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CLASS GROUPS OF CYCLIC GROUPS OF SQUARE FREE ORDER

BY

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ABSTRACT. Let G be a finite cyclic group of square free order. Let $\mathrm{Cl}(ZG)$ denote the projective class group of the integral group ring ZG. Our main theorem describes explicitly the quotients of a certain filtration of $\mathrm{Cl}(ZG)$. The description is in terms of class groups and unit groups of the rings of cyclotomic integers involved in ZG. The proof is based on a Mayer-Vietoris sequence.

1. Introduction. In this note G denotes a cyclic group of square free order b. For every positive integer d dividing b, let ζ_d be a primitive dth root of unity in a fixed algebraic closure of the field of rational numbers. Wherever the variable d appears, its domain will be some subset of the set of positive divisors of b. For any ring R, R^* will denote the group of units and $\mathrm{Cl}(R)$ the projective class group of R. For any $d \neq 1$, let π_d denote the product of all primes in $Z[\zeta_d]$ dividing d. Let ZG denote the integral group ring of G. Our result is the following. It is proved in §2.

(1.1) THEOREM. *The group* $\mathrm{Cl}(ZG)$ *can be filtered so that the quotients are* $\prod_{d|b} \mathrm{Cl}(Z[\zeta_d])$ *and the cokernels of the natural maps*

$$Z[\zeta_d]^* \to (Z[\zeta_d]/\pi_d)^*$$

for $d|b$, $d \neq 1$.

The author wishes to thank the referee for showing how to simplify the original proof of Theorem (1.1). The original proof, which was based on a canonical form theorem for matrices over Z with characteristic polynomial $x^b - 1$, proved only a slightly weaker version of the theorem. Reiner-Ullom [3] using a Mayer-Vietoris sequence different from the one in this paper obtained a lower bound on $|\mathrm{Cl}(ZG)|$ when b is the product of two odd primes.

2. Proof of Theorem (1.1). The cartesian diagram

$$\begin{array}{ccc} ZG & \to & \prod_d Z[\zeta_d] \\ \downarrow & & \downarrow \\ ZG/\prod_d bZ[\zeta_d] & \to & \prod_d Z[\zeta_d]/(b) \end{array}$$

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leads to an exact Mayer-Vietoris sequence the latter part of which (see [2]) reduces to

$$(ZG/bZG)^* \oplus \prod_d Z[\zeta_d]^* \xrightarrow{\alpha+\beta} \prod_d (Z[\zeta_d]/(b))^* \to \mathrm{Cl}(ZG) \to \prod_d \mathrm{Cl}(Z[\zeta_d]) \to 0$$

where $(ZG/\prod_d bZ[\zeta_d])^*$ has been replaced by $(ZG/bZG)^*$ which maps onto it [1, Lemma 2]. The Mayer-Vietoris sequence immediately implies a filtration of $\mathrm{Cl}(ZG)$ in which the quotients are $\prod_d \mathrm{Cl}(Z[\zeta_d])$ and $\mathrm{cok}(\alpha+\beta)$. It remains to analyze $\mathrm{cok}(\alpha+\beta)$.

The map α is the restriction of the natural map γ: $ZG/bZG \to \prod_d Z[\zeta_d]/(b)$. Let F_p be the field of p elements. The p-component of γ is

$$F_pG \to \prod_{\substack{d\\(d,p)=1}} F_p[\zeta_d] \times \prod_{\substack{d\\(d,p)=1}} F_p[\zeta_{pd}]. \tag{2.1}$$

Let $f_d(x)$ be the dth cyclotomic polynomial. Using the fact that $f_{pd} \equiv f_d^{p-1} \bmod p$, (2.1) can be rewritten

$$F_p[x]/\left(\prod_{\substack{d\\(p,d)=1}} f_d(x)^p\right) \to \prod_{\substack{d\\(p,d)=1}} F_p[x]/(f_d(x)) \times \prod_{\substack{d\\(p,d)=1}} F_p[x]/\left(f_d(x)^{p-1}\right).$$

