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A STABLE CONVERSE TO THE VIETORIS-SMALE THEOREM WITH APPLICATIONS TO SHAPE THEORY[1]

BY

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ABSTRACT. Our main result says that if $f: X \to Y$ is a map between finite polyhedra which has k-connected homotopy fiber, then there is an n such that $f \times \mathrm{id}: X \times I^n \to Y$ is homotopic to a map with k-connected point-inverses. This result is applied to give an algebraic characterization of compacta shape equivalent to locally n-connected compacta. We also show that a UV^1 compactum can be "improved" within its shape class until its homotopy theory and strong shape theory are the same with respect to finite dimensional polyhedra.

1. Introduction. In this paper we study the problem of when a map $f: X \to Y$ between nice topological spaces is homotopic to a map with some form of the homotopy lifting property. If f is a map between finite polyhedra, a necessary condition for f to be homotopic to a Hurewicz fibration is that the homotopy fiber of f be homotopy equivalent to a finite polyhedron. Thus, in order to achieve general results, we are forced to study much weaker lifting properties. We first prove that if the homotopy fiber of f is k-connected, then f is (stably) homotopic to a PL map with k-connected point-inverses. We use this result to prove that if the homotopy fiber of f has finite skeleta (i.e. is homotopy equivalent to a CW complex with finite n-skeleton for each n) then f is (stably) homotopic to a map with the approximate lifting property for m-dimensional spaces. Here m is fixed but as large as we desire.

In the sections on shape theory, we apply the results of the preceding sections to show that for many compact spaces X it is possible to 'improve' X within its shape class to obtain a compactum X' whose homotopy theory and (strong) shape theory are closely related.

We will now introduce some terminology and give more precise statements of our results. If X is a compactum imbedded in an ANR M, X is said to have property UV^k, $0 \leqslant k < \infty$, if for each neighborhood U of X there is a neighborhood $V \subset U$ of X such that the inclusion-induced map $\pi_l(V) \to \pi_l(U)$ is zero for $0 \leqslant l \leqslant k$. For $k = \infty$ we require that V contract to a point in U. It is known (see, for example, [L]) that this property is a shape property of X and is independent of

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the particular imbedding. For CW complexes, property UV^k reduces to the usual notion of k-connectedness. A map $f\colon X \to Y$ between compact ANR's is called a UV^k-map if for each $y \in Y$, $f^{-1}(y)$ is nonempty and has property UV^k. A UV^∞ compactum is called cell-like (CE) and a UV^∞-map is called a CE map. We now state a version of the Vietoris-Smale theorem of the title.

THEOREM [**Sm**]. *If X and Y are compact ANR's and $f\colon X \to Y$ is a UV^k-map then $f_*\colon \pi_l(X) \to \pi_l(Y)$ is an isomorphism for $l \leqslant k$ and an epimorphism for $l = k + 1$ (i.e., f is $(k + 1)$-connected).*

Recall that the Hilbert cube Q is the countable product of copies of the unit interval. A Hilbert cube manifold (Q-manifold) is a separable metric space in which each point has an open neighborhood homeomorphic to an open subset of Q. Here is our stable converse to Smale's theorem.

THEOREM 1. *If M and N are compact Q-manifolds and $f\colon M \to N$ is $(k + 1)$-connected, $0 \leqslant k < \infty$, then f is homotopic to a UV^k-map.*

We remark that this theorem is false for $k = \infty$. If $f\colon M \to N$ is a nonsimple homotopy equivalence, then f is not homotopic to a CE map [**Ch**]. We will now state a PL theorem which implies Theorem 1.

THEOREM 1$'$. *If K and L are finite polyhedra and $f\colon K \to L$ is a $(k + 1)$-connected map, $0 \leqslant k < \infty$, then there exist a polyhedron K', a CE-PL map $c\colon K' \to K$, and a UV^k-PL map $f'\colon K' \to L$ such that $f \circ c \simeq f'$.*

In view of Chapman's proof that CE maps between polyhedra are simple homotopy equivalences it is natural, though perhaps naive, to ask whether a nonsimple homotopy equivalence can have highly connected point-inverses. Theorem 1$'$ shows that this is indeed possible.

If X and Y are compact metric ANR's, a map $p\colon X \to Y$ is said to have the approximate homotopy lifting property (AHLP) with respect to a compact space Z if for every homotopy $f\colon Z \times I \to Y$, map $F_0\colon Z \to X$, such that $p \circ F_0 = f|Z \times 0$, and $\varepsilon > 0$ there is a map $F\colon Z \times I \to X$ such that $F_0 = F|Z \times \{0\}$ and $d(p \circ F(z, t), f(z, t)) < \varepsilon$ for each $(z, t) \in Z \times I$. f is called an *approximate fibration* if f has the AHLP for all compacta. We will call f an AF^n-map if f has the AHLP for n-dimensional compacta. Compare with [**M-R**]. If $f\colon X \to Y$ is a map, X, Y ANR's and Y connected, then the *homotopy fiber* of f is the fiber of the mapping path fibration [**S**, p. 99] of f.

THEOREM 2. *If M and N are compact Q-manifolds and the homotopy fiber of $f\colon M \to N$ has finite skeleta, then for any n f is homotopic to an AF^n-map.*

This theorem also has a stronger PL version:

THEOREM 2$'$. *If K and L are polyhedra and the homotopy fiber of f has finite skeleta, then for each n there exist a polyhedron K', a CE-PL map $c\colon K' \to K$, and a PL AF^n-map $f'\colon K' \to L$ such that $f \circ c \simeq f'$.*

As an application of Theorem 2, we show that a large class of compacta can be 'improved' within their shape classes.

THEOREM 3. *If X is a UV^1 compactum then X is shape equivalent to a compactum X' such that for every finite-dimensional compactum Z:*

(i) *Every strong shape morphism f: $Z \to X'$ contains a map.*

(ii) *If f, g: $Z \to X'$ are maps which are equivalent as strong shape morphisms, then f and g are homotopic as maps.*

A Theorem 3′, whose statement is similar to that of Theorem 3, holds for compacta which are inverse limits of ANR's with progressively more highly connected bonding maps. We should mention that our construction unavoidably yields infinite-dimensional spaces X', so we have not imbedded the strong shape category in the homotopy category. We do not know if this is possible.

We show in §5 that our improved compacta are nearly as nice locally as algebraic considerations allow. In particular, we show that the strange compacta of Edwards and Geoghegan are shape equivalent to compacta which are LC^k for all k. We also prove:

THEOREM 4. *A continuum X is shape equivalent to an LC^n continuum if and only if* pro-$\pi_l(X)$ *is stable for $0 \leqslant l \leqslant n$ and Mittag-Leffler for $l = n + 1$.*

The case $n = 0$ of this theorem is due to Krasinkiewicz [**K**].

We would like to thank John Walsh for suggesting that we attempt to prove the converse Vietoris-Smale theorem for Q-manifolds. The analogous finite-dimensional problem has been attacked by Walsh and Wilson, [**Wa₁**], [**Wa₂**], [**Wi₁**], [**Wi₂**]. The full converse is not true for maps between spheres. We would like to thank Gerard Venema for helpful conversations and we would particularly like to thank Ross Geoghegan for sharing his knowledge of shape theory.

