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DEGENERATIONS OF $K3$ SURFACES OF DEGREE 4

BY

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ABSTRACT. A generic $K3$ surface of degree 4 may be embedded as a nonsingular quartic surface in $\mathbf{P}_3$. Let f: $X \to \operatorname{Spec} \mathbf{C}[[t]]$ be a family of quartic surfaces such that the generic fiber is regular. Let Σ_0, Σ_2^0, Σ_4 be respectively a nonsingular quadric in $\mathbf{P}_3$, a cone in $\mathbf{P}_3$ over a nonsingular conic and a rational, ruled surface in $\mathbf{P}_9$ which has a section with selfintersection -4. We show that there exists a flat, *projective* morphism f': $X' \to \operatorname{Spec} \mathbf{C}[[t]]$ and a map ρ: $\operatorname{Spec} \mathbf{C}[[t]] \to \operatorname{Spec} \mathbf{C}[[t]]$ such that (i) the generic fiber of f' and the generic fiber of the pull-back of f via ρ are isomorphic, (ii) the fiber X'_0 of f' over the closed point of $\operatorname{Spec} \mathbf{C}[[t]]$ has only insignificant limit singularities and (iii) X'_0 is either a quadric surface or a double cover of Σ_0, Σ_2^0 or Σ_4. The theorem is proved using the geometric invariant theory.

The purpose of this paper is to prove projective analog of the Kulikov-Persson-Pinkham theorem [7], [11] via the geometric invariant theory in a special case. We recall that a nonsingular, projective surface, V, over $\mathbf{C}$ is called a $K3$ surface if $H^1(V, o_V) = 0$ and the canonical divisor class of the surface is trivial. It is called a $K3$ surface of degree n if V carries a line bundle L with $L \cdot L = n$. V is said to be generic if the rank of its Néron-Severi group is equal to one. If L is a line bundle on a generic $K3$ surface V such that $L \cdot L = 4$, then, the linear system $|L|$ has no fixed components and embeds V into $\mathbf{P}_3$ as a quartic surface [8]. Conversely, a nonsingular quartic surface is a $K3$ surface of degree 4.

Let S denote $\operatorname{Spec} \mathbf{C}[[t]]$. *A family of surfaces* over S is a flat, projective morphism, $\mathfrak{f}$: $X \to S$ such that the generic geometric fiber of $\mathfrak{f}$ is a nonsingular, connected surface. A family of surfaces, $\mathfrak{f}'$: $X' \to S$ is called a *modification* of the family $\mathfrak{f}$: $X \to S$ if there exists a map ρ: $S \to S$ such that the generic fiber of $\mathfrak{f}'$ and the generic fiber of the pull-back of $\mathfrak{f}$ via ρ are isomorphic. We emphasize that a modification also is a *projective* morphism.

Let Σ_0 = a nonsingular quadric surface in $\mathbf{P}_3$,
Σ_2^0 = a cone over a nonsingular conic in $\mathbf{P}_3$, and
Σ_4 = a rational, ruled surface in $\mathbf{P}_9$ which has a section whose selfintersection is equal to -4.

We prove

THEOREM 1. *Let $\mathfrak{f}$: $X \to S$ be a family of surfaces such that the generic geometric fiber of $\mathfrak{f}$ is isomorphic to a quartic surface. Then, there exists a (projective)*

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modification $\mathfrak{f}'\colon X' \to S$ *such that if* X_0' *is the fiber of* $\mathfrak{f}'$ *over the closed point of* S, *then*
(i) X_0' *is either a quartic surface or a double cover of* Σ_0, Σ_2^0 *or* Σ_4,
(ii) X_0' *has only insignificant limit singularities* **[16]**.

The theorem is easier to prove if one assumes that the generic geometric fiber is already a double cover of Σ_0, Σ_2^0 or Σ_4. The insignificant limit singularities that actually occur are isolated rational double points, simple elliptic singularities, cusp singularities, and nonnormal limits of these singularities (see §1 for explicit description). The theorem is proved using the technique described in **[17]**. It follows from the geometric invariant theory **[9]** that there exists a modification such that the fibers of the new family are semistable quartic surfaces. Moreover, we may assume that the fibers belong to minimal orbits. The trouble with the moduli space of semistable quartics is that it cannot represent $K3$ surfaces which carry a line bundle L such that $L \cdot L = 4$, L is ample, but L is not very ample. If V is such a surface, let $\varphi_L\colon V \to \mathbf{P}_3$ be the map defined by L. We have the following possibilities **[12]**:

(i) $|L|$ has no fixed components. φ_L is generically two-to-one and $\varphi_L(V)$ equals Σ_0 or Σ_2^0.

(ii) $|L|$ has a fixed component, D, which is a nonsingular rational curve. L is isomorphic to $o_V(3C + D)$ where C is a nonsingular elliptic curve. $\varphi_L(V)$ is a twisted cubic curve in $\mathbf{P}_3$.

Let $\mathfrak{g}\colon Y \to S$ be a family of $K3$ surfaces such that $\mathfrak{g}$ is *smooth* and such that its generic fiber is a generic $K3$ surface of degree 4. Let $\mathfrak{L}$ be a line bundle on Y such that $\mathfrak{L}$ induces an ample line bundle of degree 4 on the geometric fibers of $\mathfrak{g}$. Let $\varphi_{\mathfrak{L}}$ be the rational map, $\varphi_{\mathfrak{L}}\colon Y \dashrightarrow \mathbf{P}_3 \times S$, defined by $\mathfrak{L}$. Let Y_0 be the fiber of Y over the closed point of S. Let L_0 be the restriction of $\mathfrak{L}$ to Y_0. Suppose that L_0 is not very ample. If $|L_0|$ has no fixed components, then $\varphi_{\mathfrak{L}}$ is a morphism and the singular fiber of $\varphi_{\mathfrak{L}}(Y)$ equals $2\Sigma_0$ or $2\Sigma_2^0$. If $|L_0|$ has a fixed component, D, then Y must be blown up along D in order to extend $\varphi_{\mathfrak{L}}$ to a morphism $\varphi'_{\mathfrak{L}}\colon Y' \to \mathbf{P}_3 \times S$. $\varphi'_{\mathfrak{L}}(Y')$ has a singular fiber which contains a twisted cubic curve as a cuspidal curve. All of these degenerations except $2\Sigma_2^0$ are semistable. $2\Sigma_2^0$ has a quadruple point and all quartics with a quadruple point are unstable. Therefore, under the action of PGL(4), the family may be further modified so that $2\Sigma_2^0$ is replaced by a semistable quartic with *significant* limit singularities. In proving Theorem 1, we essentially reverse this phenomenon.

If we have a family of semistable quartic surface over S such that the singular fiber has significant limit singularities, we modify the family under the action of a one-parameter subgroup of PGL(4) or PGL(10) so that the singular fiber of the new family equals $2\Sigma_0$, $2\Sigma_2^0$ or $2\Sigma_4$. The singular fiber of the normalization of the family is a two-to-one cover of Σ_0, Σ_2^0 or Σ_4. The key point of the method is the simplification of the singularities of the branch locus of the double cover under the action of the stabilizer group of Σ_0, Σ_2^0 or Σ_4 via the geometric invariant theory. For this, it is essential that the equation of the family be put in a standard form. For a given type of quartic surface, X_0, with significant limit singularities, this amounts to showing the following: (i) Find a minimal subspace, N of $|H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))|$ corresponding to an appropriate subgroup G_0 of the relevant stabilizer group such that N is invariant under G_0 and the map $G \times N \to |H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))|$ is dominant, and (ii) show that any family specializing to X_0 is equivalent under the action of PGL(4) to a family induced by a map $S \to N$. Then the stage is set for applying the geometric invariant theory once more. This technique is applied repeatedly until a family whose fibers have only insignificant limit singularities is obtained.

This work was begun as the author's thesis at M.I.T. **[18]**. A weaker version of Theorem 1, based upon straightforward blowing-up of significant limit singularities was announced in **[19]**.

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1. Terminology. Throughout this paper, we will use the following terminology.

Surface singularities of embedding dimension 3. Let o be a Cohen-Macaulay, local ring of dimension 2 over **C** with embedding dimension = 3. $\hat{o} \approx \mathbf{C}[[u, v, w]]/(f)$. We will need to refer to the following types of singularities.

Insignificant limit singularities.

I. Rational double points: A_n, D_n, E_6, E_7, E_8 **[3]**.

II. A_∞: $f = uv$,

D_∞: $f = u^2 + v^2w$, (a simple pinch point).

III. Simple elliptic (or parabolic) singularities **[13]**:

$$\tilde{E}_6: f = wu^2 + v(v + w)(v + kw), \qquad k \neq 0 \text{ or } 1.$$

$$\tilde{E}_7: f = u^2 + vw(v + w)(v + kw), \qquad k \neq 0 \text{ or } 1.$$

$$\tilde{E}_8: f = u^2 + v(v + w^2)(v + kw^2), \qquad k \neq 0 \text{ or } 1.$$

IV. Cusp (or hyperbolic) singularities **[2]**:

$$T_{p,q,r}: f = kuvw + u^p + v^q + w^r, \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1, \qquad k \neq 0.$$

V. Double pinch points: $f = u^2 + v^2g$ where $g \in \mathbf{C}[[v, w]]$ such that $g = w^2$ (mod v).

Significant limit singularities: **[2]**.

$$E_{12}: f = u^2 + v^3 + w^7 + kvw^5.$$

$$E_{13}: f = u^2 + v^3 + vw^5 + kw^8.$$

$$E_{14}: f = u^2 + v^3 + w^8 + kvw^6.$$

$$J_{3,0}: f = u^2 + v^3 + bv^2w^3 + w^9 + cvw^7, \qquad 4b^3 + 27 \neq 0.$$

$$J_{3,r}: f = u^2 + v^3 + v^2w^3 + (a_0 + a_1w)w^{9+r}, \qquad r > 0, a_0 \neq 0.$$

$$J_{3,\infty}: f = u^2 + v^3 + v^2w^3.$$

$$J_{4,\infty}: f = u^2 + v^3 + v^2w^4.$$

We will also refer to types of double points as follows:

TYPE 0: Rational double points, A_∞ and D_∞.

TYPE 1: $f = u^2 + v^3 + vg + h$ where $g, h \in \mathbf{C}[[w]]$, multiplicity of $g \geqslant 4$, multiplicity of $h \geqslant 6$.

TYPE 2: $f = u^2 + g$ where $g \in \mathbf{C}[[v, w]]$ and multiplicity of $g \geqslant 4$.

The double points of Type 1 and Type 2 may be characterized as follows. In each case, $\operatorname{Proj}(\operatorname{Gr}_m o)$ consists of a single line. Let $Y \to \operatorname{Spec} o$ be the monoidal transformation with the closed point as center. Let e be the exceptional curve in Y. Then, o is of Type 2 if and only if Y is singular along e. o is of Type 1 if and only if Y is nonsingular everywhere along e except at one point which is a double point of Type 2.

Let V be a two-dimensional, Cohen-Macaulay, reduced scheme over $\mathbf{C}$. If V is singular along a curve C, then C is called a *double curve* if, for every generic point x of C, $o_{V,x}$ has multiplicity 2. A double point P of V on C is called a *pinch point* if the projective tangent cone at P consists of a single line. A double curve is called a *nodal curve* if it has only finitely many pinch points. A nodal curve, C, is called *ordinary* (respectively, *quasi-ordinary*) if V has no points on C of multiplicity $\geqslant 3$ and each pinch point on C is a simple (respectively, a simple or a double) pinch point. C is called *strictly quasi-ordinary* if it is quasi-ordinary, but not ordinary. A double curve C is called *cuspidal* if every point of V on C is a pinch point. (A cuspidal curve is a significant limit singularity.) A cuspidal curve C is called *simple* if, for every point P on C, $\hat{o}_{V,P} \approx \mathbf{C}[[u, v, w]]/(u^2 + f)$ where either $f = v^3$ or $f = v^3w$.

A singular point P on a surface V over $\mathbf{C}$ is called a *normal crossing* if $\hat{o}_{V,P} \approx \mathbf{C}[[u, v, w]]/(f)$ where either $f = uv$ or $f = uvw$.

Let V be a projective (possibly singular) surface over $\mathbf{C}$. Let $h^{p,q}(V) =$ the dimension of the (p, q)-component of the mixed Hodge structure **[5]** on $H^2(V, \mathbf{Q})$. Assume that the singularities of V are insignificant limit singularities. Then, V is called *a surface of Type* I if $h^{0,0}(V) = h^{1,0}(V) = h^{0,1}(V) = 0$, (that is, if the mixed Hodge structure on $H^2(V, \mathbf{Q})$ is a pure Hodge structure). V is called *a surface of Type* II if $h^{0,0}(V) = 0$, but, $h^{1,0}(V) \neq 0$. It is called *a surface of Type* III if $h^{0,0}(V) \neq 0$. This classification is motivated by the following. Let A be a nonsingular curve over $\mathbf{C}$. Let s be a closed point of A. Let τ: $S \to A$ be the map which induces an isomorphism $\hat{o}_{A,s} \approx \mathbf{C}[[t]]$. Let $\mathfrak{g}$: $Y \to A$ be a flat, projective morphism such that $Y \times_A S \widetilde{\to} X'$ over S where $X' \to S$ is a family of surfaces as in Theorem 1. Let T be the local Picard-Lefschetz transformation at s **[15]**. We may assume that T is unipotent. Let $N = \ln T$. Let $m = \min\{i\colon N^i = 0\}$. Since X_0' detrmines the dimensions of the (p, q)-components of the limit mixed Hodge structure at s **[16**, Theorem 2],

X_0' is of Type I if and only if $m = 1$,

X_0' is of Type II if and only if $m = 2$,

X_0' is of Type III if and only if $m = 3$.

S will denote Spec $\mathbf{C}[[t]]$ and o will denote its closed point. If $\mathfrak{g}$: $Y \to S$ is a family of surfaces over S, Y_0 will denote the fiber of $\mathfrak{g}$ over o. G will denote the group scheme PGL(4). Let $M = |H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))|$. We consider the canonical action of G on M.

2. Stability of quartic surfaces. We follow the method of computation illustrated in Chapter 4, §2, in **[9]**. Throughout this section, we will use the following notation:

R_0: the ring $\mathbf{C}[x, y, z]$.

R_1: the ring R_0, graded by assigning weights 1, 2, 3 to x, y, z respectively.

R_2: the ring R_0, graded by assigning weights 1, 1, 2 to x, y, z respectively.

$f_i, g_i, h_i, \beta_i, f_i', g_i', \ldots$; homogeneous polynomials of degree i in the variables indicated in parentheses.

$a, b, c, a', a_0, \ldots$: complex numbers.

PROPOSITION 2.1. *A quartic surface V is unstable if and only if V has an affine equation of one of the following forms:*

(i) $f = z^2 + axz^2 + f_3(y, z) + x^2zg_1(y, z) + xg_3(y, z) + g_4(y, z) = 0$. *That is, f is an element of R_1 with the initial form $z^2 + by^3$; V has a double point, P, of Type 1 such that the tangent plane at P makes a 3-fold contact with V and such that P is a significant limit singularity.*

(ii) $f = z^2 + z\{axz + f_2(y, z)\} + a'x^3z + x^2zg_1(y, z) + xg_3(y, z) + g_4(y, z) = 0$. *That is, f is an element of R_2 with the initial form $z^2 + a_0zy^2 + a_1xy^3 + a_2y^4$; V has a double point, P, of Type 2 such that either the tangent plane at P is a component of V or it makes a threefold or fourfold contact with V along a line and such that P is a significant limit singularity.*

(iii) $f = axz^2 + g_3(y, z) + \beta_4(x, y, z) = 0$. *That is, either V has a quadruple point or it has a triple point whose projective tangent cone has a singularity which is not an ordinary double point.*

PROOF. Recall that if a point p in M represents the surface V, then V is unstable if and only if there exists a one-parameter subgroup λ of G such that $\mu_\lambda(p) < 0$ where μ_λ is the numerical function defined on M by λ [9, Chapter 2]. For any one-parameter subgroup, λ, let $M_\lambda^- = \{p \in M: \mu_\lambda(p) < 0\}$. It is enough to determine all the maximal sets M_λ^-. Let λ be a one-parameter subgroup of G. Choose a basis $\{x_0, x_1, x_2, x_3\}$ of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(1))$ so that λ acts via the diagonal matrices:

$$\begin{bmatrix} \alpha^{r_0} & 0 & 0 & 0 \\ 0 & \alpha^{r_1} & 0 & 0 \\ 0 & 0 & \alpha^{r_2} & 0 \\ 0 & 0 & 0 & \alpha^{r_3} \end{bmatrix},$$

$\Sigma\, r_i = 0$ and $r_0 \geqslant r_1 \geqslant r_2 \geqslant r_3$. Let p be a point of M. Let F be the homogeneous form corresponding to p; $F = \Sigma_{|\underline{\gamma}|=4}\, a_{\underline{\gamma}} \underline{x}^{\underline{\gamma}}$ where γ is the multi-index $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ each $\gamma_i \geqslant 0$, $|\gamma| = \Sigma\, \gamma_i$ and $\underline{x}^{\underline{\gamma}} = x_0^{\gamma_0}x_1^{\gamma_1}x_2^{\gamma_2}x_3^{\gamma_3}$. $\bar{F}^{\lambda(\alpha)} = \Sigma\, \alpha^{-(\underline{r}\cdot\underline{\gamma})}a_{\underline{\gamma}}\underline{x}^{\underline{\gamma}}$ where $(\underline{r}\cdot\underline{\gamma}) = \Sigma\, r_i\gamma_i$. $\mu_\lambda(p) = \max\{(\underline{r}\cdot\underline{\gamma})$: all 4-tuples $\underline{\gamma}$ such that $a_{\underline{\gamma}} \neq 0\}$. Thus, if $a_{\underline{\gamma}} \neq 0$ in F and $\mu_\lambda(p) < 0$, then the 4-tuple $\underline{r} = (r_0, r_1, r_2, r_3)$ must satisfy the linear inequality $(\underline{r}\cdot\underline{\gamma}) < 0$. The maximal sets M_λ^- are determined by inspection of these inequalities. There are exactly three such sets. The controlling inequalities correspond to the multi-indices $\underline{\gamma} = (2, 0, 0, 2)$, $\underline{\gamma}' = (1, 0, 3, 0)$ and $\underline{\gamma}'' = (0, 3, 0, 1)$. The three cases are as follows where we have parametrized the sets M_λ^- by quartic polynomials in the variables $x = x_1/x_0, y = x_2/x_0, z = x_3/x_0$.

(1) λ satisfies $(\underline{r}\cdot\underline{\gamma}) < 0$ and $(\underline{r}\cdot\underline{\gamma}') < 0$, e.g. $\underline{r} = (5, 4, -3, -6)$.

$$M_\lambda^-: bz^2 + axz^2 + f_3(y, z) + x^2zg_1(y, z) + xg_3(y, z) + g_4(y, z).$$

(2) λ satisfies $(\underline{r}\cdot\underline{\gamma}) < 0$ and $(\underline{r}\cdot\underline{\gamma}) < 0$; e.g. $\underline{r} = (6, 2, -1, -7)$.

M_λ^-: $bz^2 + z\{axz + f_2(y, z)\} + a'x^3z + x^2zg_1(y, z) + xg_3(y, z) + g_4(y, z)$.

(3) λ satisfies $(\underline{r}\cdot\underline{\gamma}) > 0$; e.g. $\underline{r} = (8, -1, -3, -4)$.

$$M_\lambda^-: axz^2 + g_3(y, z) + \beta_4(x, y, z).$$

Note that if the coefficient of y^3 is zero in case (1), the case degenerates into case (2) and if the coefficient of z^2 is zero, then cases (1) and (2) degenerate into case (3). Q.E.D.

Let M^{ss} = the open subset of M consisting of semistable points. Let $\mathfrak{h}$: $M^{ss} \to \mathfrak{m}$ be the universal categorical quotient of M^{ss} by G. Recall that the fiber of $\mathfrak{h}$ over any closed point of $\mathfrak{m}$ contains a unique minimal orbit which lies in the closure of every orbit in the fiber. A closed point p in M^{ss} is stable if and only if $\mathfrak{h}^{-1}(\mathfrak{h}(p))$ consists of the minimal orbit. If O and O' are two orbits in M^{ss} such that $O' \subset$ the closure $\overline{O}$ of O in M^{ss}, then, there exists a one-parameter subgroup, λ: Spec $\mathbf{C}[\alpha, \alpha^{-1}] \to G$, a point p in O and a point p' in O' such that $\lim_{\alpha\to 0} p^{\lambda(\alpha)} = p'$ and $\mu_\lambda(p) = \mu_\lambda(p') = 0$.

