{s+1})$. But there are n^s elements in the set (18); hence (18) forms a Z-basis for $I^s/(N_1(s) + I^{s+1})$ (see Corollary 3.4).

Consider the E^{s-1} term of the spectral sequence E,

$$\to E^{s-1}_{s-2,4-s} \to E^{s-1}_{-1,2} \overset{d^{s-1}_{-1,2}}{\to} E^{s-1}_{-s,s} \overset{d^{s-1}_{-s,s}}{\to} E_{-2s+1,2s-2} \to \cdots,$$

where all terms of degree $\neq 0, 1$ of E^{s-1} are zero. Therefore we have

$$\to 0 \to E^{s-1}_{-1,2} \overset{d^{s-1}_{-1,2}}{\to} E^{s-1}_{-s,s} \overset{d^{s-1}_{-s,s}}{\to} 0.$$

Explicitly, we have

$$\to 0 \to (N \cap I^sF)/\left(N(2) \cap I^sF\right) \overset{d^{s-1}_{-1,2}}{\to} I^sF/\left(I^{s+1}F + N(2) \cap I^sF\right) \overset{d^{s-1}_{-s,s}}{\to} 0,$$

where $d^{s-1}_{-1,2}$ is induced from the inclusion $N \cap I^sF \to I^sF$. But

$$\begin{aligned} E^s_{-s,s} &\simeq H\left(E^{s-1}_{-s,s}\right) \simeq \ker d^{s-1}_{-s,s} / d^{s-1}_{-1,2}\left(E^{s-1}_{-1,2}\right) \\ &\simeq \frac{I^sF/\left(N(2) \cap I^sF + I^{s+1}F\right)}{(N \cap I^sF + N(2) \cap I^sF + I^{s+1}F)/\left(N(2) \cap I^sF + I^{s+1}F\right)} \\ &\simeq \frac{I^sF/\left(N(2) \cap I^sF + I^{s+1}F\right)}{(N \cap I^sF + I^{s+1}F)/\left(N(2) \cap I^sF + I^{s+1}F\right)}, \end{aligned} \tag{19}$$

since $N(2) \cap I^sF \subset N \cap I^sF$.

THEOREM (3.5) *If* $s_i - 1 \in I^sF$ $(i = 1, 2, \ldots, n)$, *then*

$$E^s_{-s,s} \simeq \frac{I^sF/\left(N_1(s) + I^{s+1}F\right)}{(N_2 + N_1(s) + I^{s+1}F)/\left(N_1(s) + I^{s+1}F\right)}, \tag{20}$$

where the set $\{(a_{i_1} - 1)(a_{i_2} - 1) \cdots (a_{i_s} - 1) + N_1(s) + I^{s+1}F\}$ $(i_1, i_2, \ldots, i_s = 1, \ldots, n)$, *gives a basis for* $I^sF/(N_1(s) + I^{s+1}F)$, *and where the set* $(s_i - 1) + N_1(s) + I^{s+1}$ $(i = 1, \ldots, n)$, *gives a basis for* $(N_2 + N_1(s) + I^{s+1}F)/(N_1(s) + I^{s+1}F)$.

PROOF. Since $s_i - 1 \in I^sF$, $N_2 \subset I^sF$. Hence $N \cap I^sF = N_1 \cap I^sF + N_2 = N_1(s) + N_2$ (see (13)). Also since $N(2) = N_1(2) + N_2(2)$ and $N_2(2) \subset I^{s+1}F$, it follows that

$$N(2) \cap I^sF = N_1(2) \cap I^sF = N_1 \cap I^2F \cap I^sF = N_1 \cap I^sF = N_1(s).$$

Substituing these equalities in (19) we get (20). The rest of Theorem (3.5) is clear.

Since

$$s_i - 1 = [a_i, w_{ik_i}] - 1 = (a_i w_{ik_i} - w_{ik_i} a_i) a_i^{-1} w_{ik_i}^{-1}$$
$$= ((a_i - 1)(w_{ik_i} - 1) - (w_{ik_i} - 1)(a_i - 1)) a_i^{-1} w_{ik_k}^{-1}.$$

Hence N_2 may be thought of as being generated by $\{\chi_i(w_{ik_i} - 1) - (w_{ik_i} - 1)\chi_i$: $i = 1, \ldots, n\}$. Thus, as a Z-module, $(N_2 + N_1(s) + I^{s+1}F)/(N_1(s) + I^{s+1}F)$ is generated by $\{\chi_i(w_{ik_i} - 1) - (w_{ik_i} - 1)\chi_i + N_1(s) + I^{s+1}F\}$.