2. The proofs of Theorems 1 and 1′. Before proving Theorem 1′, we will show that this theorem implies Theorem 1. Let f: $M \to N$ be a $(k + 1)$-connected map between Q-manifolds. By the triangulation theorem for Q-manifolds, there are polyhedra K and L such that $K \times Q \cong M$ and $L \times Q \cong N$. The map

$$\begin{tikzcd} K \arrow[r, "\times 0"] \arrow[rrr, bend right, "f"'] & K \times Q \arrow[r, "f"] & L \times Q \arrow[r, "\text{proj}"] & L \end{tikzcd}$$

is $(k + 1)$-connected, so Theorem 1′ produces a finite polyhedron K', a CE map c: $K' \to K$, and a UV^k-map f': $K' \to L$ such that $f \circ c \simeq f'$. Crossing with Q, we can approximate $c \times \mathrm{id}$ by a homeomorphism, so we obtain a homotopy commuting diagram where $(f' \times \mathrm{id}) \circ h^{-1}$ is the required UV^k-map from M to N.

$$\begin{tikzcd} K \times Q \arrow[rr, "f"] & & L \times Q \\ K' \times Q \arrow[u, "\cong"', "h"] \arrow[urr, "UV^k", "f' \times \mathrm{id}"'] & & \end{tikzcd}$$

We now proceed with the proof of Theorem 1′. We will isolate an important special case of Theorem 1′ as Lemma 2.1.

LEMMA 2.1. *Let K be a finite polyhedron and let $i\colon K \to K \cup_f D^{r+2}$ be the inclusion, where D^{r+2} is a PL $(r+2)$-cell and $f\colon \partial D^{r+2} \to K$ is a PL map. There exist a finite polyhedron K', a CE-PL map $c\colon K' \to K$, and a UV^r-map $i'\colon K' \to K$ such that $i \circ c \simeq i'$.*

$$
\begin{tikzcd}
K \arrow[r, "i"] & K \cup_f D^{r+2} \\
K' \arrow[u, "c"] \arrow[ur, "i'"'] &
\end{tikzcd}
$$

PROOF. Let $K' = M(f)$, the mapping cylinder of f. $c\colon K' \to K$ is the mapping cylinder collapse, and $i'\colon K' \to K \cup_f D^{r+2}$ is a PL map which collapses the top of the mapping cylinder to a point. i' is UV^r because the only noncontractible point-inverse of i' is a copy of S^{r+1}. $\square$

In proving Theorem 1′ we will work with finite CW complexes built up using PL cells and PL attaching maps. Thus, all of our spaces will be polyhedra. By construction, our maps will be PL. We will use the following well-known lemma frequently.

LEMMA 2.2. *If $f\colon K_1 \to L$ and $g\colon K_2 \to L$ are PL maps, then the pullback P of f and g is a polyhedron. The induced maps $g'\colon P \to K_1$ and $f'\colon P \to K_2$ are PL. If f is UV^k, then so is f'.*

PROOF. Let $G_1 = \{(x, y, z) \in K_1 \times K_2 \times L \mid f(x) = z\}$ and let $G_2 = \{(x, y, z) \in K_1 \times K_2 \times L \mid g(y) = z\}$. G_1 and G_2 are polyhedra, since the graphs of f and g are polyhedra. $G_1 \cap G_2 = \{(x, y, z) \mid f(x) = g(y) = z\}$ is therefore a polyhedron and is homeomorphic to the desired pullback. The induced maps are restrictions of PL projection maps and are therefore PL. Each point-inverse of f' is homeomorphic to some point-inverse of f, so the last statement follows. $\square$

PROOF (THEOREM 1′). If $f\colon K \to L$ is $(k+1)$-connected, then $\pi_l(M(f), K) = 0$ for $0 \leqslant l \leqslant k+1$, where $M(f)$ is the simplicial mapping cylinder of f. By Whitehead's cell-trading lemma ([**Wh**, p. 246] or [**Co**, 7.3]) there exist a polyhedron Z and CE-PL maps $c_1\colon Z \to M(f)$, $c_2\colon Z \to M$ where M is a finite PL cell complex obtained from K by attaching PL cells of dimension $\geqslant k+2$.

We write $K = K_0 \subset K_1 \subset \dots \subset K_n = M$ where each K_{i+1} is obtained from K_i by attaching a PL r-cell, $r \geqslant k+2$. We will first use induction on n to construct a finite polyhedron K'', a CE-PL map $c''\colon K'' \to K$, and a PL UV^k-map $i''\colon K'' \to M$ such that $i \circ c'' \simeq i''$, where $i\colon K \to M$ is the inclusion.

The case $n = 0$ is trivial. We assume that there exist CE-PL maps $c'''\colon K''' \to K$ and UV^k-PL $i'''\colon K''' \to K_{n-1}$ such that the composition of c''' with the inclusion $i_{n-1}\colon K \to K_{n-1}$ is homotopic to i'''. By Lemma 2.1, there exist a polyhedron K'''', a CE-PL map $c''''\colon K'''' \to K_{n-1}$, and a UV^k-PL map $i''''\colon K'''' \to M$ such that the appropriate diagram homotopy commutes. The desired polyhedron K'' is the pullback of i''' and c''''. c'' is the composition of c''' with the induced CE-PL map $K'' \to K'''$ and i'' is the composition of the induced UV^k-PL map $K'' \to K''''$ with i''''. Note that the Vietoris-Smale theorem implies that the composition of UV^k-PL maps is UV^k-PL.

We now have a homotopy commuting diagram.

$$\begin{tikzcd}
K \arrow[r] \arrow[rrr, bend left, "f"] & M \arrow[r, "c_1 \circ c_2^{-1}"] & M(f) \arrow[r, "\mathrm{CE}"] & L \\
K'' \arrow[u, "c''", "\mathrm{CE}"'] \arrow[ur, "i''", "UV^k"'] & Z \arrow[u] \arrow[ur, "c_1", "\mathrm{CE}"'] & & \\
K' \arrow[u, "\mathrm{CE}"] \arrow[ur, dashed, "UV^r"'] & & &
\end{tikzcd}$$

We complete the proof of Theorem 1 by letting K' be the pullback of i'' and c_2. c' is the composition of c'' with the induced CE map $K' \to K''$ and i' is the composition of the induced UV^k-map with c_1 and the mapping cylinder collapse. □

3. The proof of Theorem 2′. The proof of Theorem 2′ will rely on the following characterization of AF^n-maps which is a specialized version of a theorem **[D-T**, Satz 2.7] of Dold and Thom. See **[C-D]** for a Čech version.

THEOREM. *Let $f\colon K \to L$ be a PL map between finite polyhedra. f is an AF^n-map if the following condition holds: For each $x \in L$ there are arbitrarily small contractible neighborhoods U of x such that for each $y \in U$ the inclusion $f^{-1}(y) \to f^{-1}(U)$ induces isomorphisms on homotopy groups through dimension n.* □

Our strategy in proving Theorem 2′ will be to construct a homotopy commuting diagram below with p'' a PL AF^N-map and s N-connected for N large. We will then apply Theorem 1′ to the map s. We begin with the following lemma.

$$\begin{tikzcd}
K \arrow[rr, "s"] \arrow[dr, "f"'] & & K'' \arrow[dl, "p''"] \\
& L &
\end{tikzcd}$$

LEMMA 3.1. *Let B be a connected finite PL cell complex and let $\mathcal{P}\colon \mathcal{E} \to B$ be a Hurewicz fibration from an ANR to B with fiber $\mathcal{F}$. If $\mathcal{F}$ has finite skeleta then for each n there exist a finite polyhedron K, a PL AF^n-map $p\colon K \to B$, and an n-connected map $h\colon K \to \mathcal{E}$ such that $\mathcal{P} \circ h = p$.*

