QUATERNIONIC KAEHLER MANIFOLDS[1]

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ABSTRACT. The topological classification of 4- and 8- (real) dimensional compact quaternionic Kaehler manifolds is given. There is only the torus in dimension 4. In dimension 8, there are 12 homeomorphism classes; representatives are given explicitly.

In this paper, we will study compact manifolds modeled on the quaternions which admit a Kaehler metric. It is easily seen that these manifolds have zero Riemannian curvature (cf. [8]) and hence are quotients $\mathbf{R}^{4n}/\Gamma$, where Γ acts as a group of covering transformations preserving the Euclidean distance in $\mathbf{R}^{4n}$. In §2 and §4, we study Γ to give the topological classification of compact quaternionic Kaehler manifolds in real dimensions 4 and 8. In dimension 4, there is only the torus. This result was obtained by Sommese [9] under conditions weaker than ours. In dimension 8, there are 12 homeomorphism classes.

It should be noted that there are several definitions of a quaternionic Kaehler manifold. All of them, except the one considered herein, model the tangent bundle (rather than the underlying manifold) on the quaternions. For example, one can require two anticommuting complex Kaehler structures J and K, in which case the holonomy is a subgroup of the symplectic group. According to Yau [12], these manifolds need not be Riemannian flat. Now J (or K) has constant components (as a $(1, 1)$ tensor) in some coordinate neighborhood, but we cannot necessarily find a neighborhood for both J and K simultaneously. By modeling the manifold on the quaterions, we are requiring a coordinate system so that both J and K have constant components.

I would like to thank the referee for bringing Auslander [0] to my attention.

1. Preliminaries.

1.1 *Algebra.* The group $E(n)$ of rigid motions of $\mathbf{R}^n$ is, by definition, the group of diffeomorphisms of $\mathbf{R}^n$ which preserves the Euclidean distance. Relative to a choice of origin $\vec{o} \in \mathbf{R}^n$, we can write $E(n)$ as a semidirect product $O(n) \cdot \mathbf{R}^n$. If $\alpha \in E(n)$, then $\alpha = (A, a)$ with $A \in O(n)$ orthogonal, $a \in \mathbf{R}^n$ translation by a, and $(A, a)r = Ar + a$ for $r \in \mathbf{R}^n$. The product rule is $(A, a)(B, b) = (AB, Ab + a)$.

For any $\alpha \in E(n)$, there is a choice of origin for which $\alpha = (A, a)$ satisfies $A(a) = a$. In particular, if α has no fixed points, then $a \neq$ origin and A has the eigenvalue $+1$. To see this, if $\alpha = (A, x)$ for some choice of origin, then we may

choose y so that $(A - I)x = (A - I)^2 y$. Translating the origin to y, α becomes

$$(I, y)(A, x)(I, y)^{-1} = (A, x - (A - I)y).$$

One easily checks that for $a = x - (A - I)y$, we have $A(a) = a$.

Let $\Gamma \subset E(n)$ be a discrete uniform subgroup without elements of finite order. It can be shown [11, p. 99] that these are the precise requirements for the action of Γ on $\mathbf{R}^n$ to be properly discontinuous, have compact quotient space, and be free, respectively. Hence the quotient $M = \mathbf{R}^n/\Gamma$ is a compact n-manifold with universal covering space $\mathbf{R}^n$, covering transformations Γ, and fundamental group $\pi_1(M) = \Gamma$. Since Γ preserves the standard (flat) metric on $\mathbf{R}^n$, the quotient $\mathbf{R}^n/\Gamma$ has an induced (flat) metric.

The converse is also true. Every compact connected flat Riemannian manifold is a quotient $M = \mathbf{R}^n/\Gamma$ where $\Gamma \subset E(n)$ is a discrete uniform subgroup without elements of finite order. Bieberbach [2] has shown that the free abelian normal subgroup $\Gamma^* = \Gamma \cap \mathbf{R}^n$ consisting of pure translations has finite index in Γ, i.e., the (linear) holonomy group $\Psi = \Gamma/\Gamma^* \subset O(n)$ is finite. It can also be shown that Γ^* is maximal abelian in Γ [4, p. 297]. If, for example, $\Gamma^* = \Gamma$, then $\mathbf{R}^n/\Gamma$ is a torus.

Auslander and Kuranishi [1] have given an algebraic characterization of Γ. An abstract group Γ is isomorphic to the fundamental group of a compact connected flat n-manifold if (1) Γ has an abelian normal subgroup Γ^* which is free on n generators, (2) Γ^* is maximal abelian in Γ, (3) Γ/Γ^* is finite, and (4) Γ has no finite subgroups.

Let Λ be the lattice of all $a \in \mathbf{R}^n$ for which the translation $t_a \in \Gamma^*$. Since Γ^* is normal in Γ, Ψ preserves Λ and hence any minimal set of generators for Γ^* is a vector space basis of $\mathbf{R}^n$ relative to which the elements of Ψ all have integral entries. In particular, the matrix invariants (trace, determinant, etc.) are integral. For more details, see Wolf [11, Chapter 3].

Two compact flat Riemannian manifolds $\mathbf{R}^n/\Gamma$ and $\mathbf{R}^n/\Gamma'$ are *affinely equivalent* if there is an affine transformation ϕ such that $\phi\Gamma\phi^{-1} = \Gamma'$, i.e., if Γ and Γ' are conjugate in the group of affine transformations of $\mathbf{R}^n$. Bieberbach [2] proved

THEOREM 1. Γ *and* Γ' *are isomorphic as abstract groups if and only if* $\mathbf{R}^{-1}/\Gamma$ *and* $\mathbf{R}^n/\Gamma'$ *are affinely equivalent.*

It follows that two compact flat Riemannian manifolds are affinely equivalent if and only if they have isomorphic fundamental groups. In particular, affine equivalence is the same as topological equivalence.

1.2 *Geometry.* An almost quaternionic structure on a (necessarily) $4n$-dimensional C^∞ manifold consists of a pair of almost complex structures satisfying $JK = -KJ$. The quaternions provide the standard model of an (almost) quaternionic structure on $\mathbf{R}^{4n}$. Let $\mathbf{R}^{4n}$ have coordinates $\{(x_i, y_i, u_i, v_i) \mid i = 1, 2, \ldots, n\}$; the quaternionic structure is given by

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad K\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial u_i},$$

$$J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i}, \quad K\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial v_i},$$

$$J\left(\frac{\partial}{\partial u_i}\right) = -\frac{\partial}{\partial v_i}, \quad K\left(\frac{\partial}{\partial u_i}\right) = -\frac{\partial}{\partial x_i},$$

$$J\left(\frac{\partial}{\partial v_i}\right) = \frac{\partial}{\partial u_i}, \quad K\left(\frac{\partial}{\partial v_i}\right) = -\frac{\partial}{\partial y_i}.$$

A mapping f between almost quaternionic manifolds is *quaternionic* if the differential df commutes with the almost quaternionic structures. A *quaternionic manifold* is one which is covered by open sets $\{U_i\}$ with diffeomorphisms $\{\phi_i\colon U_i \to \mathbf{R}^{4n}\}$ such that $\phi_j \circ \phi_i^{-1}\colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a quaternionic mapping with respect to the above standard structure on $\mathbf{R}^{4n}$. Clearly, a quaternionic manifold has a pair of anticommuting complex structures J and K. A *quaternionic Kaehler manifold* is a quaternionic Riemannian manifold such that J and K preserve the metric g and are parallel with respect to the Riemannian connection ∇, i.e., $g(JX, JY) = g(X, Y)$, $g(KX, KY) = g(X, Y)$, $\nabla_X(JY) = J\nabla_X Y$, and $\nabla_X(KY) = K\nabla_X Y$ for all tangent vector fields X and Y.

LEMMA 1. *A quaternionic Kaehler manifold M is Riemannian flat.*

PROOF. Consider a quaternionic coordinate system $x_1, \ldots, x_{4n}$ (so that J and K have constant components as $(1, 1)$ tensors), and let X, Y be constant coefficient linear combinations of the coordinate vector fields $\partial/\partial x_i$. Hence $[Z_1, Z_2] = 0$ for Z_i equal to X, Y, JX, JY, KX, or KY, $i = 1, 2$. Now, using the symmetry of ∇, and the parallelism of J and K, we have

$$
\begin{aligned}
0 &= -\tfrac{1}{2}JK([KY, JX] - J[KY, X] - K[Y, JX] + KJ[Y, X]) \\
&= -\tfrac{1}{2}JK(J\nabla_{KY}X - K\nabla_{JX}Y - J\nabla_{KY}X + JK\nabla_X Y \\
&\qquad\qquad + K\nabla_{JX}Y + KJ\nabla_Y X - KJ\nabla_X Y) \\
&= -\tfrac{1}{2}JK(2JK\nabla_X Y) \\
&= \nabla_X Y.
\end{aligned}
$$

Similarly $\nabla_Y X = \nabla_X X = \nabla_Y Y = 0$, and hence

$$g(R(X, Y)X, Y) = g(\nabla_X \nabla_Y X - \nabla_Y \nabla_X X - \nabla_{[X,Y]}X, Y) = 0. \quad \text{Q.E.D.}$$

Next, we will define the holonomy group Ψ on an arbitrary Riemannian manifold N. It is a pleasant surprise that for a compact connected flat Riemannian manifold, this holonomy agrees with the holonomy in §1.1 (see [**11**, Lemma 3.4.4]).