A simple computation shows that for any n-link,

$$s_i - 1 = \sum_{j=1}^{n} \mu(i,j)(\chi_i\chi_j - \chi_j\chi_i) + N_1(2) + I^3F$$

where $\mu(i,j)$ is the linking number of the ith and jth components of L. Hence $E_{-2,2}^2$ gives very little information about L.

Next, we give an example where we compute $E_{-3,3}^3$ for a link whose s_i's belong to I^3F. The link is shown in the figure and one has

$$\begin{aligned} b_{1\,2j-1} &= a_{3\,4j-3}, & b_{1\,2j} &= a_{3\,4j}^{-1}, \\ b_{2\,2j-1} &= a_{3\,4j-4}^{-1}, & b_{2\,2j} &= a_{3\,4j-1}, \\ b_{3\,4j-3} &= a_{1\,2j}, & b_{3\,4j-2} &= a_{2\,2j}, \\ b_{3\,4j-1} &= a_{1\,2j}^{-1}, & b_{3\,4j} &= a_{2\,2j+2}^{-1} \end{aligned}.$$

Computing $w_{1\,2m}$, $w_{2\,2m}$ and $w_{3\,4m}$ we get

$$w_{1\,2m} = a_{31}^{-1}([a_{34}, a_{24}^{-1}][a_{38}, a_{26}^{-1}] \cdots [a_{3\,2m}, a_{22}^{-1}])a_{31},$$
$$w_{2\,2m} = a_{3\,2m}([a_{33}^{-1}, a_{12}^{-1}][a_{37}^{-1}, a_{14}^{-1}] \cdots [a_{3\,4m}^{-1}, a_{1\,2m}^{-1}])a_{3\,2m}^{-1},$$

and

$$w_{3\,4m} = a_{2\,2}^{-1}([a_{22}, a_{12}^{-1}] \cdots [a_{2\,2j}, a_{1\,2j}^{-1}] \cdots [a_{2\,2m}, a_{1\,2m}^{-1}])a_{32}.$$

Hence,

(i) $s_1 = [a_1, a_3^{-1}(\Pi_{j=1}^m [a_{3\,4j}, a_{2\,2j+2}^{-1}])a_3]$,
(ii) $s_2 = [a_2, a_{3\,2m}(\Pi_{j=1}^m [a_{3\,4j-1}^{-1}, a_{1\,2j}^{-1}])a_{3\,2m}^{-1}]$,
(iii) $s_3 = [a_3, a_{22}^{-1}(\Pi_{j=1}^m [a_{2\,2j}, a_{1\,2j}^{-1}])a_{22}]$.

Upon making use of the substitutions (17) for the different a_{ij} and a_{ij}^{-1} we obtain

$$s_1 - 1 = m[X_1, [X_2, X_3]] + N_1(3) + I^4F,$$
$$s_2 - 1 = m[X_2, [X_3, X_1]] + N_1(3) + I^4F,$$
$$s_3 - 1 = m[X_3, [X_1, X_2]] + N_1(3) + I^4F,$$

