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MONOTONE DECOMPOSITIONS OF θ_n-CONTINUA

BY

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ABSTRACT. We prove the following theorem for a compact, metric θ_n-continuum (i.e., a compact, connected, metric space that is not separated into more than n components by any subcontinuum). The continuum X admits a monotone, upper semicontinuous decomposition $\mathfrak{D}$ such that the elements of $\mathfrak{D}$ have void interiors and the quotient space $X/\mathfrak{D}$ is a finite graph, if and only if, for each nowhere dense subcontinuum H of X, the continuum $T(H) = \{x|$ if K is a subcontinuum of X and $x \in K^\circ$, then $K \cap H \neq \varnothing\}$ is nowhere dense. The elements of the decomposition are characterized in terms of the set function T. An example is given showing that the condition that requires $T(x)$ to have void interior for all $x \in X$ is not strong enough to guarantee the decomposition.

For a fixed integer n, a θ_n-continuum, i.e., a continuum X that has the property that no subcontinuum separates X into more than n components, would seemingly admit nondegenerate simplifying monotone upper semicontinuous decompositions. However, in the absence of any other property this is not so, e.g., an indecomposable continuum. In a previous paper [5] the second author investigated this decomposition problem for those continua not separated by any of their subcontinua (θ_1-continua). For those continua, the property that characterized "nice" decompositions is that for any subcontinuum H of X with void interior, $T(H)$ also has void interior, where T is the aposyndetic set function (defined below) due to Jones [2]. The precise result [5, Theorems 3 and 4, pp. 74 and 75] is the following theorem.

THEOREM (VOUGHT). *Let X be a compact, metric θ_1-continuum. Then X admits a monotone, upper semicontinuous decomposition $\mathfrak{D}$ such that the elements of $\mathfrak{D}$ have void interiors, and the quotient space $X/\mathfrak{D}$ is a simple closed curve, if and only if $[T(H)]^\circ = \varnothing$ for every subcontinuum H with void interior. Furthermore $\mathfrak{D} = \{T^2(x)|x \in X\}$.*

The main purpose of this paper is to extend the theorem to θ_n-continua where the quotient space now becomes a finite graph and the decomposition $\mathfrak{D} = \{T^{2n}(x)|x \in X\}$. The characterizing property, using the set function T, is closely related to the property that X not contain any indecomposable subcontinua with nonvoid

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interior. However, this latter property (which characterizes such decompositions in irreducible continua) is not quite strong enough to ensure the existence of the decomposition for θ_n-continua in general [**5**, p. 71]. Some of the lemmas in this paper are proved for θ-continua in general. A *θ-continuum* is a continuum X that has the property that no subcontinuum separates X into more than a finite number of components. It is an open question whether every θ-continuum is a θ_n-continuum for some n. [See added in proof.]

Clearly a θ_n-continuum is a θ_{n+1}-continuum for any $n > 1$. A simple closed curve is an example of a θ_1-continuum, as is an indecomposable continuum. Any continuum irreducible about n points is a θ_n-continuum and a θ-curve itself is a θ_2-continuum. If A is a subset of the continuum X, then $T(A)$ consists of A together with all points $x \in X \setminus A$ such that there does not exist an open set U and a continuum H such that $x \in U \subset H \subset X \setminus A$. Also, $T^0(A) = A$ and $T^n(A) = T(T^{n-1}(A))$ for any positive integer n. It is known that if A is a connected set, then $T(A)$ is a continuum. The interior and closure of A are denoted by A° and $\bar{A}$ or $\mathrm{cl}(A)$, respectively. The following result, due to R. W. FitzGerald [**1**, p. 169], is crucial in the development of this paper.

THEOREM (FITZGERALD). *If X is a compact, metric θ_n-continuum and, for some natural number k, $T^k(x) = T^{k+1}(x)$ for all $x \in X$, then X admits a decomposition $\mathfrak{D} = \{T^k(x) | x \in X\}$ such that $\mathfrak{D}$ is the unique minimal decomposition with respect to being monotone, upper semicontinuous and having a quotient space that is a finite graph.*

By the minimal decomposition is meant a decomposition of X that has the aforementioned properties and refines (in the sense of set inclusion) every other decomposition with these properties. There is nothing in FitzGerald's theorem to prevent the decomposition $\mathfrak{D}$ of X from being degenerate, e.g., if X is an indecomposable continuum. The initial goal here is to characterize nondegenerate monotone decompositions in which the elements have void interiors. This is done in Theorem 1 using FitzGerald's result and Lemma 1. The principal goal, Theorem 2, is a sharpening of Theorem 1 giving the best possible information about the decomposition elements.