PROOF. The lemma is trivial if B is a point. We will proceed by induction on PL cells. Assume that $B = B' \cup D^r$, that B' is connected (since B is connected we can assume that there is only one zero cell–an easy pullback construction shows that if Lemma 3.1 is true for B then it is true for any space homotopy equivalent to B), and that we have the commutative diagram below in which p' is an AF^n-map and the restriction of h' to any fiber is n-connected.

$$\begin{tikzcd}
K' \arrow[r, "h'"] \arrow[dr, "p'"'] & \mathcal{E} \mid B' \arrow[d, "\mathcal{P}\mid"] \\
& B'
\end{tikzcd}$$

Write $D^r = S^{r-1} \times [0, 1] \cup D_1^r$. By pulling p' back over the collapse $S^{r-1} \times [0, 1] \cup B' \searrow B'$, we reduce to the case in which ∂D^r is imbedded in B'. Let F_1 be a CW complex with finite skeleta which is homotopy equivalent to $\mathcal{F}$. Since $\mathcal{E}\,|\partial D^r$ extends over D^r, there is a map t: $\mathcal{E}\,|\partial D^r \to F_1$ which restricts to a homotopy equivalence on each fiber. The map $t \circ h'|(p')^{-1}(\partial D^r)$: $(p')^{-1}(\partial D^r) \to F_1$ is homotopic to a cellular map and therefore is homotopic to a map whose image lies in the N-skeleton $F_1^{(N)}$ of F_1 for some large N. Let F be a polyhedron homotopy equivalent to $F_1^{(N)}$. There is a PL map (coming from the trivialization t) T: $(p')^{-1}(\partial D^r) \to F$ whose restriction to each point-inverse of p' is n-connected. The desired polyhedron K is $K' \cup M(T)$ which projects onto B by the map which comes from considering D^n to be the cone on ∂D^n. That h' extends to h: $K \to \mathcal{E}$ is clear from the construction, since the trivialization of $\mathcal{E}\,|D^r$ was used to construct T. That p is an AF^n-map is clear from the criterion at the beginning of this section. That h is highly connected follows from the fact that a PL AF^n-map satisfies the homotopy sequence of a fibration through degree n. $\square$

We will now proceed with the proof of Theorem 2′. Let f: $K \to L$ be a map between finite polyhedra as in the statement of Theorem 2′. Let $\mathcal{P}: \mathcal{E} \to L$ be the mapping path fibration of f. Let $N \gg \dim K$. According to Lemma 3.1, there exist a finite polyhedron K'', an AF^N-map p'': $K'' \to L$, and a map h'': $K'' \to \mathcal{E}$ such that $\mathcal{P} \circ h'' = p''$ and such that h'' is N-connected. We therefore have a homotopy commuting diagram.

$$\begin{tikzcd} K \arrow[r] \arrow[dr, "f"'] & \mathcal{E} \arrow[d, "\mathcal{P}"] & K'' \arrow[l, "h''"'] \arrow[dl, "p''"] \arrow[ll, bend right, "d"', "\simeq"] \\ & L & \end{tikzcd}$$

The map d from K'' to K is a homotopy domination since the obstructions to finding a homotopy section of d lie in groups $H^l(K, \pi_{l-1}(\mathcal{F}_1))$, where $\mathcal{F}_1$ is the homotopy fiber of d. $\mathcal{F}_1$ is $N - 1$ connected, so these groups are all zero. If s: $K \to K''$ is a map such that $d \circ s \simeq \mathrm{id}$, then s is $(N - 1)$-connected, so Theorem 1′ produces a polyhedron K', a CE-PL map c: $K' \to K$, and a UV^{N-2}-PL map s': $K' \to K''$ such that $s \circ c \simeq s'$. The desired polyhedron and CE map are K' and c. The desired map f' is $p'' \circ s'$, which is an AF^{N-2}-map. $\square$

LEMMA 3.2.[2] *If K and L are finite polyhedra with $\pi_1 K = 0 = \pi_1 L$ and f: $K \to L$ is a map, then the homotopy fiber of $\mathcal{F}$ has finite skeleta.*

PROOF. According to Wall [W, p. 61] it suffices to show that the group ring Λ of $\pi_1(\mathcal{F})$ is Noetherian and that $H_n(\tilde{\mathcal{F}})$ is a finitely generated Λ-module for each n.

The homology groups of K and L are finitely generated. By the Hurewicz theorem modulo Serre classes of Abelian groups [S, p. 509], the homotopy groups of K and L are finitely generated Abelian groups in each dimension. It follows from the homotopy sequence of the mapping path fibration that $\pi_1(\mathcal{F})$ is a finitely generated Abelian group and that $\pi_n(\mathcal{F}) = \pi_n(\tilde{\mathcal{F}})$ is a finitely generated Abelian

[2]This also holds for *finite* fundamental groups.

group for each n. A second application of the mod-$\mathcal{C}$ Hurewicz theorem shows that $H_n(\tilde{\mathfrak{F}})$ is a finitely generated Abelian group for each n. According to Wall [**W**, p. 61] the group ring of a finitely generated Abelian group is Noetherian. This completes the proof. □

Theorem 2 follows from Theorem 2′ as Theorem 1 follows from Theorem 1′.

COROLLARY 3.3.[3] *If M and N are compact simply connected Q-manifolds and f: $M \to N$ is a map then for each n f is homotopic to an AF^n-map.* □

There is, of course, an analogous PL Corollary 3.3′ whose statement we leave to the reader.

4. A lemma concerning inverse limits. In order to apply Theorems 1 and 2 to the study of compacta, we will need the following easy lemma.

LEMMA 4.1. *Let $X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \xleftarrow{f_4} \dots$ be an inverse sequence of compact metric spaces and let $X = \varprojlim (X_i, f_i)$. There is a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive real numbers such that for any N and any space Z a sequence of maps α_i: $Z \to X_i$ such that $d(f_i \circ \alpha_i, \alpha_{i-1}) < \varepsilon_{i-1}$ for all $i > N$ determines a map α: $Z \to X$.*

PROOF. For each n, consider the sequence of maps

$$\{f_{n+1} \circ \cdots \circ f_{n+k} \circ \alpha_{n+k}: Z \to X_n\}_{k=1}^{\infty}.$$

The distance between the kth and $(k + 1)$st maps in this sequence is

$$d(f_{n+1} \circ \cdots \circ f_{n+k} \circ (f_{n+k+1} \circ \alpha_{n+k+1}), f_{n+1} \circ \cdots \circ f_{n+k} \circ (\alpha_{n+k})).$$

If ε_{n+k} is small enough, depending on the modulus of continuity of $f_{n+1} \circ \cdots \circ f_{n+k}$, this distance will be very small. Thus, for a suitable choice of $\{\varepsilon_i\}$, the sequence of maps described above will be Cauchy for each n and each sequence will converge to a map α'_n: $Z \to X_n$. It is easy to verify that $f_n \circ \alpha'_n = \alpha'_{n-1}$, so the sequence $\{\alpha'_i\}$ induces a map α: $Z \to X$ defined by $\alpha(z) = (\alpha'_1(z), \alpha'_2(z), \dots)$ for all z. □

This lemma is a version 'for maps' of a well-known theorem of Morton Brown [**B**]. Our first proof of this result used Brown's theorem. The next corollary will be superceded by the results of §5. It is included here because it exhibits the main idea of §5 without getting the reader bogged down in the technicalities associated with strong shape theory.