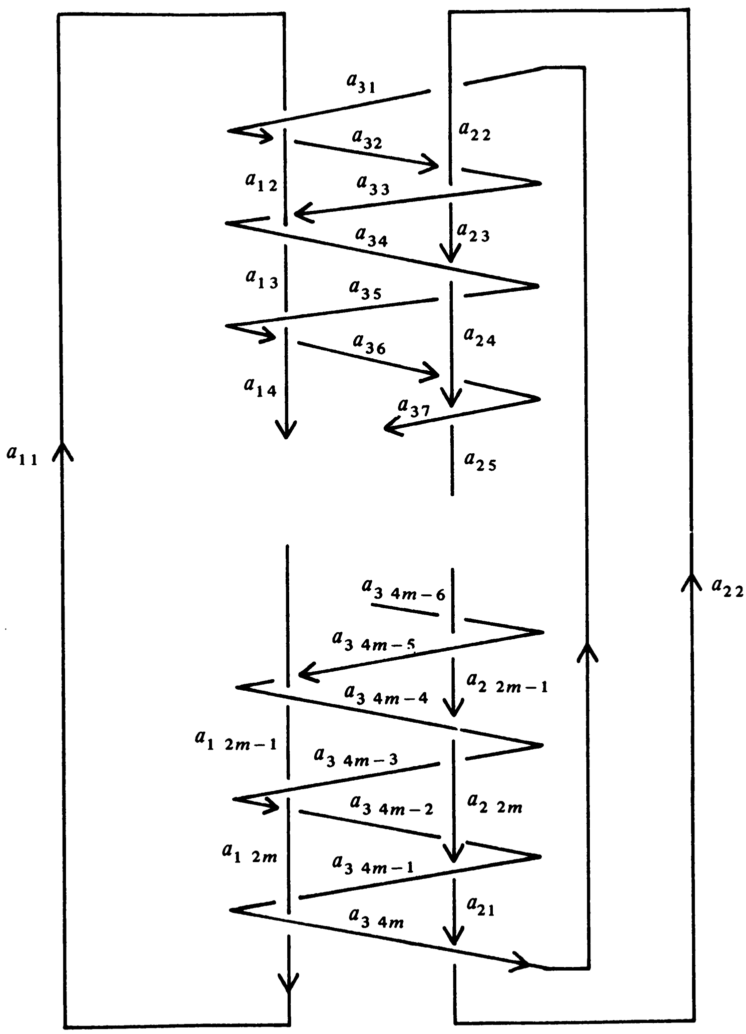
where by $[X, Z]$ we mean the usual Lie bracket, $[X, Z] = XZ - ZX$. Thus $(N_2 + N_1(3) + I^4F)/(N_1(3) + I^4F)$ is generated by $s_1 - 1$, $s_2 - 1$ and $s_3 - 1$ as in Theorem (3.5). But $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$ so that $s_1 - 1$ and $s_2 - 1$ form a basis for $(N_2 + N_1(3) + I^4F)/(N_1(3) + I^4F)$. Therefore

$$E_{-3,3}^3 \simeq Z \oplus \cdots \oplus Z \oplus Z_m \oplus Z_m,$$

where there are twenty-five copies of Z in the above sum; since

$$(I^sF)/\left(N_1(s) + I^{s+1}\right) \simeq Z \oplus \cdots \oplus Z,$$

there are twenty-seven copies of Z. Thus 3-links of the type shown in the figure whose m's differ are distinguishable links.



Finally, we point out how some of the Milnor invariants show up in computing the $E^s_{-s,s}$ terms. Here then is a brief account of Milnor's work.

In [7] Milnor showed that *the group* G/G_{s+1}, *for any nonnegative integer s, may be presented by* $\langle \alpha_1, \ldots, \alpha_n: [\alpha_i, \omega_i], F_{s+1}\rangle$ $(i = 1, \ldots, n)$, *where* $\alpha_i = a_{i1} = a_i$ *represents an ith meridian of L,* ω_i *is a word in* $\alpha_1, \ldots, \alpha_n$ *that represents an ith longitude of L in* G/G_{s+1} *and F is the free group on* $\{\alpha_i: i = 1, \ldots, n\}$.

The Magnus expansion of ω_i is obtained by substituting $\alpha_j = 1 + X_j$, $\alpha_j^{-1} = 1 - X_j + X_j^2 - X_j^3 + \cdots$ in the word ω_i. Thus ω_i can be expressed as a formal, noncommutative power series in the indeterminants $X_1, \ldots, X_n$. Namely,

$$\omega_i = 1 + \sum_{j_1 = 1, \ldots, n} \mu(j_1, i) X_{j_1} + \sum_{j_1, j_2 = 1, \ldots, n} \mu(j_1, j_2, i) X_{j_1} X_{j_2} + \cdots$$
$$+ \sum_{j_1, j_2, \ldots, j_t = 1, \ldots, n} \mu(j_1, j_2, \ldots, j_t, i) X_{j_1} X_{j_2} \cdots X_{j_t} + \cdots .$$

Thus a coefficient is defined for each sequence $j_1, j_2, \ldots, j_t, i$ $(t > 1)$ of integers between 1 and n.