LEMMA 1. *If X is a θ-continuum, P the projection map of X onto a monotone, upper semicontinuous decomposition of X the elements of which are nowhere dense, and $[p, q]$ is an arc in $P(X)$ such that $(p, q) = [p, q] \setminus \{p, q\}$ is open, then any subcontinuum M of X such that $P(M) \supset (p, q)$, contains $P^{-1}((p, q))$ and hence $\mathrm{cl}[P^{-1}((p, q))]$ is a continuum that is irreducible between the subcontinua $P^{-1}(p)$ and $P^{-1}(q)$ of X.*

PROOF. Suppose there is a subcontinuum M of X such that $P(M) \supset (p, q)$, but $M \not\supset P^{-1}((p, q))$. Then $U = P^{-1}((p, q)) \setminus M$ is a nonvoid, open subset of X and so there is a sequence $t_1, t_2, \ldots$ of members of (p, q) that is increasing in the separation order from p to q such that $P^{-1}(t_i) \not\subset M$, for $i = 1, 2, \ldots$. Furthermore, the sequence can be chosen such that the collection $\{P^{-1}((t_i, t_{i+1})) \setminus M | i$ is a natural number} is an infinite collection of nonvoid, disjoint open subsets of X. It follows that the complement of the union of this collection is a subcontinuum of X but this is impossible since X is a θ-continuum. So $M \supset P^{-1}((p, q))$ and this then implies that $\mathrm{cl}[P^{-1}((p, q))]$ is irreducible from $P^{-1}(p)$ to $P^{-1}(q)$.

THEOREM 1. *Let X be a compact, metric θ_n-continuum. Then X admits a monotone, upper semicontinuous decomposition $\mathfrak{D}$ such that the elements of $\mathfrak{D}$ have void interior and the quotient space $X/\mathfrak{D}$ is a finite graph if and only if $[T(H)]^\circ = \varnothing$ for every subcontinuum H with void interior. Furthermore $\mathfrak{D} = \{T^{n(n+1)}(x) | x \in X\}$.*

PROOF. Suppose X has the required decomposition and let H be a subcontinuum of X such that $H^\circ = \varnothing$. Since $X/\mathfrak{D}$ is a finite graph, it follows from Lemma 1 that there exists $d \in \mathfrak{D}$ such that $H \subset d$. Then $T(H) \subset T(d)$. Due to the fact that $X/\mathfrak{D}$ is a finite graph (hence locally connected) it must be that $T(d) \subset d$. Since $d^\circ = \varnothing$ then $T(H)^\circ = \varnothing$.

Now assume that whenever H is a continuum with void interior, it follows that $T(H)^\circ = \varnothing$. We will show that $T^{n(n+1)}(x) = T^{n(n+1)+1}(x)$ for all $x \in X$. Then according to FitzGerald's theorem, $\{T^{n(n+1)}(x) | x \in X\}$ will be the unique minimal monotone, upper semicontinuous decomposition such that the quotient space is a finite graph. Furthermore, by hypothesis,

$$[T(x)]^\circ = \varnothing,$$
$$[T^2(x)]^\circ = [T(T(x))]^\circ = \varnothing, \ldots,$$
$$[T^{n(n+1)}(x)]^\circ = [T(T^{n(n+1)-1}(x))]^\circ = \varnothing$$

and so the elements $T^{n(n+1)}(x)$ of the decomposition will have void interiors.

First let us show that X contains no indecomposable subcontinuum with nonvoid interior. Suppose H is such a continuum. Let $x \in H^\circ$ and denote the components of $X \setminus H$ by $Q_1, \ldots, Q_k$. Since $T(x)^\circ = \varnothing$, let $y \in H^\circ \setminus T(x)$ and let N be a subcontinuum of X such that $y \in N^\circ \subset N \subset X \setminus \{x\}$. Because H is indecomposable $N \not\subset H$ and therefore $N \cap \bigcup_{i=1}^{k} Q_i \neq \varnothing$. Without loss of generality suppose there is an integer j such that $N \cap Q_i \neq \varnothing$, for $1 \leqslant i \leqslant j \leqslant k$ and $N \cap Q_i = \varnothing$, for $j < i \leqslant k$. For each i such that $j < i \leqslant k$, let C_i be a composant of H such that $C_i \cap \overline{Q_i} \neq \varnothing$. Take C_i' to be a subcontinuum of C_i such that $C_i' \cap N \neq \varnothing$ and $C_i' \cap \overline{Q_i} \neq \varnothing$. It follows that

$$N' = N \cup \left(\bigcup_{i=1}^{k} \overline{Q_i} \right) \cup \left(\bigcup_{i=j+1}^{k} C_i' \right)$$

is a continuum, and $X \setminus N'$ is a nonempty subset of H°. But $X \setminus N'$ has at most n components. If K denotes one of these components, then $K = K^\circ \neq \varnothing$. Since $K^\circ \subset X \setminus N' \subset H^\circ$ it follows that $\overline{K}$ is a subcontinuum of H that has a nonvoid interior. But $\overline{K} \subset H \setminus \{y\}$ and this contradicts the indecomposability of H.