COROLLARY. 4.2. *If $X = \varprojlim (K_i, f_i)$, K_i a finite polyhedron, and*

(i) *each f_i is an approximate fibration, then for every compact Z each shape morphism $Z \to X$ is represented by a map;*

(ii) *each f_i is an AF^i-map, then for each compact finite-dimensional Z each shape morphism $Z \to X$ is represented by a map.*

[3]Again, this holds for finite fundamental groups.

PROOF. A shape morphism $Z \to X$ is represented by maps $\alpha_i: Z \to K_i$ such that $f_i \circ \alpha_i \simeq \alpha_{i-1}$. Altering each α_i by a homotopy does not change the shape morphism. In either case, given a sequence $\{\varepsilon_i\}$ of positive numbers, we will construct maps $\{\alpha_i'\}$ as in Lemma 4.1.

Consider case (ii). Let $N = \dim Z$. Let $\alpha_i' = \alpha_i$ for $i < N$. Using the AF^N property of f_N, we can 'cover' the homotopy from $\alpha_{N-1} = \alpha_{N-1}'$ to $f_N \circ \alpha_N$ by a homotopy from α_N to a new map α_N' such that $f_N \circ \alpha_N'$ is ε_{N-1}-close to α_{N-1}'. We construct α_{N+1}' in a similar fashion, using the AF^{N+1}-property of f_{N+1} and continue. By Lemma 4.1 or, better, by its proof, we are done. $\square$

The proof above will not generalize to show that maps into X which are equivalent as shape maps are homotopic as maps. The interested reader is urged to work out the case of maps of a point into the dyadic solenoid as an exercise.

If X is a UV^1 compactum, then X can be written as an inverse limit $X = \varprojlim (K_i, f_i)$ with $\pi_1(K_i) = 0$ for each i. By Corollary 3.3, $f_i \times \mathrm{id}: K_i \times Q \to K_{i-1} \times Q$ is homotopic to an AF^i-map g_i for each i. $X' = \varprojlim (K_i \times Q, g_i)$ has the shape of X and condition (ii) of Corollary 4.2 applies. Thus, every UV^1 compactum X can be 'improved' within its shape class to a compactum X' for which shape maps $Z \to X'$, Z compact and finite dimensional, are represented by maps.

There is an interpretation of Lemma 4.1 which will be useful in the sequel. If $\{(X_i, f_i)\}_{i=1}^{\infty}$ is an inverse sequence of compacta, we can form a new inverse sequence $\{(X_i', f_i')\}_{i=1}^{\infty}$ of compacta by letting X_k' be the disjoint union $\amalg_{i=1}^{k} X_i$ and letting $f_i': X_i' \to X_{i-1}'$ be the map induced by identity maps and f_i. $\varprojlim (X_i', f_i')$ is a compactification of $\amalg_{i=1}^{\infty} X_i'$ obtained by adding on a copy of $\varprojlim (X_i, f_i)$. The condition that a sequence of maps $\alpha_i: Z \to X_i$ converge to a map $\alpha: Z \to \varprojlim (X_i, f_i)$ is precisely the condition which appears in the proof of Lemma 4.1.

5. Approaching maps and strong shape. The reader who has worked out the suggested exercise involving the dyadic solenoid has discovered that it is not sufficient for our purposes to call sequences of maps $\{\alpha_i\}$ and $\{\beta_i\}$, $\alpha_i, \beta_i: Z \to X_i$ homotopic if there are independent level-preserving homotopies $h_i: \alpha_i \simeq \beta_i$. The idea of strong shape theory is to require some form of coherence between the various homotopies. Several equivalent notions of strong shape theory have been defined (**[E-H]**, **[D-S]**, **[K-O]**, **[Q]**). We will use a slight modification of the construction of **[Q]**, where strong shape theory is defined based on a modification of Borsuk's definition of fundamental sequence. Quigley's theory is further developed in **[K-O]**.

DEFINITION 5.1. If X and Y are compact metric spaces and $i: Y \to W$ is an imbedding, W an absolute retract (AR), then an *approaching map* $\underline{f}: X \to Y$ is a pair (f, i) where f is a map $f: X \times [0, \infty) \to W$ such that for each neighborhood U of $i(Y)$ in W there is an N such that $f(X \times [N, \infty)) \subset U$. Two approaching maps $\underline{f}$, $\underline{g}: X \to Y$ ($\underline{f} = (f, i)$, $\underline{g} = (g, i)$) are *homotopic* if there is an approaching map $\underline{H}: X \times I \to Y$ ($\underline{H} = (H, \bar{i})$) such that $\underline{H}|X \times \{0\} = \underline{f}$ and $\underline{H}|X \times \{1\} = \underline{g}$.

If $i': Y \to W'$ is a second imbedding of Y in an AR, then there are maps t: $W \to W'$ and t': $W' \to W$ extending $i' \circ i^{-1}|i(Y)$ and $i \circ (i')^{-1}|i'(Y)$. Composing an approaching map f: $X \times [0, \infty) \to W$ with t yields an approaching map $\underline{f}'$: $t \circ f$: $X \times [0, \infty) \to W'$. Since W' is an AR, the space of maps t: $W \to W'$ extending $i' \circ i^{-1}|i(Y)$ is contractible, so the association of $t \circ f$ to f is independent of the choice of t up to homotopy. If $[\underline{f}]$ and $[\underline{f}']$ are homotopy classes of approaching maps $X \to Y$ in AR's W and W', we will say that $[\underline{f}]$ and $[\underline{f}']$ are *equivalent* if $[\underline{tf}] = [\underline{f}']$ for some t as above. A *strong shape morphism* $\underline{s}$: $X \to Y$ is an equivalence class of homotopy classes of approaching maps from X to Y. In practice, then, a strong shape morphism will be a homotopy class of approaching maps where i: $Y \to W$ is an imbedding of Y into some convenient AR.

DEFINITION 5.2. If $\{(X_i, f_i)\}$ is a finite or infinite inverse system, we will define $\operatorname{Map}\{(X_i, f_i)\}$ to be the union of the mapping cylinders $M(f_i)$ with the base of $M(f_{i+1})$ identified with the top of $M(f_i)$. There is a natural map p: $\operatorname{Map}\{(X_i, f_i)\} \to R^1$ which sends each X_i to i and interpolates linearly between the various X_i's. If $\{(X_i, f_i)\}_{i=1}^{\infty}$ is an infinite inverse system, we can define a new inverse system $\{(X_i', f_i')\}_{i=0}^{\infty}$ by setting $X_k' = \operatorname{Map}(\mathrm{pt} \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots \leftarrow X_k)$ and letting f_k': $X_k' \to X_{k-1}'$ be induced from collapsing the last mapping cylinder to its base. $\varprojlim (X_i', f_i')$ is a compactification of $\operatorname{Map}(\mathrm{pt} \leftarrow X_1 \leftarrow X_2 \leftarrow \dots)$ obtained by adding on a copy of $\varprojlim (X_i, f_i)$. We will call this space (the completed contractible mapping cylinder) $\overleftarrow{C}\operatorname{Map}^+((X_i, f_i))$. If the spaces X_i are ANR's, then $\operatorname{CMap}^+((X_i, f_i))$ is an AR since it is ε-dominated by the open mapping telescope for each $\varepsilon > 0$. Compare with [**Ch-S**, p. 181]. This is often a convenient AR containing the inverse limit.