For each one-parameter subgroup, λ, let $M_\lambda = \{p \in M: \mu_\lambda(p) = 0\}$, $\overline{M}_\lambda$ = the points in M_λ which are fixed under the action of λ, $M_\lambda^{ss} = M_\lambda \cap M^{ss}$, $\overline{M}_\lambda^{ss} = \overline{M}_\lambda \cap M^{ss}$. If $p \in M_\lambda$, then $\lim_{\alpha\to 0} p^{\lambda(\alpha)} \in \overline{M}_\lambda$. A point p does not belong to a minimal orbit in M^{ss} if and only if there exists a one-parameter subgroup λ such that $p \in M_\lambda^{ss} - \overline{M}_\lambda^{ss}$ and such that p and $\lim_{\alpha\to 0} p^{\lambda(\alpha)}$ do not belong to the same orbit. If we partially order the sets M_λ^{ss} by the relation $M_{\lambda_1}^{ss} > M_{\lambda_2}^{ss}$ if and only if $M_{\lambda_1}^{ss} \supset M_{\lambda_2}^{ss}$ and for every point $p \in \overline{M}_{\lambda_2}^{ss}$, the closure of the orbit of p in M^{ss} contains a point of $\overline{M}_{\lambda_1}^{ss}$, then, in order to determine the minimal orbits in M^{ss}, it is enough to determine all the maximal sets M_λ^{ss}. If $a_{\underline{\gamma}} \neq 0$ for a generic member of M_λ, then $(\underline{r}\cdot\underline{\gamma}) \leqslant 0$. By inspecting these inequalities, we get

Proposition 2.2. *Let λ be a one-parameter subgroup of G such that M_λ^{ss} is maximal. Assume that λ is diagonalized as in the proof of Proposition* 2.1. *Then, we have the following cases where we have parametrized M_λ^{ss} and $\overline{M}_\lambda^{ss}$ by quartic polynomials f and $\bar{f}$ respectively*:

(1) $r_1 + r_2 = r_0 + r_3 = 0$; $\underline{r} = (n, m, -m, -n)$, $1 \geqslant m/n \geqslant 0$.

(1.1) $m/n = 1/3$ *or* $r_0 + 3r_2 = 0$.

$$f = a_1z^2 + xzf_1(y, z) + f_3(y, z) + a_5x^3z + x^2g_2(y, z) + xg_3(y, z) + g_4(y, z)$$
$$= a_1z^2 + a_2y^3 + a_3xyz + a_4x^2y^2 + a_5x^3z + \textit{terms of weight} > 6 \textit{ in } R_1.$$

Either $a_1a_2a_5 = 0$ and the quartic belongs to one of the cases below or else, $a_1a_2a_5 \neq 0$ and the quartic contains the line $y = z = 0$ and has a double point of Type 1 *at the origin.*

$$\bar{f} = a_1z^2 + a_2y^3 + a_3xyz + a_4x^2y^2 + a_5x^3z.$$

If $a_1a_2a_5 \neq 0$, the quartic contains two lines, $x_2 = x_3 = 0$ and $x_0 = x_1 = 0$ and has double points of Type 1 *at the points $x_1 = x_2 = x_3 = 0$ and $x_0 = x_1 = x_2 = 0$.*

(1.2) $m/n = 0$.

$$f = az^2 + z\beta_2(x, y, z) + \beta_4(x, y, z),$$

that is, f consists of elements of weight $\geqslant 4$ in R_2. If $a = 0$, the quartic belongs to one of the cases below. If $a \neq 0$, the quartic has a double point of Type 2 at the origin.

$$\bar{f} = az^2 + zg_2(x, y) + g_4(x, y),$$

that is, $\bar{f}$ consists of elements of weight $= 4$ in R_2. If $a \neq 0$, the quartic has double points of Type 2 at the points $x_1 = x_2 = x_3 = 0$ and $x_0 = x_1 = x_2 = 0$.

(1.3) $m/n = 1$.

$$f = f_2(y, z) + xg_2(y, z) + x^2h_2(y, z) + g_3(y, z) + xh_3(y, z) + h_4(y, z).$$

M_λ^{ss} consists of the quartics which are singular along the line $y = z = 0$.

$$\bar{f} = f_2(y, z) + xg_2(y, z) + x^2h_2(y, z).$$

$\overline{M_\lambda^{ss}}$ consists of the quartics which are singular along the lines $x_2 = x_3 = 0$ and $x_0 = x_1 = 0$.

(1.4) $1 > m/n > 1/3$.

$$f = a_1z^2 + a_2y^3 + a_3xyz + a_4x^2y^2 + \text{terms of weight} > 6 \text{ in } R_1$$

$$= a_1z^2 + xzg_1(y, z) + x^2h_2(y, z) + g_3(y, z) + xh_3(y, z) + h_4(y, z).$$

If $a \neq 0$ the quartic has a nodal line, $y = z = 0$, which has a double pinch point at the origin.

$$\bar{f} = a_1z^2 + a_3xyz + a_4x^2y^2.$$

If $a_1a_4 \neq 0$, the quartic is either a nonsingular quadric with multiplicity two or the union of two distinct nonsingular quadrics which intersect in the four lines, $\{x_1x_2 = 0, x_0x_3 = 0\}$.

(1.5) $1/3 > m/n > 0$.

$$f = a_1z^2 + yzg_1(x, y) + y^2g_2(x, y) + \text{terms of weight} > 4 \text{ in } R_2$$

$$= a_1z^2 + a_2xyz + a_3x^2y^2 + a_5x^3z + \text{terms of weight} > 6 \text{ in } R_2.$$

M_λ^{ss} is a subset of the sets in cases (1.1) and (1.3). Let λ_1 be a one-parameter subgroup of G with $m/n = 0$ and let λ_2 be a one-parameter subgroup with $m/n = 1/3$. Then, a point p in $M_{\lambda_1}^{ss}$ belongs to case (1.5) if and only if $\lim_{\alpha\to 0} p^{\lambda_1(\alpha)}$ corresponds to a quartic which is singular along the line $x_0 = x_1 = 0$. A point p in $M_{\lambda_2}^{ss}$ belongs to case (1.5) if and only if $\lim_{\alpha\to 0} p^{\lambda_2(\alpha)}$ corresponds to a quartic which is singular along the line $x_2 = x_3 = 0$.

$$\bar{f} = a_1z^2 + a_3xyz + a_4x^2y^2.$$

(2) $r_1 + r_2 > 0$ *and* $r_1 = r_2$; $\underline{r} = (n, m, m, -n - 2m)$, $1 \geqslant m/n > 0$.

(2.1) $m/n = 1$.

$$f = z\{a + \beta_1(x, y, z) + \beta_2(x, y, z) + \beta_3(x, y, z)\}.$$

Each quartic in M_λ^{ss} is the union of a cubic surface and a plane not contained in the cubic surface.

$$\bar{f} = z\{a + h_1(x, y) + h_2(x, y) + h_3(x, y)\}.$$

Each quartic in $\overline{M_\lambda^{ss}}$ is the union of a plane and a cone over a cubic curve in the plane.

(2.2) $1 > m/n > 0$.

$$f = z\{az + \beta_2(x, y, z) + \beta_3(x, y, z)\}.$$

Each quartic in M_λ^{ss} is the union of a plane and a cubic surface such that their intersection is a cubic curve with a double point.

$$\bar{f} = zh_2(x, y).$$

Each quartic in $\overline{M}_\lambda^{ss}$ consists of four planes.

(3) $r_1 + r_2 > 0$, $r_1 > r_2$. *M_λ^{ss} is maximal if and only if either $r_0 = r_1$ and $r_0 + 3r_2 < 0$ or $3r_1 + r_3 < 0$ and $r_2 < 0$.*

(3.1) $r_0 = r_1$, $r_1 + r_2 > 0$, $r_1 > r_2$, $r_0 + 3r_2 < 0$. $\underline{r} = (n, n, m, -2n - m)$ *where* $-1/3 > m/n > -1$.

$$f = z\{f_1(y, z) + xg_1(y, z) + x^2h_1(y, z)\} + g_3(y, z) + xh_3(y, z) + h_4(y, z).$$

Each quartic in M_λ^{ss} is singular along the line $y = z = 0$ such that either the plane $z = 0$ is a component of the quartic or the plane makes a 3-fold or 4-fold contact with the quartic.

$$\bar{f} = yz(a_1 + a_2x + a_3x^2).$$

Each quartic in $\overline{M}_\lambda^{ss}$ consists of four planes.

(3.2) $r_0 + r_2 > 0$, $r_1 > r_2$, $r_2 < 0$, $3r_1 + r_3 < 0$; *e.g.* $\underline{r} = (6, 2, -1, -7)$.

$$f = a_1z^2 + a_3xyz + a_5x^3z + \text{terms of weight} > 6 \text{ in } R_1.$$

M_λ^{ss} is a subset of the set in case (1.5). *Let λ_1 be a one-parameter subgroup of G corresponding to case* (1.5). *A point p in $M_{\lambda_1}^{ss}$ belongs to case* (3.2) *if and only if* $\lim_{\alpha\to 0} p^{\lambda_1(\alpha)}$ *belongs to case* (3.1).

$\bar{f} = xyz$ *(four planes)*.

(4) $r_1 + r_2 < 0$, $r_2 = r_3$. $\underline{r} = (3n - m, -n + m, -n, -n)$, $2 > m/n \geq 0$.

(4.1) $m/n = 0$.

$$f = \beta_3(x, y, z) + \beta_4(x, y, z).$$

M_λ^{ss} consists of quartics with a triple point.

$$\bar{f} = \beta_3(x, y, z).$$

Each quartic in $\overline{M}_\lambda^{ss}$ is the union of a plane and a cone over a cubic curve in the plane.

(4.2) $2 > m/n > 0$.

$$f = xg_2(y, z) + g_3(y, z) + \beta_4(x, y, z).$$

Each quartic has a triple point whose projective tangent cone is singular.

$$\bar{f} = xg_2(y, z) \quad (\textit{four planes}).$$

(5) $r_1 + r_2 < 0$, $r_2 > r_3$. *Each quartic in this case also belongs to case* (4.2).

COROLLARY 2.3. *A quartic surface V is not stable if and only if V has either*

(i) *an isolated, nonrational, double point of Type* 1 *through which passes a line contained in V, or*

(ii) *an isolated, nonrational, double point of Type* 2, *or*

(iii) *a double line, or*

(iv) *a nodal curve with a pinch point through which passes a line contained in V, or*

(v) *a plane as a component, or*

(vi) *a point of multiplicity* ≥ 3.

The following theorem describes the semistable quartics in more geometric detail.

THEOREM 2.4. *Let V be quartic surface. Let $\Delta =$ the singular locus of V.*

A. *V is stable if and only if V is one of the following surfaces.*

Type I: *Δ is empty or consists of rational double points.*

Type II: (i) *Δ consists of a double point P of type $\tilde{E}_8$ and some rational double points such that no line in V passes through P.*

(ii) *Δ consists of an ordinary nodal curve, C, and some rational double points. Either V is irreducible and C is a nonsingular curve of degree* 2 *or* 3 *with four simple pinch points or V consists of two quadric surfaces which intersect transversely along a nonsingular elliptic curve of degree* 4.

Type III: (i) *Δ consists of a double point, P, of type $T_{2,3,r}$ and some rational double points such that no line in V passes through P.*

(ii) *Δ consists of a strictly quasi-ordinary nodal curve, C, and some rational double points such that no line in V passes through a double pinch point. C is a nonsingular, rational curve of degree* 2. *V has either two double pinch points on C or one double pinch point and two simple pinch points on C.*

Surfaces with significant limit singularities: (i) *Δ consists of a double point, P, of type E_{12}, E_{13}, E_{14} or $J_{3,r}$ and some rational double points such that no line in V passes through P.*

(ii) *Δ consists of a nodal curve, C, and rational double points such that no line in V passes through a nonsimple pinch point. C is a nonsingular, rational curve of degree* 2. *Every point of V on C is a double point and the set of pinch points consists of either a point of type $J_{3,\infty}$ and a simple pinch point or a point of type $J_{4,\infty}$.*

B. *V is strictly semistable and belongs to a minimal orbit if and only if V is one of the following surfaces.*

Type II: (i) *Either Δ consists of two double points of type $\tilde{E}_8$ or it consists of two double points of type $\tilde{E}_7$ and some rational double points.*

(ii) *Δ consists of two skew lines, each of which is an ordinary nodal curve with four simple pinch points.*

(iii) *V consists of a plane and a cone over a nonsingular cubic curve in the plane.*

Type III: (i) *Δ consists of a nonsingular, rational curve of degree* 2 *or* 3, *and some rational double points. C is a strictly quasi-ordinary, nodal curve and the set of pinch points consists of two double pinch points. Each double pinch point lies on a line in V.*

(ii) *V consists of two, nonsingular, quadric surfaces which intersect in a reduced curve, C, of arithmetic genus* 1. *C consists of two or four lines such that its singularities consist of* 2 *or* 4 *ordinary double points*; *the dual graph of C is homeomorphic to a circle.*

(iii) *V consists of four planes with normal crossings.*

Surfaces with significant limit singularities: (i) *Δ consists of a nonsingular, rational curve, C, of degree* 2 *or* 3; *C is a simple cuspidal curve. The normalization of V has exactly two rational double points if C is of degree* 2; *it is nonsingular otherwise.*

(ii) *V consists of two quadric surfaces, V_1 and V_2, tangent to each other along a nonsingular, rational curve of degree* 2 *such that $V_1 \cap V_2 = 2C$.*

(iii) *V consists of a nonsingular, quadric surface with multiplicity equal to* 2.

PROOF. We first describe the representation of quartics with a double point as double planes. Let P be a double point on a reduced quartic, V, which contains only finitely many lines through P. Let V' be the monoidal transformation of V with center P. Let $\pi: V' \to \mathbf{P}_2$ be the morphism defined by the projection of V from P. Let

$$\begin{array}{ccc} V' & \xrightarrow{\varnothing} & V^* \\ & \pi \searrow \quad \swarrow \pi^* & \\ & \mathbf{P}_2 & \end{array}$$

be the Stein factorization. Clearly, V^* is reduced. Suppose that V^* is irreducible. Let $\tilde{V}^*$ be the normalization of V^*. The canonical map, $\tilde{V}^* \to \mathbf{P}_2$ is flat [**1**, Proposition V-3.5]. It follows that π^* must be flat. Similar argument shows that the same conclusion holds when V^* is not irreducible. Thus, V^* is a double plane, ramified over a plane curve, Ω. If V is defined by the affine equation $\beta_2(x, y, z) + \beta_3(x, y, z) + \beta_4(x, y, z) = 0$ such that P is the origin, then Ω is defined by the equation, $\beta_3^2 - \beta_2\beta_4 = 0$. The map π contracts the proper transform of the lines in V through P and is an isomorphism everywhere else. Let E be the exceptional curve in V'. Let e be the (reduced) image of E in $\mathbf{P}_2$; $e =$ the algebraic set defined by the equation $\beta_2 = 0$.

We consider now the stable quartics. Corollary 2.3 gives us the following cases.

S-1. V is nonsingular.

S-2. Δ consists of isolated, rational, double points.

S-3. Δ consists of isolated, rational, double points and an ordinary nodal curve, C. Since V is stable, C cannot have a line as a component. If V is irreducible, then the degree of C must be less than 4 since the generic plane section of V is then an irreducible plane quartic curve and such a curve cannot have more than 3 double points. If degree$(C) = 3$, then C cannot be a planar curve since, otherwise, the plane containing C would intersect V in a curve of degree > 4. Thus, C must be a nonsingular, rational curve of degree 2 or 3. In general, if W is an irreducible, reduced, surface of degree n in $\mathbf{P}_3$, and if the singular locus of W consists of a nonsingular curve, D, and some isolated rational double points, such that D is an ordinary nodal curve, we have the formula [**10**]

$$\gamma_i = 2(n-4)d_i - 4g_{D_i} + 4$$

where $\gamma_i =$ the number of pinch points on a connected component D_i of D, $d_i =$ the degree of the connected component D_i of D, $g_{D_i} = \dim(H^1(D_i, o_{D_i}))$. The formula is proved as follows. From Grothendieck's duality theory, it follows that if $\tilde{W}$ is the normalization of W and D', h' are inverse images of D and a generic plane h in $\mathbf{P}_3$, respectively, then, $(n-4)h' - D'$ is a canonical divisor on $\tilde{W}$. Now apply the formula

$$D_i'(D_i' + K_{\tilde{W}}) = 2g_{D_i'} - 2 = 2(2g_{D_i} - 2) + \gamma_i.$$

It follows that, in our case, C has four simple pinch points. The normalization of V^* is either a quartic or quadric double plane and hence, a rational surface.

If V is reducible, then, since it can have neither a triple point nor a plane as a component, it must consist of two irreducible quadric surfaces, intersecting in a nonsingular curve, C. C must, therefore, be a nonsingular connected curve of degree 4 and hence, an elliptic curve.

S-4. Suppose now that V has a double point, P, of Type 1 such that no line in V passes through P so that $V' = V^*$. We may assume that $\beta_2 = z^2$ so that Ω has the homogeneous equation of the form $\beta_3^2 - z^2\beta_4 = 0$. e is defined by the equation $z = 0$. Therefore, e is not a component of Ω. Every point in $e \cap \Omega$ is a singular point of Ω. Since V' has exactly one singular point on E and that point is a double point of Type 2, $e \cap \Omega$ consists of a single point, p, with multiplicity equal to 6 and p is a quadruple point of Ω. It follows that e must be a component of the tangent cone of Ω at p. Choose coordinates so that p has the coordinates, $y = z = 0$ and $x = 1$. Then, the coefficient of x^3 in β_3 must be zero. Since no line in V passes through P, the coefficient of x^4 in β_4 cannot be zero and may be assumed to be 1. Since z does not divide β_3, the coefficient of y^3 in β_3 may be assumed to equal 1. The equation of the tangent cone at p must have the form

$$z\left(y^3 + a_1y^2z + a_2yz^2 + a_3z^3\right) = 0.$$

Choose y so that $a_3 = 0$. Then, by comparing the coefficients, it is easily checked that V is defined by an affine equation of the form

$$\{z + h_2(x,y) + z\beta_1'(x,y,z)\}^2 + y^3 + x^2yg_1(y,z) + xg_3(y,z) + g_4(y,z) = 0$$

where $h_2(x,y) = x^2 + bxy + ay^2$. By replacing x_0 by $x_0 - \beta_1'(x_1, x_2, x_3)$, we may assume that $\beta_1' = 0$. Finally, by replacing x by $x - (b/2)y$, we may assume that $b = 0$. Thus, the affine equation of V takes the form

$$f = \{z + x^2 + ay^2\}^2 + y^3(1 + \beta_1(x,y,z)) + y^2f_2'(x,z) + yzf_2''(x,z) + z^3f_1'(x,z) = 0.$$

Since V is stable, f_2', f_2'' and f_1' cannot all be identically zero. If they were, V would have a double point of Type 2 at a point in the locus of $x_3 + x_1^2 = x_2 = x_0 + \beta_1(x_1, x_2, x_3) = 0$. The tangent cone of Ω at p has the equation

$$yz(y^2 - yzf_2'(1,0) - z^2f_2''(1,0)) = 0.$$

The next task is to classify the singularities that V can have at P. Let $Z = z + x^2 + ay^2$. Consider the ring $\hat{R} = \mathbf{C}[[x,y,z]]$, filtered by assigning weights w_1, w_2, w_3, to x, y, Z, respectively, as follows. The weights satisfy the inequalities $w_2 \geqslant 2w_1$ and $w_3 \geqslant 3w_1$. Let d_6, d_3, d_2 denote the weights of the initial forms in $\hat{R}$ of f_2', zf_2'' and z^3f_1'. Note that the initial form of z is x^2 and $2 \leqslant d_6/w_1 \leqslant 4$, $4 \leqslant d_3/w_1 \leqslant 6$, $7 \leqslant d_2/w_1 \leqslant 8$. Let $d = \min_i\{id_i\}$. Let $w_1 =$ the smallest positive integer such that 6 divides d. Let $w_2 = d/3$ and $w_3 = d/2$. Then the initial form of f in $\hat{R}$ is

$$\bar{f} = Z^2 + y^3 + a_6x^{n_6}y^2 + a_3x^{n_3}y + a_2x^{n_2}$$

where $a_i = 0$ if iw_1 does not divide d and $n_i = d/iw_1$ if iw_1 divides d. Note that $2 \leqslant n_6 \leqslant 4$, $4 \leqslant n_3 \leqslant 6$ and $7 \leqslant n_2 \leqslant 8$. If the discriminant δ of $y^3 + a_6x^{n_6}y^2 + a_3x^{n_3}y + a_2x^{n_2}$ is not identically zero, then, δ is homogeneous of weight d.