Suppose $T^{n(n+1)+1}(x) \setminus T^{n(n+1)}(x) \neq \varnothing$ for some x and let

$$y \in T^{n(n+1)+1}(x) \setminus T^{n(n+1)}(x).$$

Let $E_0 = \{y\}$ and let B_1 be a continuum such that

$$y \in B_1^\circ \subset B_1 \subset X \setminus T^{n(n+1)-1}(x).$$

Denote the closure of the component of $X \setminus B_1$ that contains $T^{n(n+1)-1}(x)$ by F_1'. Let E_1 be the closure of the component of $X \setminus F_1'$ that contains y. Assume that the continuum E_k has been defined for some $k < n(n+1)$, so that $E_k \subset X \setminus T^{n(n+1)-k}(x)$. There exists a continuum B_{k+1} such that $E_k \subset B_{k+1}^\circ \subset B_{k+1} \subset X \setminus T^{n(n+1)-(k+1)}(x)$. Denote the closure of the component of $X \setminus B_{k+1}$ that contains $T^{n(n+1)-(k+1)}(x)$ by F_{k+1}'. Let E_{k+1} be the closure of the component of $X \setminus F_{k+1}'$ that contains E_k. By induction, continua $E_1, \ldots, E_{n(n+1)}$ have been defined such that $y \in [E_1]^\circ$ and, for $1 \leqslant i \leqslant n(n+1)$, $E_{i-1} \subset [E_i]^\circ$ and $E_i \cap T^{n(n+1)-i}(x) = \varnothing$. For $1 \leqslant i \leqslant n(n+1)$, let F_i be the continuum $\operatorname{cl}(X \setminus E_i)$. For $2 \leqslant i \leqslant n(n+1)$, let $H_i = \operatorname{cl}[X \setminus (E_{i-1} \cup F_i)]$. Also, let $H_1 = E_1$ and $H_{n(n+1)+1} = F_{n(n+1)}$. Note that each component of H_i intersects both E_{i-1} and F_i, for $i = 2, \ldots, n(n+1)$.

If, for some j such that $1 \leqslant j \leqslant n(n+1)$, all the components of H_j are irreducible between E_{j-1} and F_j, then $y \notin T^{n(n+1)+1}(x)$. To see this let $C_1, C_2, \ldots, C_h$ be the components of H_j. Since C_i contains no indecomposable subcontinuum with nonvoid interior (relative to C_i), for $1 \leqslant i \leqslant h$, there exists a decomposition $C_i = K_i \cup L_i$ such that $K_i \cap E_{j-1} \neq \varnothing$, $K_i \cap F_j = \varnothing$ and $L_i \cap F_j \neq \varnothing$, $L_i \cap E_{j-1} = \varnothing$. But $T^{n(n+1)-j}(x) \subset F_j^\circ$ and $T^{n(n+1)-(j-1)}(x) \subset F_j$, so

$$T^{n(n+1)-(j-2)}(x) \subset F_j \cup \left(\bigcup_{i=1}^{h} L_i \right).$$

It follows that

$$T^{n(n+1)-(j-3)}(x) \subset F_j \cup \left(\bigcup_{i=1}^{h} L_i \right) \cup \left(\bigcup_{i=1}^{h} K_i \right) \subset F_{j-1}$$

if $j \neq 1$ and $y \notin T^{n(n+1)+1}(x)$ if $j = 1$. Consequently $T^{n(n+1)+1}(x) \subset F_1$ and because $y \notin F_1$ it follows that $y \notin T^{n(n+1)+1}(x)$, a contradiction. Thus, for $1 \leqslant j \leqslant n(n+1)$, there must be some component of H_j that is not irreducible between E_{j-1} and F_j.

Let k be the maximum number of components of any of $H_2, H_3, \ldots, H_{n(n+1)}$ and let us show that a subcontinuum can be found that separates X into at least $n+1$ components. If $k > n$, let H_j be one of the sets that has k components. Then let C be a proper subcontinuum of one of the components of H_j that is not irreducible between E_{j-1} and F_j such that $E_{j-1} \cap C \neq \varnothing$ and $F_j \cap C \neq \varnothing$. It follows that $X \setminus (E_{j-1} \cup C \cup F_j) \subset H_j$ and has at least k components, where $k > n$. Since X is a θ_n-continuum this is impossible and so $k \leqslant n$.