Call this $A \to B \times C$. Identify the isomorphic rings B and $C/\mathrm{rad}\, C$, and define a map $B^* \times C^* \to B^*$ by $(b, c) \mapsto b^{-1}c$. Then

$$A^* \to (B \times C)^* \to B^* \to 1$$

is exact. Putting p-components back together gives an exact sequence

$$(ZG/bZG)^* \to \prod_d (Z[\zeta_d]/(b))^* \xrightarrow{\delta} \prod_{\substack{p,d\\(p,d)=1}} (F_p[\zeta_d])^* \to 1.$$

Using the fact that $F_p[\zeta_d] \cong F_p[\zeta_{pd}]/\mathrm{rad}$ when $(p, d) = 1$, and combining p-components by the Chinese Remainder Theorem (CRT hereafter) yields an isomorphism

$$\phi\colon \prod_{\substack{p,d\\(p,d)=1}} F_p[\zeta_d]^* \to \prod_{\substack{d\\d\neq 1}} (Z[\zeta_d]/\pi_d)^*.$$

Thus there is an exact sequence

$$(ZG/bZG)^* \xrightarrow{\alpha} \prod_d (Z[\zeta_d]/(b))^* \xrightarrow{h} \prod_{\substack{d\\d\neq 1}} (Z[\zeta_d]/\pi_d)^* \to 1$$

where $h = \phi \circ \delta$. This determines $\mathrm{cok}\,\alpha$.

Next the map $h \circ \beta$: $\prod_d Z[\zeta_d]^* \to \mathrm{cok}\,\alpha$ must be determined. Fix d and let p be a prime dividing d. Set $D = d/p$. Define a ring isomorphism

$$\tau_{Dd}\colon Z[\zeta_D]/(p) \to Z[\zeta_d]/(\pi_p) = p\text{-component of } Z[\zeta_d]/\pi_d$$

by $\zeta_D \pmod{p} \mapsto \zeta_d \pmod{\pi_p}$ where (π_p) denotes the ideal π_p (in $Z[\zeta_p]$) induced up to $Z[\zeta_d]$. For $x \in Z[\zeta_d]/\pi_d$, let x_p denote the image of x in the p-component of $Z[\zeta_d]/\pi_d$. By the CRT, x is completely determined by the x_p, p ranging over all primes dividing d. Let $u = (u_d) \in \prod_d Z[\zeta_d]^*$, and $w = h \circ \beta(u)$. It is straightforward to check that

$$(w_d)_p = u'_d \mu_{Dd} \tag{2.2}$$

where u'_d is the image of u_d in $Z[\zeta_d]/(\pi_p)$ under the canonical map, and μ_{Dd} is the image in $Z[\zeta_d]/(\pi_p)$ of u_D^{-1} under

$$Z[\zeta_D] \to Z[\zeta_D]/(p) \xrightarrow{\tau_{Dd}} Z[\zeta_d]/(\pi_p).$$

Fix an ordering $d_1, d_2, \ldots, d_t$ of the distinct positive divisors of b such that the number of primes dividing d_i is less than or equal to the number of primes dividing d_j when $i < j$. Make the obvious notation change in subscripts. For example ζ_{d_i} is now denoted ζ_i. Let $G_i = Z[\zeta_i]^*$, $1 \leqslant i \leqslant t$, and $H_i = (Z[\zeta_i]/\pi_i)^*$, $2 \leqslant i \leqslant t$. Let $G = \prod G_i$, $H = \prod H_i$.

(2.3) LEMMA. *Let $a \in G_k$. If $k > 1$, let $a \equiv 1 \bmod \pi_k$. Then there is some $g = (g_i)$ in the kernel of $h \circ \beta$: $G \to H$ with $g_i = 1$ for $i < k$, and $g_k = a$.*

PROOF. Define a ring homomorphism T_{ji}: $Z[\zeta_j] \to Z[\zeta_i]$ when $d_j | d_i$ by $\zeta_j \to \zeta_i^n$ where $n \equiv 1 \bmod d_j$ and $n \equiv 0 \bmod (d_i/d_j)$. Let $g_i = T_{ki}(a)$ if $d_k | d_i$, $g_i = 1$ otherwise. Clearly $g_i = 1$ for $i < k$. Let $w = h \circ \beta(g)$. It remains to show that $w = 1$.