DEFINITION 5.3. If $\underline{f}$: $X \to Y$ is an approaching map and $A \subset X$ is a closed set, we say that $\underline{f}|A$ *represents a map*, f^*: $A \to Y$, on A if $f|A \times [0, \infty)$: $A \times [0, \infty) \to W$ extends to a continuous function f^+: $A \times [0, \infty] \to W$ by defining $f^+(a, \infty) = f^*(a)$ for all $a \in A$.

The following lemma is the main technical result of this section.

LEMMA 5.4. *Let X be a compactum and let $A \subset X$ be a closed set. Let $Y = \varprojlim (Y_i, f_i)$ where each Y_i is an ANR and each f_i is an approximate fibration for X. If $\underline{\alpha}$: $X \to Y$ is an approaching map (with respect to $Y \hookrightarrow \operatorname{C\,Map}^+(Y_i, f_i)$) which represents a map on A, then $\underline{\alpha}$ is homotopic through approaching maps representing the same map on A to an approaching map $\underline{\beta}$: $X \to Y$ which represents a map on X.*

PROOF. Let p: $\operatorname{CMap}^+(Y_i, f_i) \to [0, \infty]$ be the natural map and let φ_t: $\operatorname{CMap}^+(Y_i, f_i) \to \operatorname{CMap}^+(Y_i, f_i)$ be the map which retracts $\operatorname{CMap}^+(Y_i, f_i)$ onto $p^{-1}([0, t])$ via the projection of Y to the Y_i's and mapping cylinder collapses.

Since $\underline{\alpha}$ is an approaching map, α: $X \times [0, \infty) \to \operatorname{CMap}^+(Y_i, f_i)$ has the property that for each k there exists N_k such that $\alpha(X \times [N_k, \infty)) \subset p^{-1}([k + 1, \infty])$. We may clearly assume that $N_k < N_{k+1}$ for all k and that $\lim_{k\to\infty} N_k = \infty$. Let ρ: $[0, \infty) \to [0, \infty)$ be a homeomorphism into with $\rho(k) = N_k$ for all k. α: $X \times [0, \infty) \to \operatorname{CMap}^+(Y_i, f_i)$ is homotopic to α' where $\alpha'(x, t) = \alpha(x, \rho(t))$. α' has the property that $p \circ \alpha'(x, t) > t$ for all t. α', in turn, is homotopic to α'' defined by $\alpha''(x, t) = \varphi_t \circ \alpha'(x, t)$. Both homotopies preserve the map represented by α on A. We may drop the primes and assume that $p \circ \alpha(x, t) = t$ for all t.

The map $\alpha\colon X \times [0, \infty) \to \mathrm{Map}(* \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots)$ is proper. If $\alpha^*\colon A \to Y$ is the map represented by $\underline{\alpha}|A$, then there is a proper homotopy from $\alpha|A \times [0, \infty)$ to the map

$$\alpha^{**}\colon A \times [0, \infty) \to \mathrm{Map}(* \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots)$$

given by $\alpha^{**}(a, t) = \varphi_t \circ \alpha^*(a)$. The restriction of the homotopy to $A \times \{t\}$ is obtained from $\varphi_t \circ (\alpha|A \times [t, \infty])$ by reparameterizing from $[t, \infty]$ to $[0, 1]$. We may use the proper homotopy extension theorem to alter α so that $\varphi_t \circ \alpha(a, t) = (\alpha^*(a), t)$ for all t. We may repeat the initial types of homotopies to regain the condition $p \circ \alpha(x, t) = t$ for all t.

We now define a homotopy $\underline{A}$ of approaching maps by

$$\begin{aligned} A_s(x, t) &= \varphi_t \circ \alpha\left(x, n - \frac{1}{2}\right) + \frac{2}{2 - s}\left[t - \left(n - \frac{1}{2}\right)\right] \\ &\qquad\qquad\qquad\qquad \text{for } t \in \left[n - \frac{1}{2}, n + \frac{1 - s}{2}\right], \\ &= \varphi_t \circ \alpha\left(x, n + \frac{1}{2}\right) \quad \text{for } t \in \left[n + \frac{1 - s}{2}, n + \frac{1}{2}\right]. \end{aligned}$$

Again, we write $\alpha(x, t) = A_1(x, t)$. α now has the property that $\alpha(x, t) = \varphi_t \circ \alpha(x, n + \frac{1}{2})$ for $t \in [n, n + \frac{1}{2}]$. We can think of α as being constructed from maps $\alpha_n\colon X \to Y_n$ (here identified with $p^{-1}(n - \frac{1}{2})$) and homotopies $h_n\colon X \times I \to Y_{n-1}$ from $f_n \circ \alpha_n$ to α_{n-1}. The formulas are $\alpha_n(x) = \alpha(x, n - \frac{1}{2})$ and $h_n(x, t) = \varphi_{n-1/2} \circ \alpha(x, n - t/2)$. By construction, $h_n(a, t) = \varphi_{n-1/2} \circ \alpha^*(a)$ for all $a \in A$ and for all $t \in [0, 1]$.

Let $\{\varepsilon_i'\}$ be a sequence of real numbers as in Lemma 4.1. Now choose $\varepsilon_i < \varepsilon_i'$ so small that maps into Y_i which are ε_i-close are canonically ε_i'-homotopic. We start our construction by considering the diagram below.

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\alpha_2"] \arrow[d] & Y_2 \arrow[d, "f_2"] \\ X \times I \arrow[r, "h_2"] \arrow[ur, dashed, "H_2"] & Y_1 \end{tikzcd}$$

$h_2(x, 0) = f_2 \circ \alpha_2$, so there is a homotopy $H_2\colon X \times I \to Y_2$ so that $H_2(x, 0) = \alpha_2(x)$ and $d(f_2 \circ H_2, h_2) < \varepsilon_1$. Let $H_2(x, 1) = \alpha_2'(x)$. Since $h_2|(A \times I)$ is a constant homotopy, the usual regularizing trick for Hurewicz fibrations **[D]** allows us to assume that $H_2(a, t) = \alpha_2(a)$ for all $a \in A$ and $t \in [0, 1]$. Set $\alpha_1' = \alpha_1$ and let $h_2'\colon X \times I \to Y_1$ be a small canonical homotopy from $f_2 \circ \alpha_2'$ to $\alpha_1' = \alpha$. Note that there is a canonical homotopy G_2 from $f_2 \circ H_2$ to h_2 which extends h_2'.

We have a homotopy K_2 from $f_3 \circ \alpha_3$ to α_2' given by

$$K_2(x, t) = \begin{cases} h_3(x, 2t), & 0 \leqslant t \leqslant \frac{1}{2}, \\ H_2(x, 2t - 1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

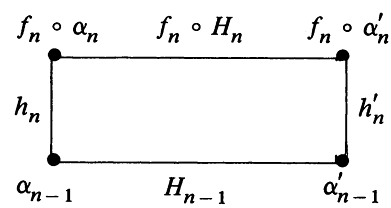
Let $H_3\colon X \times I \to Y_3$ be an ε_2-lifting of K_2 starting at α_3 and ending at $\alpha_3'\colon X \to Y_3$. As before, let h_3' be a canonical homotopy from $f_3 \circ \alpha_3'$ to α_2' and let G_3 be a canonical homotopy from $f_3 \circ H_3$ to K_2. Continue in this fashion, letting $K_{n-1} = h_n * H_{n-1}: X \times I \to Y_{n-1}$, letting H_n be an ε_{n-1}-lifting of K_{n-1}, and letting G_n be a canonical homotopy from $f_n \circ H_n$ to K_{n-1} extending a canonical homotopy h_n' from $f_n \circ \alpha_n'$ to α_{n-1}', where $\alpha_n' = H_n(x, 1) = K_n(x, 1)$.