Consider the sets $H_1, \ldots, H_{n+1}$. Since each of these sets has at most n components, there exists an integer j such that $1 \leqslant j \leqslant n$ and H_j has at least as many components as H_{j+1}. If each component of H_j intersects at most one component of H_{j+1}, then let C be a component of H_j that is not irreducible between E_{j-1} and F_j. Let C' be a proper subcontinuum of C that intersects both E_{j-1} and F_j. Then

$Q_1 = E_{j-1} \cup (H_j \setminus C) \cup C' \cup H_{j+1}$ is a continuum, $E_j \setminus Q_1 \subset C$ and $E_{j+1} \setminus Q_1 = C^\circ \setminus C' \neq \varnothing$. If, on the other hand, some component H_j intersects more than one component of H_{j+1} then some component C of H_j intersects only components of H_{j+1} that are also intersected by some other component of H_j. Then $Q_1 = E_{j-1} \cup (H_j \setminus C) \cup H_{j+1}$ is a continuum and $H_j \supset E_{j+1} \setminus Q_1 = C^\circ \neq \varnothing$. In either case, it follows that $R_{n+1} = Q_1 \cup (H_{j+2} \cup \cdots \cup H_{n+1})$ is a continuum (if $j = n$, then $R_{n+1} = Q_1$) and $E_{n+1} \setminus R_{n+1}$ is a nonvoid subset of the interior of E_{n+1}.

Next consider the $n + 1$ sets $H_{n+2}, \ldots, H_{2n+2}$. Repeating the above argument, we can find an integer j such that $n + 2 \leqslant j \leqslant 2n + 1$ and continuum Q_2 such that $Q_2 \supset H_{j+1}$, $E_{j+1} \setminus Q_2 \subset H_j$ and $E_{j+1} \setminus Q_2 \neq \varnothing$. Now $R_{n+1} \supset H_{n+1}$ and each component of $Q_2 \setminus E_{n+1}$ has a limit point in H_{n+1}. Hence $R_{2n+2} = R_{n+1} \cup (Q_2 \setminus E_{n+1}) \cup (H_{j+2} \cup \cdots \cup H_{2n+2})$ is a continuum and $E_{2n+2} \setminus R_{2n+2}$ has at least two components, each in the interior of E_{2n+2}. After this procedure has been followed n times, a continuum $R_{n(n+1)}$ is obtained such that $E_{n(n+1)} \supset R_{n(n+1)}$ and $E_{n(n+1)} \setminus R_{n(n+1)}$ has at least n components, all lying in the interior of $E_{n(n+1)}$. But $X \setminus E_{n(n+1)} \neq \varnothing$, so $X \setminus R_{n(n+1)}$ has at least $n + 1$ components, which is impossible. This contradiction shows that $T^{n(n+1)+1}(x) \setminus T^{n(n+1)}(x) = \varnothing$ and, therefore, that $T^{n(n+1)+1}(x) = T^{n(n+1)}(x)$, for all $x \in X$. This completes the proof of the theorem.

As corollaries of Theorem 1 are the following two theorems.

THEOREM (KURATOWSKI). *If X is an irreducible continuum, then X has a monotone upper semicontinuous decomposition, the elements of which have void interior and whose quotient space is an arc, if and only if X contains no indecomposable subcontinuum with nonvoid interior.*

PROOF. For an irreducible continuum X, the condition that X contains no indecomposable subcontinuum with nonvoid interior is equivalent to the condition that if H is a subcontinuum of X with void interior, then $T(H)$ has empty interior also. Now X is a θ_2-continuum and Theorem 1 yields the conclusion that the quotient space is a finite graph. However, the quotient space is also irreducible between two points so in fact must be an arc.

More generally, if X is a continuum irreducible about n points, then the two conditions are equivalent and since X is a θ_n-continuum, Theorem 1 yields the desired conclusion with a finite graph for the quotient space. Since the quotient space is also irreducible about n points, it is a dendrite (a metric tree). This general result is a theorem of Vought [**6**, Theorem 1].

THEOREM (VOUGHT). *If X is a continuum that is separated by no subcontinuum then X admits a monotone, upper semicontinuous decomposition the elements of which have void interiors and for which the quotient space is a simple closed curve if and only if for every subcontinuum H with void interior it follows that $T(H)$ has void interior.*

PROOF. Since X is a θ_1-continuum the conclusion follows with a finite graph for the quotient space. However, the quotient space is clearly a θ_1-continuum itself so it must be a simple closed curve.