Given $p \in N$ and a C^∞ closed curve $\sigma\colon [0, 1] \to N$, $\sigma(0) = \sigma(1) = p$, we construct the parallel translation $\tau_\sigma\colon T_pN \to T_pN$ as follows. If $v \in T_pN$, then we may extend v along σ to $\tilde{v}$ so that $\nabla_{\sigma'}\tilde{v} = 0$, where σ' is the tangent field to σ (in local coordinates, the equation $\nabla_{\sigma'}\tilde{v} = 0$ is a system of linear differential equations which can be solved uniquely given the initial data v). We define $\tau_\sigma(v) = \tilde{v}_{\sigma(1)}$ where $\tilde{v}_{\sigma(0)} = v$. Since the defining equations of $\tilde{v}$ are linear, it follows that τ_σ is a linear transformation. In fact, τ_σ is an orthogonal transformation for the connection ∇ is Riemannian.

These τ_σ form the (linear) holonomy group Ψ_p. The product is composition, or equivalently addition of curves, and the inverse τ_σ^{-1} is obtained by reversing the parametrization of σ. If $q \in N$ and β is a curve from p to q, then $\tau \to \tau_\beta \tau \tau_\beta^{-1}$ is an isomorphism of Ψ_p and Ψ_q. Hence, the holonomy group Ψ is independent of the point.

Observe that if J is an almost complex structure on a Riemannian manifold satisfying $\nabla J = 0$, then J commutes with Ψ. Hence, if $\mathbf{R}^{4n}/\Gamma$ is a quaternionic Kaehler manifold, then Γ/Γ^* and $\{J, K\}$ commute. In the next section, we will use this to show that for $n = 1$, Ψ is the trivial group $\{e\}$, or equivalently, $\Gamma = \Gamma^*$. In §4, we will show that for $n = 2$, Ψ is either $\{e\}$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, or $\mathbf{Z}_6$.

2. Classification in 4 real dimensions. According to Lemma 1, a quaternionic Kaehler manifold M is Riemannian flat. If M is compact and connected, then it follows that $M = \mathbf{R}^{4n}/\Gamma$, where $\Gamma \subset E(4n)$ is a discrete uniform subgroup without elements of finite order. To prove our classification theorems, we will first classify the holonomy $\Psi = \Gamma/\Gamma^*$, and then determine the (affine equivalence classes of) lattices Λ invariant under Ψ. In dimension 4, Ψ is the trivial group and all lattices are invariant and equivalent.

THEOREM 2. *If M is a compact connected quaternionic Kaehler manifold of real dimension* 4, *then M is a torus.*

PROOF. By the above remarks, we have $M = \mathbf{R}^4/\Gamma$ where $\Gamma \subset E(4)$. For $\alpha \in \Gamma$, we write $\alpha = (A, a)$ with $A(a) = a \neq 0$, and $A \in \Psi$. Choose $\{a, Ja, Ka, JKa\}$ as a basis for $\mathbf{R}^4$. Since J and K commute with $A \in \Psi$, it follows that $A = I$. Hence $\Psi = \{e\}$, Γ consists of pure translations, and M is a torus.   Q.E.D.

Conversely, every 4-torus inherits the standard quaternionic Kaehler structure on $\mathbf{R}^4$ described in §1.2 (since this structure is invariant under translation). We remark that Sommese [9] has shown that if J is complex Kaehler and K is complex (but not necessarily Kaehler), then M is still a torus.

3. Algebra.

PROPOSITION 1. *Let ζ_n be a primitive nth root of unity. If the real part of ζ_n is rational, then* $n = 1, 2, 3, 4,$ *or* 6.

PROOF (IMPROVED BY THE REFEREE). We adjoin ζ_n to the rationals $\mathbf{Q}$ and obtain the extension $\mathbf{Q}(\zeta_n)$. The degree of this extension is given by $[\mathbf{Q}(\zeta_n): \mathbf{Q}] = \phi(n)$, the Euler phi functions. If $\zeta_n = \alpha + i\beta$, with α rational, then $\zeta_n^2 - 2\alpha\zeta_n + 1 = 0$, and hence $\phi(n) \leqslant 2$. The phi function is given by $\phi(n) = \prod_{p|n} p^{r-1}(p - 1)$ where p^r divides n, but p^{r+1} does not. For $\phi(n) \leqslant 2$, it is easy to see that $n = 1, 2, 3, 4,$ or 6. The roots are 1, -1, $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $\pm i$, or $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, respectively.   Q.E.D.

Let $f_n(x)$ be the nth cyclotomic polynomial; the roots of f_n are precisely the primitive nth roots of unity and $\deg f_n = \phi(n)$. This polynomial is irreducible and has rational coefficients. If $g(x)$ is a polynomial (with rational coefficients) satisfying $g(\zeta_n) = 0$, then f_n divides g. In particular, the roots of g include all $\phi(n)$ primitive nth roots of unity. These considerations are important in §4 and §5A.

4. Classifications in 8 real dimensions. In this section, we give the affine (or equivalently topological—see Theorem 1) classification of compact quaternionic Kaehler 8-manifolds M. Our theorem appears quite lengthy because we have included, for $M = \mathbf{R}^8/\Gamma$, generators and relations for Γ as well as a representation of Γ as a discrete uniform subgroup of the rigid motions $E(8)$.

THEOREM 3. *There are* 12 *affine diffeomorphism classes of compact connected quaternionic Kaehler 8-manifolds. They are represented by the manifold* $\mathbf{R}^8/\Gamma$, *where* $\Gamma \subset E(8)$ *is one of the* 12 *groups given below. Here* Λ *is the translation lattice with generators* $\{a_1, \ldots, a_8\}$, $t_k = t_{a_k}$ *are translations, and* $\Psi = \Gamma/\Gamma^*$ *is the holonomy. Let* $i = 1, 2, 3, 4$ *and* $j = 5, 6, 7, 8$, *and let* E *be the span of* $\{a_1, a_2, a_3, a_4\}$.

(1) $\Psi = \{e\}$ *and* Γ *is generated by the translations* $\{t_1, \ldots, t_8\}$.

(2.1) $\Psi = \mathbf{Z}_2$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^2 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, *and* $\alpha t_j \alpha^{-1} = t_j^{-1}$; $\langle a_i, a_j \rangle = 0$ *and* $\alpha = (A, a_1/2)$ *with* $A(a_i) = a_i$ *and* $A(a_j) = -a_j$.

(2.2) $\Psi = \mathbf{Z}_2$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^2 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_k \alpha^{-1} = t_k^{-1}$, $k = 5, 6, 7$, $\alpha t_8 \alpha^{-1} = t_1 t_2 t_3 t_4 t_8^{-1}$; $\langle a_i, a_k \rangle = 0$, $k = 5, 6, 7$, *the projection of* a_8 *on* E *is* $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$, *and* $\alpha = (A, a_1/2)$ *with* $A(a_i) = a_i$, $A(a_k) = -a_k$, $k = 5, 6, 7$, *and* $A(a_8) = a_1 + a_2 + a_3 + a_4 - a_8$.

(2.3) $\Psi = \mathbf{Z}_2$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^2 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_5^{-1}$, $\alpha t_6 \alpha^{-1} = t_6^{-1}$, $\alpha t_7 \alpha^{-1} = t_1 t_2 t_3 t_4 t_7^{-1}$, $\alpha t_8 \alpha^{-1} = t_2 t_8^{-1}$; $\langle a_i, a_5 \rangle = \langle a_i, a_6 \rangle = 0$, *the projections of* a_7 *and* a_8 *on* E *are* $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$ *and* $\frac{1}{2}a_2$, *respectively, and* $\alpha = (A, a_1/2)$ *with* $A(a_i) = a_i$, $A(a_5) = -a_5$, $A(a_6) = -a_6$, $A(a_7) = a_1 + a_2 + a_3 + a_4 - a_7$, *and* $A(a_8) = a_2 - a_8$.