Use the maps α_n' and homotopies h_n' to define an approaching map $\alpha': X \times [0, \infty) \to \mathrm{C\,Map}^+(X_i, f_i)$ by the formula

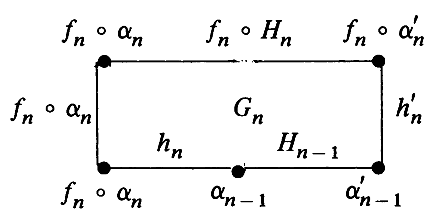
$$\alpha'(x, t) = \begin{cases} \left(h_n'\left(x, 1 - 2\left(t - n + \tfrac{1}{2}\right)\right), t\right), & t \in \left[n - \tfrac{1}{2}, n\right], \\ \left(\alpha_{n+1}'(x), t\right), & t \in \left[n, n + \tfrac{1}{2}\right]. \end{cases}$$

This simply reverses the process by which the original α_n's and h_n's were defined. By Lemma 4.1, α' represents a map which exends α^*.

It remains to show that $\underline{\alpha}'$ is homotopic to $\underline{\alpha}$. We have appropriate maps H_n: $X \times I \to Y_n$. We need only produce homotopies M_n from $f_n \circ H_n$ to H_{n-1} extending h_n and h_n' and use them to produce an approaching map as above. This amounts to extending a map from $X \times \partial I^2$ to $X \times I^2$, where the map on ∂I^2 is given schematically by the following diagram which shows only the I^2-coordinates.



The extension is given by reparameterizing the homotopy G_n.



The unpleasant formula is left to the reader. This completes the proof of Lemma 5.4. □

Corollary 5.5. *Let* $Y = \varprojlim(Y_i, f_i)$ *with each* Y_i *a compact ANR.*

(i) *If each* f_i *is an approximate fibration, then for any compactum* Z*, homotopy classes of maps* $Z \to Y$ *are in* 1-1 *correspondence with strong shape morphisms* $Z \to Y$.

(ii) *If each* f_i *is an* AF^i*-map, then the same conclusion holds for compact finite-dimensional* Z.

Proof. We prove part (ii). If $\dim Z = N$, then

$$Y = \varprojlim(Y_N \leftarrow Y_{N+1} \leftarrow \dots),$$

so Lemma 5.4 applies to maps of Z or $Z \times I$ into Y. Maps in a strong shape class are obtained by applying the lemma to Z. Homotopies are obtained by applying the lemma to $Z \times I$ relative to $Z \times \{0, 1\}$. $\square$

REMARK. One can define simplicial sets of maps $Z \to Y$ and approaching maps $Z \to Y$. The proof above shows that the inclusion is a homotopy equivalence.

Theorem 3 now follows as in the discussion following Corollary 4.2. In fact, we can prove the slightly sharper result:

THEOREM 3′. *If $Y = \varprojlim(Y_i, f_i)$ with each Y_i an ANR and the homotopy fiber of each map f_i has finite skeleta, then there is a compactum Y' shape equivalent to Y which is improved in the sense of Theorem* 3 *and Corollary* 5.5(ii).

PROOF. Apply Theorem 2 to $f_i \times \mathrm{id}\colon Y_i \times Q \to Y_{i-1} \times Q$. $\square$

We can prove a result which does not follow from Theorem 3′ by appealing to Theorem 1 rather than to Theorem 2. A compactum X is LC^k if for each $x \in X$ and neighborhood U of x there is a neighborhood V of x so that every map $S^l \to V$, $0 \leqslant l \leqslant k$, is nullhomotopic in U.

THEOREM 3″. *If $Y = \varprojlim(Y_i, f_i)$ with each Y_i an ANR and each f_i $(i+1)$-connected, then Y is shape equivalent to a compactum Y' which is improved in the sense of Theorem* 3′. *In addition, Y' is LC^k for each $k > 0$.*

PROOF. Apply Theorem 1 to construct a homotopy from $f_i \times \mathrm{id}\colon Y \times Q_i \to Y_{i-1} \times Q$ to a UV^i-map f_i' and let $Y' = \varprojlim(Y_i \times Q, f_i')$. This proves Theorem 3″ modulo the last assertion, which follows from the next proposition.

PROPOSITION 5.6. *Let $Y = \varprojlim(Y_i, f_i)$ with each Y_i an ANR. If all but finitely many f_i's are UV^k-maps, then Y is LC^k.*

PROOF. Let $y = (y_1, y_2, \dots) \in Y$. Let $p_i\colon Y \to Y_i$ be the natural projection. The collection $\{p_i^{-1}(U) | U$ is open in Y_i, $i \geqslant K\}$ is a basis of Y for each K, so given a neighborhood U of y we may find a neighborhood U_K of y_K in Y_K so that $y \in p_K^{-1}(U_K) \subset U$. K may be chosen so that f_i is a UV^k-map for each $i > K$. Choose $V_K \subset U_K$ so that maps $S^l \to V$ are nullhomotopic in U_K for $0 \leqslant l \leqslant k$. V_K exists because Y_K is LC^k. Set $V = p_K^{-1}(V_K)$.

If $\alpha\colon S^l \to V$, $0 \leqslant l \leqslant k$, let $\alpha_n = p_n \circ \alpha$ for each n. Since $\alpha_K\colon S^l \to V_K$, there is an extension $\beta_K | D^{l+1} \to U_K$ of α_K. If a sequence $\{\varepsilon_i\}$ of positive reals is chosen as in Lemma 4.1, we can find a map $\beta_{K+1}\colon D^{l+1} \to Y_{K+1}$ extending α_{K+1} so that $d(f_{K+1} \circ \beta_{K+1}, \beta_K) < \varepsilon_K$. This lifting is possible because f_K is UV^k. Continuing this process, we obtain a sequence $\{\beta_i\colon D^{l+1} \to Y_i\}_{i=K}^{\infty}$ of maps extending the maps α_i such that $d(f_i \circ \beta_i, \beta_{i-1}) < \varepsilon_{i-1}$ for $i \geqslant K+1$. As in Lemma 4.1, this defines a map β into Y extending α. $\square$

REMARK. The condition that a space be LC^k for all k is not as nice as it seems at first glance. The one-point compactification of $\bigvee_{i=1}^{\infty} S^i$ has this property and is, in fact, an 'improved' compactum.

A compactum is *strange* if it has the shape of an infinite CW complex. Strange compacta were first constructed by Edwards and Geoghegan in **[E-G]**. (See **[F$_1$]** for the analogous construction in homotopy theory.)

COROLLARY 5.7. *If X is a strange compactum, then X is shape equivalent to an improved compactum which is LC^k for each $k \geq 0$.*

PROOF. Let K be a CW complex shape equivalent to X. K is finitely dominated in the sense of Wall [E-G], so for each i there exist a finite polyhedron K_i and $(i + 1)$-connected maps d_i: $K_i \to K$, u_i: $K \to K_i$ such that $d_i \circ u_i \simeq \mathrm{id}$. See [W] for a proof of this. The inverse system $\{(K_i, u_i \circ d_{i+1})\}$ satisfies the hypothesis of Theorem 3″ and has inverse limit shape equivalent to X by [E-G]. Theorem 3″ produces the desired compactum. □

We remark that the compacta generated by Theorem 3′ do not have good local properties. The dyadic solenoid is already improved in the sense of Theorem 3′.