As seen in the proof of Theorem 1, the characterizing condition for the decomposition using the set function T implies that X contains no indecomposable subcontinuum with nonvoid interior. For an example of a θ_1-continuum in which this latter condition will not ensure the nondegenerate decomposition, see [5, p. 71]. However, it might seem plausible to expect the condition that $[T(x)]^\circ = \varnothing$ for all $x \in X$ to yield the desired decomposition. This is stronger than requiring that X not contain any indecomposable subcontinuum with nonvoid interior (the implication between these two conditions can be proved by means of the third paragraph in the proof above, where all that is needed is that $[T(x)]^\circ = \varnothing$ for all x). But even so, this additional strengthening of the hypothesis will still not be enough to give the decomposition, as the following example shows.

EXAMPLE 1. A θ_1-continuum X such that $[T(x)]^\circ = \varnothing$ but $T^2(x) = X$, for all x in X.

In the xy-plane, let M be the plane indecomposable continuum with one end point, described by Knaster [3, p. 204]. Let K be a (topological) Cantor set in M intersecting each composant of M in at most one point but not intersecting all composants of M. Such a set exists by [4, p. 305]. Let C be the standard Cantor set in the interval [0, 1] on the z-axis. Let h be a homeomorphism from $\{(c_1, c_2)$ in $C \times C | c_1 \leqslant c_2\}$ onto K and extend h to $C \times C$ symmetrically, i.e., let $h(c_2, c_1) = h(c_1, c_2)$ if $c_1 \leqslant c_2$. Let X be the decomposition space of $M \times C$ whose set of nondegenerate elements is $\{\{(h(c_1, c_2), c_1), (h(c_1, c_2), c_2)\} | c_1 < c_2\}$ and let P be the projection function from $M \times C$ onto X. If $(q, c) \in M \times C$ is not a member of some nondegenerate element of X, then we will consider (q, c) rather than $\{(q, c)\}$ to be a point of X.

The decomposition is easily seen to be, not only upper semicontinuous, but continuous and to result in a connected space. Hence, X is a compact metric continuum.

We wish to show, by an indirect proof, that X is a θ_1-continuum. Assume X has a subcontinuum Y that separates X. First we will show if $c \in C$ then either Y contains the whole "level" $L(c) = P(M \times \{c\})$ of X or some component of $[X \setminus Y] \cap L(c)$ is dense in $L(c)$. Let c be a member of C such that $Y \not\supset L(c)$. Then there is an open subset C' of C and an open subset D' of M such that $(D' \times C') \cap Y = \varnothing$. Let A be a composant of M that does not intersect K. If $Y \cap P(A \times \{c\}) = \varnothing$, then some component of $[X \setminus Y] \cap L(c)$ is dense in $L(c)$. Suppose $Y \cap P(A \times \{c\}) \neq \varnothing$ and let $(p, c) \in Y \cap P(A \times \{c\})$. Let Z be the closure of the (p, c)-component of $Y \cap [X \setminus P(K \times C)]$. Clearly, $Z \subset L(c)$ and $Z \cap P(K \times C) \neq \varnothing$. Then $\pi_1(P^{-1}(Z))$, where π_1 is the projection function from $M \times C$ onto M, is a subcontinuum of M containing p and a point of K but not intersecting D'. This is impossible since $p \in A$ and $K \cap A = \varnothing$. Since the intersection of any two dense open subsets of M is a dense open subset of M, it follows that Y separates X between two points (p, c_1) and (p, c_2), where $p \in M \setminus K$. From this it follows that Y separates X between two sets $\overline{D} \times C_1$ and $\overline{D} \times C_2$ where (1) C_1 and C_2 are "intervals" in C that are open and closed relative to C, (2) each member of C_1 is less than each member of C_2, (3) D is an open set in

$M \setminus P(K \times C)$ and (4) $Y \not\subset P(M \times [C_1 \cup C_2])$. For each c_1 in C_1 and each c_2 in C_2, let $A(c_1, c_2)$ be the $(h(c_1, c_2), c_1)$-component of $[M \setminus D] \times \{c_1\}$ and let $A(c_2, c_1)$ be the $(h(c_1, c_2), c_2)$-component of $[M \setminus D] \times \{c_2\}$. Then $Y \cap P(A(c_1, c_2) \cup A(c_2, c_1)) \neq \emptyset$, since $P(A(c_1, c_2) \cup A(c_2, c_1))$ is a connected set that intersects both $\overline{D} \times C_1$ and $\overline{D} \times C_2$, and Y separates X between them.