(2.4) $\Psi = \mathbf{Z}_2$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^2 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_5^{-1}$, $\alpha t_6 \alpha^{-1} = t_1 t_2 t_3 t_4 t_6^{-1}$, $\alpha t_7 \alpha^{-1} = t_2 t_7^{-1}$, $\alpha t_8 \alpha^{-1} = t_3 t_8^{-1}$; $\langle a_i, a_5 \rangle = 0$, *the projections of* a_6, a_7 *and* a_8 *on* E *are* $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$, $\frac{1}{2}a_2$, *and* $\frac{1}{2}a_3$ *respectively, and* $\alpha = (A, a_1/2)$ *with* $A(a_i) = a_i$, $A(a_5) = -a_5$, $A(a_6) = a_1 + a_2 + a_3 + a_4 - a_6$, $A(a_7) = a_2 - a_7$, *and* $A(a_8) = a_3 - a_8$.

(3.1) $\Psi = \mathbf{Z}_3$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^3 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_5^{-1} t_6^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_7^{-1} t_8^{-1}$; $\langle a_i, a_j \rangle = 0$, $\{a_5, a_6\}$ *and* $\{a_7, a_8\}$ *generate hexagonal plane lattices, and* $\alpha = (A, a_1/3)$ *with* $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = -a_5 - a_6$, $A(a_7) = a_8$, *and* $A(a_8) = -a_7 - a_8$.

(3.2) $\Psi = \mathbf{Z}_3$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^3 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_5^{-1} t_6^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_1 t_2 t_3 t_4 t_7^{-1} t_8^{-1}$; $\langle a_i, a_5 \rangle = \langle a_i, a_6 \rangle = 0$, $\{a_5, a_6\}$ *generates a hexagonal plane lattice, the projection of* a_7 *and* a_8 *on* E *is* $\frac{1}{3}(a_1 + a_2 + a_3 + a_4)$, *and* $\alpha = (A, a_1/3)$ *with* $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = -a_5 - a_6$, $A(a_7) = a_8$, *and* $A(a_8) = a_1 + a_2 + a_3 + a_4 - a_7 - a_8$.

(3.3) $\Psi = \mathbf{Z}_3$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^3 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_1 t_2 t_3 t_4 t_5^{-1} t_6^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_2 t_7^{-1} t_8^{-1}$; *the projection of* a_5 *and* a_6 *on* E *is* $\frac{1}{3}(a_1 + a_2 + a_3 + a_4)$, *the projection of* a_7 *and* a_8 *on* E *is* $\frac{1}{3}a_2$, *and* $\alpha = (A, a_1/3)$ *with* $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = a_1 + a_2 + a_3 + a_4 - a_5 - a_6$, $A(a_7) = a_8$, *and* $A(a_8) = a_2 - a_7 - a_8$.

(4.1) $\Psi = \mathbf{Z}_4$ *and* Γ *is generated by* $\{\alpha, t_1, \ldots, t_8\}$ *where* $\alpha^4 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_5^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_7^{-1}$; $\langle a_i, a_j \rangle = 0$, $\{a_5, a_6\}$ *and*

$\{a_7, a_8\}$ generate a square plane lattice, and $\alpha = (A, a_1/4)$ with $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = -a_5$, $A(a_7) = a_8$, and $A(a_8) = -a_7$.

(4.2) $\Psi = \mathbf{Z}_4$ and Γ is generated by $\{\alpha, t_1, \ldots, t_8\}$ where $\alpha^4 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_5^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_1 t_2 t_3 t_4 t_7^{-1}$; $\langle a_i, a_5 \rangle = \langle a_i, a_6 \rangle = 0$, $\{a_5, a_6\}$ generates a square plane lattice, the projection of a_7 and a_8 on E is $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$, and $\alpha = (A, a_1/4)$ with $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = -a_5$, $A(a_7) = a_8$, and $A(a_8) = a_1 + a_2 + a_3 + a_4 - a_7$.

(4.3) $\Psi = \mathbf{Z}_4$ and Γ is generated by $\{\alpha, t_1, \ldots, t_8\}$ where $\alpha^4 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_1 t_2 t_3 t_4 t_5^{-1}$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_2 t_7^{-1}$; the projection of a_5 and a_6 on E is $\frac{1}{2}(a_1 + a_2 + a_3 + a_4)$, the projection of a_7 and a_8 on E is $\frac{1}{2}a_2$, and $\alpha = (A, a_1/4)$ with $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = a_1 + a_2 + a_3 + a_4 - a_5$, $A(a_7) = a_8$, and $A(a_8) = a_2 - a_7$.

(6) $\Psi = \mathbf{Z}_6$ and Γ is generated by $\{\alpha, t_1, \ldots, t_8\}$ where $\alpha^6 = t_1$, $\alpha t_i \alpha^{-1} = t_i$, $\alpha t_5 \alpha^{-1} = t_6$, $\alpha t_6 \alpha^{-1} = t_5^{-1} t_6$, $\alpha t_7 \alpha^{-1} = t_8$, $\alpha t_8 \alpha^{-1} = t_7^{-1} t_8$; $\langle a_i, a_j \rangle = 0$, $\{a_5, a_6\}$ and $\{a_7, a_8\}$ generate hexagonal plane lattices, and $\alpha = (A, a_1/6)$ with $A(a_i) = a_i$, $A(a_5) = a_6$, $A(a_6) = a_6 - a_5$, $A(a_7) = a_8$, and $A(a_8) = a_8 - a_7$.

PROOF. The proof consists of three parts. In part I, we show that each of the 12 types admits a quaternionic Kaehler structure (induced from $\mathbf{R}^8$). In part II, we show that no two of these types are affinely diffeomorphic. Finally, in part III, we show that any compact connected quaternionic Kaehler 8-manifold is affinely diffeomorphic to one of the 12 types.

I. First observe that each of the 12 types of $\Gamma \subset E(8)$ is a discrete uniform subgroup without elements of finite order. Hence $\mathbf{R}^8/\Gamma$ is a compact flat Riemannian manifold.

We must construct a quaternionic Kaehler structure $\{J, K\}$ on $\mathbf{R}^8$, invariant under Γ. Let E be the span of $\{a_1, a_2, a_3, a_4\}$ and $E^\perp$ the orthogonal complement. Since $\mathbf{R}^8 = E \oplus E^\perp$, we obtain the required structure by identifying each summand with $\mathbf{R}^4$ (provided with the standard quaternionic Kaehler structure). The identifications may be arbitrary except that we can assume J preserves the span E_+ of the projection of $\{a_5, a_6\}$ on $E^\perp$, and $\langle Ja_5, a_6 \rangle \langle Ja_7, a_8 \rangle < 0$, where $\langle \ , \ \rangle$ is the standard inner product on Euclidean space.

To see that $\{J, K\}$ is invariant under Γ, it suffices to show that $A \in \Psi$ commutes with $\{J, K\}|_E$ and $\{J, K\}_{E^\perp}$ (the standard structure on $\mathbf{R}^4$ is invariant under translation). This is obvious on E for $A|_E =$ Identity, and also on $E^\perp$ whenever $A|_{E^\perp} = \pm$ Identity. A calculation shows that $A|_{E_+}$ is well-defined and is a rotation by $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$ depending on the holonomy type 3, 4, 6, respectively. Since $J|_{E_+}$ and $A|_{E_+}$ are both rotations, they commute on E_+. Let E_- be the orthogonal complement, i.e., $E^\perp = E_+ + E_-$. Since A and J are orthogonal, they both preserve E_-. A calculation shows that $A|_{E_-}$ is also a rotation by $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$ depending on the type. Hence, A and J commute on E_-, $E^\perp$, and finally $\mathbf{R}^8$.

We need to show that A and K commute; we will do this for holonomy type 4 and leave types 3 and 6 to the reader. If $X \in E_+$, then either $AX = JX$ or $AX = -JX$. We assume, without loss of generality, that $AX = JX$ (for we may replace J by $-J$ everywhere). Let $X \in E_+$ and $Y \in E_-$ satisfy $KX = Y$. The condition

$\langle Ja_5, a_6 \rangle \langle Ja_7, a_8 \rangle < 0$ forces $AY = -JY$, and hence $AKX = -JKX = KJX = KAX$. A similar argument establishes the equality $AK = KA$ on E_-, so A and K commute.

II. Let $M = \mathbf{R}^8/\Gamma$, so that $\pi_1(M) = \Gamma$ and $H_1(M; \mathbf{Z}) = \Gamma/[\Gamma, \Gamma]$, where $[\Gamma, \Gamma]$ is the commutator subgroup generated by $[\Gamma, \Gamma] = \{\sigma\tau\sigma^{-1}\tau^{-1} \mid \sigma, \tau \in \Gamma\}$. In the table below, we record $H_1(M; \mathbf{Z})$ for all 12 types; the calculations follow the table.