V is normal if and only if Ω is reduced. Suppose that Ω is reduced. Then, the singularities of Ω consist of the quadruple point p, triple points which have at least two distinct tangents and double points. Therefore, Δ consists of the point P and some rational double points. Suppose that Ω is not reduced. Ω cannot equal $2B$ where B is a cubic curve since Ω has a quadruple point with a simple tangent. Ω cannot have a nonsingular conic as a component with multiplicity two. If it did, Ω would equal $2B_1 + B_2$ where B_1 is nonsingular. But, since p is a quadruple point, B_2 must consist of two lines passing through p, one of which then must be e. Contradiction! Ω cannot have a line with multiplicity three since, if $\Omega = 3L + B$ where L is a line and B is a cubic, then, V would have double points of Type 2 above $L \cap B$. Thus, $\Omega = 2L + B$ where L is a line, B is a quartic and $L \not\subset B$. B cannot have a quadruple point since then, Ω would consist of lines and would contain e. Hence $p \in L$. If Ω is not reduced, choose y so that L is the line $y = 0$; then, f_2'' and f_1' are identically zero. V has a nodal curve which is a nonsingular, plane curve of degree 2 and V is defined by an affine equation of the form

$$f = (z + x^2 + ay^2)^2 + y^2(y + y\beta_1'(x, y, z) + f_2'(x, z)) = 0.$$

The pinch points are given by the equations

$$x_2 = x_0x_3 + x_1^2 = f_2'(x_1, x_3) = 0.$$

It is now easy to verify that we have the following possibilities.

S-4.1. z does not divide f_2'', $d = 12$ and $\delta \neq 0$, $w_1 = 1$, $n_6 = 2$, $n_3 = 4$, $a_2 = 0$. There are four distinct tangents at p. P is of type $\tilde{E}_8$. V is a rational surface.

S-4.2. $d = 12$ and $\delta = 0$. There are exactly three distinct tangents at p. Choosing y so that the line $y = 0$ is a double tangent at p, we may assume that z divides f_2'', but not f_2'. Then, $n_6 = 2$, $a_3 = a_2 = 0$.

S-4.2.1. P is an isolated singularity. Then, P is a cusp singularity of type $T_{2,3,r}$ and V is a rational surface.

S-4.2.2. $\Omega = 2L + B$.

S-4.2.2(a). $f_2'(x, z)$ has distinct factors. $L \cap B = 2p + q_1 + q_2$ where q_1 and q_2 are distinct points. The nodal curve in V has one double pinch point and two simple pinch points. V is a rational surface.

S-4.2.2(b). f_2' is a perfect square. $L \cap B = 2p + 2q$. The nodal curve in V has two double pinch points. V is a rational surface.

S-4.3. $z|f_2'$ and $z|f_2''$. There are exactly two distinct tangents at p. V has a significant limit singularity at P. The affine equation of V has the form

$$f = \{z + x^2 + ay^2\}^2 + y^3\{1 + \beta_1(x, y, z)\} + z\{y^2 f_1(x, z) + yzg_1(x, z) + z^2 h_1(x, z)\} = 0.$$

S-4.3.1. $z \nmid h_1$; $n_2 = 7$, $a_6 = a_3 = 0$. P is of type E_{12}.

S-4.3.2. $z|h_1$, $z \nmid g_1$; $n_3 = 5$, $a_6 = a_2 = 0$. P is of type E_{13}.

S-4.3.3. $z|g_1$ and $h_1 = cz \neq 0$; $n_2 = 8$, $a_6 = a_3 = 0$. P is of type E_{14}.

S-4.3.4. $h_1 = 0$ and $g_1 = bz \neq 0$; $n_6 = 3$, $n_3 = 6$, $a_2 = 0$. P is of type $J_{3,r}$.

S-4.3.5. $h_1 = g_1 = 0$ and $z \nmid f_1$; $n_6 = 3$, $a_3 = a_2 = 0$. V has a nodal curve through P and P is of type $J_{3,\infty}$.

S-4.3.6. $h_1 = g_1 = 0$ and $f_1 = a'z \neq 0$; $n_6 = 4$, $a_3 = a_2 = 0$. V has a nodal curve through P and P is of type $J_{4,\infty}$.

We now turn to strictly semistable quartics. Let V be a quartic surface which is strictly semistable and belongs to a minimal orbit. V is defined by one of the equations, $\bar{f} = 0$, of Proposition 2.2.

SS-1. $\bar{f} = z^2 + y^3 + a_3xyz + a_4x^2y^2 + a_5x^3z$.
The quartic has two points of Type 1, one at P with coordinates $x_1 = x_2 = x_3 = 0$ and the other at P' with coordinates $x_0 = x_1 = x_2 = 0$. Ω consists of three (not necessarily distinct), nonsingular conics with consecutive triple points at p with coordinates $y = z = 0$ and at p' with coordinates $y = x = 0$. There is a single line, l in V through P, mapping onto a point p_1 in $\mathbf{P}_2$ and a single line, l' in V through P'. l and l' do not intersect. Clearly, V is nonsingular everywhere along l and l' except at P and P'.

SS-1.1. Ω consists of 3 distinct, nonsingular conics which are mutually tangent at p and p'. Δ consists of the points P and P' which are of type $\tilde{E}_8$. V is birationally a ruled variety with the base curve of genus 1.

SS-1.2. $\Omega = 2B + B'$ where B and B' are nonsingular conics, mutually tangent at p and p'. Δ consists of a nonsingular, rational curve of degree 3 which is a strictly quasi-ordinary nodal curve with two double pinch points. V is a rational surface.

SS-1.3. $\Omega = 3B$ where B is a nonsingular conic. Δ consists of a nonsingular, rational curve, C, of degree 3 which is a simple cuspidal curve. The normalization of V is nonsingular. Choose coordinates so that B is defined by the equation $xz - y^2 = 0$. Then, V is defined by the affine equation

$$z^2 + 4y^3 - 6xyz - 3x^2y^2 + 4x^3z = 0$$

and C is defined by the equations $y = x^2$ and $z = x^3$.

SS-2. $\bar{f} = z^2 + 2zh_2(x, y) + h_4(x, y) = \{z + h_2(x, y)\}^2 + h_4(x, y) - h_2^2(x, y)$.
The quartic has two points of Type 2, one at P with coordinates $x_1 = x_2 = x_3 = 0$ and the other at P' with coordinates $x_0 = x_1 = x_2 = 0$. $y|h_2$ and $y^2|h_4$ if and only if the quartic has a double line, $x_2 = x_3 = 0$. If the quartic has the double line, $x_2 = x_3 = 0$, it also has the double line, $x_0 = x_2 = 0$. If the quartic has a double line, it belongs to one of the cases SS-i, $i \geqslant 3$. We assume then that the quartic does not have a double line. The quartic has four lines, l_1, l_2, l_3, l_4 through P which intersect the four lines l_1', l_2', l_3', l_4' through P' at points defined by the equations $x_0 = x_3 = h_4(x_1, x_2) = 0$. It is easily checked that V has at most a rational double point at such a point and that V is nonsingular along these lines except at their intersections. Ω has the equation $z^2(h_4 - h_2^2) = 0$ so that $\Omega = 2e + B$ where B is a quartic cone. The lines l_i are mapped onto the points defined by $z = h_4 = 0$.

SS-2.1. B has no multiple component. Δ consists of the points P and P' which are of type $\tilde{E}_7$ and some rational double points. V is birationally a ruled variety with the base curve of genus 1.

SS-2.2. B has three distinct components. Δ consists of some rational double points and a nonsingular rational curve of degree 2 which is a strictly quasi-ordinary nodal curve with two double pinch points. V is a rational surface.

SS-2.3. $B = 2L_1 + 2L_2$ where L_1 and L_2 are the lines $x = 0$ and $y = 0$. Therefore, $x \nmid h_2$ and $y \nmid h_2$ and $\bar{f}$ factors as $(z + x^2 + y^2 + a_1xy)(z + x^2 + y^2 + a_2xy)$ where $a_1 \neq a_2$. V consists of two quadric surfaces intersecting in a curve C. C consists of two nonsingular conics and the singularities of C consist of two ordinary double points.

SS-2.4. $B = 3L_1 + L_2$ where L_1 is the line $y = 0$ and $L_1 \neq L_2$. Δ consists of some rational double points and a nonsingular rational curve of degree 2 which is a simple cuspidal curve. V is a rational surface and its normalization has two rational double points over the cuspidal curve. The affine equation of V has the form

$$\{z + x^2 + ay^2\}^2 + y^3h_1(x, y) = 0 \quad \text{where } y \nmid h_1.$$

SS-2.5. $B = 4L$. V is defined by an equation of the form

$$\{z + x^2 + ay^2\}^2 + y^4 = 0.$$

V consists of two quadric surfaces, V_1 and V_2, which are tangent to each other along a nonsingular plane curve, C, of degree 2. $V_1 \cap V_2 = 2C$.

SS-3. $\bar{f} = z^2 + xg_2(y, z) + x^2h_2(y, z)$.

The quartic has double lines, $x_2 = x_3 = 0$ and $x_0 = x_1 = 0$. V has a double pinch point at the point $x = y = z = 0$ if and only if $z | g_2$. Therefore, if $z | g_2$, V belongs to one of the cases that follow. If $z \nmid g_2$, then the double lines are ordinary nodal curves, each with four simple pinch points. V is birationally a ruled surface with an elliptic curve as its base curve.

SS-4. $\bar{f} = (z + xy)(z + axy)$, $a \neq 0$.

SS-4.1. $a \neq 1$. V consists of two nonsingular, quadric surfaces, intersecting in 4 lines.

SS-4.2. $a = 1$. V consists of a nonsingular, quadric surface with multiplicity two.

SS-5. V consists of a plane and a cone over a nonsingular cubic curve in the plane.

SS-6. V consists of 4 planes with normal crossings.

This finishes the geometric description. The mixed Hodge structure of a quartic with insignificant limit singularities is computed via its dual complex as in **[17**, §3].

3. Double covers of Σ_4. Let $\mathfrak{f}$: $X \to S$ be a family of quartic surfaces such that the singular locus of X_0 consists of a twisted cubic curve, C, which is a simple cuspidal curve in X_0. We will show that there exists a modification $\mathfrak{f}^*$: $X^* \to S$ such that X_0^* is a double cover of Σ_4 and has only insignificant limit singularities.

From inspection of the affine equation of X_0, it can be immediately seen that X_0 contains all lines which are tangent to C and thus, X_0 is traced out by the tangent lines of C.

STEP I. We embed the family in $\mathbf{P}_9 \times S$ and deform it under the action of a one-parameter subgroup of PGL(10) such that we get a family whose singular fiber equals $2\Sigma_4$.

Let $\mathbf{A} = H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(2))$. Let ι: $\mathbf{P}_3 \to \mathbf{P}_9$ be the embedding defined by the linear system $\mathbf{A}$. Let $W = \iota(\mathbf{P}_3)$.

LEMMA 3.1. *W is projectively Gorenstein.*

PROOF.[2] The homogeneous coordinate ring of W is isomorphic with the ring of invariants in $\mathbf{C}[x_0, x_1, x_2, x_3]$ under the involution which sends x_i to $-x_i$. Therefore, W is projectively Gorenstein by a theorem of Watanabe **[20]**, **[21]**. Q.E.D.

Next, we describe the defining equations of W. Let G_0 be the stabilizer of the twisted cubic, C, in PGL(4). G_0 acts on $\mathbf{A}$. We have a unique, G_0-invariant decomposition $\mathbf{A} \approx \mathbf{A}_3 \oplus \mathbf{A}_7$ where $\mathbf{A}_3$ consists of the quadrics vanishing on C. For $i = 2$ and 6, let L_i be the G_0-invariant subspace in $\mathbf{P}_9$, defined by vanishing of the elements of $\mathbf{A}_{9-i}$. Let $D = L_6 \cap W$. D is an embedding of C in $\mathbf{P}_9$ as a sextic curve, defined by the linear system $\mathbf{A}_7$, restricted to C. Choose a basis $\{q_0, \ldots, q_9\}$ of $\mathbf{A}$ such that $\{q_0, q_1, q_2\}$ is a basis of $\mathbf{A}_3$ and $\{q_3, \ldots, q_9\}$ is a basis of $\mathbf{A}_7$. Let Λ be the graded ring $\mathbf{C}[q_0, \ldots, q_9] = \bigoplus_i \operatorname{Symm}^i \mathbf{A}$. Let I be the ideal of W in Λ; $I = \bigoplus I_k$. Let $\bar{\Lambda}$ be the graded subring $\mathbf{C}[q_3, \ldots, q_9] \subset \Lambda$. We have a canonical surjection $\Lambda \to \bar{\Lambda}$ which sends q_0, q_1, q_2 to zero. If u is an element or a subset of Λ, we let $\bar{u}$ denote its image in $\bar{\Lambda}$. Note that $L_6 \approx \operatorname{Proj} \bar{\Lambda}$ and $\bar{I}$ is the ideal of the rational, normal, sextic curve, D.

LEMMA 3.2. (i) *I_2 generates I and dimension of $I_2 = 20$.*

(ii) *$\bar{I}_2$ generates $\bar{I}$ and dimension of $\bar{I}_2 = 15$.*

(iii) *Choose a basis $\{Q_1, \ldots, Q_{20}\}$ of I_2 such that $\{\bar{Q}_6, \ldots, \bar{Q}_{20}\}$ is a basis of $\bar{I}_2$. For $1 \leqslant i \leqslant 5$, let*

$$Q_i = \sum_{0<j<2} l_{ij} q_j + \sum_{0<j<k<2} a_{ijk} q_j q_k$$

where each l_{ij} is a linear form in variable $q_3, \ldots, q_9$ and each $a_{ijk} \in \mathbf{C}$. Then, for $0 \leqslant j \leqslant 2$, the set $\{l_{ij}\}_{1 \leqslant i \leqslant 5}$ is linearly independent.

PROOF. Since Λ/I is Gorenstein, has multiplicity $e = 8$ at the origin, and has embedding dimension $= e + \dim - 2 = 10$, (i) follows from **[14]**. D is projectively normal **[4]**. Since the multiplicity $\bar{e}$ of $\bar{\Lambda}/\bar{I}$ at the origin is 6 and the embedding dimension $= \bar{e} - \dim - 1 = 6$, (ii) follows from **[14]** also. Let $\{j, k, m\}$ be an ordered set of integers which is a permutation of the ordered set $\{0, 1, 2\}$. Let E be the curve in $\mathbf{P}_3$, defined by the equations $q_k = q_m = 0$. E contains C and is a reduced curve of arithmetic genus 1. Let L' and L'' be the hyperplanes in $\mathbf{P}_9$, corresponding to q_k and q_m. Then, $W \cap L' \cap L''$ is the image of E in $\mathbf{P}_9$ and is reduced and projectively Gorenstein. Therefore, its ideal is generated by 20 linearly independent, quadratic forms and, for $1 \leqslant i \leqslant 5$,

$$Q_i = (l_{ij} + a_{ijj} q_j) q_j \quad \operatorname{mod}(q_k, q_m).$$

Hence, the set $\{l_{ij} + a_{ijj} q_j\}_{1 < i < 5}$ must be linearly independent. Suppose that $\{l_{ij}\}_{1 < i < 5}$ is linearly dependent. But, then, we may choose Q_i's so that some $l_{ij} = 0$. That would mean that $W \cap L' \cap L''$ is not reduced. Contradiction! Q.E.D.

For an integer $n \geqslant 1$, let λ_n be the one-parameter subgroup of PGL(10) which acts on $\mathbf{C}[q_0, \ldots, q_9]$ via transformation: $q_i \to t^n q_i$ if $0 \leqslant i \leqslant 2$ and $q_i \to q_i$ if $3 \leqslant i \leqslant 9$. Note that λ_n commutes with G_0. Deform W in $\mathbf{P}_9$ under the action of λ_n. Let $A = \operatorname{Spec} \mathbf{C}[t]$. In the graded algebra $\Lambda[t]$ over $\mathbf{C}[t]$, let I_t be the ideal

[2]This proof was supplied to me by D. Eisenbud.

generated by the 20 quadratic forms, $\{Q_{it}\}_{i \leqslant i \leqslant 20}$, obtained as follows. For $6 \leqslant i \leqslant 20$, Q_{it} is obtained from Q_i by replacing q_0, q_1, q_2 by $t^n q_0, t^n q_1, t^n q_2$. For $1 \leqslant i \leqslant 5$, Q_{it} is obtained from Q_i by replacing each coefficient a_{ijk} by $t^n a_{ijk}$. We have a scheme $\mathfrak{W}_n \subset A \times \mathbf{P}_9$, defined by the ideal I_t and a canonical projection, $\mathfrak{p}$: $\mathfrak{W}_n \to A$. Let $W_0 = \mathfrak{p}^{-1}$ (origin).

LEMMA 3.3. *W_0 is projectively Gorenstein.*

PROOF. Let N_4 denote the four-dimensional cone over the sextic D with a two-dimensional vertex, L_2. The ideal of N_4 is generated by $\bar{Q}_6, \ldots, \bar{Q}_{20}$. The ideal of W_0 is generated by $Q_1^*, \ldots, Q_5^*, \bar{Q}_6, \ldots, \bar{Q}_{20}$ where, for $1 \leqslant i \leqslant 5$, $Q_i^* = \Sigma_{0 \leqslant j \leqslant 2} l_{ij} q_j$. Thus, $W_0 \subset N_4$ and $3 \leqslant \dim W_0 \leqslant 4$. For $1 \leqslant i \leqslant 5$, $Q_i^* = l_{i0} q_0 \bmod(q_1, q_2)$. Since the set $\{l_{i0}\}_{1 \leqslant i \leqslant 5}$ is linearly independent, we may choose coordinates so that for $1 \leqslant i \leqslant 5$, $l_{i0} = q_{i+4}$. Let L' and L'' be the hyperplanes in $\mathbf{P}_9$ corresponding to q_1 and q_2. Let $E = W \cap L' \cap L''$ and $E_0 = W_0 \cap L' \cap L''$. Let $\Lambda' = \mathbf{C}[q_0, q_3, \ldots, q_9]$. The ideal of E in Λ' is generated by $Q_1', \ldots, Q_{20}'$ where, for $1 \leqslant i \leqslant 20$, Q_i' is obtained from Q_i by setting $q_1 = q_2 = 0$. For $1 \leqslant i \leqslant 5$, $Q_i' = q_0 q_{i+4}'$ where $q_{i+4}' = q_{i+4} + a_{i00} q_0$. The equations $q_1 = q_2 = 0$ define a curve in $\mathbf{P}_3$, consisting of C and a line, l; $C \cap l$ is a divisor of degree 2 on C. Under the embedding ι: $\mathbf{P}_3 \to \mathbf{P}_9$, l is mapped onto a plane conic, B, which is contained in the plane defined by $q_1 = q_2 = q_4' = \cdots = q_8' = 0$. Let $B \cap D = \{p_1, p_2\}$. It follows that the linear system spanned by $q_5, \ldots, q_9$ cuts out a system of divisors on D with its fixed component equal to $p_1 + p_2$. We turn now to E_0. The ideal of E_0 in Λ' is generated by the linearly independent forms, $q_0 q_5, \ldots, q_0 q_9$, $\bar{Q}_6, \ldots, \bar{Q}_{20}$. The ideal $(\bar{Q}_6, \ldots, \bar{Q}_{20})$ defines a two-dimensional cone, N_2, over D with the vertex at a point in L_2 with coordinates $q_0 = 1$, $q_1 = q_2 = 0$. Clearly, $D \subset E_0 \subsetneqq N_2$ and E_0 contains the two lines, l_1 and l_2, connecting the points p_1 and p_2 to the vertex. But the curve $D \cup l_1 \cup l_2$ is projectively Gorenstein. This follows from a general (unpublished) theorem of D. Eisenbud which asserts in our case that the curves on N_2 of degree 8 are precisely the curves which are projectively Gorenstein. Therefore, the ideal of $D \cup l_1 \cup l_2$ is generated by 20 linearly independent, quadratic forms in Λ' and hence must equal the ideal of E_0. It follows that W_0 must be three-dimensional and projectively Gorenstein. Q.E.D.