Theorem 1 and Proposition 5.6 allow us to give a complete algebraic characterization of metric continua which are shape equivalent to LC^n continua.

THEOREM 4. *A continuum X is shape equivalent to an LC^n continuum if and only if* pro-$\pi_l(X)$ *is stable for* $0 \leq l \leq n$ *and Mittag-Leffler for* $l = n + 1$.

PROOF. The argument of Kozlowski and Segal [K-S] shows that an LC^n metric spaces have stable pro-π_l for $l \leq n$. Borsuk [D-S, 5.2.8] has shown that LC^n metric continua have Mittag-Leffler pro-π_{n+1}. For completeness and variety we sketch a proof which does not explicitly use refinements of nerves. Let $X = \varprojlim(K_i, f_i)$ and form $\mathrm{CMap}^+(K_i, f_i)$. Triangulate the open telescope by finer and finer triangulations going out toward X. Use the LC^n property to define a retraction of the $(n + 1)$-skeleton to X in some neighborhood of X. Since $\mathrm{CMap}^+(K_i, f_i)$ is an ANR, given any neighborhood U of X there is a neighborhood V of X so that $r|(V \cap (n + 1)\text{-skeleton}) \cup X$ is homotopic to the identity by a homotopy which fixes X. We claim that $\mathrm{im}(\pi_l(V) \to \pi_l(U))$ is isomorphic to $\pi_l(X)$ for $l \leq n$ and that $\pi_{n+1}(X) \to \mathrm{im}(\pi_{n+1}(V) \to \pi_{n+1}(U))$ is a surjection. If $(S^l, *) \xrightarrow{\alpha} (V, *)$, then by simplicial approximation we may assume that $\alpha(S^l) \subset (V \cap (n + 1)\text{-skeleton}) \cup X$ for $l \leq n + 1$. $r \circ \alpha(S^l) \subset X$ and $r \circ \alpha \sim \alpha$ rel $*$ in U. This establishes the surjectivity of $\pi_1(X) \to \mathrm{im}(\pi_1(V) \to \pi_1(U))$. If α: $(S^l, *) \to (X, *)$ is nullhomotopic as a map α: $(S^l, *) \to (U, *)$, $l \leq n$, let β: $(D^{l+1}, *) \to (U, *)$ be an extension of α. We assume as before that $\beta(D^{l+1}) \subset ((n + 1)\text{-skeleton}) \cap U$, so $r \circ \beta$: $D^{l+1} \to X$, $*$ is an extension of α. This proves injectivity.

Thus, we need only prove the reverse implication. The idea is to show that a continuum X satisfying the stated conditions can be represented as an inverse limit of ANR's with $(n + 1)$-connected bonding maps.[4] We will then cross with Q and apply Theorem 1 to construct an inverse sequence with UV^n bonding maps and inverse limit shape equivalent to X. We proceed by induction on n.

Case $n = 0$. We can represent X as an inverse limit $\varprojlim(K_i, f_i)$ of connected polyhedra where f_{i*} maps $f_{i+1*}(\pi_1(K_{i+1}))$ onto $f_{i*}(\pi_1(K_i))$ for each i, the last part being essentially the Mittag-Leffler condition. For each i, let $\{{}^i g_j\}_{j=1}^{N_i}$ be a finite generating set for $\pi_1(K_i)$ and for each j choose ${}^i h_j \in f_{i+1*}(\pi_1(K_{i+1}))$ such that

[4] Compare with [H-I].

$f_{i*}({}^ig_j) = f_{i_*}({}^ih_j)$. Attach 2-cells to K_i to kill the elements ${}^ig_j({}^ih_j)^{-1}$ and call the resulting complex K_i'.

Since $f_{i*}({}^ig_j{}^ih_j^{-1}) = 1$ for each j, f_i extends to a map k_i': $K_i' \to K_{i-1}$. Setting f_i' equal to the composition of k_i' with $K_{i-1} \to K_{i-1}'$, we have a strictly commuting diagram of polyhedra.

$$\begin{tikzcd}
K_0 & K_1 \arrow[l, "f_1"'] \arrow[dl, "f_2'"] & K_2 \arrow[l, "f_2"'] \arrow[dl, "f_3'"] & \cdots \arrow[l, "f_3"'] \\
K_1' \arrow[u, "k_1'"] & K_2' \arrow[l] \arrow[u, "k_2'"] & K_3' \arrow[l] \arrow[u, "k_3'"] & \cdots \arrow[l]
\end{tikzcd}$$

One can check that each f_{i*}': $\pi_1(K_i') \to \pi_1(K_{i-1}')$ is surjective. A word represented by a string of ${}^ig_j$'s is the same as the word represented by the same string of ${}^ih_j$'s in $\pi_1(K_i')$. Since the diagram strictly commutes, $\underleftarrow{\lim}(K_i', f_i') = X$. This completes the case $n = 0$. Note that if pro-π_1 is stable (i.e. if $\operatorname{im} f_{i*}$ maps isomorphically to $\operatorname{im} f_{i*} \circ f_{i+1*}$ for large i), as in the succeeding cases, then all but finitely many of the maps f_i' induce isomorphisms on π_1.

Case $n = 1$. In this case we have (dropping primes) isomorphisms $\pi_1(K_i) \xrightarrow{\approx} \pi_1(K_{i-1})$ for all i and a commuting diagram.

$$\begin{tikzcd}
\pi_2(K_0) & \pi_2(K_1) \arrow[l] \arrow[dl] & \pi_2(K_2) \arrow[l] \arrow[dl] & \cdots \arrow[l] \\
G_0 \arrow[u] & G_1 \arrow[l, two heads] \arrow[u] & G_2 \arrow[l, two heads] \arrow[u] & \arrow[l, two heads]
\end{tikzcd}$$

We have an exact sequence $\pi_2(K_i) \to \pi_2(K_{i-1}) \to \pi_2(M(f_i), K_i) \to 0$. Since we have π_1-isomorphisms, we can pass to universal covers and get:

$$\pi_2(M(f_i), K_i) = \pi_2(\widetilde{M(f_i)}, \tilde{K}_i) = H_2(M(\tilde{f}_i), \tilde{K}_i).$$

$C_*(M(\tilde{f}_i), \tilde{K}_i)$ is a complex of finitely generated free $Z\pi_1(K)$-modules. The first nonvanishing homology group of such a complex is always finitely generated. See **[W]**, **[Si]**, or the proof of **[Co**, 13.1] for a proof of this. Choose a finite set of generators $\{\bar{\alpha}_j\}$ for $\pi_2(M(f_i), K_i)$ and for each j let $\{\alpha_j'\}$ be a representative for $\bar{\alpha}_j$ in $\pi_2(K_{i-1})$. Using the Mittag-Leffler condition, we see that for each j there is a $\beta_j \in \pi_2(K_i)$ such that $f_{i-1*} \circ f_{i*}(\beta_j) = f_{i-1*}(\alpha_j')$. Let $\alpha_j = \alpha_j' - f_{i*}\beta_j$. α_j maps to $\bar{\alpha}_j$ in the cokernel and $f_{i-1*}(\alpha_j) = 0$ for each j. We can now attach finitely many 3-cells to K_i to kill $\{\alpha_j\}$ and create a system $\{(K_i', f_i')\}$ with inverse limit shape equivalent to X and 2-connected bonding maps.