Let $K_1 = \{h(c_1, c_2) | c_1 \in C_1 \text{ and } c_2 \in C_2\}$ and let $K_2 = K \setminus K_1$. Then K_1 and K_2 are topological Cantor sets. For each $k_1 \in K_1$ and $k_2 \in K_2$, the k_1-component of $M \setminus D$ is the k_1-quasicomponent of $M \setminus D$ and does not contain k_2, and so, there is a separation of $M \setminus D$ between k_1 and k_2. Since K_1 and K_2 are compact, there is a separation of $M \setminus D$ between K_1 and K_2, i.e., $M \setminus D = R \cup S$, separated, where $R \supset K_1$ and $S \supset K_2$. Then $P(R \times [C_1 \cup C_2]) \supset P(A(c_1, c_2) \cup A(c_2, c_1))$, if $c_1 \in C_1$ and $c_2 \in C_2$, so $P(R \times [C_1 \cup C_2]) \cap Y \neq \emptyset$. Also, $Y \not\subset P(M \times [C_1 \cup C_2])$ so $P(K_2 \times C) \cap Y \neq \emptyset$ and, hence, $[X \setminus P(R \times [C_1 \cup C_2])] \cap Y \neq \emptyset$. But $P(R \times [C_1 \cup C_2])$ is both open and closed in $X \setminus (D \times [C_1 \cup C_2])$, since $R \times [C_1 \cup C_2]$ is open and closed in $[M \times C] \setminus (D \times [C_1 \cup C_2])$ and any element of X that intersects $R \times [C_1 \cup C_2]$ is contained in it. It follows from this that Y is not connected, contrary to the assumption that Y is a subcontinuum of X.

We now wish to show that, if $x \in X$, then $T(x) = P(M \times [\pi_2(P^{-1}(x))])$, where π_2 is the projection function from $M \times C$ onto C, i.e., $T(x)$ is the union of the "levels" of X that x lies on. Since any "level" intersects every other level and since $T^2(x) = T(T(x))$ contains $T(y)$ for each y in $T(x)$, it will follow immediately that $T^2(x) = X$, for each x in X.

Suppose X is aposyndetic at $P(a, c)$ with respect to $P(b, c)$ and let H be a continuum contained in $X \setminus P(b, c)$ and containing $P(a, c)$ in its interior. Without loss of generality we assume that $P(a, c) = (a, c)$ and $P(b, c) = (b, c)$. Then there exist open sets D_1 and D_2 in $M \setminus K$ and an open and closed "interval" C_1 in C such that $D_1 \times C_1 \subset H$ and $[D_2 \times C_1] \cap H = \emptyset$. Let $c \in C_1$ and let q be a point in D_2 and in the $h(c, c)$-composant of M but not in the $h(c, c)$-component of $M \setminus D_1$. Then, since components of $M \setminus D_1$ are quasicomponents and K is compact, there is a point set D_3 in $M \setminus D_1$ that is open and closed relative to $M \setminus D_1$, contains q and misses K and part of D_2. Then $D_3 \times C_1$ is open and closed relative to $[M \times C] \setminus [D_1 \times C_1]$. Consequently, $P(D_3 \times C_1) = D_3 \times C_1$ is open and closed relative to $X \setminus [D_1 \times C_1]$ and so $[P(D_3 \times C_1)] \setminus H$ is open and closed relative to $X \setminus H$. But $[P(D_3 \times C_1)] \setminus H$ is a proper subset of $X \setminus H$ and $X \setminus H$ is connected since X is a θ_1-continuum. From this contradiction we conclude that $T(P(a, c)) \supset \{P(d, c) | d \in M\}$ for all $c \in C$. This implies that $T(x) \supset P(M \times [\pi_2(P^{-1}(x))])$, for all x in X.

If C_1 is a subset of C, then $P(M \times C_1)$ is connected since, for each two points c_1 and c_2 of C_1, the "levels" $P(M \times \{c_1\})$ and $P(M \times \{c_2\})$ intersect in the point $\{(h(c_1, c_2), c_1), (h(c_1, c_2), c_2)\}$. If C_1 is an open and closed set then any boundary point of $P(M \times C_1)$ is $\{(h(c_1, c), c_1), (h(c_1, c), c)\}$ for some c_1 in C_1 and some c in $C \setminus C_1$. From this it follows that $T(P(q, c)) \subset P(M \times \{c\})$, if $q \neq h(c, c_1)$ for all c_1 in $C \setminus \{c\}$, and $T(P(q, c)) \subset [P(M \times \{c\}) \cup P(M \times \{c_1\})]$, if $q = h(c, c_1)$ for some c_1 in $C \setminus \{c\}$.

Let $x = P(q, c)$. From the above we see that $T(x) \subset [P(M \times \{c\}) \cup P(M \times \{c_1\})] = P(M \times \{c, c_1\})$, if $q = h(c, c_1)$, and otherwise $T(x) \subset P(M \times \{c\})$. In either case $T(x) \subset P(M \times [\pi_2(P^{-1}(x))])$ and so we now have $T(x) = P(M \times [\pi_2(P^{-1}(x))])$. Clearly $[T(x)]^\circ = \varnothing$ and, as we saw above, $T^2(x) = X$.