If $\mathbf{R}^8/\Gamma_1$ and $\mathbf{R}^8/\Gamma_2$ are affinely diffeomorphic, then Γ_1 and Γ_2 are isomorphic (Theorem 1). In particular, $\Gamma_1^* \simeq \Gamma_2^*$ (unique maximal abelian subgroups), $\Gamma_1/\Gamma_1^* \simeq \Gamma_2/\Gamma_2^*$, and $\Gamma_1/[\Gamma_1, \Gamma_1] \simeq \Gamma_2/[\Gamma_2, \Gamma_2]$, i.e., the holonomy and first homology groups are isomorphic. From Table 1, it then follows that $\mathbf{R}^8/\Gamma_1$ and $\mathbf{R}^8/\Gamma_2$ are of the same type.

TABLE 1

Type	$H_1(M; \mathbf{Z})$	Type	$H_1(M; \mathbf{Z})$
1	$\mathbf{Z}^8$	3.2	$\mathbf{Z}^4 \times \mathbf{Z}_3$
2.1	$\mathbf{Z}^4 \times \mathbf{Z}_2^4$	3.3	$\mathbf{Z}^4$
2.2	$\mathbf{Z}^4 \times \mathbf{Z}_2^3$	4.1	$\mathbf{Z}^4 \times \mathbf{Z}_2^2$
2.3	$\mathbf{Z}^4 \times \mathbf{Z}_2^2$	4.2	$\mathbf{Z}^4 \times \mathbf{Z}_2$
2.4	$\mathbf{Z}^4 \times \mathbf{Z}_2$	4.3	$\mathbf{Z}^4$
3.1	$\mathbf{Z}^4 \times \mathbf{Z}_3^2$	6	$\mathbf{Z}^4$

(As usual, the subscript on $\mathbf{Z}$ is the modulus and the superscript is the multiplicity of the product.)

(1) $[\Gamma, \Gamma]$ is trivial, so $H_1 = \Gamma = \mathbf{Z}^8$.

(2.1) $[\Gamma, \Gamma]$ is generated by t_k^2 for $k = 5, 6, 7, 8$; $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2^4$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 2, 3, \ldots, 8$.

(2.2) $[\Gamma, \Gamma]$ is generated by t_k^2 for $k = 5, 6, 7$, and $t_1 t_2 t_3 t_4 t_8^{-2}$; we may choose generators for Γ as α, $t_1 t_2 t_3 t_4 t_8^{-2}$, t_3, $t_4, \ldots, t_8$, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2^3$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 3, 4, \ldots, 8$.

(2.3) $[\Gamma, \Gamma]$ is generated by t_5^2, t_6^2, $t_1 t_2 t_3 t_4 t_7^{-2}$, $t_2 t_8^{-2}$; we may choose generators for Γ as α, $t_1 t_2 t_3 t_4 t_7^{-2}$, $t_2 t_8^{-2}$, $t_4, \ldots, t_8$, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2^2$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 4, \ldots, 8$.

(2.4) $[\Gamma, \Gamma]$ is generated by t_5^2, $t_1 t_2 t_3 t_4 t_6^{-2}$, $t_2 t_7^{-2}$, $t_3 t_8^{-2}$; we may choose generators for Γ as α, $t_1 t_2 t_3 t_4 t_6^{-2}$, $t_2 t_7^{-2}$, $t_3 t_8^{-2}$, t_5, t_6, t_7, t_8, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 5, 6, 7, 8$.

(3.1) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_5 t_6^2$, $t_8 t_7^{-1}$, $t_7 t_8^2$, or equivalently by $t_6 t_5^{-1}$, t_6^3, $t_8 t_7^{-1}$, t_8^3; we may choose generators for Γ as α, t_1, t_2, t_3, t_4, $t_6 t_5^{-1}$, t_6, $t_8 t_7^{-1}$, t_8, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_3^2$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 2, 3, 4, 6, 8$.

(3.2) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_5 t_6^2$, $t_8 t_7^{-1}$, $t_1 t_2 t_3 t_4 t_7^{-1} t_8^{-2}$, or equivalently by $t_6 t_5^{-1}$, t_6^3, $t_8 t_7^{-1}$, $t_1 t_2 t_3 t_4 t_8^{-3}$; we may choose generators for Γ as α, $t_1 t_2 t_3 t_4 t_8^{-3}$, t_3, t_4, $t_6 t_5^{-1}$, t_6, $t_8 t_7^{-1}$, t_8, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_3$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 3, 4, 6, 8$.

(3.3) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_1 t_2 t_3 t_4 t_5^{-1} t_6^{-2}$, $t_8 t_7^{-1}$, $t_2 t_7^{-1} t_8^{-2}$; we may choose generators for Γ as α, $t_6 t_5^{-1}$, $t_1 t_2 t_3 t_4 t_5^{-1} t_6^{-2}$, $t_8 t_7^{-1}$, $t_2 t_7^{-1} t_8^{-2}$, t_4, t_6, t_8, and hence $H_1 = \mathbf{Z}^4$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 4, 6, 8$.

(4.1) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_5 t_6$, $t_8 t_7^{-1}$, $t_7 t_8$, or equivalently by $t_6 t_5^{-1}$, t_6^2, $t_8 t_7^{-1}$, t_8^2; we may choose generators for Γ as α, t_2, t_3, t_4, $t_6 t_5^{-1}$, t_6, $t_8 t_7^{-1}$, t_8, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2^2$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 2, 3, 4, 6, 8$.

(4.2) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_5 t_6$, $t_8 t_7^{-1}$, $t_1 t_2 t_3 t_4 t_7^{-1} t_8^{-1}$, or equivalently by $t_6 t_5^{-1}$, t_6^2, $t_8 t_7^{-1}$, $t_1 t_2 t_3 t_4 t_8^{-2}$; we may choose generators for Γ as α, $t_1 t_2 t_3 t_4 t_8^{-2}$, t_3, t_4, $t_6 t_5^{-1}$, t_6, $t_8 t_7^{-1}$, t_8, and hence $H_1 = \mathbf{Z}^4 \times \mathbf{Z}_2$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 3, 4, 6, 8$.

(4.3) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, $t_1 t_2 t_3 t_4 t_5^{-1} t_6^{-1}$, $t_8 t_7^{-1}$, $t_2 t_7^{-1} t_8^{-1}$; we may choose generators for Γ as α, $t_6 t_5^{-1}$, $t_1 t_2 t_3 t_4 t_5^{-1} t_6^{-1}$, $t_8 t_7^{-1}$, $t_2 t_7^{-1} t_8^{-1}$, t_4, t_6, t_8, and hence $H_1 = \mathbf{Z}^4$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 4, 6, 8$.

(6) $[\Gamma, \Gamma]$ is generated by $t_6 t_5^{-1}$, t_5, $t_8 t_7^{-1}$, t_7, or equivalently by t_5, t_6, t_7, t_8; hence $H_1 = \mathbf{Z}_4$ with generators $\alpha[\Gamma, \Gamma]$ and $t_k[\Gamma, \Gamma]$ for $k = 2, 3, 4$.

III. Let M be a compact connected quaternionic Kaehler manifold. From §1.1 and Lemma 1, it follows that $M = \mathbf{R}^8/\Gamma$, where $\Gamma \subset E(8)$.

We begin by classifying Ψ. If $A \in \Psi$, then A preserves the lattice Λ (since $\Gamma^* \subset \Gamma$ is normal) and hence has a matrix representation with integral entries. In particular, the matrix invariants (trace, determinant, etc.) are integers. The vanishing of ∇J and ∇K implies $AJ = JA$ and $AK = KA$.

Let $a \in \mathbf{R}^8$ be a nonzero vector such that $A(a) = a$, and let E be the span of $\{a, Ja, Ka, JKa\}$ with 4-dimensional orthogonal complement $E^\perp$. Clearly $A\mid_E = I$. Since A, J, and K are orthogonal, $E^\perp$ is quaternionic and $A\mid_{E^\perp}$ is well defined. If $A\mid_{E^\perp} \neq I$, then $A - I: \mathbf{R}^8 \to E^\perp$ is onto, and hence $\Lambda \cap E^\perp$ is a 4-dimensional lattice in $E^\perp$. In particular, $A\mid_{E^\perp}$ has integral matrix invariants. Below, we will use the fact $\operatorname{tr}(A\mid_{E^\perp}) \in \mathbf{Z}$.