Now $w = (w_i)$, $i = 2, 3, \ldots, t$. Taking p-components of $Z[\zeta_i]/\pi_i$, it suffices to show that $(w_i)_p = 1$ for $p | d_i$. By (2.2), $(w_i)_p$ is the product of the images in $Z[\zeta_i]/(\pi_p)$ of g_i and g_j^{-1} where $d_j = d_i/p$. The following facts are now needed.

(1) $[T_{ji}(x)] \bmod (\pi_p) = \tau_{ji}(x \bmod p)$.

(2) $T_{ji} \circ T_{kj} = T_{ki}$ when $d_k | d_j$.

(3) The composite

$$Z[\zeta_k] \xrightarrow{T_{ki}} Z[\zeta_i] \to Z[\zeta_i]/(\pi_p)$$

factors through $Z[\zeta_k] \to Z[\zeta_k]/\pi_k$ when $d_k | d_i$, $p | d_k$.

If $d_k | d_j$, then by (2.2) and facts (1) and (2) the images of g_j^{-1} and g_i cancel in $Z[\zeta_i]/(\pi_p)$. If $d_k | d_i$, and $d_k \nmid d_j$, then the image of g_i is 1 by (2.2), fact (3), and the fact that $a \equiv 1 \bmod \pi_k$, while $g_j = 1$ by definition. If $d_k \nmid d_i$, $g_i = 1$ and $g_j = 1$ by definition. Thus, in all cases, $(w_i)_p = 1$, and the lemma is proved.

Define filtrations on $X = G, H$ by

$$F_k(X) = \{x \in X | x_i = 1 \text{ for } i < k\},$$

and let $F_k(Q)$ be the filtration induced on $Q = \operatorname{cok}(h \circ \beta)$.

(2.4) PROPOSITION. *For $k = 2, 3, \ldots, t$,*

$$F_k(Q)/F_{k+1}(Q) \cong H_k/G'_k$$

where G'_k denotes the image of G_k in H_k under the canonical map.

PROOF. One verifies that

$$F_k(Q)/F_{k+1}(Q) \cong F_k(H)/[F_{k+1}(H) + (F_k(H) \cap \operatorname{Im} F_1(G))] \tag{2.5}$$

where Im means image under $h \circ \beta$. By Lemma (2.3),

$$F_k(H) \cap \operatorname{Im} F_1(G) = \operatorname{Im} F_k(G).$$

Therefore the right-hand side of (2.5) becomes

$$F_k(H)/F_{k+1}(H) + \operatorname{Im} F_k(G).$$

This is isomorphic to H_k / G'_k, and the proof is complete.

Theorem (1.1) follows from Proposition (2.4) because $Q \cong \operatorname{cok}(\alpha + \beta)$.

REMARK. Theorem (1.1) can be strengthened slightly. Let Q_d be the cokernel of the natural map $Z[\zeta_d]^* \to (Z[\zeta_d]/\pi_d)^*$. For $d|b$, let $\Delta(d)$ be the number of distinct primes dividing d. Then the filtration of Q in the proof of Theorem (1.1) can be made coarser so that the quotients are the groups $\prod\{Q_d|\Delta(d) = k\}$ for $1 \leqslant k \leqslant \Delta(b)$. This is because if $\Delta(d_i)$ is constant for $k \leqslant i < r$, then, modulo $F_r(G)$ and $F_r(H)$, the restriction of $h \circ \beta$ to $F_k(G)$ splits as the direct product of the natural maps $G_i \to H_i$, $k \leqslant i < r$. The required analogue of Proposition (2.4) is proved without difficulty by imitating the proof of Proposition (2.4).

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