Similar examples of θ_n-continua, for $n > 1$, can be constructed from n copies of X by identifying all of the bottom "levels" (copies of $P(M \times \{0\})$). In these examples $[T(x)]^\circ = \varnothing$ and $T^3(x)$ is the whole continuum, for each point x, but $T^2(x)$ is the whole continuum if and only if x is in the identified level.

Question. Is there a θ-continuum X, that is not a θ_1-continuum, such that $[T(x)]^\circ = \varnothing$ and $T^2(x) = X$, for each x in X?

Next some lemmas are established for the purpose of proving the main theorem which is the result of replacing $T^{n(n+1)}(x)$ by $T^{2n}(x)$ in Theorem 1.

Notation and terminology. If $\mathcal{C}$ is a collection of point sets and $\mathcal{C}'$ is a subcollection of $\mathcal{C}$ (is a point of $\mathcal{C}^*$), then $I(\mathcal{C}', \mathcal{C})$ is the collection of all members of $\mathcal{C}$ that intersect some member of $\mathcal{C}'$ (contain the point $\mathcal{C}'$, respectively), $I^2(\mathcal{C}', \mathcal{C}) = I(I(\mathcal{C}', \mathcal{C}), \mathcal{C})$, etc., and $L(\mathcal{C})$ is the minimum number i such that $I^i(x, \mathcal{C}) = \mathcal{C}$ for some point x in $\mathcal{C}^*$. Let $N(\mathcal{C})$ be the number of elements of $\mathcal{C}$.

LEMMA 2. *If $\mathcal{C}$ is a nonvoid, coherent collection of point sets having either $2n$ or $2n - 1$ members, for some positive integer n, then $L(\mathcal{C}) \leqslant n$.*

PROOF. The proof is by induction on $N(\mathcal{C})$. The conclusion obviously holds if $N(\mathcal{C}) = 1$ or 2. Assume the conclusion holds for all $\mathcal{C}$ such that $N(\mathcal{C}) < k$ and assume $\mathcal{C}$ is a coherent collection of sets such that $N(\mathcal{C}) = k$. Let n be such that $k = 2n$ or $k = 2n - 1$. Let $h_1 \in \mathcal{C}$ be such that $\mathcal{C} \setminus \{h_1\}$ is coherent. Let h_2 and h_2' be such that $\mathcal{C} \setminus \{h_1, h_2\}$ and $\mathcal{C} \setminus \{h_1, h_2'\}$ are coherent. If either $h_1 \cap h_2$ or $h_1 \cap h_2'$ is void, then rename if necessary so that $h_1 \cap h_2$ is void. In this case or if both intersections are nonvoid, $\mathcal{C}' = \mathcal{C} \setminus \{h_1, h_2\}$ is a coherent collection and each of h_1 and h_2 intersects some member of $\mathcal{C}'$. Now $N(\mathcal{C}') = N(\mathcal{C}) - 2 < k$, so there is a point x in $(\mathcal{C}')^*$ such that $I^m(x, \mathcal{C}') = \mathcal{C}'$, for some $m \leqslant n - 1$. But $I^{m+1}(x, \mathcal{C}) = I(\mathcal{C}', \mathcal{C}) = \mathcal{C}$ and $m + 1 \leqslant n$, so $L(\mathcal{C}) \leqslant m + 1 \leqslant n$.

LEMMA 3. *Let X be a θ-continuum, P be the projection map of X onto a monotone, upper semicontinuous decomposition of X with nowhere dense elements, and $[p, q]$ be an arc in $P(X)$ such that $(p, q) = [p, q] \setminus \{p, q\}$ is open. Then if r and s are two points in* $\text{cl}[P^{-1}((p, q])] \setminus P^{-1}((p, q])$, *then $r \in T(s)$.*

PROOF. Suppose $r \notin T(s)$, i.e., suppose there is a subcontinuum H of X such that $r \in H^\circ \subset H \subset X \setminus \{s\}$. Let z be a point of $P^{-1}((p, q)) \cap H$ such that the z-component Z of $H \cap \text{cl}[P^{-1}((p, q))]$ contains a point of $P^{-1}(p)$. Let $p' = P(z)$ and $M = Z \cup P^{-1}([p', q])$. By construction, $M \not\supset P^{-1}((p, q))$, since $s \notin M$ and $s \in \text{cl}[P^{-}((p, q])]$. This contradicts Lemma 1, so $r \in T(s)$.