COROLLARY 3.4. *W_0 is of pure dimension 3. $\mathfrak{W}_n$ is flat over A so that $W_0 = \lim_{t \to 0} W^{\lambda_n(t)}$. Moreover, W_0 is invariant under λ_n and G_0.*

PROOF. W_0 is equidimensional and without embedded primes by the Cohen-Macaulay theorem [**1**, Proposition III-4.3.]. Therefore, $\mathfrak{W}_n$ is flat over A by Proposition V-3.5 in [**1**]. Since W is invariant under G_0 and since λ_n and G_0 commute, W_0 is invariant under λ_n and G_0. Q.E.D.

COROLLARY 3.5. *Let q', q'' be distinct elements of $\mathbf{A}_3$. Let p_0 be the point in L_2 defined by the equations $q' = q'' = 0$. Let l be the line in $\mathbf{P}_3$ such that $C \cup l$ is the curve in $\mathbf{P}_3$ defined by the equations $q' = q'' = 0$. Let $\iota(C \cap l) = \{p', p''\} \subset D$ and let l' and l'' be the lines in $\mathbf{P}_9$ joining p_0 to p' and p''. Then, $\lim_{t \to 0} \iota(l)^{\lambda_n(t)} = l' \cup l''$.*

PROOF. Clear from the proof of Lemma 3.3.

The next lemma describes the geometry of W_0.

LEMMA 3.6. (i) *L_2 is canonically isomorphic to the space of divisors of degree 2 on D. The isomorphism is G_0-invariant. The isomorphism canonically determines a G_0-invariant conic, D_0, in L_2, corresponding to the divisors on D of the form $2p$.*

(ii) *Each point p on D determines a line l_p in L_2, corresponding to the divisors on D of the form $p + p'$. l_p is tangent to D_0. W_0 contains the plane determined by p and l_p; W_0 is in fact the set-theoretic union of all such planes. It follows that $W_0 - L_2$ is a vector bundle of rank* 2 *over D. The multiplicity of W_0 at every point of L_2 is equal to* 2.

PROOF. (i) We omit the construction of the actual isomorphism since we do not need it here. We show only a one-to-one correspondence. Let p_0 be a point in L_2. Let $q' = q'' = 0$ be the equations defining p_0 in L_2. Then, as in Corollary 3.5, q' and q'' determine a secant l (which may actually be a tangent) of C and hence a divisor of degree 2 on D. Conversely, a divisor of degree 2 on D determines a secant, l, of C. The ideal of $C \cup l$ is generated by two quadratic forms, q' and q'', which, in turn, determine a point in L_2. The G_0-invariance is obvious. Note that the divisor corresponding to a point p_0 in L_2 is of the form $2p$ if and only if the corresponding secant of C is actually a tangent.

(ii) If $p' + p''$ is the divisor of D corresponding to a point p_0 in L_2, then, by Corollary 3.5, W_0 contains the lines, l' and l'' which join p_0 to p' and p''. Therefore, W_0 contains the plane determined by p and l_p. As in the proof of Lemma 3.3, let N_4 be the cone over D with vertex L_2. Recall that $W_0 \subset N_4$. Let p_0, q', q'', l', l'', be as above. Let N_2 be the cone over D with vertex p_0. N_2 is defined in N_4 by the equations $q' = q'' = 0$. Therefore, $N_2 \cap W_0 = D \cup l' \cup l''$. It follows that W_0 is the set-theoretic union of the lines joining points of L_2 to their corresponding divisors on D. The rest of the assertion is now clear. Q.E.D.

LEMMA 3.7. *W_0 contains a $\lambda_n \times G_0$-invariant, rational, ruled surface, Σ_4, such that $\lim_{t\to 0}(X_0)^{\lambda_n(t)} = 2\Sigma_4$. Σ_4 contains the curves D and D_0 as sections such that $D_0 \cdot D_0 = -4$ and $D \cdot D = 4$. Σ_4 is not contained in a hyperplane of $\mathbf{P}_9$ and its degree is equal to* 8.

PROOF. Let $F_0 \in H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))$ be a quartic form which vanishes on X_0. F_0 is G_0-invariant. Let $\mathcal{Q}_0$ be the quadric hypersurface in $\mathbf{P}_9$ defined by F_0. The equation $F_0 = 0$ defines a divisor $\mathcal{V}$ on $\mathcal{W}_n$ which is flat over A such that the fibers of $\mathcal{V}$ are projectively Gorenstein. Let $V_0 = \mathcal{Q}_0 \cap W_0$. $V_0 = \lim_{t\to 0} \iota(X_0)^{\lambda_n(t)}$. Let $q \in \mathbf{A}_3$ be a nonzero element. The equation $q = 0$ defines a line, l, in L_2 such that $l \cap D_0 = \{p_0', p_0''\}$. Therefore, the quadric in $\mathbf{P}_3$ corresponding to q contains exactly two tangents, l' and l'' of C, touching C at points p' and p'' respectively, and intersecting X_0 in the divisor $2C + l' + l''$. By Corollary 3.5,

$$\lim_{t\to 0} \iota(2C + l' + l'')^{\lambda_n(t)} = 2D + 2l_0' + 2l_0''$$

where l_0' and l_0'' are the lines joining p_0' and p_0'' to $\iota(p')$ and $\iota(p'')$ respectively. Therefore, if H_q is the hyperplane in $\mathbf{P}_9$ corresponding to q, then $H_q \cap V_0$ equals $2(D \cup l_0' \cup l_0'')$. Since X_0 is traced out by the tangents of C, it follows that if we let $\Sigma_4 = V_{0,\mathrm{red}}$, then Σ_4 is traced out by the lines joining points on D_0 to the corresponding reduced divisor on D and $V_0 = 2\Sigma_4$. Since the degree of V_0 = the degree of $\iota(X_0) = 16$, $\deg(\Sigma_4) = 8$. Now, $H_q \cap \Sigma_4 = D \cup l_0' \cup l_0''$. For any point p on D, we can find $q \in \mathbf{A}_3$ such that l_0' and l_0'' are distinct and p lies on l_0'. Therefore, Σ_4 can have singularities only on D. But, if Σ_4 has a singularity, then, by homogeneity under G_0, it must be singular everywhere along D. Then, since $D \subset H_q$, $H_q \cap \Sigma_4$ must contain D with multiplicity > 1. Since D has degree 6, this is a contradiction. Therefore, Σ_4 must be nonsingular. Since Σ_4 contains D_0 and D which span L_2 and L_6 respectively, Σ_4 cannot be contained in a hyperplane. Σ_4 is therefore a nonsingular, rational, scroll **[12]** with D_0 and D as sections.

$$(D + l_0' + l_0'') \cdot (D + l_0' + l_0'') = \deg(\Sigma_4) = 8.$$

Therefore, $D \cdot D = 4$. Let q' be a nonzero element in $\mathbf{A}_7$ and let H' be the corresponding hyperplane in $\mathbf{P}_9$. $D_0 \subset H'$. Therefore, $H' \cap \Sigma_4 = D_0 + D'$ where D' is a curve in Σ_4 of degree 6. $D \cdot D' = D \cdot (D_0 + D') = 6$. Hence, D' is linearly equivalent to $6l + ks$ where $k \geqslant 0$, l is a fiber of Σ_4 and s is a section of Σ_4 with the smallest selfintersection number. Since $\deg(D') = 6$, k must be zero and $H' \cap \Sigma_4$ is linearly equivalent to $D_0 + 6l$. Since $(D_0 + 6l) \cdot (D_0 + 6l) = 8$, $D_0 \cdot D_0 = -4$ and $D_0 = s$. Q.E.D.

ANOTHER DESCRIPTION OF $\mathcal{W}_n$. We need to describe $\mathcal{W}_n$ in another way in order to calculate the limit cycles of subvarieties under the action of λ_n. Let o_A denote the origin in A. Let λ: $A - o_A \to \mathrm{PGL}(10)$ be the one-parameter subgroup such that t corresponds to the transformation which sends q_i to tq_i if $0 \leqslant i \leqslant 2$ and sends q_i to q_i if $3 \leqslant i \leqslant 9$. For any positive integer n, the one-parameter subgroup λ_n is the composition

$$A - o_A \xrightarrow{\rho_n} A - o_A \xrightarrow{\lambda} \mathrm{PGL}(10)$$

where the first morphism sends t to t^n. Let $A^\times$ denote $A - o_A$. Let $\sigma_n^\times$ denote the composition

$$A^\times \times \mathbf{P}_3 \xrightarrow{\lambda_n \times \iota} \mathrm{PGL}(10) \times \mathbf{P}_9 \to \mathbf{P}_9$$

where the second morphism is the canonical action of PGL(10) on $\mathbf{P}_9$. As a rational map from $A \times \mathbf{P}_3$ to $\mathbf{P}_9$, $\sigma_n^\times$ has the fundamental set $o_A \times C$. Let $\mathcal{V}' \to A \times \mathbf{P}_3$ be the monoidal transformation with $o_A \times C$ as center and let $\mathcal{V}_n'$ be the pull-back of $\mathcal{V}'$ via the morphism ρ_n. Let $\mathbf{P}_3^*$ be the proper transform of $o_A \times \mathbf{P}_3$ in $\mathcal{V}_n'$. Let V be the exceptional divisor in $\mathcal{V}_n'$. Let $E = V \cap \mathbf{P}_3^*$.

LEMMA 3.8. *$\sigma_n^\times$ extends to a morphism σ_n: $\mathcal{V}_n' \to \mathbf{P}_9$. σ_n maps $\mathbf{P}_3^*$ onto L_2 and maps $V - E$ isomorphically onto $W_0 - L_2$.*

PROOF. Let P be a point on C. Choose coordinates so that P is the point $x_1 = x_2 = x_3 = 0$. Let $q_0 = x_0x_3 - x_1x_2$, $q_1 = x_0x_2 - x_1^2$, $q_2 = x_1x_3 - x_2^2$. Choose a basis $\{q_3, \ldots, q_9\}$ of $\mathbf{A}_7$ such that, $\mathrm{mod}(q_0, q_1, q_2)$, $q_3 = x_0^2$, $q_4 = x_0x_1$, $q_5 = x_1^2$, $q_6 = x_1x_2$, $q_7 = x_2^2$, $q_8 = x_2x_3$, $q_9 = x_3^2$. Let P' denote the point $x_0 = x_1 = x_2 = 0$.

Then, x_0, x_1, x_2, x_3 cut out divisors $3P', 2P' + P, P' + 2P, 3P$, respectively, on C. For $3 \leqslant i \leqslant 9$, q_i cuts out the divisor $(9 - i)P' + (i - 3)P$ on C. Let the embedding ι be given by the equations $q_i = u_i$ where $u_i \in o_{\mathbf{P}_3, P}$. Let $x = x_1/x_0$, $y = x_2/x_0$ and $z = x_3/x_0$. Then $u_0 = z - xy$, $u_1 = y - x^2$, $u_2 = xz - y^2$ and, for $3 \leqslant i \leqslant 9$, $u_i = x^{3-i} \bmod(u_0, u_1)$. Note that, $u_2 = xu_0 - x^2u_1 - u_1^2$. Let $\mathcal{O}$ denote the complete local ring of $A \times \mathbf{P}_3$ at $o_A \times P$; $\mathcal{O} \approx \mathbf{C}[[u_0, u_1, u_4, t]] \approx \mathbf{C}[[u_0, u_1, x, t]]$. The map σ_n^x is given at $o_A \times P$ by the equations:

$$q_i = u_i/t^n, \qquad 0 \leqslant i \leqslant 2,$$
$$q_i = u_i, \qquad 3 \leqslant i \leqslant 9.$$

Let $\mathfrak{U}$ be the ideal $(u_0, u_1, u_2, t^nu_3, \ldots, t^nu_9)$; $\mathfrak{U}$ is generated by u_0, u_1, t^n. The map σ_n^x then extends to the monoidal transformation $\mathcal{V}'_{n,P} \to \operatorname{Spec} \mathcal{O}$ of $\operatorname{Spec} \mathcal{O}$ with center $\mathfrak{U}$. Clearly, $\mathcal{V}'_{n,P}$ is the fiber of $\mathcal{V}'_n$ over $\operatorname{Spec} \mathcal{O}$. Since λ_n projects $\mathbf{P}_9 - L_6$ onto L_2, it follows that σ_n maps $\mathbf{P}_3^*$ onto L_2. (A point p in $\mathbf{P}_3 - C$ lies on a unique secant, l_p, of C since the projection from p maps C onto a plane cubic curve with exactly one double point. σ_n maps the proper transform of l_p in $\mathbf{P}_3^*$ onto a point in L_2.)

Let p_r: $\mathcal{V}'_n \to A$ be the projection. The map $p_r \times \sigma_n$: $\mathcal{V}'_n \to A \times \mathbf{P}_9$ is proper since p_r is. Hence, the image of $p_r \times \sigma_n$ equals $\mathcal{W}_n$ and we get a proper, surjective, birational, A-morphism π: $\mathcal{V}'_n \to \mathcal{W}_n$ which is an isomorphism over $A - o_A$. It follows that $\sigma_n(V) = W_0$.

The fiber of $V - E$ over the point P of C is the affine Spec R_0 where $R_0 = \mathbf{C}[[x]][u_0/t^n, u_1/t^n]$. In R_0, $u_2 = 0$ and, for $3 \leqslant i \leqslant 9$, $u_i = x^{3-i}$. The map σ_P: Spec $R_0 \to \mathbf{P}_9$ is defined by sending q_0 to u_0/t^n, q_1 to u_1/t^n, q_2 to $xu_0/t^n - x^2u_1/t^n$, and, for $3 \leqslant i \leqslant 9$, q_i to x^{3-i}. Therefore, $\sigma_P(\operatorname{Spec} R_0)$ does not meet the hyperplane $q_3 = 0$. For $0 \leqslant i \leqslant 9$, let $s_i = q_i/q_3$. Then the map σ_P is induced by the homomorphism $\mathbf{C}[s_0, \ldots, s_9] \to R_0$ which sends s_0 to u_0/t^n, s_1 to u_1/t^n, s_2 to $xu_0/t^n - x^2u_1/t^n$ and s_i to x^{3-i} for $3 \leqslant i \leqslant 9$. Therefore, σ_P is injective. It follows that π is one-to-one over $\mathcal{V}_n - \mathbf{P}_3^*$ and, hence, by Zariski's Main Theorem, π is an isomorphism when restricted to $\mathcal{V}_n - \mathbf{P}_3^*$. Q.E.D.

REMARK 3.9. Let $\mathcal{C}$ = the proper transform of $A \times C$ in $\mathcal{V}'_n$. Then, $\mathcal{C} \cap V$ = the inverse image of D under the restriction of σ_n to V.

STEP II. Standardization of the equation of the family.

LEMMA 3.10. *Let l denote a fiber of Σ_4. Let $|ml|$ denote $|H^0(\Sigma_4, o_{\Sigma_4}(ml))|$. There is a unique, $\lambda_n \times G_0$-invariant decomposition*

$$H^0\big(\Sigma_4, o_{\Sigma_4}(2)\big) \approx \Theta \oplus \Phi \oplus \Xi$$

such that

(i) $|\Theta|$ *has $2D$ as the fixed component and* $|\Theta| - 2D = |4l|$, $\dim \Theta = 5$.

(ii) $|\Phi|$ *has $D + D_0$ as the fixed component,* $|\Phi| - D - D_0 = |8l|$, $\dim \Phi = 9$,

(iii) $|\Xi|$ *has $2D_0$ as the fixed component and* $|\Xi| - 2D_0 = |12l|$, $\dim \Xi = 13$.

PROOF. From the proof of Lemma 3.7, we have that D is linearly equivalent to $D_0 + 4l$ and $2D + 4l$, $D + D_0 + 8l$, $2D_0 + 12l$ are elements of $|H^0(\Sigma_4, o_{\Sigma_4}(2))|$. Clearly, Θ, Φ, Ξ are $\lambda_n \times G_0$-invariant and have the indicated dimensions. From

[12], the canonical divisor on Σ_4 belongs to $|-2D_0 - 6l|$ and $H^1(\Sigma_4, o_{\Sigma_4}(m)) = 0$ for $m \geq 1$. It follows from duality that $H^2(\Sigma_4, o_{\Sigma_4}(m)) = 0$ for $m \geq 1$. By Riemann-Roch, dim $H^0(\Sigma_4, o_{\Sigma_4}(2)) = 27$. Therefore, Θ, Φ and Ξ span $H^0(\Sigma_4, o_{\Sigma_4}(2))$. The uniqueness follows from the irreducibility of Θ, Φ and Ξ. Q.E.D.

LEMMA 3.11. *There is a* $\lambda_n \times G_0$*-invariant decomposition* $H^0(\mathbf{P}_9, o_{\mathbf{P}_9}(2)) \approx \mathbf{J}_5 \oplus \mathbf{J}_{15} \oplus \mathbf{B}_1 \oplus \mathbf{B}_5 \oplus \mathbf{B}_7 \oplus \mathbf{B}_9 \oplus \mathbf{B}_{13}$ *such that*

(i) $\mathbf{J}_{15}$ *generates the ideal of* N_4, *the cone over* D *with vertex* L_2,

$\mathbf{J}_5$, *together with* $\mathbf{J}_{15}$ *generates the ideal of* W_0,

$\mathbf{B}_1 = \mathbf{C} \cdot F_0$ *where* $F_0 \in \mathrm{Symm}^2(\mathbf{A}_3)$ *and* F_0 *vanishes on* X_0 *in* $\mathbf{P}_3$,

$\mathbf{B}_7$, *together with* $\mathbf{J}_5$, $\mathbf{J}_{15}$, $\mathbf{B}_1$ *generates the ideal of* Σ_4, *and there are* $\lambda_n \times G_0$*-linear isomorphisms* $\mathbf{B}_5 \approx \Theta$, $\mathbf{B}_9 \approx \Phi$, $\mathbf{B}_{13} \approx \Xi$.

(ii) λ_n *acts as follows*:

if $F \in \mathbf{J}_{15} \oplus \mathbf{B}_{13}$, $F^{\lambda_n(t)} = F$,

if $F \in \mathbf{J}_5 \oplus \mathbf{B}_7 \oplus \mathbf{B}_9$, $F^{\lambda_n(t)} = t^nF$,

if $F \in \mathbf{B}_1 \oplus \mathbf{B}_5$, $F^{\lambda_n(t)} = t^{2n}F$.

PROOF. Under the $\lambda_n \times G_0$-linear restriction, $H^0(\mathbf{P}_9, o_{\mathbf{P}_9}(1)) \to H^0(\Sigma_4, o_{\Sigma_4}(1))$, $|\mathbf{A}_3|$ restricts to divisors on Σ_4 with D as the fixed component such that $|\mathbf{A}_3| - D = |2l|$ where l is a fiber of Σ_4. If $q \in \mathbf{A}_3$, $q^{\lambda_n(t)} = t^nq$. $\mathbf{A}_7$ restricts to divisors on Σ_4 with D_0 as the fixed component such that $|\mathbf{A}_7| - D_0 = |6l|$. λ_n acts trivially on $\mathbf{A}_7$.

Let $F_0 \in H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))$ be a quartic form, vanishing on X_0. Since X_0 is singular along C, $F_0 \in \mathrm{Symm}^2(\mathbf{A}_3)$. F_0 vanishes on Σ_4 and hence on D_0. Therefore, we have a $\lambda_n \times G_0$-linear exact sequence

$$0 \to \mathbf{C} \cdot F_0 \to \mathrm{Symm}^2(\mathbf{A}_3) \to H^0(D_0, o_{D_0}(2)) \to 0.$$

Choose a $\lambda_n \times G_0$-linear section s: $H^0(D_0, o_{D_0}(2)) \to \mathrm{Symm}^2(\mathbf{A}_3)$ and let $\mathbf{B}_1 = \mathbf{C} \cdot F_0$, $\mathbf{B}_5$ = image of s. Clearly, $\mathbf{B}_5 \approx \Theta$ under the $\lambda_n \times G_0$-linear restriction to Σ_4. If $F \in \mathrm{Symm}^2(\mathbf{A}_3)$, $F^{\lambda_n(t)} = t^{2n}F$.