If we fix the complex structure J, then $E^\perp$ becomes a 2-dimensional complex vector space and we may diagonalize $A\mid_{E^\perp}$. Let $X \in E^\perp$ be a J-eigenvector with J-eigenvalue $\rho = \alpha + i\beta$, i.e., $AX = \alpha X + \beta JX$. The calculation $AKX = KAX = \alpha KX - \beta JKX$ shows that KX is also a J-eigenvector with J-eigenvalue $\bar\rho = \alpha - i\beta$. Since A has finite order, say n, it follows that ρ is a primitive nth root of unity. As a real 4×4 matrix relative to the basis $\{X, JX, KX, JKX\}$,

$$
A\mid_{E^\perp} = \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}
$$

with $\operatorname{tr}(A\mid_{E^\perp}) = 4\alpha \in \mathbf{Z}$. By the algebraic proposition in §3, the only possible values for n are $n = 1, 2, 3, 4$, and 6. In particular, we have $\rho = \pm 1$, $\pm i$, $e^{\pm i\pi/3}$, or $e^{\pm i\pi/6}$.

We can now complete the proof of

LEMMA 2. *The holonomy group* Ψ *is either* $\{1\}$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_4$, *or* $\mathbf{Z}_6$.

PROOF. By the above remarks, it suffices to show that Ψ is cyclic. We will first establish that Ψ is simultaneously diagonalizable (or equivalently Ψ is abelian), from which it easily follows that Ψ is cyclic.

Let A, $B \in \Psi$ be nontrivial. If $AB = I$, then $A = B^{-1}$ and $AB = BA$. $AB \neq I$ is harder. We will show (1) A and B fix a common 4-dimensional subspace E (and hence the orthogonal complement $E^{\perp}$), and (2) A and B are simultaneously diagonalizable on these subspaces. Let E_C denote the 4-dimensional $+1$ eigenspace of nontrivial $C \in \Psi$.

(1) If $X \in E_{AB} \cap BE_{AB}$ and $X \neq 0$, then $E_{AB} = BE_{AB}$, for X, JX, KX, JKX generate E_{AB} and the quaternionic structure commutes with B. If $BX = Y \in E_{AB}$, then $AY = X \in E_{AB}$, and hence $E_{AB} = AE_{AB}$.

If $E_{AB} \cap BE_{AB} = 0$, then we may write $Z \in E_{BA}$ as $Z = X_1 + BX_2$ where X_1, $X_2 \in E_{AB}$. Now, $Z = BAZ = BAX_1 + BX_2$, and hence $X_1 = BAX_1$, i.e., $E_{BA} = E_{AB}$. If $W \in E_A$, then we may write $W = Y_1 + BY_2$ where Y_1, $Y_2 \in E_{AB}$. Now, $BW = BAW = BAY_1 + BABY_2 = Y_1 + BY_2$, and hence $W = BW$, i.e., $E_A = E_B = E_{AB}$.

(2) We will show that A and B commute on $E = E_A = E_B$ and $E^{\perp}$. This is trivial on E. On $E^{\perp}$, we may fix a complex structure, say J, and diagonalize

$$A\,|_{E^{\perp}} = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix} \quad B\,|_{E^{\perp}} = \begin{bmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{bmatrix}$$
$$\mathrm{rel}\{X_1, X_2\} \qquad \mathrm{rel}\{Y_1, Y_2\}$$

where ρ and τ are J-eigenvalues, the J-eigenvectors have unit length, and $X_2 = KX_1$ and $Y_2 = KY_1$.

We have an S^2-action on $E^{\perp}$ given by $(a, b, c) \rightarrow aJ + bK + cJK$, with $a^2 + b^2 + c^2 = 1$. Recall that $F = aJ + bK + cJK$ satisfies $F^2 = -I$. For $X \in E^{\perp}$ and $\|X\| = 1$, the orbit $S^2 \cdot X$ is a standard 2-sphere of radius 1. Since the dimension of $E^{\perp}$ is 4, it follows that span $\{Y_1, JY_1\}$ and $S^2 \cdot X_1$ have a common element, which we may choose to be $Y_1 = FX_1$. Now,

$$AY_1 = AFX_1 = FAX_1 = F\rho X_1 = \rho^{-1}FX_1 - 2\beta a X_1 = \left(\rho^{-1} + 2\beta a F\right)Y_1$$
$$= \left[\alpha I + (2\beta a^2 - \beta)J + 2\beta ab K + 2\beta ac JK\right]Y_1$$

and

$$AY_1 = \tau Y_1 = \gamma Y_1 + \delta JY_1$$

where $\tau = \gamma + i\delta$. Thus

$$\alpha = \gamma,$$
$$2\beta\alpha^2 - \beta = \delta,$$
$$2\beta\alpha b = 0,$$
$$2\beta\alpha c = 0.$$

If $\beta = 0$, then $\delta = 0$ and $\tau = \pm 1 = \rho$, i.e., $B\,|_{E^{\perp}} = I = A\,|_{E^{\perp}}$ and they commute on $E^{\perp}$. If $a = 0$, then $AY_1 = \rho^{-1}Y_1$, and hence $Y_1 = \nu X_2$ for some J-complex number ν. By orthogonality, $Y_2 = \mu X_1$ for some J-complex number μ, and it is clear that A and B commute on $E^{\perp}$. Finally, if $a \neq 0$ and $\beta \neq 0$, then $b = c = 0$ and $F = \pm J$, i.e., $Y_1 = \pm JX_1$. We may choose Y_2 so that $Y_2 = \pm JX_2$, and it is clear that A and B commute on $E^{\perp}$.

This completes the proofs of statements (1) and (2). We have shown that there is a decomposition $\mathbf{R}^8 = E \oplus E^\perp$ and a J-complex basis $\{X_1, X_2\}$ for $E^\perp$ so that for any $A \in \Psi$, we have $A\,|_E = I$ and

$$A\,|_{E^\perp} = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix}.$$

Furthermore, ρ is an nth root of unity where $n = 1, 2, 3, 4,$ or 6. It follows easily that $\Psi = \{e\}$, Z_2, Z_3, Z_4, or Z_6. This completes the proof of the lemma.   Q.E.D.

It remains to classify (up to an affine diffeomorphism commuting with Ψ) the lattices on $\mathbf{R}^8$ invariant under each Ψ:

$\Psi = \{e\}$.

In this case, $\Gamma = \Gamma^*$ and $\mathbf{R}^8/\Gamma$ is a torus. Clearly any two lattices are invariant and equivalent under Ψ, and any two tori are affinely diffeomorphic.

$\Psi = \mathbf{Z}_2$.

Choose $(A, a) \in \Gamma$ such that A generates Ψ, $A(a) = a$, and $\|a\|$ minimal for this. Note that a is not in the lattice Λ. Otherwise, $(I, -a) \in \Gamma$ and $(A, a)(I, -a) = (A, 0) \in \Gamma$; but $(A, 0)$ has 0 as a fixed point; contradiction. For $\Psi = \mathbf{Z}_2$, we have $A\,|_{E^\perp} = -I$ and $A + I \colon \mathbf{R}^8 \to E$. Since A preserves Λ, it follows that $\Lambda \cap E$ is a 4-dimensional lattice with generators, say $a_1, a_2, a_3,$ and a_4. Now $(A, a)^2 = (I, 2a)$ shows that $2a \in \Lambda \cap E$, and by the minimality of $\|a\|$, we may choose these generators so that $a = a_1/2$. If we complete $\{a_1, a_2, a_3, a_4\}$ to a full set of lattice generators for Λ, by adding $a_5, a_6, a_7,$ and a_8, then we have

$$a_j = \alpha_j^1 a_1 + \alpha_j^2 a_2 + \alpha_j^3 a_3 + \alpha_j^4 a_4 + \beta_j b_j$$

where $j = 5, 6, 7, 8$, $b_j \in E^\perp$, and the coefficients are real numbers. Since $A(a_j) + a_j = 2\alpha_j^1 a_1 + \cdots + 2\alpha_j^4 a_4 \in \Lambda$, we may reduce $\mathrm{mod}\{a_1, a_2, a_3, a_4\}$ to see that α_j^i is 0 or $\frac{1}{2}$, for $i = 1, 2, 3, 4$ and $j = 5, 6, 7, 8$.

For $j = 5, 6, 7, 8$, we will examine the possible 4-tuples $(\alpha_j^1, \alpha_j^2, \alpha_j^3, \alpha_j^4)$ whose entries are either 0 or $\frac{1}{2}$. The easiest case is $\alpha_j^i = 0$ for all i and j, that is, $a_j \in E^\perp$ and $A(a_j) = -a_j$. Hence (A, a) and (I, a_k), $k = 2, 3, \ldots, 8$, generate Γ and the relations are given by type 2.1 (the quaternionic Klein bottle).