Case $n \geqslant 2$. We assume that we have produced a system with n-connected bonding maps. The system $\{(\pi_n(K_i), f_{i*})\}$ is a system of surjections and is stable, so an easy argument shows that it is a sequence of isomorphisms beyond some point. We now proceed to the $(n + 1)$st homotopy and kill the cokernels as in the case $n = 1$. $\square$

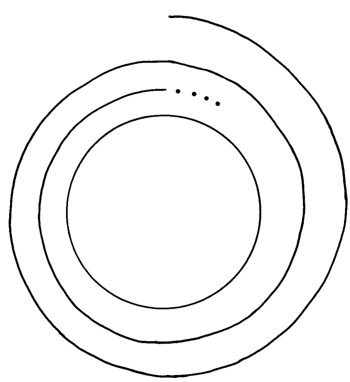
The case $n = 0$ of Theorem 4 is a result of Krasinkiewicz **[K]**. Our proof of Theorem 1 is somewhat crude, so the LC^n compacta we produce are infinite-dimensional. In case $n = 0$, a theorem of John Walsh **[Wa$_1$]** can be substituted to give finite-dimensional LC^0 continua when X is finite dimensional.

For $n = \infty$ we see that Y has the shape of a continuum which is LC^k for all $k \Leftrightarrow$ pro-$\pi_l(Y)$ is stable for all $l \Leftrightarrow$ the conditions of Theorem 3″ are satisfied. Are these spaces shape equivalent to locally contractible spaces?

6. An application to CE equivalence.

DEFINITION 6.1. If X and Y are compacta, we say that X is CE *equivalent* to Y if there exist compacta $X = X_0, X_1, \ldots, X_n = Y$ and for each i either $X_{i+1} \xrightarrow{\text{CE}} X_i$ or $X_i \xrightarrow{\text{CE}} X_{i+1}$. (The sophisticate will want to use hereditary shape equivalences **[D-S]** in this definition. We will work with the simpler notion and point out that the problems discussed here are still interesting when all of the spaces involved are required to be finite dimensional, in which case CE maps are hereditary shape equivalences.)

In **[F$_1$]**, the author showed that homotopy equivalent compacta are CE equivalent. Our attempt to prove that shape equivalent compacta are CE equivalent was blocked by a counterexample; in **[F$_2$]** we showed that the spiral X is not CE equivalent to S^1.

$X =$ 

Note that X is the 'mapping cylinder' of an approaching map $* \to S^1$. Our interest in approaching maps was stimulated by an attempt to develop a calculus of mapping cylinders in the shape category and imitate the proof in **[F$_1$]**.

DEFINITION 6.2. If f: $X \to Y$ is an approaching map, $Y \subset W^{\text{AR}}$, such that $f(X \times [0, \infty)) \subset W - Y$ and f: $X \times [0, \infty) \to W - Y$ is an imbedding, then the set $f(X \times [0, \infty)) \cup Y$ is called the *mapping cylinder* of f and is denoted by $M(f)$. If Y is imbedded as a Z-set **[Ch]** in Q, then every approaching map f: $X \to Y$ can be perturbed slightly to one which admits a mapping cylinder. It is easy to extend this definition so that all approaching maps have well-defined mapping cylinders by using such perturbations, since mapping cylinders obtained by different perturbations are homeomorphic.

PROPOSITION 6.3. *If K and L are finite polyhedra with $\pi_1 L = 0$ and $\underline{f}: K \to L$ is an approaching map which is homotopic to a constant map, then $M(\underline{f})$ is CE equivalent to L.*

REMARK. No pair of shape equivalent CE-inequivalent UV^1 compacta is known. Proposition 6.3 shows that one cannot construct such an example by, for example, letting a ray spiral toward a copy of S^2. Note that the spiral X shows that the condition $\pi_1 L = 0$ is necessary.

PROOF (PROPOSITION 6.3). Let $\dim K = k$. By Theorem 2′ there exist a contractible polyhedron P and a PL AF^k-map $P \to L$. By taking a regular neighborhood of P in R^N, N large, we may assume that P is a disc. Let $r: U \to L$ be a retraction of a neighborhood of L onto L. We may extend r to be defined on $f(X \times [0, \infty))$. We have a diagram.

$$\begin{tikzcd} & P \arrow[d, "p"', "UV^k"] \\ K \times [0, \infty) \arrow[ur, dashed, "F"] \arrow[r, "r \circ f"] & L \end{tikzcd}$$

Since $r \circ f | K \times \{0\}$ is nullhomotopic, there is a lifting $F: K \times [0, \infty) \to P$ such that $\lim_{t \to \infty} d(p \circ F(x, t), r \circ f(x, t)) = 0$.

Imbed P as a Z-set in Q and perturb the approaching map $\underline{F}: K \to P$ so that it admits a mapping cylinder. Now consider the space $M(p) \cup_P M(\underline{F})$. $M(\underline{F})$ is cell-like, so there is a CE-map $M(p) \cup_P M(\underline{F}) \to M(p)/P \cong M(p)$, since P is a disc. Since $M(p) \overset{\text{CE}}{\to} L$, $M(p) \cup_P M(\underline{F})$ is CE equivalent to L. On the other hand, one can collapse the rays of $M(p)$ without first contracting $M(\underline{F})$. The resulting quotient space is homeomorphic to $M(\underline{f})$. This completes the proof of Proposition 6.3. $\square$

If L is an $(r + 1)$-connected polyhedron then a map $pt \to L$ is $(r + 1)$-connected and, by Theorem 2′, L is the UV^r-image of a contractible polyhedron. If Y is a compactum CE equivalent to L (using hereditary shape equivalences) then an argument using pullbacks [**F$_2$**] shows that Y must be the UV^r-image of a cell-like set. This property was used in [**F$_2$**] to show that the spiral X is not CE equivalent to S^1. The next proposition shows that this invariant cannot distinguish a mapping cylinder from its base when the base is simply connected.

PROPOSITION 6.4. *If $\underline{f}: K \to L$ is an approaching map, and L is $(s + 1)$-connected, $s \geq 0$, then $M(\underline{f})$ is the UV^s-image of a cell-like set.*

PROOF. Let $r: U \to L$ be a retraction as before. Since L is $(s + 1)$-connected, the restriction $r \circ f | K^{(s+1)} \times \{0\}$ is nullhomotopic. The inclusion $K^{(s+1)} \to K$ is $(s + 1)$-connected, so there exist a polyhedron $\bar{K}$ and a UV^s-map $\bar{i}: \bar{K} \to \bar{K}$ so that $r \circ f \circ \bar{i}$ is nullhomotopic. Letting P be as before, we have a diagram.

$$\begin{tikzcd}
 & & P \arrow[d] \\
K \times [0, \infty) \arrow[r, "\bar{i} \times \mathrm{id}"'] \arrow[urr, dashed, "F"] & K \times [0, \infty) \arrow[r, "r \circ f"'] & L
\end{tikzcd}$$

As before, we lift to F and form $M(\underline{F})$. There is an obvious UV^s-map from the cell-like set $M(\underline{F})$ onto $M(f)$. $\square$

The place where these arguments break down when $L = S^1$ is that there is no AF^n-map from a contractible polyhedron to L for $n \geqslant 0$. That is because the homotopy fiber of $pt \to S^1$ is an infinite discrete set. This is exploited by the covering space argument in [$\mathbf{F_2}$]. This suggests (to the author, at least) that UV^1, shape equivalent, CE inequivalent compacta may well exist but that to detect them one will need to exploit the fact that the homotopy fiber of $pt \to L$ frequently has infinitely generated homology when $\pi_1 L = 0$. This will require much more delicate arguments than those of [$\mathbf{F_2}$].

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