Similarly, we have the $\lambda_n \times G_0$-linear exact sequence

$$0 \to \mathbf{J}_{15} \to \mathrm{Symm}^2(\mathbf{A}_7) \xrightarrow{j'} H^0(D, o_D(2)) \to 0$$

where λ_n acts trivially on $\mathrm{Symm}^2(\mathbf{A}_7)$. $\mathbf{J}_{15}$ consists of elements which vanish on D and, hence, vanish on N_4. Since dim $\mathbf{J}_{15} = 15$, $\mathbf{J}_{15}$ in fact, generates the ideal of N_4. Let s' be a $\lambda_n \times G_0$-linear section of j' and set $\mathbf{B}_{13}$ = the image of s'. $\mathbf{B}_{13} \approx \Xi$ under restriction to Σ_4.

Let $\mathbf{B}''$ = the image of $\mathbf{A}_3 \otimes \mathbf{A}_7$ in $H^0(\Sigma_4, o_{\Sigma_4}(2))$ under the restriction map. Let $\mathbf{J}''$ = the kernel of the restriction map. The exact sequence

$$0 \to \mathbf{J}'' \to \mathbf{A}_3 \otimes \mathbf{A}_7 \xrightarrow{j''} \mathbf{B}'' \to 0$$

is $\lambda_n \times G_0$-linear. If $F \in \mathbf{A}_3 \otimes \mathbf{A}_7$, $F^{\lambda_n(t)} = t^nF$. Therefore, $\mathbf{B}'' \cap \Theta = \{0\}$ and $\mathbf{B}'' \cap \Xi = \{0\}$. Since the restriction $H^0(\mathbf{P}_9, o_{\mathbf{P}_9}(2)) \to H^0(\Sigma_4, o_{\Sigma_4}(2))$ is surjective, dim $\mathbf{B}'' = 9$. By the uniqueness of decomposition, $\mathbf{B}'' \approx \Phi$. Let s'' be a $\lambda_n \times G_0$-linear section of j'' and let $\mathbf{B}_9 = s''(\mathbf{B}'')$. $\mathbf{J}''$ vanishes on Σ_4 and has dimension 12. $\mathbf{J}''$ contains the $\lambda_n \times G_0$-invariant, 5-dimensional subspace, $\mathbf{J}_5$, consisting of elements which vanish on W_0. Let $\mathbf{B}_7$ be a $\lambda_n \times G_0$-invariant complement of $\mathbf{J}_5$ in $\mathbf{J}''$.

According to [12], the ideal of Σ_4 is generated by quadratic elements and, hence, by $\mathbf{J}'' \oplus \mathbf{J}_{15} \oplus \mathbf{B}_1$. Q.E.D.

LEMMA 3.12. *There is a G_0-linear isomorphism*

$$r\colon \mathbf{B}_1 \oplus \mathbf{B}_5 \oplus \mathbf{B}_7 \oplus \mathbf{B}_9 \oplus \mathbf{B}_{13} \to H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4)).$$

PROOF. The map r is the composition

$$H^0(\mathbf{P}_9, o_{\mathbf{P}_9}(2)) \to H^0(W, o_W(2)) \to H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4)).$$

Let $\mathbf{B}$ denote the space on the left side of the map r. Let β be a nonzero element of $\mathbf{B}$. Let Z be the quadric hypersurface in $\mathbf{P}_9$ defined by the equation $\beta = 0$. Let $\lim_{t\to 0} Z^{\lambda_n(t)} = Z_0$. Since the projective space $|\mathbf{B}|$ is invariant under λ_n, there exists β_0 in $\mathbf{B}$ such that Z_0 is defined by the equation $\beta_0 = 0$. Now suppose that $r(\beta) = 0$. Then, $W = \iota(\mathbf{P}_3) \subset Z$ and $W^{\lambda_n(t)} \subset Z^{\lambda_n(t)}$. Therefore, $W_0 \subset Z_0$, β_0 must vanish on W_0 and $\beta_0 \in \mathbf{J}_5 \oplus \mathbf{J}_{15}$. Contradiction. Therefore, r must be injective. Since dim $\mathbf{B}$ = dim $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4)) = 35$, r must be an isomorphism. Q.E.D.

For $i = 1, 5, 7, 9, 13$, let $\mathbf{D}_i = r(\mathbf{B}_i)$. Fix a nonzero element F_0 in $\mathbf{D}_1$. Let $\mathbf{N} = \mathbf{D}_1 \oplus \mathbf{D}_9 \oplus \mathbf{D}_{13}$.

LEMMA 3.13. *The morphism η: $G \times |\mathbf{N}| \to M$, induced by the G-action on M is smooth in a neighborhood of $G \times |\mathbf{D}_1|$.*

PROOF. Let e denote the identity in G. Let $p = \eta(e \times |\mathbf{D}_1|)$. By homogeneity, it is enough to show that the tangent space at $e \times |\mathbf{D}_1|$ maps surjectively onto the tangent space at p. Let $O = \eta(G \times |\mathbf{D}_1|)$, the orbit of p in M. Let T_G = the tangent space of G at e and $T_{O,p}$ = the tangent space of O at p. The canonical map $G \times |\mathbf{D}_1| \to O$ maps T_G surjectively onto $T_{O,p}$, sending tangent vectors along G_0 to zero. Also, $e \times |\mathbf{N}|$ maps isomorphically into M. Therefore, it is enough to show that no nonzero tangent vector in the image of T_G lies along $|\mathbf{N}|$ in M.

Let τ: Spec $\mathbf{C}[\varepsilon]/(\varepsilon^2) \to G$ be a morphism which maps the closed point on the identity. τ determines an infinitesimal automorphism of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))$ under which F_0 transforms into $F_0 + \Delta F_0$. Let P be a point of $\mathbf{P}_3$ on C. τ determines a derivation d: $o_{\mathbf{P}_3,P} \to o_{\mathbf{P}_3,P}$ such that if f_0 and Δf_0 are the images of F_0 and ΔF_0 in $o_{\mathbf{P}_3,P}$, then $d(f_0) = \Delta f_0$. Choose a basis of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(2))$ as in the proof of Lemma 3.8. We use the same notation. We may assume that $F_0 = q_0^2 - 4q_1q_2$ so that $f_0 = (u_0 - 2xu_1)^2 + 4u_1^3$. Let $\zeta = u_0 - 2xu_1$. Then, $\Delta f_0 = \zeta d(\zeta) + 12u_1^2 d(u_1)$. The proper transform f_0' of f_0 in $\mathcal{O}[u_0/t^n, u_1/t^n]$ is $(\zeta/t^n)^2 + 4t^n(u_1/t^n)^3$ and the image of f_0' in R_0 is $(\zeta/t^n)^2$. Since $\lim_{t\to 0} F_0^{\lambda_n(t)}$ vanishes on Σ_4, ζ/t^n must vanish on $\pi^{-1}(\Sigma_4) \cap V$. Let $\Sigma' = \pi^{-1}(\Sigma_4) \cap V$. Let $\Delta f_0'$ = the proper transform of Δf_0 in $\mathcal{O}[u_0/t^n, u_1/t^n]$. Then, the restriction of $\Delta f_0'$ to Σ' either is zero or else vanishes to the order $\geqslant 2$ on the inverse image of D in Σ'. It follows that $\lim_{t\to 0}(\Delta f_0')^{\lambda_n(t)} \in \mathbf{D}_1 \oplus \mathbf{D}_5 \oplus \mathbf{D}_7$ and hence, Δf_0 must be in $\mathbf{D}_1 \oplus \mathbf{D}_5 \oplus \mathbf{D}_7$. Therefore, the image of the tangent vector lies along $|\mathbf{N}|$ if and only if $\Delta f_0 \in \mathbf{D}_1$. But, then the infinitesimal automorphism of M determined by τ fixes p. Hence τ must factor as

$$\begin{array}{ccc} & G_0 & \\ \nearrow & & \searrow \\ \operatorname{Spec} \mathbf{C}[\varepsilon]/(\varepsilon^2) & \to & G \end{array}$$

and the image of the tangent vector must be zero. Q.E.D.

COROLLARY 3.14. *Fix a nonzero element F_0 of $\mathbf{D}_1$. There exist elements, $F_t' \in \mathbf{D}_9 \otimes \mathbf{C}[[t]]$ and $F_t'' \in \mathbf{D}_{13} \otimes \mathbf{C}[[t]]$ such that the family X may be defined by an equation of the form*

$$F(t) = F_0 + F_t' + F_t'' = 0.$$

PROOF. Let p be the point of M corresponding to F_0. The family X is defined by a map τ: $S \to M$, mapping the closed point onto p. The map τ lifts to a map μ: $S \to G \times |\mathbf{N}|$ which maps the closed point onto $e \times |\mathbf{D}_1|$ where e is the identity in G. Let μ' be the composition

$$S \xrightarrow{\mu} G \times |\mathbf{N}| \xrightarrow{pr_2} N \to e \times |\mathbf{N}| \to G \times |\mathbf{N}|$$

and let $\tau' = \eta \circ \mu'$. Clearly, the maps μ and μ' are G-equivalent and hence, so are the maps τ and τ'. Q.E.D.

STEP III. Modification of the family via the geometric invariant theory.

Let N denote the affine in $|\mathbf{N}|$ which is the complement of the hyperplane in $|\mathbf{N}|$, $F_0 = 0$. A closed point of N corresponds to an element of $\mathbf{N}$ which can be written uniquely as $F_0 + F' + F''$ where $F' \in \mathbf{D}_9$ and $F'' \in \mathbf{D}_{13}$. Let

$$\Omega = \operatorname{Symm}(\mathbf{D}_9^* \oplus \mathbf{D}_{13}^*)$$

where the superscript * denotes the dual vector space. Grade Ω by assigning weight 2 to $\mathbf{D}_9^*$ and weight 3 to $\mathbf{D}_{13}^*$. G_0 acts on Spec Ω and Proj Ω. Proj Ω contains G_0-invariant subspaces $\mathbf{D}_9$ and $\mathbf{D}_{13}$. Let p_1: Proj $\Omega \dashrightarrow \mathbf{D}_9$ and p_2: Proj $\Omega \dashrightarrow \mathbf{D}_{13}$ be the rational maps defined by the canonical projections. If $\omega \in$ Proj Ω and if p_i is not defined at ω, we let $p_i(\omega)$ denote the empty set. By Lemma 3.11 there are G_0-linear isomorphisms $\mathbf{D}_9 \approx \Phi$ and $\mathbf{D}_{13} \approx \Xi$. If $\omega \in |\mathbf{D}_9|$ (respectively, $|\mathbf{D}_{13}|$), let $\bar{\omega}$ denote the corresponding element in $|\Phi|$ (respectively, $|\Xi|$). It is easy to verify the following [17]:

PROPOSITION 3.15. *Let $\omega \in (\text{Proj } \Omega)^{ss}$ such that ω belongs to a minimal orbit. Then, ω is stable if and only if no fiber of Σ_4 has multiplicity $\geqslant 4$ in $\overline{p_1(\omega)}$ and multiplicity $\geqslant 6$ in $\overline{p_2(\omega)}$. ω is not stable if and only if there exist two distinct fibers of Σ_4 such that each has multiplicity of 4 in $\overline{p_1(\omega)}$ if it is not empty and multiplicity of 6 in $\overline{p_2(\omega)}$ if it is not empty.*

Let o_N denote the origin in N. Let p_r: $N - o_N \to$ Proj Ω be the canonical projection.

LEMMA 3.16. *The family may be modified so that the new family is induced by a map u: $S \to N$, mapping the closed point o onto o_N, such that, if u^* is the restriction of u to* Spec $\mathbf{C}((t))$, *then the composition $p_r \circ u^*$ extends to a map v: $S \to (\text{Proj } \Omega)^{ss}$ which maps o onto a point in a minimal orbit.*

PROOF. By Corollary 3.15, we may assume that the given family of quartics is induced by a map u_0: $S \to N$ such that $u_0(o) = o_N$. Let τ: $N \to \mathfrak{N}$ be the universal categorical quotient of N by G_0. It is enough to find a section of τ over $\tau \circ u_0$ after replacing t by a suitable root of t such that the section has the required properties. We define a blow-up of N and $\mathfrak{N}$ as follows. Let $\Omega = \bigoplus \Omega_i$ where Ω_i is the graded component of weight i. Let $\Omega^\#$ be the graded ring $\bigoplus_{k>0} \Omega^\#_k$ where $\Omega^\#_k = \bigoplus_{i \geq k} \Omega_i$. If we regard Ω as an ungraded ring, then $\Omega^\#$ is a graded algebra over Ω. Let $N' = \operatorname{Proj} \Omega^\#$ and let π: $N' \to N$ be the canonical projection. π is an isomorphism everywhere except over the origin o_N. Let E be the exceptional divisor in N'. The projection p_r extends to a morphism p'_r: $N' \to \operatorname{Proj} \Omega$ which maps E isomorphically onto $\operatorname{Proj} \Omega$. Since the blow-up is equivariant with respect to the action of G_0, G_0 acts on N' and p'_r is equivariant also.

Let $\partial = \tau(o_N)$. Let $\Delta = (\tau \circ \pi)^{-1}(\partial)$. If p is a closed point of $N' - \Delta$, the closure of its orbit lies in $N' - \Delta$ since the closure of the orbit of $\pi(p)$ lies in $N - \tau^{-1}(\partial)$. p is semistable since $\pi(p)$ is. Suppose that $p \in \Delta - E$. Then, $\pi(p)$ lies in $\tau^{-1}(\partial) - o_N$ and o_N lies in the closure of the orbit of $\pi(p)$. That is, there exists a one-parameter subgroup $\lambda(t)$ of G_0 such that $\lim_{t\to 0}(\pi(p))^{\lambda(t)} = o_N$. Therefore, $\pi(p)$ is represented by a quartic form $F_0 + F' + F''$ such that $F' \in \mathbf{D}_9$, $F'' \in \mathbf{D}_{13}$ and $\lim_{t\to 0}(F', F'')^{\lambda(t)} = (0, 0)$. In other words, (F', F'') represents an unstable point of $\operatorname{Proj} \Omega$. There exists a positive integer m such that $(F', F'')^{\lambda(t)} = (t^{2m}F'_t, t^{3m}F''_t)$ where $\lim_{t\to 0}(F'_t, F''_t) = (F'_0, F''_0) \neq (0, 0)$ and $\mu((F'_0, F''_0), \lambda) > 0$. Therefore, p is unstable. Hence, $\Delta^{ss} = E^{ss}$.

Let $\mathfrak{N}'$ denote the categorical quotient of $(N')^{ss}$ by G_0. We have a canonical commutative diagram

$$\begin{array}{ccc} N' & \xrightarrow{\pi} & N \\ \tau'\downarrow & & \downarrow\tau \\ \mathfrak{N}' & \xrightarrow[\mathfrak{p}]{} & \mathfrak{N} \end{array}$$

The morphism $\mathfrak{p}$ is an isomorphism over $\mathfrak{N}' - \partial$ and by Proposition 5.2 in **[17]**, the exceptional divisor $\mathfrak{E}$ in $\mathfrak{N}'$ over ∂ is the universal categorical quotient of $(\operatorname{Proj} \Omega)^{ss}$ by G_0. The map $\tau \circ u_0$ lifts uniquely to a map w: $S \to \mathfrak{N}'$. From the properties of universal quotients (Proposition 2.1 in **[17]**) it follows that there is a positive integer n and a map ρ: $S \to S$ which sends t to t^n such that $w \circ \rho$ lifts to a section u': $S \to N'$ and $u'(o)$ belongs to a minimal orbit. $\pi \circ u'$ now provides the required map. Q.E.D.

Assume now that the family of quartics, $\mathfrak{f}$: $X \to S$ is defined by the equation

$$F = F_0 + t^{4m}F'_t + t^{6m}F''_t = 0$$

where $F'_t \in \mathbf{D}_9 \otimes \mathbf{C}[[t]]$ and $F''_t \in \mathbf{D}_{13} \otimes \mathbf{C}[[t]]$ such that $\lim_{t\to 0}(F'_t, F''_t) = (F'_0, F''_0)$ is not zero and defines a semistable point ω of $(\operatorname{Proj} \Omega)^{ss}$ belonging to a minimal orbit. The quadric hypersurface in $\mathbf{P}_9 \times S$, corresponding to F, transforms under the action of λ_{2m} to the quadric surface defined by the equation

$$F_0 + t^{2m}F'_t + t^{2m}F''_t = 0.$$

The latter hypersurface defines a divisor, Y, on $\mathcal{W}_{2m} \times_A S$ which is flat over S. Y is a modification of the family X. Let $\tilde{Y}$ be the normalization of Y.

THEOREM 3.17. *The special fiber* Y_0 *is a two-to-one cover of* Σ_4, *ramified over a curve with two connected components,* D_0 *and* B, *such that* B *is linearly equivalent to* $3D$. Y_0 *has only insignificant limit singularities.*

PROOF. We use the description of $\mathcal{W}_{2m}$ given above the statement of Lemma 3.8 and the local description given in the proof of the lemma. We keep the same notation. Let X' be the proper transform of X in $\mathcal{V}'_{2m} \times_A S$. Let $\tilde{X}'$ be the normalization of X'. Let $X_\#$ be the proper transform of X_0 in X'. Let $\tilde{X}_\#$ be the proper transform of X_0 in $\tilde{X}'$. We have a proper, surjective, birational map, ρ: $X' \to Y$ which is an isomorphism when restricted to $X' - X_\#$. Hence, we have a proper, surjective, birational map $\tilde{\rho}$: $\tilde{X}' \to \tilde{Y}$ which is an isomorphism when restricted to $\tilde{X}' - \tilde{X}_\#$. Let P be a point of $\mathbf{P}_3$ on C. As before, let $\mathcal{O}$ be the complete local ring of $S \times \mathbf{P}_3$ at $o \times \mathbf{P}_3$. Recall that $\mathcal{O} \approx \mathbf{C}[[u_0, u_1, x, t]]$ in terms of the basis of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(2))$ chosen as in Lemma 3.8. $F_0 = q_0^2 - 4q_1q_2$ and its image in $\mathcal{O}$ is $f_0 = \zeta^2 + 4u_1^3$ where $\zeta = u_0 - 2xu_1$. Let f_t' and f_t'' denote the images of F_t' and F_t'' respectively in $\mathcal{O}$. Then the image of F in $\mathcal{O}$ is:

$$f = \zeta^2 + 4u_1^3 + t^{4m}f_t' + t^{6m}f_t''.$$

The fiber of $\mathcal{V}'_{2m} - \mathbf{P}_3^*$ over Spec $\mathcal{O}$ is isomorphic to Spec R where $R = \mathbf{C}[[\zeta, u_1, x, t]][\zeta/t^{2m}, u_1/t^{2m}]$. The proper transform of f in R is

$$f_\# = (\zeta/t^{2m})^2 + t^{2m}\left(4(u_1/t^{2m})^3 + f_t' + f_t''\right).$$

Let

$$g = (\zeta/t^{3m})^2 + 4(u_1/t^{2m})^3 + f_t' + f_t''.$$

Let $T = R/(f_\#)$ and $T^* =$ the normalization of T; $T^* \approx R[\zeta/t^{3m}]/(g)$. Let

$$T_0 = [T/(t)]_{\text{reduced}} \approx \mathbf{C}[[x]][u_1/t^{2m}];$$

let

$$T_0^* = T^*/(t).$$

$T_0^* \approx T_0[\zeta/t^{3m}]/(g_0)$ where $g_0 = (\zeta/t^{3m})^2 + 4(u_1/t^{2m})^3 + \bar{f}_0' + \bar{f}_0''$, $\bar{f}_0'$ and $\bar{f}_0''$ are the images of f_t' and f_t'' in T_0.