Assume that $a_5, a_6, a_7 \in E^\perp$, i.e., $\alpha_5^i = \alpha_6^i = \alpha_7^i = 0$ for $i = 1, 2, 3, 4$, and $a_8 \notin E^\perp$. If $\alpha_8^i = \frac{1}{2}$ for exactly one i, then by renumbering, we may assume that $\alpha_8^1 = \frac{1}{2}$ or $\alpha_8^2 = \frac{1}{2}$ (note that a_1 is distinguished in $\{a_1, a_2, a_3, a_4\}$ by the requirement $a = a_1/2$). If $(\alpha_8^1, \alpha_8^2, \alpha_8^3, \alpha_8^4) = (\frac{1}{2}, 0, 0, 0)$, that is, $a_8 = \frac{1}{2}a_1 + \beta_8 b_8$, then $(I, -a_8)(A, a) = (A, a - a_8) = (A, -\beta_8 b_8)$ and this rigid motion fixes $-\frac{1}{2}\beta_8 b_8$; contradiction. For $\alpha_8^2 = \frac{1}{2}$, we change the lattice generator a_2 by $a_2 \to a_2 - a_1 - a_3 - a_4$ to obtain $\alpha_8^i = \frac{1}{2}$ for all i. In this way we see that if one or more of α_8^i equal $\frac{1}{2}$, then either there is a contradiction or, by a suitable change of the lattice generators in E, we may assume $a_8 = \frac{1}{2}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3 + \frac{1}{2}a_4 + \beta_8 b_8$, where $b_8 \in E$. Hence $A(a_8) = \frac{1}{2}a_1 + \cdots + \frac{1}{2}a_4 - \beta_8 b_8 = a_1 + \cdots + a_4 - a_8$ and this flat manifold is of type 2.2.

Assume that $a_5, a_6 \in E^\perp$, i.e., $\alpha_5^i = \alpha_6^i = 0$ for $i = 1, 2, 3, 4$, and $a_7, a_8 \notin E^\perp$. By the above argument, we may assume $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, in particular,

$A(a_7) = a_1 + \cdots + a_4 - a_7$. Below we list the possibilities for $(\alpha_8^1, \alpha_8^2, \alpha_8^3, \alpha_8^4)$, up to a permutation of $\{a_2, a_3, a_4\}$ (always a_1 is fixed by the condition $a = a_1/2$). Most of the entries reduce to the third entry by a change of lattice generators; we indicate this change next to each 4-tuple. Two of the entries give contradictions for we cannot obtain a lattice vector with associated 4-tuple $(\frac{1}{2}, 0, 0, 0)$ (as mentioned above this contradicts the freeness of the Γ-action). The last entry may be reduced to type 2.2 by the indicated change of lattice vector. In changing generators, we must be careful to preserve the assumption $\alpha_7^i = \frac{1}{2}$, for all i, or equivalently, the sum $a_1 + a_2 + a_3 + a_4$. If a lattice generator does not appear, then it is left unchanged.

$(0, 0, 0, 0)$ $a_8 \to a_7, a_7 \to a_8$ (reduces to type 2.2)

$(\frac{1}{2}, 0, 0, 0)$ contradiction

$(0, \frac{1}{2}, 0, 0)$ type 2.3

$(\frac{1}{2}, \frac{1}{2}, 0, 0)$ $a_2 \to a_2 + a_1, a_3 \to a_3 - a_1$ (reduces to type 2.3)

$(0, \frac{1}{2}, \frac{1}{2}, 0)$ $a_2 \to a_2 + a_3, a_4 \to a_4 - a_3$ (reduces to type 2.3)

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ $a_2 \to a_2 + a_1 + a_3, a_4 \to a_4 - a_1 - a_3$ (reduces to type 2.3)

$(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $a_7 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $a_7 \to a_7 - a_8$ (reduces to type 2.2)

For those entries that reduce to $(0, \frac{1}{2}, 0, 0)$, we have $a_8 = \frac{1}{2}a_2 + \beta_8 b_8$ and $A(a_8) = \frac{1}{2}a_2 - \beta_8 b_8 = a_2 - a_8$; this is of type 2.3.

Assume now that $a_5 \in E^\perp$, i.e., $\alpha_5^i = 0$ for $i = 1, 2, 3, 4$, and $a_6, a_7, a_8 \notin E^\perp$. The above argument shows that we may assume $(\alpha_6^1, \alpha_6^2, \alpha_6^3, \alpha_6^4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4) = (0, \frac{1}{2}, 0, 0)$. Below we list the possible 4-tuples for a_8, up to a permutation of $\{a_3, a_4\}$. In changing generators, we must preserve a_1, a_2, and $a_1 + \cdots + a_4$.

$(0, 0, 0, 0)$ $a_6 \to a_7, a_7 \to a_8, a_8 \to a_6$ (reduces to type 2.3)

$(\frac{1}{2}, 0, 0, 0)$ contradiction

$(0, \frac{1}{2}, 0, 0)$ $a_8 \to a_8 - a_7$ (reduces to type 2.3)

$(0, 0, \frac{1}{2}, 0)$ type 2.4

$(\frac{1}{2}, \frac{1}{2}, 0, 0)$ $a_8 - a_7$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction

$(\frac{1}{2}, 0, \frac{1}{2}, 0)$ $a_3 \to a_3 + a_1, a_4 \to a_4 - a_1$ (reduces to type 2.3)

$(0, \frac{1}{2}, \frac{1}{2}, 0)$ $a_3 \to a_3 + a_2, a_4 \to a_4 - a_2$ (reduces to type 2.3)

$(0, 0, \frac{1}{2}, \frac{1}{2})$ $a_6 - a_7 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$ $a_3 \to a_3 + a_2 + a_1, a_4 \to a_4 - a_2 - a_1$ (reduces to type 2.3)

$(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$ $a_8 \to a_6 - a_7 - a_8$ (reduces to type 2.2)

$(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $a_6 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction

$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ $a_8 \to a_8 - a_6$ (reduces to type 2.2)

The fourth entry in this list gives the only new manifold. It is characterized by $A(a_8) = \frac{1}{2}a_3 - \beta_8 b_8 = a_3 - a_8$ where $a_8 = \frac{1}{2}a_3 + \beta_8 b_8$; this is of type 2.4.

Finally, by the above arguments, we may assume $(\alpha_5^1, \alpha_5^2, \alpha_5^3, \alpha_5^4) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\alpha_6^1, \alpha_6^2, \alpha_6^3, \alpha_6^4) = (0, \frac{1}{2}, 0, 0)$, and $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4) = (0, 0, \frac{1}{2}, 0)$. The possibilities for a_8 are listed below. The generator changes must preserve a_1, a_2, a_3, and a_4.

$(0, 0, 0, 0)$	$a_5 \to a_6, a_6 \to a_7, a_7 \to a_8, a_8 \to a_5$ (reduces to type 2.4)
$(\frac{1}{2}, 0, 0, 0)$	contradiction
$(0, \frac{1}{2}, 0, 0)$	$a_8 \to a_6 - a_8$ (reduces to type 2.4)
$(0, 0, \frac{1}{2}, 0)$	$a_8 \to a_7 - a_8$ (reduces to type 2.4)
$(0, 0, 0, \frac{1}{2})$	$a_5 - a_6 - a_7 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	$a_8 - a_6$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, 0, \frac{1}{2}, 0)$	$a_8 - a_7$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, 0, 0, \frac{1}{2})$	$a_8 \to a_5 - a_6 - a_7 - a_8$ (reduces to type 2.4)
$(0, \frac{1}{2}, \frac{1}{2}, 0)$	$a_8 \to a_8 - a_6 - a_7$ (reduces to type 2.4)
$(0, \frac{1}{2}, 0, \frac{1}{2})$	$a_8 \to a_5 - a_7 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(0, 0, \frac{1}{2}, \frac{1}{2})$	$a_8 \to a_5 - a_6 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$	$a_8 \to a_8 - a_6 - a_7$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$	$a_8 \to a_5 - a_7 - a_8$ (reduces to type 2.4)
$(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2})$	$a_8 \to a_5 - a_6 - a_8$ (reduces to type 2.4)
$(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$a_8 \to a_5 - a_8$ gives $(\frac{1}{2}, 0, 0, 0)$; contradiction
$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$a_8 \to a_5 - a_8$ (reduces to type 2.4)

In this list, no new types appear. In particular, we may always choose one of the lattice generators to be in $E^\perp$. This completes the classification with holonomy $\mathbf{Z}_2$.

$\Psi = \mathbf{Z}_3$.

Choose $(A, a) \in \Gamma$ such that A generates Ψ, $A(a) = a$, and $\|a\|$ minimal for this. As noted in the $\mathbf{Z}_2$ case, $a \notin \Lambda$. We again fix J on $E^\perp$ and diagonalize

$$A\,|_{E^\perp} = \begin{bmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{bmatrix}$$

where ρ is a cube root of unity. Hence $A^2 + A + I$: $\mathbf{R}^8 \to E$ preserves A, and it follows that $\Lambda \cap E$ is a 4-dimensional lattice with generators a_1, a_2, a_3, and a_4. Now $(A, a)^3 = (I, 3a)$ shows that $3a \in \Lambda$, and by the minimality of $\|a\|$, we may choose these generators so that $a = a_1/3$.