Let $\bar{\Delta}(i_1, \ldots, i_r) =$ g.c.d. $\mu(j_1, \ldots, j_t)$, where $j_1, \ldots, j_t$ $(2 < t < r - 1)$ is to range over all sequences obtained by cancelling at least one of the indices $i_1, \ldots, i_r$ and permuting the remaining indices cyclically. Then Milnor proved that: *the residue classes*

$$\bar{\mu}(j_1, \ldots, j_t, k) \equiv \mu(j_1, \ldots, j_t, k) \mod \bar{\Delta}(j_1, \ldots, j_t, k)$$

are isotopy invariants of L provided that $t \leqslant s$.

If we restrict ourselves to links whose ω_i's belong to F_{s-1} for $(i = 1, \ldots, n)$, then $\mu(j_1, \ldots, j_t, i) = 0$ for $1 \leqslant t \leqslant s - 2$. But then $\bar{\mu}(j_1, \ldots, j_{s-1}, i) = \mu(j_1, \ldots, j_{s-1}, i)$, and hence $\mu(j_1, \ldots, j_{s-1}, i)$ are isotopy invariants for such links.

Let $I\bar{F}$ be the kernel of $Z\bar{F} \to Z$. Let $\bar{N}$ be the ideal of $Z\bar{F}$ generated by $[\alpha_i, \omega_i] - 1$ $(i = 1, \ldots, n)$, and $\bar{F}_{s+1} - 1$. Let $\bar{E}$ be the spectral sequence associated with the presentation given by Milnor for the group G/G_{s+1}. Now

$$\bar{E}^s_{-s,s} = I^s\bar{F}/(\bar{N} \cap I^s\bar{F} + I^{s+1}\bar{F}).$$

If $\omega_i \in \bar{F}_{s-1}$, then $[\alpha_i, \omega_i] - 1 \in I^s\bar{F}$ $(i = 1, \ldots, n)$ and $\bar{N} \cap I^s\bar{F} = \bar{N}$. Hence for this case,

$$\bar{E}^s_{-s,s} = I^s\bar{F}/(\bar{N} + I^{s+1}\bar{F}) \simeq \frac{I^s\bar{F}/I^{s+1}\bar{F}}{(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}} .$$

Where $I^s\bar{F}/I^{s+1}\bar{F}$ is a free Z-module write

$$\left\{ X_{i_1} X_{i_2} \cdots X_{i_s} + I^{s+1}\bar{F} : i_1, \ldots, i_s = 1, \ldots, n \right\}$$

as a basis, and where $(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$ is a free Z-module generated by $\{[\alpha_i, \omega_i] - 1 + I^{s+1}\bar{F} : i = 1, \ldots, n\}$. But,

$$[\alpha_i, \omega_i] - 1 = \sum_{j_1, \ldots, j_{s-1} = 1, \ldots, n} [X_i, \mu(j_1, \ldots, j_{s-1}, i) X_{j_1} X_{j_2} \cdots X_{j_{s-1}}] + I^{s+1}\bar{F}.$$

Therefore we can replace the set of generators above of the Z-module $(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$ by the set

$$\left\{ \sum_{j_1, \ldots, j_{s-1} = 1, \ldots, n} [X_i, \mu(j_1, \ldots, j_{s-1}, i) X_{j_1} \cdots X_{j_{s-1}}] + I^{s+1}\bar{F} : i = 1, \ldots, n \right\}. \tag{21}$$

We already proved $E^s_{-s,s} \simeq \bar{E}^s_{-s,s}$ (see, Theorem (2.11)). We shall describe a precise isomorphism for the case at hand (see, Theorem (3.5)).

$$E^s_{-s,s} = \frac{I^sF/\big(N_1(s) + I^{s+1}F\big)}{\big(N_2 + N_1(s) + I^{s+1}F\big)/\big(N_1(s) + I^{s+1}F\big)}$$
$$\to \frac{I^s\bar{F}/I^{s+1}\bar{F}}{(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}} = \bar{E}^s_{-s,s}$$

is an isomorphism. From the Wirtinger presentation of G we have $r_{ij} = b_{ij}a_{ij}b_{ij}^{-1}a_{ij+1}^{-1}$. Thus $a_{ij+1} = b_{ij}a_{ij}b_{ij}^{-1} = b_{ij}b_{ij-1} \cdots b_{i1}a_{i1}b_{i1}^{-1} \cdots b_{ij-1}^{-1}b_{ij}^{-1}$. Let $z_{ij} = b_{ij}b_{ij-1} \cdots b_{i1}$.