Spec T_0 maps isomorphically onto the fiber of $\Sigma_4 - D_0$ over Spec $\hat{o}_{\Sigma_4,P}$ where P is considered a point of D by identifying C with D. The equation $\bar{f}_0' = 0$ (respectively, $\bar{f}_0'' = 0$) is the local equation of the divisor on Σ_4 defined by F_0' (respectively, F_0''). The equation $(u_1/t^{2m}) = 0$ is the local equation of D. Therefore, $\bar{f}_0' = (u_1/t^{2m})p'(x)$ and $\bar{f}_0'' = p''(x)$ where p' and p'' are polynomials whose order of vanishing at P is less than 5 and 7 respectively. Therefore, the ramification curve of the double cover $\tilde{Y}_0 \to \Sigma_4$ equals $B + kD_0$ where $k \geq 0$, $B \cap D_0$ is empty and B is a three-to-one cover of D. B is linearly equivalent to $3D + nl$, where l is a fiber of Σ_4 and $n \geq 0$. Since $(3D + nl) \cdot D_0 = 0$, $n = 0$ and B is linearly equivalent to $3D$. The ramification curve is linearly equivalent to $(3 + k)D_0 + 12l$ where k is an odd, positive integer. Let $2j = 3 + k$ and $B_0 = jD_0 + 6l$. Now, $\chi(\tilde{Y}_0) = 2\chi(\Sigma_4) + 1/2B_0 \cdot (B_0 + K_{\Sigma_4})$ where K_{Σ_4} is a canonical divisor on Σ_4 and is linearly equivalent to $-2(D_0 + 3l)$. (For a proof of this formula see **[6**, §2**]**. The proof given there extends to our case.) Since $\chi(\tilde{Y}_0) = 2$ and $\chi(\Sigma_4) = 1$, $0 = (j-2)D_0 \cdot (jD_0 + 6l) = (j-2)(6-4j)$. Therefore, $j = 2$ and $k = 1$. From the local description of B, it is clear that $\tilde{Y}_0$ has only insignificant limit singularities. Q.E.D.

4. Double covers of Σ_0. Suppose now that $\mathfrak{f}\colon X \to S$ is a family of quartic surfaces such that X_0 consists of a nonsingular quadric Σ_0 with multiplicity two. Let G_0 = the stabilizer of X_0 in G. If we normalize X after replacing t by $t^{1/2}$ if necessary, we obtain a new family whose special fiber is a double cover of Σ_0, ramified over a curve B. As in the previous section, we modify the family by applying geometric invariant theory so that B is semistable with respect to the action of G_0. Unfortunately, the double cover may still have significant limit singularities. These cases are dealt with in the next section where we further modify these families so that we get families specializing to double covers of Σ_2^0 with insignificant limit singularities.

Fix a quadratic form, q, which vanishes on Σ_0. Since G_0 is semisimple, we have

LEMMA 4.1. *There are G_0-invariant decompositions*

$$H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(2)) \approx \mathbf{C} \cdot q \oplus \Theta, \qquad H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4)) \approx \mathbf{C} \cdot q^2 \oplus q \cdot \Theta \oplus \Phi$$

and G_0-linear isomorphisms

$$\Theta \xrightarrow{\sim} H^0(\Sigma_0, o_{\Sigma_0}(2)), \qquad \Phi \xrightarrow{\sim} H^0(\Sigma_0, o_{\Sigma_0}(4)).$$

By the methods of the previous section, one may show

LEMMA 4.2. *We may modify a given family of quartics, specializing to $2\Sigma_0$, such that the new family is defined by an equation of the form $F = q^2 + t^{2m}\varphi_t$ where*

(i) $\varphi_t \in \Phi \otimes \mathbf{C}[[t]]$ *and* $\varphi_0 = \lim_{t\to 0} \varphi_t \neq 0$,

(ii) *the point in* $|H^0(\Sigma_0, o_{\Sigma_0}(4))|$ *corresponding to* φ_0 *is semistable and belongs to a minimal orbit.*

PROPOSITION 4.3. *Suppose that the family of quartics, $\mathfrak{f}\colon X \to S$, is defined by an equation of the form given in the previous lemma. Let $\tilde{X}$ be the normalization of X. Then, $\tilde{X}_0$ is a double cover of Σ_0, ramified over a curve B in Σ_0 of bidegree* $(4, 4)$ *defined by the equation* $\varphi_0 = 0$.

It remains to describe the minimal orbits in $|H^0(\Sigma_0, o_{\Sigma_0}(4))|^{ss}$ and describe the geometry of $\tilde{X}_0$. Let $\mathbf{H}$ denote $H^0(\Sigma_0, o_{\Sigma_0}(4))$ and let $H = |\mathbf{H}|$. Let $\iota\colon \mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_3$ be the Segre embedding with Σ_0 as its image. Let $G_0' = \mathrm{SL}_2 \times \mathrm{SL}_2$. G_0' acts on $\mathbf{P}_3$ via the embedding ι. Since G_0' is isogenous to the component of G_0 containing the identity, we may determine the stability of curves on Σ_0 by considering the action of G_0' instead of the action of G_0. Let λ be a one-parameter subgroup of G_0'. It is the product of two one-parameter subgroups, λ_1 and λ_2, of SL_2. Choose a basis $\{u_0, u_1\}$ so that λ_1 acts on $H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(1))$ via the diagonal matrices

$$\begin{bmatrix} t^r & 0 \\ 0 & t^{-r} \end{bmatrix}.$$

Similarly, choose a basis $\{v_0, v_1\}$ so that λ_2 acts via the matrices

$$\begin{bmatrix} t^{r'} & 0 \\ 0 & t^{-r'} \end{bmatrix}.$$

We may assume that $r > 0$ and $0 \leqslant r'/r \leqslant 1$.

$\{u_0v_0, u_0v_1, u_1v_0, u_1v_1\}$ is a basis of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(1))$. Let $x_0 = u_0v_0$, $x_1 = u_0v_1$, $x_2 = u_1v_0$, $x_3 = u_1v_1$. Then, $q = x_0x_3 - x_1x_2$. Since, $H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(4)) \otimes H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(4)) \to \mathbf{H}$ is an isomorphism, the set

$$\{u_0^{4-i}u_1^iv_0^{4-j}v_1^j\}_{0 \leqslant i \leqslant 4; 0 \leqslant j \leqslant 4}$$

forms a basis of $\mathbf{H}$. In order to determine the minimal orbits, we proceed as in §2. Let H_λ, H_λ^{ss}, $\overline{H}_\lambda$, $\overline{H}_\lambda^{ss}$, be the sets analogous to the sets M_λ, M_λ^{ss}, $\overline{M}_\lambda$, $\overline{M}_\lambda^{ss}$ in §2. Let $u = u_1/u_0$ and $v = v_1/v_0$. The two propositions below are easy to verify.

PROPOSITION 4.4. *A curve B on Σ_0 of bidegree* (4, 4) *is unstable if and only if B has an affine equation, $f = 0$, where f is of one of the following forms.*

(i) *With* weight$(u) = 2$ *and* weight$(v) = 1$, $f = u^3 +$ *terms of weight* > 6,

(ii) $f = av^4 + buv^3 +$ *terms of higher degree.*

PROPOSITION 4.5. *Let λ be a one-parameter subgroup of G_0', diagonalized as above such that H_λ^{ss} is maximal. Then, we have the following cases where we have parametrized H_λ^{ss} and $\overline{H}_\lambda^{ss}$ by polynomials f and $\bar{f}$ of the form*

$$\sum_{\substack{0 \leqslant i \leqslant 4 \\ 0 \leqslant j \leqslant 4}} a_{ij}u^iv^j:$$

1. $r'/r = 1/2$. *Let* weight$(u) = 2$ *and* weight$(v) = 1$. $f = a_1u^3 + a_2u^2v^2 + a_3uv^4$ $+$ *terms of weight* > 6. *Either $a_1a_3 = 0$ and the curve belongs to case* 4 *or case* 5 *below or $a_1a_3 \neq 0$ and the curve has an isolated singularity at the origin $u = v = 0$, consisting of consecutive triple points with the line $u = 0$ as the tangent.*

$\bar{f} = a_1u^3 + a_2u^2v^2 + a_3uv^4$. *If $a_1a_3 \neq 0$, the curve consists of two skew lines and two twisted cubics such that it has two isolated, consecutive triple points at $u_1 = v_1 = 0$ and at $u_0 = v_0 = 0$.*

2. $r'/r = 1$. $f = f_4(u, v) +$ *terms of degree* > 4 *where $f_4(u, v)$ is a homogeneous polynomial of degree* 4. *Either f_4 has multiple factors and belongs to case* 5 *or the curve has an isolated quadruple point at the origin with four distinct tangents.*

$\bar{f} = f_4(u, v)$. *The curve consists of four (some possibly singular) conics, each of which passes through the points $u_1 = v_1 = 0$ and $u_0 = v_0 = 0$.*

3. $r'/r = 0$.

$$f = u^2 \sum_{\substack{0 \leqslant i \leqslant 2 \\ 0 \leqslant j \leqslant 4}} b_{ij}u^iv^j.$$

The corresponding curves have a line as a component with multiplicity 2. *If a curve here has consecutive triple points or a quadruple point on the double line, it belongs to one of the cases below.*

$\bar{f} = u^2\Sigma_{0\leqslant j<4} c_j v^j$. *The curves consist of the lines* $u_1 = 0$ *and* $u_0 = 0$, *each with multiplicity* 2, *and four other lines.*

4. $0 < r'/r < 1/2$. *Let* weight(u) = 2 *and* weight(v) = 1, $f = a_1u^3 + a_2u^2v^2 +$ *terms of weight* $> 6 = u^2g$ *where* g *consists of terms of weight* $\geqslant 2$. *The curves have the line* $u = 0$ *as a component with multiplicity* 2 *and have consecutive triple points or a quadruple point at the origin.*

$\bar{f} = u^2v^2$. *The curves are of the form* $2B'$ *where* B' *consists of four distinct lines.*

5. $1/2 < r'/r < 1$. $f = u^2f_2(u, v)$ + *terms of degree* > 4 *where* $f_2(u, v)$ *is a polynomial of degree* 2 *such that* u *does not divide* f_2. *The curves have the line* $u = 0$ *as a component with multiplicity* 2 *and have a quadruple point at the origin.*

$\bar{f} = u^2v^2$.

The following two lemmas are needed to describe the geometry of stable curves on Σ_0 of bidegree (4, 4).

LEMMA 4.6. *Let* B *be a curve on* Σ_0 *of bidegree* (4, 4).

(i) *Suppose that* B *has consecutive triple points at a point* P *such that no line in* Σ_0 *is tangent to* B *at* P. *Then, we may choose homogeneous coordinates,* x_0, x_1, x_2, x_3 *in* $\mathbf{P}_3$ *such that if we let* $x = x_1/x_0$, $y = x_2/x_0$ *and* $z = x_3/x_0$, *then, the affine equations of* B *have the form* $z + x^2 + y^2 = 0$ (*equation of* Σ_0)

$$y^3 + x^2g_2(y, z) + xg_3(y, z) + g_4(y, z) = 0$$

where for $2 \leqslant i \leqslant 4$, $g_i(y, z)$ *is a homogeneous polynomial of degree* 4 *in* y, z.

(ii) *The quadratic transform of* Spec $o_{B,P}$ *has a triple point with a single tangent if and only if in the second equation, we may assume that* $g_2 = 0$.

PROOF. Let $x_2 = 0$ define the plane containing a conic in Σ_0 which is tangent to B at P. We may choose coordinates so that $z + x^2 + y^2 = 0$ is the affine equation of Σ_0. In the affine

$$\operatorname{Spec} \mathbf{C}[x, y] \approx \operatorname{Spec} \mathbf{C}[x, y, z]/(z + x^2 + y^2) \subset \Sigma_0,$$

B has consecutive triple points at the origin with the line $y = 0$ as the tangent if and only if its equation in $\mathbf{C}[x, y]$ has the form

$$f = y^3 + y^2p_2(x, y) + yp_4(x, y) + \text{terms of higher degree} = 0$$

where $p_i(x, y)$ denotes a homogeneous polynomial of degree i in variables x, y. Part (i) of the lemma now follows easily by lifting f to a polynomial of degree 4 in $\mathbf{C}[x, y, z]$.

To prove (ii), note that

$$y^3 + x^2g_2(y, z) + xg_3(y, z) + g_4(y, z)$$
$$= y^3 - zg_2(y, z) + xg_3(y, z) + g_4'(y, z) \mod(z + x^2 + y^2).$$

It is easily seen that the quadratic transform of Spec $o_{B,P}$ has a triple point whose tangent cone consists of a line if and only if $y^3 - zg_2$ is of the form $(y + az)^3$. If $y^3 - zg_2 = (y + az)^3$, then, replacing x_2 by $x_2 - ax_3$ and then, replacing x_0 by $x_0 + 2ax_2 - a^2x_3$, we get the desired result. (The form $x_0x_3 + x_1^2 + x_2^2$ is invariant under the above coordinate change.) Q.E.D.

LEMMA 4.7. *Let B be a reduced curve on Σ_0 of bidegree (4, 4). If B has consecutive triple points at a point P such that no line in Σ_0 is tangent to B at P, then, B does not have another singular point which either consists of consecutive triple points or has multiplicity $\geqslant 4$.*

PROOF. Project B from P onto $\mathbf{P}_2$. The image of B is a reduced quintic curve, C, which has a triple point, p, such that $B - P \approx C - p$. Suppose that B has another point P' of multiplicity $\geqslant 3$. Let p' be the image of P' in C. Let L be the line joining p and p'. Then, L must be a simple component of C and $C = L \cup C'$ where C' is a quartic curve not containing L. C' must have a double point at p. Hence, its singularity at p' must also be a double point and L cannot be tangent to C' at p'. Q.E.D.

It is now easy to check

THEOREM 4.8. *Let π: $Y \to \Sigma_0$ be a double cover, ramified over a curve B of bidegree (4, 4). Assume that B is semistable and belongs to a minimal orbit. Let Δ denote the singular locus of Y.*

A. *B is stable if and only if Y is one of the following surfaces:*

Type I: *Δ is empty or consists of rational double points.*

Type II: (i) *Δ consists of a double point, P, of type $\tilde{E}_8$ and some rational double points; no line in Σ_0 is tangent to B at $\pi(P)$.*

(ii) *Δ consists of an ordinary nodal curve and some rational double points; $B = 2C \cup D$ where C is a nonsingular conic and $C \cap D$ consists of* 4 *distinct points.*

Type III: (i) *Δ consists of a double point, P, of type $T_{2,3,r}$ and some rational double points; no line in Σ_0 is tangent to B at $\pi(P)$.*

(ii) *Δ consists of some rational double points and a strictly quasi-ordinary nodal curve which either has two double pinch points or has one double pinch point and two simple pinch points. $B = 2C \cup D$ where C is a nonsingular conic, D is reduced and B does not have a quadruple point. (Note that a line in Σ_0 cannot be tangent to C.)*

Surfaces with significant limit singularities: (i) *Δ consists of a double point, P, of type E_{12}, E_{13}, E_{14} or $J_{3,r}$ and some rational double points; no line in Σ_0 is tangent to B at $\pi(P)$.*

(ii) *Δ consists of some rational double points and a nodal curve which either has a pinch point of type $J_{3,\infty}$ and a simple pinch point or has a pinch point of type $J_{4,\infty}$. $B = 2C \cup D$, C is a nonsingular conic, D is reduced, B does not have a quadruple point and $B \cap D$ has a point of multiplicity* 3 *or* 4.

B. *B is strictly semistable if and only if Y is one of the following surfaces:*

Type II: (i) *Δ consists of two double points of type $\tilde{E}_8$. B has an affine equation of the form $u(u + a_1v^2)(u + a_2v^2) = 0$ where a_1 and a_2 are nonzero and unequal.*

(ii) *Δ consists of two double points of type $\tilde{E}_7$ and some rational double points. B has an affine equation of the form $\prod_{1 \leqslant i \leqslant 4}(a_iu + b_iv) = 0$.*

(iii) *Δ consists of two ordinary nodal curves. B consists of two skew lines, each with multiplicity two, and four other distinct lines.*

Type III: (i) *B has an affine equation of the form $(u + v)^2(u + av)(bu + v) = 0$ where a, b and ab are unequal to* 1. *Δ consists of some rational double points and a quasi-ordinary nodal curve with two double pinch points.*

(ii) *B has an affine equation of the form $(u+v)^2(u+av)^2 = 0$ where $a \neq 1$ and $a \neq 0$. Δ consists of two nodal curves which intersect transversely.*

(iii) *$B = 2B'$ where B' consists of four distinct lines. Y is the union of two nonsingular surfaces, Y_1 and Y_2, and $Y_1 \cap Y_2$ consists of four nonsingular rational curves intersecting transversely such that the dual graph of $Y_1 \cap Y_2$ is homeomorphic to a circle.*

Surfaces with significant limit singularities: (i) *B has an affine equation of the form $(u+v)^3(u+av) = 0$ where $a \neq 1$. Δ consists of a simple cuspidal curve and possibly a rational double point.*

(ii) *$B = 4B'$ where B' is a nonsingular conic.*

5. Double covers of Σ_2^0. It remains to consider the following cases of the families of quartics, $\mathfrak{f}$: $X \to S$. We use the following notation: $\pi_{14} = x_1x_3^3$, $\pi_{15} = x_1x_2x_3^2$, $\pi_{16} = x_3^4$, $\pi_{24} = x_2^2x_3^2$. For $i = 14, 15, 16, 24$, let $\Pi_i = \mathbf{C} \cdot \pi_i$. Let $\pi'_{18} = x_1x_2^2x_3$, $\pi''_{18} = x_2x_3^3$, $\Pi_{18} = \mathbf{C} \cdot \pi'_{18} \oplus \mathbf{C} \cdot \pi''_{18}$. Let π_{18} denote a nonzero element of Π_{18}. $\mathbf{M} = H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(4))$, $\mathbf{M}_t = \mathbf{M} \otimes \mathbf{C}[[t]]$.

CASE 1. X_0 is a stable quartic with significant limit singularities. X has an equation of the form

$$\left(x_0x_3 + x_1^2 + ax_2^2\right)^2 + x_2^3(x_0 + a_1x_1 + a_2x_2 + a_3x_3) + \pi + t^nF_t = 0$$

where $\pi = \Sigma a_i\pi_i \neq 0$ and $F_t \in \mathbf{M}_t$.

CASE 2. X_0 is a strictly semistable, reduced quartic, singular along a nonsingular curve of degree 2 which is a simple cuspidal curve. X has an equation of the form

$$\left(x_0x_3 + x_1^2 + ax_2^2\right)^2 + x_2^3f_1 + t^nF_t = 0$$

where f_1 equals either $x_1 + bx_2$ or x_2 and $F_t \in \mathbf{M}_t$.

CASE 3. $X_0 = 2\Sigma_0$ such that if $\tilde{X}$ is the normalization of X, then $\tilde{X}_0$ has significant limit singularities and is a double cover of Σ_0, ramified over a curve B such that either B is a stable curve or $B = 3B' + B''$ or $B = 4B'$ where B' is a nonsingular conic and $B' \cap B''$ = two distinct points. X has an equation of the form $(x_0x_3 + x_1^2 + x_2^2)^2 + t^nF_t = 0$ where $F_t \in \mathbf{M}_t$ such that

$$\lim_{t\to 0} F_t = \begin{cases} x_2^3x_0 + \pi,\ \pi = \Sigma a_i\pi_i \neq 0 & \text{if } B \text{ is stable,} \\ x_2^3(x_1 + bx_2) \text{ or } x_2^4 & \text{if } B \text{ is strictly semistable.} \end{cases}$$

We will prove

THEOREM 5.1. *Let $\mathfrak{f}$: $X \to S$ be a family of quartics belonging to one of the cases above. Then, there exists a modification $\mathfrak{g}$: $Y \to S$ such that Y_0 has insignificant limit singularities and is a double cover of Σ_2^0.*

LEMMA 5.2. *Let $\mathfrak{f}$: $X \to S$ be a family of quartics as above. Then there exists a modification $\mathfrak{f}'$: $X' \to S$ such that $X'_0 = 2\Sigma_2^0$. Moreover, if $\tilde{X}'$ is the normalization of X', then $\tilde{X}'_0$ is a double cover of Σ_2^0, ramified over a curve defined by the equations $q = f = 0$ where $q = x_0x_3 + x_1^2 = 0$ is the equation of Σ_2^0 and f is one of the following types of quartic polynomials*:

Type 1: $f = x_2^4$,

Type 2: $f = x_2^3x_1$,

Type 3: $f = x_2^3x_0 + a_i\pi_i$, $a_i \neq 0$, $i = 14, 15, 16, 18$ *or* 24.