LEMMA 4. *If X is a θ-continuum such that $\{\bigcup_{i=1}^{\infty} T^i(x) | x \in X\}$ is an upper semicontinuous decomposition of X with nowhere dense elements onto a locally* *connected θ_n-space, then for each x in X, $\bigcup_{i=1}^{\infty} T^i(x) = T^{2n}(x)$ and there is a y in $T^{2n}(x)$ such that $T^n(y) = T^{2n}(x)$.*

PROOF. Let P be the projection function from X onto the decomposition space, let $x \in X$ and let $p = P(x)$. By [**1**, Theorem 4.9, p. 158] any locally connected θ_n-continuum has order $\leqslant 2n$ at each of its points so there exist points $q_1, \ldots, q_k$ of $P(X) \setminus \{p\}$, for some $k \leqslant 2n$, such that $[p, q_i] \cap [p, q_j] = \{p\}$ unless $i = j$ and $\bigcup_{j=1}^{k}[p, q_i)$ is an open neighborhood of p. For $i = 1, \ldots, k$, let $C_i = \mathrm{cl}[P^{-1}((p, q_i])] \setminus P^{-1}((p, q_i])$ and let $\mathcal{C} = \{C_1, \ldots, C_k\}$. But $\mathcal{C}^* = P^{-1}(p)$, since $P^{-1}(p)$ is nowhere dense and the decomposition is upper semicontinuous. Also, $\mathcal{C}$ is a coherent collection since $P^{-1}(p)$ is connected. Since $N(\mathcal{C}) < 2n$, it is clear that $I^{2n}(x, \mathcal{C}) = \mathcal{C}$. Also, it is clear, from Lemma 3, that $[I^i(z, \mathcal{C})]^* \subset T^i(z)$, for each z in $\mathcal{C}^* = P^{-1}(p)$ and each natural number i. But $T^{2n}(x) \subset \bigcup_{i=1}^{\infty} T^i(x) = P^{-1}(p) = \mathcal{C}^*$ so $T^{2n}(x) = \bigcup_{i=1}^{\infty} T^i(x)$.

It can be seen using Lemma 2 that $L(\mathcal{C}) \leqslant n$, i.e., it can be seen that there is a point y in $\mathcal{C}^*$ such that $I^n(y, \mathcal{C}) = \mathcal{C}$ and hence such that $[I^n(y, \mathcal{C})]^* = P^{-1}(p)$. It then follows that $T^n(y) = T^{2n}(x)$.

The next theorem, the main result of the paper, is an immediate consequence of Theorem 1 and Lemma 4.

THEOREM 2. *Let X be a compact, metric θ_n-continuum. Then X admits a monotone, upper semicontinuous decomposition $\mathfrak{D}$ such that the elements of $\mathfrak{D}$ have void interior and the quotient space $X/\mathfrak{D}$ is a finite graph, if and only if $[T(H)]^\circ = \varnothing$ for every subcontinuum H of X with void interior. Furthermore, $\mathfrak{D} = \{T^{2n}(x) | x \in X\}$.*

To see that no exponent smaller than $2n$ will suffice in Theorem 2, consider the following example.

EXAMPLE 2. For any positive integer n, there is a θ_n-continuum X_n and a point x in X_n such that $T^{2n}(x) \neq T^{2n-1}(x)$ and such that $[T(H)]^\circ = \varnothing$ for every subcontinuum H of X with void interior.

Let $K_1, \ldots, K_{2n}$ be $2n$ disjoint copies of the $\sin 1/x$ continuum, and, for $i = 1, \ldots, 2n$, let a_i and b_i be the end points of the limiting arc in K_i, and let c_i be the other end point of K_i. Let X_n be the union of $K_1, \ldots, K_{2n}$ with the following identifications: b_i and a_{i+1} are identified, for $i = 1, \ldots, 2n - 1$ and c_i and c_{2n+1-i} are identified for $i = 1, \ldots, n$. The unique minimal decomposition of X_n with respect to being upper semicontinuous with T-closed elements yields a space homeomorphic to the union of n circles of different radii that are all tangent at one point. The only nondegenerate "point" in the decomposition space corresponds to the common point of tangency and is the union of the limiting arcs (with identifications). The decomposition space and X_n are readily seen to be θ_n-spaces. Let $x = a_1$, then it is easily seen that $T^i(x) = \bigcup_{j=1}^{i}[a_j, b_j]$, for $i = 1, \ldots, 2n$, where $[a_i, b_i]$ is the projection in X_n of the limiting arc in K_i, and, hence $T^{2n}(x) \neq T^{2n-1}(x)$ (but, of course, $T^{2n+1}(x) = T^{2n}(x)$).

ADDED IN PROOF. Jo Ford has constructed a θ-continuum in R^3 that is not a θ_n-continuum for any n [*On n-ods* (to appear)].

H. S. Davis called θ-continua weakly irreducible continua [*A note on connectedness im kleinen*, Proc. Amer. Math. Soc. **19** (1968), 1237–1241].

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