Define a sequence of homomorphisms $M_k\colon F \to \bar{F}$ as follows, by induction on k:

$$M_1(a_{ij}) = a_{i1}, \qquad M_{k+1}(a_{ij+1}) = M_k\big(z_{ij}a_{i1}z_{ij}^{-1}\big), \qquad M_{k+1}(a_{i1}) = a_{i1}.$$

Then it can be proved by induction on k that

$$M_k(a_{ij}) = a_{ij} \mod(F_kR), \qquad M_k(a_{ij}) = M_{k+1}(a_{ij}) \mod\big(\bar{F}_k\big).$$

We claim that $\bar{M}_{s+1}\colon IF \to I\bar{F}$, where $\bar{M}_{s+1}$ is the map induced from $M_{s+1}\colon F \to \bar{F}$, induces the required isomorphism. Because

$$M_{s+1}(a_{ij}) = M_{s+1}\big(z_{ij-1}a_{i1}z_{ij-1}^{-1}\big) = M_{s+1}(z_{ij-1})a_{i1}M_{s+1}\big(z_{ij-1}^{-1}\big) \equiv a_{i1} \mod F_2;$$

it follows that $\bar{M}_{s+1}(u_{ij} - a_{ij}a_i^{-1}) \in I^2\bar{F}$. Hence $M_{s+1}(N_1(s)) \subset I^{s+1}\bar{F}$, moreover, because of $M_{s+1}(w_{ik_i}) = M_{s+1}(z_{ik_i}) = w_{ik_i} \mod(F_{s+1}R)$. Since w_{ik_i} represents an ith longitude of G, $M_{s+1}(w_{ik_i})$ represents an ith longitude in G/G_{s+1}. Let M_{s+1}: $(I^sF)/(N_1(s) + I^{s+1}F) \to I^s\bar{F}/I^{s+1}\bar{F}$ be the canonical homomorphism induced from $\bar{M}_{s+1}$. Then M_{s+1} is an isomorphism, since $I^sF/(N_1(s) + I^{s+1}F)$ and $I^s\bar{F}/I^{s+1}\bar{F}$ both have rank n^s. Also,

$$\bar{M}_{s+1}\colon \big(N_2 + N_1(s) + I^{s+1}F\big)/\big(N_1(s) + I^{s+1}F\big) \to (\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$$

is an isomorphism.

So if one can extract a basis from the generating set (21) of the free Z-module $(\bar{N} + I^{s+1}\bar{F})/I^{s+1}\bar{F}$, one can then express $E^s_{-s,s} \simeq \bar{E}^s_{-s,s}$ as a direct sum of a finite number of infinite cyclic groups and cyclic groups of finite order; hence obtaining an explicit demonstration of how the μ's appear in $E^s_{-s,s}$. For example, for the link described in the figure we have

$$E^s_{-s,s} \simeq \bar{E}^s_{-s,s} \simeq Z \oplus \cdots \oplus Z \oplus Z_m \oplus Z_m,$$

where $m = \mu(1, 2, 3) = \mu(3, 2, 1) = \mu(2, 3, 1)$.

Here are some properties of the Milnor invariants that we will need (see [7]).

(A) *The* $\bar{\mu}$ *satisfy a cyclic symmetry, that is,* $\bar{\mu}(i_1, i_2, \ldots, i_s) = \bar{\mu}(i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(s)})$, *where* σ *is a cyclic permutation of* $1, 2, \ldots, s$. By an *invariant* $\bar{\mu}(i_1, \ldots, i_{r+s})$ *of type* $[r, s]$ will be meant one which involves the index '1' r-times and the index '2' s-times. Then

(B) (i) All invariants of type $[r, 0]$ and $[r, 1]$ ($r > 2$) are zero. The invariants of type $[1, 1]$ are the linking numbers and these are not necessarily zero.

(ii) All invariants of type $[2m + 1, 2]$ are also zero.

(iii) For the invariants of type $[2m, 2]$ we have

$$\bar{\mu}(1, \ldots, 1, 2, 1, 2) = -\binom{2m}{1}\bar{\mu}(1, \ldots, 1, 1, 2, 2),$$

$$\bar{\mu}(1, \ldots, 1, 2, 1, 1, 2) = \binom{2m}{2}\bar{\mu}(1, \ldots, 1, 1, 2, 2),$$

$$\bar{\mu}(1, \ldots, 1, 2, 1, 1, 1, 2) = -\binom{2m}{3}\bar{\mu}(1, \ldots, 1, 1, 2, 2), \quad \text{etc.}$$

In view of cyclic symmetry (see (A)) this means that all of the invariants of type $[2m, 2]$ are completely determined by $\bar{\mu}(1, \ldots, 1, 1, 2, 2)$.