PROOF. By replacing t by an appropriate root of t, we may assume that X is defined by an equation of the form $(q + ax_2^2)^2 + t^{2n}F_0' + t^{2n+k}F_t' = 0$ where $n \geqslant 0$, $k > 144$, $F_t' \in \mathbf{M}_t$ and $F_0' = x_2^3\{a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3\} + \sum a_i\pi_i$. Modify the family under the action of the one-parameter subgroup of G which acts via the transformations $x_0 \to t^{-r}x_0$, $x_1 \to x_1$, $x_2 \to t^s x_2$, $x_3 \to t^r x_3$ where the positive integers r, s are chosen as follows:

if $F_0' = x_2^4$ or $x_2^3(x_1 + bx_2)$, let $r = s = 2$,

if $a_0 \neq 0$, let $m = \min\{i: a_i\pi_i \neq 0\}$ and let $r = 12$, $s = 2m - 12$.

The new family is defined by the equation

$$\left(q + at^{2s}x_2^2\right)^2 + t^{2n+2p}F_0'' + t^{2n+2p+1}F_t'' = 0$$

where $F_t'' \in \mathbf{M}_t$ and

if $F_0' = x_2^4$, then $F_0'' = x_2^4$ and $p = 4$,

if $F_0' = x_2^3(x_1 + bx_2)$, then $F_0'' = x_2^3x_1$ and $p = 3$, and

if $a_0 \neq 0$, then $F_0'' = a_0x_2^3x_0 + a_m\pi_m$ and $p = 3m - 24$.

To verify this, note that the term of maximum negative weight in F_t' with respect to the action of the subgroup is x_0^4. $t^{2n+k}x_0^4$ transforms into $t^{2n+k-48}$ and check that $2n + k - 48 > 2n + 2p$ in all cases. Now, blow up the ideal $(q + at^{2s}x_2^2, t^{n+p})$. Q.E.D.

We have to adopt more terminology. From now on, we will say that a curve on Σ_2^0 cut out by a quartic surface is *a curve of type i*, where $1 \leqslant i \leqslant 3$, if the curve may be defined by the equations $q = f = 0$ where $q = x_0x_3 + x_1^2 = 0$ is the equation of Σ_2^0 and f is of type i as in the previous lemma. If $\mathfrak{f}$: $X \to S$ is a family of quartics, we say that it is of type i, $1 \leqslant i \leqslant 3$, if $X_0 = 2\Sigma_2^0$ and the special fiber of the normalization of X is a double cover of Σ_2^0, ramified over a curve of type i. We will indicate the normalization of a variety X by $\tilde{X}$.

We begin by decomposing $\mathbf{M}$ under the action of the stabilizer of Σ_2^0. Let $\mathcal{G}$ = the stabilizer of Σ_2^0 in G. Let $q = x_0x_3 - x_2^2 = 0$ be the equation of Σ_2^0. Let $\mathbf{A}_1$ = the subspace of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(1))$ consisting of the elements which vanish at the vertex of Σ_2^0. There exist subgroups G_u, G_m, G_s of $\mathcal{G}$ such that $\mathcal{G} \approx G_u \cdot G_m \cdot G_s$ **[17]**. G_u is the unipotent subgroup of $\mathcal{G}$ and consists of transformations which act trivially on $\mathbf{A}_1$ and take x_2 to an element of the form $x_2 + h$ where $h \in \mathbf{A}_1$. The spaces $\mathbf{C} \cdot x_2$ and $\mathbf{A}_1$ are invariant under G_m and G_s. Let C denote the conic on Σ_2^0 defined by the equation $x_2 = 0$. Then, $G_m \cdot G_s$ is the stabilizer of C in $\mathcal{G}$. Let L denote the plane $x_2 = 0$. Note that $\mathbf{A}_1 \xrightarrow{\sim} H^0(L, o_L(1))$. Let ι: $\mathbf{P}_1 \xrightarrow{\sim} C \subset L$ be an embedding. Via ι, we embed PGL(2) as a subgroup G_s of $\mathcal{G}$. The subgroup G_m is isomorphic to the one-dimensional multiplicative group. It acts trivially on $\mathbf{A}_1$ and its action on $\mathbf{C} \cdot x_2$ determines a character of G_m. Dim $G_u = 3$ and dim $G_s = 3$. Let G_r denote the subgroup $G_m \cdot G_s$. G_r is reductive and it is, in fact, isomorphic to the direct product $G_m \times G_s$.

For $n \geqslant 1$, let $\mathbf{A}_n$ be the $\mathcal{G}$-invariant subspace of $H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(n))$ consisting of the elements which have multiplicity n at the vertex of Σ_2^0. Let $\mathbf{A}_0 = \mathbf{C}$. For $n \geqslant 1$, we have the following G_r-invariant decompositions:

$$H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(n)) \approx \bigoplus_{0 \leqslant i \leqslant n} x_2^{n-i}\mathbf{A}_i.$$

G_m acts trivially on each $\mathbf{A}_n$. By restricting each $\mathbf{A}_n$ to L, we get canonical, G_r-linear isomorphisms, $\mathbf{A}_n \tilde{\to} H^0(L, o_L(n))$, $n \geq 0$.

Next, we have G_s-invariant decompositions $\mathbf{A}_n \approx q \cdot \mathbf{A}_{n-2} \oplus \mathbf{B}_n$ for $n > 2$, such that the pull-back via ι yields G_s-linear isomorphisms $\mathbf{B}_n \tilde{\to} H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(2n))$. Let $\mathbf{B}_1 = \mathbf{A}_1 \tilde{\to} H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(2))$. We get G_r-invariant decompositions

$$\mathbf{A}_2 \approx \mathbf{C} \cdot q \oplus \mathbf{B}_2, \qquad \mathbf{A}_3 \approx q \cdot \mathbf{B}_1 \oplus \mathbf{B}_3,$$

$$\mathbf{A}_4 \approx \mathbf{C} \cdot q^2 \oplus q \cdot \mathbf{B}_2 \oplus \mathbf{B}_4$$

and

$$\mathbf{M} \approx \mathbf{C} \cdot q^2 \oplus q \cdot \left(\mathbf{C} \cdot x_2^2 \oplus x_2 \cdot \mathbf{B}_1 \oplus \mathbf{B}_2\right) \oplus \mathbf{C} \cdot x_2^4 \oplus x_2^3 \cdot \mathbf{B}_1 \oplus x_2^2 \cdot \mathbf{B}_3 \oplus \mathbf{B}_4.$$

If $f \in \mathbf{B}_n$, let $\bar{f}$ denote its restriction to C. Similarly, if $\alpha \in |\mathbf{B}_n|$, let $\bar{\alpha}$ denote its restriction to C.

We are now ready to consider the three types of families. We omit proofs of the propositions which are analogous to the propositions in §§3 and 4.

FAMILIES OF TYPE 1. X is defined by an equation of the form

$$\left(q + a_t' x_2^2\right)^2 + t^{2n} x_2^4 + t^{2n+1} F_t' = 0$$

where a_t' is a nonunit in $\mathbf{C}[[t]]$ and $F_t' \in \mathbf{M}_t$. Let $\mathbf{N} = \mathbf{C} \cdot q^2 \oplus \mathbf{C} \cdot q \cdot x_2^2 \oplus \mathbf{C} \cdot x_2^4 \oplus x_2^2 \cdot \mathbf{B}_2 \oplus x_2 \cdot \mathbf{B}_3 \oplus \mathbf{B}_4$, and $\mathbf{B} = x_2^2 \cdot \mathbf{B}_2 \oplus x_2 \cdot \mathbf{B}_3 \oplus \mathbf{B}_4$.

LEMMA 5.3 (STANDARDIZATION). *The family X may be defined by an equation of the form $(q + a_t x_2^2)^2 + t^{2n} x_2^4 + t^{2n+1} F_t = 0$ where a_t is a nonunit in $\mathbf{C}[[t]]$ and $F_t \in \mathbf{B} \otimes \mathbf{C}[[t]]$.*

PROOF. We prove this inductively. Suppose that X is defined by an equation of the form

$$F^{(k-1)} = \left(q + a_t^{(k-1)} x_2^2\right)^2 + t^{2n}\left\{u_t^{(k-1)} x_2^4 + F_t^{(k-1)}\right\} = 0$$

where $a_t^{(k-1)} = a t^{2m} + b_t$ such that $b_t \in \mathbf{C}[[t]]$ and t^{2n+1} divides b_t, and $u_t^{(k-1)}$ is a unit in $\mathbf{C}[[t]]$, $F_t^{(k-1)} = \sum_{j>0} f_j^{(k-1)} t^j$ such that $f_0^{(k-1)} = 0$ and for $1 < j < k-1$, $f_j^{(k-1)} \in \mathbf{C} \cdot x_2^4 \oplus \mathbf{B}$. Replace t by a square root of t if necessary. Then this is true for $k = 1$. We show that X may be defined by $F^{(k)}$ of the same form such that $F^{(k)} = F^{(k-1)} \bmod t^{2n+k}$.

Let Λ denote the graded ring $\mathbf{C}[x_0, x_1, x_2, x_3]$. Let d be a derivation of Λ into itself. d induces an automorphism of $\Lambda \otimes \mathbf{C}[t]/(t^2)$ over $\mathbf{C}[t]/(t^2)$ which sends x_i to $x_i + t dx_i$. Thus, d defines a tangent vector τ: Spec $\mathbf{C}[t]/(t^2) \to \mathrm{GL}(4)$ at the identity and hence, a tangent vector $|\tau|$: Spec $\mathbf{C}[t]/(t^2) \to G$. Let g: $S \to \mathrm{GL}(4)$ be a lifting of τ. Let $|g|$: $S \to G$ denote the corresponding S-valued point of G. For a given positive integer p, let g_p be the composition:

$$S \xrightarrow{\rho_p} S \xrightarrow{g} \mathrm{GL}(4)$$

where ρ_p is the map obtained by extracting a pth root of t. Let $|g_p|$ denote the S-valued point of G corresponding to $|g_p|$.

Let Σ denote the quadric surface defined by the equation $q + a x_2^2 = 0$. Let σ: $S \to \mathrm{GL}(4)$ be the map induced by the transformation: $x_2 \to t^m x_2$, and, for $i = 0$, 1 and 3, $x_i \to x_i$.

We will consider three types of transformations:

TYPE 1. $dx_2 = h \in \mathbf{B}_1$ and, for $i = 0$, 1 and 3, $dx_i = b_i x_2$ such that $d(q + ax_2^2)^2 = 0$. $|\tau|$ is then a tangent vector along the stabilizer, Stab(Σ). Let T_1 = the vector space spanned by such $|\tau|$; dim $T_1 = 3$. The lifting g is chosen so that g factors as

$$\begin{array}{ccc} S & \xrightarrow{g} & G \\ & \searrow \quad \nearrow & \\ & \mathrm{Stab}(\Sigma) & \end{array}$$

Let $\tilde{g}_p = \sigma \circ g_p \circ \sigma^{-1}$ if $a \neq 0$ and g_p if $a = 0$. If $a \neq 0$, then $\tilde{g}_p$ acts via the transformation: $x_2 \to x_2 + t^{p-m}h \bmod t^{p-m+1}$, and, for $i = 0, 1, 3$, $x_i \to x_i + t^{p+m}b_i x_2 \bmod t^{p+m+1}$. Therefore, $\tilde{g}_p$ is defined over S if $p > m$. The form $q + at^{2m}x_2^2$ is invariant under $\tilde{g}_p$.

TYPE 2. $dx_2 = 0$ so that $|\tau|$ is a tangent vector along Stab(L). Let T_2 = the vector space spanned by such $|\tau|$. Dim $T_2 = 12 =$ dim Stab(L). Note that $T_1 \oplus T_2$ span the tangent vector space of G at the identity. g is chosen so that $|g|$ factors through Stab(L).

TYPE 3. $dq = dx_2 = 0$ so that $|\tau|$ is tangent to G_s. Let T_3 = the space of such $|\tau|$. Dim $T_3 = 3 =$ dim G_s. Choose g so that $|g|$ factors through G_s.

If $g: S \to \mathrm{GL}(4)$ is a morphism, we let g^* denote the correspnding automorphism of $\Lambda \otimes \mathbf{C}[[t]]$ over $\mathbf{C}[[t]]$. If $\alpha \in \Lambda$, let $\delta\alpha = g^*(\alpha) - \alpha \bmod t^{2n+k+1}$.

We are now ready to modify $F^{(k-1)}$. First, we use a transformation of Type 1. Let $p = k + m$ if $a \neq 0$ and $p = k$ otherwise. Then, $\delta F^{(k-1)} = t^{2n+k}d(x_2^4) = 4t^{2n+k}x_2^3 dx_2$ where $dx_2 \in \mathbf{B}_1$. Therefore, there exists a derivation d of Type 1 such that $-\delta F^{(k-1)}/t^{2n+k}$ equals the component of $f_k^{(k-1)}$ along $x_2^3 \cdot \mathbf{B}_1$. Let $F_\sharp = \tilde{g}_p^*(F^{(k-1)})$.

Next, we apply a transformation of Type 2. Let $p = 2n + k$. Then, $\delta F_\sharp = \delta(q^2) = 2t^{2n+k}q\, dq$ where $dq \in \mathbf{C} \cdot q \oplus x_2 \cdot \mathbf{B}_1 \oplus \mathbf{B}_2$. q divides dq if and only if $|\tau|$ is a tangent vector along Stab(L) $\cap$ Stab(Σ_2^0), that is, along G_r. Therefore, $dq = 0$ if and only if $|\tau|$ is a tangent vector along G_s. Since dim T_2 − dim $G_s = 9$, T_2 maps onto $\mathbf{C} \cdot q \oplus x_2 \cdot \mathbf{B}_1 \oplus \mathbf{B}_2$. Therefore, there exists a transformation g' of Type 2 such that $g'^*(F_\sharp)$ is the required form $F^{(k)}$. (Transformations of Type 3 are needed in the proof of Lemmas 5.9 and 5.15 which are analogous.) Q.E.D.

Let $\Omega = \mathrm{Symm}(\mathbf{B}_2^* \oplus \mathbf{B}_3^* \oplus \mathbf{B}_4^*)$. Grade Ω by assigning weight 2 to $\mathbf{B}_2^*$, weight 3 to $\mathbf{B}_3^*$ and weight 4 to $\mathbf{B}_4^*$. G_s acts on Spec Ω and Proj Ω.

LEMMA 5.4. *In Lemma* 5.3, *we may assume that* $tF_t = t^{2k}x_2^2\varphi_t + t^{3k}x_2\xi_t + t^{4k}\psi_t$ *where* $\varphi_t \in \mathbf{B}_2 \otimes \mathbf{C}[[t]]$, $\xi_t \in \mathbf{B}_3 \otimes \mathbf{C}[[t]]$, $\psi_t \in \mathbf{B}_4 \otimes \mathbf{C}[[t]]$ *such that* $\{\varphi_0, \xi_0, \psi_0\} = \lim_{t\to 0}\{\varphi_t, \xi_t, \psi_t\} \neq 0$ *and* $\{\varphi_0, \xi_0, \psi_0\}$ *determines a point in* (Proj $\Omega)^{ss}$, *belonging to a minimal orbit.*

COROLLARY 5.5. *The family X may be modified so that the new family X′ is defined by an equation of the form*

$$F' = \left(q + a_t t^{2k}x_2^2\right)^2 + t^{2m+4k}\left\{x_2^4 + x_2^2\varphi_t + x_2\xi_t + \psi_t\right\} = 0$$

where φ_t, ξ_t, ψ_t *are as in Lemma* 5.4. *It follows that* X_0' *is a double cover of* Σ_2^0, *ramified over a curve B which is determined by the equation* $x_2^4 + x_2^2\varphi_0 + x_2\xi_0 + \psi_0 = 0$. *B is semistable and belongs to a minimal orbit.*

PROOF. Assume that the family X is defined by an equation of the form given in the previous lemma. Modify the family under the action of the one-parameter subgroup of G_m which takes x_2 to $t^k x_2$. Q.E.D.

It remains to describe the semistable curves on Σ_2^0 which are defined by an equation of the form $x_2^4 + x_2^2\varphi + x_2\xi + \psi = 0$. For $2 \leqslant i \leqslant 4$, let p_{r_i}: Proj Ω--→ $|\mathbf{B}_i|$ be the rational map defined by the canonical projection. If $\omega \in$ Proj Ω, let $p_{r_i}(\omega)$ denote the empty set if $p_{r_i}(\omega)$ is not defined at ω.

LEMMA 5.6. *Let ω be a point in* (Proj $\Omega)^{ss}$, *belonging to a minimal orbit. ω is stable if and only if C does not have a point p such that for $2 \leqslant i \leqslant 4$, p has multiplicity $\geqslant i$ in $\overline{p_{r_i}(\omega)}$. ω is strictly semistable if and only if there exist two distinct points in C such that each has multiplicity $= i$ in $\overline{p_{r_i}(\omega)}$ if $p_{r_i}(\omega)$ is not empty.*

LEMMA 5.7. *Let B be a curve on Σ_2^0 defined by an equation of the form $f = x_2^4 + x_2^2\varphi + x_2\xi + \psi = 0$ where $\{\varphi, \xi, \psi\}$ determines a point ω of* (Proj $\Omega)^{ss}$.

(i) *ω is strictly semistable if and only if B has a quadruple point. B cannot have a quadruple point with a single tangent. Suppose ω belongs to a minimal orbit. Then, B has a quadruple point with tangent of multiplicity* 3 *if and only if there exists $g \in G_u$ such that f^g is of the form $x_2^4 + ax_2^3x_1$* (mod q).

(ii) *Suppose that B is stable. Then, B has consecutive triple points at a point P if and only if there exists $g \in G_u$ which sends x_2 to $x_2 + h$ where $\nu_P(h) =$ the multiplicity of h at $P = 0$, such that $f^g = x_2^4 + 4x_2^4h + x_2^2\varphi' + x_2\xi' + \psi'$ where $\nu_P(\varphi') \geqslant 2$, $\nu_P(\xi') \geqslant 4$, $\nu_P(\psi') \geqslant 6$. Moreover, B has a triple point which remains a triple point with a single tangent after one quadratic transformation if and only if B may be defined by an equation of the form $x_2^3(x_0 + a_1x_1 + a_2x_2 + a_3x_3) + \sum a_i\pi_i$ such that $\sum a_i\pi_i \neq 0$.*

PROOF. (i) Suppose that ω is strictly semistable. Then there exists a point P on C such that $\nu_P(\varphi) \geqslant 2$, $\nu_P(\xi) \geqslant 3$, $\nu_P(\psi) \geqslant 4$ and at least one equality holds. We may assume that P is the point $x_1 = x_2 = x_3 = 0$. Let $x = x_1/x_0$, $y = x_2/x_0$ and $z = x_3/x_0$. In the affine Spec $\mathbf{C}[x, y] \approx$ Spec $\mathbf{C}[x, y, z]/(z + x^2) \subset \Sigma_2^0$, B has an equation of the form $y^4 + y^2x^2p(x) + yx^3p'(x) + x^4p''(x) = 0$ where p, p', p'' are polynomials, at least one of which does not vanish at the origin. Conversely, suppose that B has a quadruple point at P. We may assume that P has the coordinates $x_1 = x_2 = 0$ and $x_2/x_0 = a$. In the affine Spec $\mathbf{C}[x, y]$, B has an equation of the form $y^4 + y^2p_4(x) + yp_6(x) + p_8(x) = 0$ where $p_i(x)$ denotes a polynomial of degree $\leqslant i$. Since the y^3-term is missing, B has a quadruple point with x-coordinate zero if and only if $a = 0$ and p_{2i} vanishes to the order i at P. Hence, B is strictly semistable. The rest of the statement is clear.

(ii) Clearly, if f^g is of the indicated form, then B has consecutive triple points. Suppose that B has consecutive triple points at P. We may assume that the x-coordinate of P is zero. In $\mathbf{C}[x, y]$, $f = y^4 + y^2p_4(x) + yp_6(x) + p_8(x)$ as above. Since B is a four-to-one cover of C, the line $x = 0$ cannot be the tangent at P. From the form of the equation, it is clear that P cannot have coordinates $x = y = 0$ since the tangent line at P must have an equation of the form $y + ax = 0$ and the y^3-term is missing. Choose $h \in \mathbf{A}_1$ such that the conic defined by the equation $x_2 + h = 0$ is tangent to B at P. Let $g \in G_u$ which sends x_2 to $x_2 + h$. Then, in $\mathbf{C}[x, y]$, f^g has the form $y^4 + y^3p_2'(x) + y^2p_4'(x) + yp_6'(x) + p_8'(x)$ where $p_i'(x)$ is a polynomial of degree $\leqslant i$ and $p_2'(0) \neq 0$. Since B has consecutive triple points at P, for $2 \leqslant i \leqslant 4$, x^{2i-2} must divide p_{2i}'. It follows that the homogeneous form of f^g must be as stated above. P remains a triple point with a single tangent after one quadratic transformation if and only if for $2 < i < 4$, x^{2i-1} divides p_{2i}'. Assume that P is such a point. Then, homogenizing the affine form of f^g appropriately, we get

$$f^g = x_2^3(x_0 + a_1x_1 + a_2x_2 + a_3x_3) + \sum a_i\pi_i \pmod{q}.$$

Moreover, $\sum a_i\pi_i \neq 0$ since ω is stable. Q.E.D.