To complete the set of lattice generators, we will use the following lemma.

LEMMA 3. *For* $\Psi = \mathbf{Z}_3$, *there is a set of lattice generators* $\{a_1, \ldots, a_8\}$ *for* Λ *satisfying* $a_1, a_2, a_3, a_4 \in E$, $A(a_5) = a_6$, *and* $A(a_7) = a_8$.

PROOF. It suffices to choose a_5, a_6, a_7, a_8 satisfying the above condition. Let P_V be the projection onto the vector subspace $V \subset \mathbf{R}^8$. Choose $a_5 \notin E \cap \Lambda$ which minimizes the length $\|P_{E^\perp} a_5\|$ among all nonzero lattice vectors *not* in $E \cap \Lambda$ (the discreteness of Λ allows the minimum). Let Λ_1 be the sublattice generated by $\{a_1, a_2, a_3, a_4, a_5, A(a_5)\}$, and let E_1 be the span of these vectors. Choose $a_7 \notin \Lambda_1$ which minimizes the length $\|P_{E_1^\perp} a_7\|$ among all nonzero lattice vectors *not* in Λ_1.

We claim that $S = \{a_1, \ldots, a_5, A(a_5), a_7, A(a_7)\}$ generates Λ. To see this, assume the existence of a lattice vector y not in the integral span of S. It is easy to see that S spans $\mathbf{R}^8$, and we may write

$$y = r_1 a_1 + \cdots + r_5 a_5 + r_6 A(a_5) + r_7 a_7 + r_8 A(a_7),$$

where at least one of the coefficients is not integral. By adding integral multiples of S, we may assume that $|r_i| \leqslant \frac{1}{2}$, $i = 1, \ldots, 8$. Also we may assume $y \notin E \cap \Lambda$ since $\{a_1, a_2, a_3, a_4\}$ were chosen to generate $E \cap \Lambda$. Hence, at least one of r_5, r_6, r_7, r_8 is nonzero. If r_7 or r_8 is nonzero, then $y \notin \Lambda_1$ and

$$P_{E_1^\perp} y = r_7 P_{E_1^\perp} a_7 + r_8 P_{E_1^\perp} A(a_7)$$

and

$$\|P_{E_1^\perp} y\| = \left(r_7^2 + 2 r_7 r_8 \cos(\pm \pi/3) + r_8^2\right)^{\frac{1}{2}} \|P_{E_1^\perp} a_7\| < \|P_{E_1^\perp} a_7\|.$$

This contradicts our choice of a_7. If r_7 and r_8 are both zero, then

$$P_{E^\perp} y = r_5 P_{E^\perp} a_5 + r_6 P_{E^\perp} A(a_5)$$

and, as above,

$$\|P_{E^\perp} y\| < \|P_{E^\perp} a_5\|.$$

This contradicts our choice of a_5, and hence y does not exist.   Q.E.D.

Let us choose generators for Λ from Lemma 3 and set

$$a_j = \alpha_j^1 a_1 + \alpha_j^2 a_2 + \alpha_j^3 a_3 + \alpha_j^4 a_4 + \beta_j b_j$$

where $j = 5$ or 7 (a_6 and a_8 are determined by Lemma 3), $b_j \in E$, and the coefficients are real numbers. Since $A^2(a_j) + A(a_j) + a_j = 3\alpha_j^1 a_1 + \cdots + 3\alpha_j^4 a_4 \in \Lambda$, we may reduce $\mathrm{mod}\{a_1, a_2, a_3, a_4\}$ to see that the only possibilities for α_j^i are 0, $\frac{1}{3}$, $\frac{2}{3}$, where $i = 1, 2, 3, 4$ and $j = 5, 7$.

For $j = 5, 7$, we will examine the possible 4-tuples $(\alpha_j^1, \alpha_j^2, \alpha_j^3, \alpha_j^4)$ whose entries are either 0, $\frac{1}{3}$, or $\frac{2}{3}$. The easiest case is $\alpha_j^i = 0$ for $i = 1, 2, 3, 4$ and $j = 5, 7$, i.e., $a_j \in E$, $A(a_j) = a_{j+1}$, and $A(a_{j+1}) = -a_j - a_{j+1}$ for $j = 5, 7$. Hence (A, a) and (I, a_k), $k = 2, \ldots, 8$, generate Γ and the relations are given by type 3.1.

Assume that $a_5 \in E^\perp$, i.e., $\alpha_5^i = \alpha_6^i = 0$ for $i = 1, 2, 3, 4$, and $a_7 \notin E^\perp$. If $\alpha_7^i \in \{0, \frac{1}{3}\}$ for all i, then by an argument analogous to the $\mathbf{Z}_2$ case, we choose $\{a_1, a_2, a_3, a_4\}$ so that $\alpha_7^i = \frac{1}{3}$ for all i (similarly if $\alpha_7^i \in \{0, \frac{2}{3}\}$). If $\alpha_7^i = \frac{2}{3}$ for all i, then the change of basis $a_7 \to a_1 + a_2 + a_3 + a_4 - a_7$ gives $\alpha_7^i = \frac{1}{3}$ for all i (we must change a_8 accordingly). It is obvious that the new basis will generate Λ.

If both $\frac{1}{3}$ and $\frac{2}{3}$ appear in $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4)$, we have 3 possibilities: $(\frac{1}{3}, \frac{2}{3}, *, *)$, $(\frac{2}{3}, \frac{1}{3}, *, *)$ or $(*, \frac{1}{3}, \frac{2}{3}, *)$, where $* \in \{0, \frac{1}{3}, \frac{2}{3}\}$ and no two $*$ are necessarily equal. The first 4-tuple becomes $(\frac{1}{3}, \frac{1}{3}, *, *)$ by the change $a_2 \to a_1 + a_2$ and $a_7 \to a_1 + a_2 - a_7$. The second becomes $(\frac{1}{3}, \frac{1}{3}, *, *)$ by the change $a_2 \to a_1 + a_2$, and the third becomes $(*, \frac{1}{3}, \frac{1}{3}, *)$ by the change $a_2 \to a_2 + a_3$. It is clear from this argument that we may eliminate all $\frac{2}{3}$'s.

So for $a_5 \in E^\perp$ and $a_7 \notin E^\perp$, we may change the lattice generators (preserving a_1 and satisfying the conditions in Lemma 3) so that $\alpha_7^i = \frac{1}{3}$ for all i, i.e.,

$$a_7 = \tfrac{1}{3}a_1 + \cdots + \tfrac{1}{3}a_4 + \beta_7 b_7, \qquad b_7 \in E^\perp,$$
$$(a_7) = \tfrac{1}{3}a_1 + \cdots + \tfrac{1}{3}a_4 + \beta_7 A(b_7),$$
$$A(a_8) = A^2(a_7) = \tfrac{1}{3}a_1 + \cdots + \tfrac{1}{3}a_4 + \beta_7 A^2(b_7)$$
$$= \tfrac{1}{3}a_1 + \cdots + \tfrac{1}{3}a_4 - \beta_7 b_7 - \beta_7 A(b_7)$$
$$= a_1 + \cdots + a_4 - a_7 - a_8.$$

This is of type 3.2.

Finally, we assume that $a_5, a_7 \notin E^\perp$. By the above argument, we may assume $\alpha_5^i = \frac{1}{3}$ for all i, i.e., $a_5 = \tfrac{1}{3}a_1 + \cdots + \tfrac{1}{3}a_4 + \beta_5 b_5$. For $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4)$, we shall reduce each possibility to $(0, \frac{1}{3}, 0, 0)$ by a suitable change of lattice generators (preserving a_1 and $a_1 + \cdots + a_4$, and satisfying the conditions in Lemma 3). If $\alpha_7^i \in \{0, \frac{1}{3}\}$ for all i, then the arguments in the $\mathbf{Z}_2$ case show that we may reduce $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4)$ to $(0, \frac{1}{3}, 0, 0)$. If $\alpha_7^i \in \{0, \frac{2}{3}\}$ for all i, we may replace $\frac{2}{3}$ by $\frac{1}{3}$ as the following example illustrates: $(\frac{2}{3}, \frac{2}{3}, 0, 0)$ becomes $(\frac{1}{3}, \frac{1}{3}, 0, 0)$ by the change $a_7 \to a_1 + a_2 - a_7$. If both $\frac{1}{3}$ and $\frac{2}{3}$ appear, and if $\frac{2}{3}$ occurs more often than $\frac{1}{3}$, we may reverse the roles of $\frac{1}{3}$ and $\frac{2}{3}$ by $a_7 \to \delta_1 a_1 + \delta_2 a_2 + \delta_3 a_3 + \delta_4 a_4 - a_7$, where $\delta_i = 0$ or 1 whenever $\alpha_7^i = 0$ or $\frac{2}{3}$ respectively. With these simplifications, we list below, up to a permutation of $\{a_2, a_3, a_4\}$, the possibilities for $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4)$. The given lattice changes replace the given 4-tuple with $(0, \frac{1}{3}, 0, 0)$.