Let L be a two-link. Then

$$E^2_{-2,2} \simeq \frac{I^2\bar{F}/I^3\bar{F}}{(\bar{N} + I^2\bar{F})/I^3\bar{F}}.$$

The set $\{X_1^2 + I^3\bar{F}, X_2^2 + I^3\bar{F}, X_1X_2 + I^3\bar{F}, X_2X_1 + I^3\bar{F}\}$ is a basis for $I^2\bar{F}/I^3\bar{F}$; while $(\bar{N} + I^2\bar{F})/I^3\bar{F}$ is generated by

$$[\alpha_1, \omega_1] - 1 = [X_1, \mu(2, 1)X_2] + I^3\bar{F}_1, \qquad [\alpha_2, \omega_2] - 1 = [X_2, \mu(1, 2)X_1] + I^3\bar{F}.$$

But $[X_1, \mu(2, 1)X_2] = -[X_2, \mu(1, 2)X_1]$. Therefore $(\bar{N} + I^2\bar{F})/I^3\bar{F}$ is a free Z-module with basis $\{\mu(1, 2)(X_1X_2 - X_2X_1) + I^3\bar{F}\}$. But the set $\{X_1^2 + I^3F, X_2^2 + I^3F, X_1X_2 + I^3F, X_1X_2 - X_2X_1 + I^3\bar{F}\}$ may be taken as a basis for $I^2\bar{F}/I^3\bar{F}$; it follows that

$$\bar{E}^2_{-2,2} \simeq E^2_{-2,2} \simeq Z \oplus Z \oplus Z \oplus Z_{\mu(1,2)}.$$

Next assume $[\alpha_i, \omega_i] - 1 \in I^3\bar{F}$ $(i = 1, 2)$. Then $(\bar{N} + I^3\bar{F})/I^4\bar{F}$ generated by

$$[\alpha_1, \omega_1] - 1 = \sum_{j_1, j_2 = 1,2} [X_1, \mu(j_1, j_2, 1)X_{j_1}X_{j_2}] + I^4\bar{F},$$

$$[\alpha_2, \omega_2] - 1 = \sum_{j_1, j_2 = 1,2} [X_2, \mu(J_1, j_2, 2)X_{j_1}X_{j_2}] + I^4\bar{F}.$$

But, all the $\mu(j_1, j_2, i)$, $i = 1, 2$, appearing above are zero, due to properties (B) (i) and (B) (ii). Hence nothing could be said about such a link by looking at $\bar{E}^3_{-3,3}$. So we consider the case $[\alpha_i, \omega_i] - 1 \in I^4\bar{F}$ $(i = 1, 2)$. Then $(\bar{N} + I^4\bar{F})/I^5\bar{F}$ is generated by

$$[\alpha_1, \omega_1] - 1 = \mu(1, 1, 2, 2)(X_1^2X_2^2 + 2X_2X_1X_2X_1 - 2X_1X_2X_1X_2 - X_2^2X_1^2) + I^5\bar{F},$$

$$[\alpha_2, \omega_2] - 1 = \mu(1, 1, 2, 2)(X_2^2X_1^2 + 2X_1X_2X_1X_2 - 2X_2X_1X_2X_1 - X_1^2X_2^2) + I^5\bar{F}.$$

Hence $(\bar{N} + I^4\bar{F})/I^5\bar{F}$ is a free Z-module with basis the vector

$$\mu(1, 1, 2, 2)(X_2^2X_1^2 + 2X_1X_2X_1X_2 - 2X_2X_1X_2X_1 - X_1^2X_2^2) + I^5\bar{F}.$$

The free Z-module $I^4\bar{F}/I^5\bar{F}$ has rank 16. Hence the spectral sequence term

$$\bar{E}^4_{-4,4} \simeq E^4_{-4,4} \simeq Z \oplus \cdots \oplus Z \oplus Z_{\mu(1,1,2,2)},$$

where there are fifteen copies of Z in the summand.

Thus, for the special links whose longitudes belong to I^sF the term $E^s_{-s,s}$ sheds light on the Milnor invariants. Naturally one would like to do this study for more general links. The calculations are similar to those in [8].

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