COROLLARY 5.8. *Let* $\mathfrak{f}'$: $X' \to S$ *be a family of quartics as in Corollary* 5.5. *Then, either* $\tilde{X}_0'$ *has only insignificant limit singularities or there exists a modification* $X'' \to S$ *which is a family of quartics of Type* 2 *or* 3.

PROOF. $\tilde{X}_0'$ is a double cover of Σ_0^0, ramified over a curve B. From the previous lemma, it is clear that if $\tilde{X}_0'$ has significant limit singularities, then X' must be defined by an equation of the form $(q + t^mF_t')^2 + t^{2n}F_0 + t^{2n+k}F_t'' = 0$ where $F_t' \in H^0(\mathbf{P}_3, o_{\mathbf{P}_3}(2)) \otimes \mathbf{C}[[t]]$, $F_t'' \in \mathbf{M}_t$, m, n, k are positive integers and either

$$F_0 = ax_2^4 + x_2^3x_1$$

or

$$F_0 = x_2^3(x_0 + a_1x_1 + a_2x_2 + a_3x_3) + \sum a_i\pi_i$$

such that $\sum a_i\pi_i \neq 0$. Now apply the method used in the proof of Lemma 5.2. Q.E.D.

FAMILIES OF TYPE 2. Let P be the point with coordinates $x_1 = x_2 = x_3 = 0$ and let Q be the point $x_0 = x_1 = x_2 = 0$. Let G_m' be the stabilizer of the divisor $P + Q$ in G_s. Let $G_r' = G_m \times G_m'$. G_r' acts trivially on x_1 and x_0x_3. Let $\mathbf{V}_0 = \mathbf{C}$ and $\mathbf{V}_i = \mathbf{C} \cdot x_0^i \oplus \mathbf{C} \cdot x_3^i$. Let $\mathbf{B}_n' = \bigoplus_{0 \leqslant i \leqslant n} x_1^i\mathbf{V}_{n-i}$. This is a G_r'-invariant decomposition of $\mathbf{B}_n'$ into one-dimensional subspaces. The pull-back via the embedding ι of $\mathbf{P}_1$ gives us the G_m'-linear isomorphisms $\mathbf{B}_n' \xrightarrow{\sim} H^0(\mathbf{P}_1, o_{\mathbf{P}_1}(2n))$. We also have G_r'-invariant decompositions $\mathbf{A}_n \approx \bigoplus_{0 \leqslant i \leqslant n} q^i\mathbf{B}_{n-i}'$ and

$$\mathbf{M} \approx \mathbf{C} \cdot q^2 \oplus q \cdot \left(\mathbf{C} \cdot x_2^2 \oplus x_2 \cdot \mathbf{B}_1' \oplus \mathbf{B}_2'\right)$$
$$\oplus\, \mathbf{C} \cdot x_2^4 \oplus x_2^3\mathbf{B}_1' \oplus x_2^2 \cdot \mathbf{B}_2' \oplus x_2 \cdot \mathbf{B}_3' \oplus \mathbf{B}_4'.$$

Let

$$\mathbf{N}' = \mathbf{C} \cdot q^2 \oplus \mathbf{C} \cdot q \cdot x_2^2 \oplus \mathbf{C} \cdot x_2^4 \oplus \mathbf{C} \cdot x_2^3x_1 \oplus x_2^2 \cdot \mathbf{V}_2 \oplus x_2 \cdot \mathbf{B}_3' \oplus \mathbf{B}_4'.$$

Let

$$\mathbf{B}' = \mathbf{C} \cdot x_2^4 \oplus x_2^2 \cdot \mathbf{V}_2 \oplus x_2 \cdot \mathbf{B}_3' \oplus \mathbf{B}_4'.$$

LEMMA 5.9 (STANDARDIZATION). *The family* X *may be defined by an equation of the form* $(q + a_tx_2^2)^2 + t^{2n}x_2^3x_1 + t^{2n+1}F_t = 0$ *where* a_t *is a nonunit in* $\mathbf{C}[[t]]$ *and* $F_t \in \mathbf{B}' \otimes \mathbf{C}[[t]]$.

PROOF. We proceed inductively as in the proof of Lemma 5.3. Suppose that X is defined by an element $F^{(k-1)}$ in $\mathbf{M}_t$ which is of the right form mod t^{2n+k}. First, pick a transformation g_p of Type 1 such that, mod t^{2n+k+1}, $\tilde{g}_p^*$ kills off the component of $F^{(k-1)}$ along $\mathbf{C} \cdot x_2^2x_1^2 \oplus x_2^2x_1 \cdot \mathbf{V}_1$. Let $F_\# = \tilde{g}_p^*(F^{(k-1)})$. Next, pick a transformation g_k' of Type 3 so that $g_k'^*(F_\#)$ does not have a component along $x_2^3 \cdot \mathbf{V}_1$ mod t^{2n+k+1}. Now apply a transformation of Type 2 as in Lemma 5.3. Q.E.D.

Let $\Omega' = \mathrm{Symm}(\mathbf{V}_2^* \oplus \mathbf{B}_3'^* \oplus \mathbf{B}_4'^*)$. Grade Ω' by assigning weight 1 to $\mathbf{V}_2^*$, weight 2 to $\mathbf{B}_3'^*$ and weight 3 to $\mathbf{B}_4'^*$. G_m' acts on Spec Ω' and Proj Ω'.

LEMMA 5.10. *In Lemma 5.9, we may assume that* $tF_t = b_tx_2^4 + t^{2k}x_2^2\varphi_t + t^{4k}x_2\xi_t + t^{6k}\psi_t$ *where* b_t *is a nonunit in* $\mathbf{C}[[t]]$, $\varphi_t \in \mathbf{V}_2 \otimes \mathbf{C}[[t]]$, $\xi_t \in \mathbf{B}_3' \otimes \mathbf{C}[[t]]$, $\psi_t \in \mathbf{B}_4' \otimes \mathbf{C}[[t]]$ *such that* $\{\varphi_0, \xi_0, \psi_0\} = \lim_{t\to 0}\{\varphi_t, \xi_t, \psi_t\} \neq 0$ *and* $\{\varphi_0, \xi_0, \psi_0\}$ *determines a point in* $(\mathrm{Proj}\ \Omega')^{ss}$, *belonging to a minimal orbit.*

(Note that $x_2^3x_1$ is not stable under G_m'. Therefore, the generic $\{\varphi_t, \xi_t, \psi_t\}$ must be stable under G_m' since the generic quartic is stable.)

COROLLARY 5.11. *The family X may be modified so that the new family X' is defined by an equation of the form*

$$F' = (q + a_tt^{4k}x_2^2)^2 + t^{2n+6k}\{b_tt^{2k}x_2^4 + x_2^3x_1 + x_2^2\varphi_t + x_2\xi_t + \psi_t\} = 0$$

where φ_t, ξ_t, ψ_t *are as in Lemma 5.10. It follows that* $\tilde{X}_0'$ *is a double cover of* Σ_2^0, *ramified over a curve B which is defined by an equation of the form* $x_2^3x_1 + x_2^2\varphi_0 + x_2\xi_0 + \psi_0 = 0$. *$B$ is semistable and belongs to a minimal orbit.*

Let p_{r_2}: Proj $\Omega' \dashrightarrow \mathbf{V}_2$, p_{r_3}: Proj $\Omega' \dashrightarrow \mathbf{B}_3'$ and p_{r_3}: Proj $\Omega' \dashrightarrow \mathbf{B}_4'$ be the rational maps defined by the canonical projections.

LEMMA 5.12. *Let ω be a semistable point of* (Proj Ω') *belonging to a minimal orbit. Then, ω is stable if and only if neither P nor Q is a point of C which, for each i,* $2 \leqslant i \leqslant 4$, *has multiplicity* $\geqslant i$ *in* $\overline{p_{r_i}(\omega)}$. *$\omega$ is strictly semistable if and only if* $p_{r_2}(\omega)$ *is empty and for* $i = 3$ *and* 4, *both P and Q have multiplicity i in* $\overline{p_{r_i}(\omega)}$ *if it is not empty.*

LEMMA 5.13. *Let B be a curve on Σ_2^0 defined by the equation* $f = x_2^2x_1 + x_2^2\varphi + x_2\xi + \psi$ *where* $\varphi \in \mathbf{V}_2$, $\xi \in \mathbf{B}_3'$, $\psi \in \mathbf{B}_4'$ *such that* $\{\varphi, \xi, \psi\}$ *determines ω in* $(\mathrm{Proj}\ \Omega')^{ss}$.

(i) *B has a quadruple point if and only if ω is strictly semistable. If B has a quadruple point, it must be the point P or Q. If ω is strictly semistable and belongs to a minimal orbit, then $\varphi = 0$ and*

$$f = x_1(x_2 + a_1x_1)(x_2 + a_2x_1)(x_2 + a_3x_1)$$

such that $\sum a_i = 0$. *Hence, each quadruple point has at least three distinct tangents.*

(ii) *Suppose that ω is stable and B has consecutive triple points at a point P'. Then P' is distinct from P and Q. If P' is a triple point which remains a triple point with a single tangent after one quadratic transformation, then there exists* $g \in \mathcal{G}$ *such that*

$$f^g = x_2^3(x_0 + a_1x_1 + a_2x_2 + a_3x_3) + \sum a_i\pi_i \pmod{q}$$

such that $\sum a_i\pi_i \neq 0$.

COROLLARY 5.14. *Let $X' \to S$ be the family of quartics as in Corollary 5.11. Then, either $\tilde{X}'_0$ has only insignificant limit singularities or there exists a modification $X'' \to S$ which is a family of quartics of Type 3.*

FAMILIES OF TYPE 3. Let $X \to S$ be a family of quartics of Type 3. Call X a family of Type (3-i) if $\tilde{X}_0$ is a double cover whose ramification curve is defined by the equation $x_2^3 x_0 + a_i \pi_i = 0$, $i = 14, 15, 16, 18$ or 24, $a_i \neq 0$. Let P, Q, G'_m, G'_r, $\mathbf{B}'_n$, $\mathbf{V}_n$ be as in the previous case. We blow up the point P under the action of a one-parameter subgroup of $\mathcal{G}$ such that the singularity at P is replaced by a milder singularity. (This is done in Lemma 5.16. The modification is actually done in two steps.) Let

$\alpha_i =$ the quadratic monomial in $\mathbf{B}'_2$ which vanishes to the order i at P, $0 \leq i \leq 4$,
$\beta_i =$ the cubic monomial in $\mathbf{B}'_3$ which vanishes to the order i at P, $0 \leq i \leq 6$,
$\gamma_i =$ the quartic monomial in $\mathbf{B}'_4$ which vanishes to the order i at P, $0 \leq i \leq 8$.

Let $\mathbf{B}_1^0 = \mathbf{C} \cdot x_0 \oplus \mathbf{C} \cdot x_3 \subset \mathbf{B}'_1$, $\mathbf{B}_2^0 = \mathbf{C} \cdot x_1 x_3 \oplus \mathbf{C} \cdot x_3^2 \subset \mathbf{B}'_2$, (note: $\alpha_3 = x_1 x_3$, $\alpha_4 = x_3^2$). $\mathbf{B}'_3 \approx \oplus\, \mathbf{C} \cdot \beta_i$ and $\mathbf{B}'_4 \approx \oplus\, \mathbf{C} \cdot \gamma_i$. Let $\mathbf{D} = x_2^2 \cdot \mathbf{B}_2^0 \oplus x_2 \cdot \mathbf{B}'_3 \oplus \mathbf{B}'_4$. For $i = 7$ or 8, let $\mathbf{D}_{2i} =$ the subspace of $\mathbf{D}$ obtained by leaving out $\mathbf{C} \cdot \gamma_{i-1}$. Let $\mathbf{D}_{15} =$ the subspace of $\mathbf{D}$ obtained by leaving out $\mathbf{C} \cdot x_2 x_1^2 x_3$. Let $\mathbf{D}_{24} =$ the subspace of $\mathbf{D}$ obtained by leaving out $\mathbf{C} \cdot x_2^2 x_1 x_3$. We have three subcases in the case of families of Type (3-18):

(3-18a)	$\pi_{18} = a_{18}\pi'_{18}$	$\mathbf{D}_{18a} = \mathbf{D}_{18b} =$ the subspace of $\mathbf{D}$ obtained by leaving out $\mathbf{C} \cdot x_2 x_1 x_3^2$.
(3-18b)	$\pi_{18} = a_{18}\pi''_{18}$	
(3-18c)	$\pi_{18} = a'_{18}\pi'_{18} + a''_{18}\pi''_{18}$, $a'_{18}a''_{18} \neq 0$	$\mathbf{D}_{18c} =$ the subspace of $\mathbf{D}$ obtained by leaving out $\mathbf{C} \cdot x_3^4$.

LEMMA 5.15 (STANDARDIZATION). *Let X be a family of Type (3-i), $i = 14, 15, 16, 18$ or 24. Then X may be defined by an equation of the form $(q + a_t x_2^2)^2 + t^{2n}\{b_t x_2^4 + x_2^3 h_t + F_t\} = 0$ where a_t, b_t are nonunits in $\mathbf{C}[[t]]$,*
$h_t \in \mathbf{B}_1^0 \otimes \mathbf{C}[[t]]$ such that $\lim_{t\to 0} h_t = x_0$ and
$F_t \in \mathbf{D}_i \otimes \mathbf{C}[[t]]$ such that $\lim_{t \to 0} F_t = a_i \pi_i$, $a_i \neq 0$.

PROOF. Similar to the proof of Lemma 5.9. Let X be defined by an element $F^{(k-1)}$ in M_t which is of the right form mod t^{2n+k}. First, apply a transformation g_p of Type 1. If $i \neq 18$ or if $i = 18$b, use this to kill off mod t^{2n+k+1}, the component of $F^{(k-1)}$ along $\mathbf{C} \cdot x_2^2 x_0^2 \oplus \mathbf{C} \cdot x_2^2 x_0 x_1 \oplus \mathbf{C} \cdot x_2^2 x_1^2$. In Case (3-18a), use g_p to kill off, mod t^{2n+k+1}, the component along $\mathbf{C} \cdot x_2^2 x_0^2 \oplus \mathbf{C} \cdot x_2^2 x_0 x_1 \oplus \mathbf{C} \cdot x_2 x_1 x_3^2$. In Case (3-18c), kill off, mod t^{2n+k+1}, the component along $\mathbf{C} \cdot x_2^2 x_0^2 \oplus \mathbf{C} \cdot x_2^2 x_0 x_1 \oplus \mathbf{C} \cdot x_3^4$. Let $F_\# = \tilde{g}_p^*(F^{(k-1)})$. Next, apply a transformation g'_k of Type 3 so that the component of $g'_p{}^*(F_\#)$ along $x_2^3 \cdot \mathbf{B}'_1 \oplus x_2^2 \cdot \mathbf{B}'_2 \oplus x_2 \cdot \mathbf{B}'_3 \oplus \mathbf{B}'_4$ has the right form. Now apply a transformation of Type 2. Q.E.D.

Define a grading of $\mathbf{D}$ by weights as follows: weight$(x_2^2\alpha_i) = 6i$, weight$(x_2\beta_i) = 3i$, weight$(\gamma_i) = 2i$. Then, $\mathbf{D} \approx \oplus\, \Pi_i$ where Π_i is the piece of weight i. For $i = 14, 15, 16, 18$ and 24, the new definition of Π_i agrees with the old definition.

LEMMA 5.16. *Let X be a family of type* (3-i), $i = 14, 15, 16, 18$ *or* 24. *Then X may be modified so that the new family X' is defined by an equation of the form as in Lemma* 5.15 *except that*

$$\lim_{t\to 0} F_t = F_0 = a_i\pi_i + \pi, \quad F_0 \in \mathbf{D}_i, \qquad 0 \neq \pi \in \bigoplus_{j<i} \Pi_j, \quad a_i \neq 0.$$

PROOF. We use the following notation. If η is an element in $\mathbf{D} \otimes \mathbf{C}[[t]]$, and $\eta = x_2^2\alpha_t + x_2\beta_t + \gamma_t$ where $\alpha_t \in \mathbf{B}_2^0 \otimes \mathbf{C}[[t]]$, $\beta_t \in \mathbf{B}_3' \otimes \mathbf{C}[[t]]$ and $\gamma_t \in \mathbf{B}_4' \otimes \mathbf{C}[[t]]$, then, for any positive integer k,

$$t^k * \eta = t^k x_2^2\alpha_t + t^{2k}x_2\beta_t + t^{3k}\gamma_t.$$

Suppose that the given family of quartics is defined by an equation of the form $(q + a_t'x_2^2)^2 + t^{2n}\{b_t'x_2^4 + x_2^3h_t' + F_t'\} = 0$ where a_t', b_t' are nonunits in $\mathbf{C}[[t]]$, $h_t' \in \mathbf{B}_1^0 \otimes \mathbf{C}[[t]]$ such that $\lim_{t\to 0} h_t' = x_0$ and $F_t' \in \mathbf{D}_i \otimes \mathbf{C}[[t]]$ such that $\lim_{t\to 0} F_t' = a_i\pi_i$, $a_i \neq 0$. $F_t' = \sum t^{m_j} * \eta_j(t)$ such that $m_i = 0$, $\eta_i(0) = a_i\pi_i$ and $m_j > 0$ if $j \neq i$. Replacing t by its appropriate root, we may assume that, for $0 \leqslant j < i$, $2(i-j)|m_j$. Let $m = \min_{j<i}\{m_j/2(i-j)\}$. Let λ be the one-parameter subgroup of G which acts via the transformation $x_0 \to x_0$, $x_1 \to t^{12m}x_1$, $x_2 \to x_2$, $x_3 \to t^{24m}x_3$. If $\eta \in \Pi_j$, $\eta^\lambda = t^{2mj} * \eta$. $F'^\lambda = (t^{24m}q + a_t'x_2^2)^2 + t^{2n}\{b_t'x_2^4 + x_2^3h_t + \sum t^{m_j+2mj} * \eta_j(t)\}$ where $h_t = h_t'^\lambda$ so that $\lim_{t\to 0} h_t = x_0$. Now, $m_j + 2mj \geqslant 2mi$. There exists $j_0 < i$ such that $m_{j_0} + 2mj_0 = 2mi$. Moreover, if $j > i$, $m_j + 2mj > 2mi$. Therefore, $\sum t^{m_j+2mj} * \eta_j(t) = t^{2mi} * \eta(t)$ such that $\eta(0) = a_i\pi_i + \pi$ where $0 \neq \pi \in \bigoplus_{j<i}\Pi_j$. Let $F'' = F'$. F'' may be rewritten as

$$F'' = \left(t^{24m}q + a_t'x_2^2\right)^2 + t^{2n}\left\{b_t'x_2^4 + x_2^3h_t + t^{2mi}x_2^2\alpha_t + t^{4mi}x_2\beta_t + t^{6mi}\gamma_t\right\}$$

where $\{\alpha_0, \beta_0, \gamma_0\} = \lim_{t\to 0}\{\alpha_t, \beta_t, \gamma_t\} \neq 0$ such that $x_2^2\alpha_0 + x_2\beta_0 + \gamma_0 = a_i\pi_i + \pi$. Transform F'' now under the action of the one-parameter subgroup λ' which acts via the transformation $x_0 \to x_0$, $x_1 \to x_1$, $x_2 \to t^{2mi}x_2$, $x_3 \to x_3$. Then, $F'' = t^{24m}F$ where F has the required form. Q.E.D.

It is now easy to check

COROLLARY 5.17. *Let $X' \to S$ be the family of quartics as in Lemma* 5.16. *Then, either $\tilde{X}_0'$ has insignificant limit singularities or there exists a modification $X'' \to S$ which is a family of Type* (3-j), $j < i$.

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