$(0, \frac{1}{3}, \frac{2}{3}, 0)$ $a_2 \to a_2 + 2a_3, a_4 \to a_4 - 2a_3$ (reduces to type 3.3)

$(\frac{1}{3}, \frac{2}{3}, 0, 0)$ $a_2 \to a_2 + 2a_1, a_3 \to a_3 - 2a_1, a_7 \to a_1 + a_2 - a_7$

        (reduces to type 3.3)

$(0, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ $a_2 \to a_2 - a_1, a_3 \to a_3 + a_1, a_7 \to a_7 - a_5$ (reduces to type 3.3)

$(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0)$ $a_2 \to a_2 + a_1 + 2a_3, a_4 \to a_4 - a_1 - 2a_3$ (reduces to type 3.3)

$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ $a_2 \to a_2 + 2a_1 + a_3, a_4 \to a_4 - 2a_1 - a_3$ (reduces to type 3.3)

$(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ $a_2 \to a_2 + a_1, a_3 \to a_3 - a_1, a_7 \to a_7 - a_5$ (reduces to type 3.3)

$(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ $a_7 \to a_7 - a_5$ (reduces to type 3.3)

$(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ $a_7 - a_5$ gives $(\frac{1}{3}, 0, 0, 0)$; contradiction

Hence, for $a_5, a_7 \notin E^\perp$, we may change the lattice generators so that $(\alpha_5^1, \alpha_5^2, \alpha_5^3, \alpha_5^4) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\alpha_7^1, \alpha_7^2, \alpha_7^3, \alpha_7^4) = (0, \frac{1}{3}, 0, 0)$, i.e.,

$$a_7 = \tfrac{1}{3}a_2 + \beta_7 b_7,$$
$$a_8 = A(a_7) = \tfrac{1}{3}a_2 + \beta_7 A(b_7),$$
$$A(a_8) = \tfrac{1}{3}a_2 + \beta_7 A^2(b_7) = \tfrac{1}{3}a_2 - \beta_7 b_7 - \beta_7 A(b_7)$$
$$= a_2 - a_7 - a_8.$$

This is of type 3.3. This completes the classification with $\mathbf{Z}_3$ holonomy.

$\Psi = \mathbf{Z}_4$.

Choose $(A, a) \in \Gamma$ such that A generates Ψ, $A(a) = a$, and $\|a\|$ minimal for this. As noted in the previous cases, $a \notin \Lambda$. We can diagonalize $A\mid_{E^{\perp}}$ with J-eigenvalues $\pm i$. Hence $A^2 + I$: $\mathbf{R}^8 \to E$ and $\Lambda \cap E$ is a 4-dimensional lattice with generators a_1, a_2, a_3, a_4. Now $(A, a)^4 = (I, 4a)$ shows that $4a \in \Lambda$, and by the minimality of $\|a\|$, we may choose these generators so that $a = a_1/4$. To complete the set of lattice generators we use Lemma 3; the proof goes through in this case replacing $\pm\frac{\pi}{3}$ by $\pm\frac{\pi}{2}$.

Set $a_j = \alpha_j^1 a_1 + \alpha_j^2 a_2 + \alpha_j^3 a_3 + \alpha_j^4 a_4 + \beta_j b_j$ where $j = 5$, 7, $b_j \in E^{\perp}$, and the coefficients are real numbers. Since $A^2 a_j + a_j = 2\alpha_j^1 a_1 + \cdots + 2\alpha_j^4 a_4 \in \Lambda$, we may reduce mod $\{a_1, a_2, a_3, a_4\}$ and see that the only possibilities for α_j^i are 0 or $\frac{1}{2}$ for $i = 1, 2, 3, 4$ and $j = 5, 7$. By an argument similar to the $\mathbf{Z}_3$ case, we can show that the only compact connected quaternionic Kaehler 8-manifolds with holonomy $\mathbf{Z}_4$ which can occur are of type 4.1, 4.2, or 4.3.

$\Psi = \mathbf{Z}_6$.

Choose $(A, a) \in \Gamma$ such that A generates Ψ, $A(a) = a$, and $\|a\|$ minimal for this. As noted in the previous cases, $a \notin \Lambda$. We can diagonalize $A\mid_{E^{\perp}}$ with J-eigenvalues $e^{(\pm\pi/6)i}$. Hence $A^2 - A + I$: $\mathbf{R}^8 \to E$ and $\Lambda \cap E$ is a 4-dimensional lattice with generators a_1, a_2, a_3, a_4. Now $(A, a)^6 = (I, 6a)$ shows that $6a \in \Lambda$, and by the minimality of $\|a\|$, we may choose these generators so that $a = a_1/6$. To complete the set of lattice generators, we use Lemma 3; the proof goes through in this case replacing $\pm\frac{\pi}{3}$ by $\pm\frac{\pi}{6}$.

Set $a_j = \alpha_j^1 a_1 + \alpha_j^2 a_2 + \alpha_j^3 a_3 + \alpha_j^4 a_4 + \beta_j b_j$ where $j = 5$, 7, $b_j \in E^{\perp}$, and the coefficients are real numbers. Since $A^2(a_j) - A(a_j) + a_j = \alpha_j^1 a_1 + \cdots + \alpha_j^4 a_4 \in \Lambda$, we may reduce modulo $\{a_1, a_2, a_3, a_4\}$ and see that the only possibility for α_j^i is 0, for all i and j, i.e., $A(a_6) = a_6 - a_5$ and $A(a_8) = a_8 - a_7$. Hence (A, a) and (I, a_k), $k = 2, 3, \ldots, 8$, generate Γ and the relations are given by type 6.

5. Remarks. A. In higher dimensions, Γ is more complicated. For example, Ψ is not cyclic, in general, and is probably not even abelian. A positive statement we can make is that if $A \in \Psi$ and ζ_n (primitive nth root of unity) is a (J-) eigenvalue of A, then all primitive nth roots of unity are (J-) eigenvalues of A. To see this, fix J and diagonalize A with J-eigenvalues $\{\zeta = \alpha + i\beta\}$. Since A has finite order, ζ is a primitive (say, nth) root of unity.

As a real matrix, A consists of the 2×2 diagonal blocks

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}.$$

Hence the characteristic polynomial $\det(xI - A)$ is a product of the polynomials $x^2 - 2\alpha x + 1$. Note that ζ is a root of this quadratic. Since $\det(xI - A)$ has integer coefficients (A preserves a lattice), it follows that the nth cyclotomic polynomial (see §3) divides $\det(xI - A)$.

B. It is fortunate that only 12 equivalence classes occur in dimension 8. However it is no surprise that the number of classes is finite. According to Bieberbach [2], there is a finite number of affine diffeomorphism classes of the compact connected flat Riemannian n-manifolds (for any n).

C. A complex manifold is projective if it can be complex analytically imbedded in a complex projective space. If M is a quaternionic manifold then M has a 2-sphere of complex structures $\{aJ + bK + cJK \mid a^2 + b^2 + c^2 = 1\}$, and it is natural to ask which structures are projective. Sommese [9] has shown that at most countably many are projective.

D. A Hodge manifold M is a Kaehler manifold whose Kaehler 2-form is in the image of the inclusion $H^2(M; \mathbf{Z}) \hookrightarrow H^2(M; \mathbf{R})$. The imbedding theorem of Kodaira [5] states that a compact Hodge manifold is projective. If M is a quaternionic Kaehler manifold, then at most countably many of the complex structures are Hodge. In light of this, we shall say that a quaternionic Kaehler manifold is Hodge if at least one of the complex structures is Hodge.

It would be increasing to know if each of our 12 equivalence classes in Theorem 3 admits a Hodge representative. To see if a representative has a chance to be Hodge, first notice that if $\mathbf{R}^8/\Gamma$ is Hodge, then $\mathbf{R}^8/\Gamma^*$ is Hodge. Now one uses the Riemann conditions (see [3, p. 62]) to determine if the torus $\mathbf{R}^8/\Gamma^*$ is Hodge. The given representatives in Theorem 3 (i.e., the given representations of Γ in $E(8)$) are not all Hodge.

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