

AMPLENESS AND CONNECTEDNESS IN COMPLEX G/P

BY

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ABSTRACT. This paper determines the “ampleness” of the tangent bundle of the complex homogeneous space, G/P , by calculating the maximal fibre dimension of the desingularization of a nilpotent subvariety of the Lie algebra of G .

1. Introduction. Let G be a connected semisimple complex Lie Group. Let P be a parabolic subgroup of G , and define the homogeneous space $Z = G/P$. Let $\phi: \mathbf{P}(T^*Z) \rightarrow \mathbf{P}^N$ be the map determined by the global sections of TZ , the tangent bundle of Z ; see §3. Define the ampleness of TZ , $\text{amp}(TZ)$, to be the maximum fibre dimension of ϕ , and the coampleness, $\text{ca}(Z)$, to be $\dim Z - \text{amp}(TZ)$.

In this paper, I calculate $\text{ca}(Z)$. The results are in Table 1 at the end of this section. I have also determined the number of irreducible components in a largest fibre of the ampleness map ϕ . This result is described in §5.

The following theorem of Sommese [21A, §3] is a generalization of the Barth-Larsen Lefschetz theorems.

LEFSCHETZ THEOREM. *Let $f: X \rightarrow Z$ be a regular immersion of the connected compact complex manifold X into Z . Let Y be a connected compact complex submanifold of Z . Let k be the ampleness of NY , the normal bundle of Y , i.e. k is the largest fibre dimension of the restriction of ϕ to $\mathbf{P}(N^*Y)$. Sommese [21C, Corollary 1.4] shows that $Z \setminus Y$ is $k + \text{cod } Y$ convex in the sense of Andreotti-Grauert. Assume that $2 \cdot \dim Y \geq k + \dim Z$ and $\dim X \leq \dim Y + 1$. Then $\pi_i(X, f^{-1}(Y), x) = 0$ for $i \leq \dim X - \text{cod } Y - k$.*

It is difficult to compute k exactly, but the inequality $k \leq \text{amp}(TZ)$ may be used.

The following theorem of Faltings [10, p. 148, Satz 5, Korollar] is a generalization of a theorem of Fulton and Hansen [11] on $Z = \mathbf{P}^r$. See also Hansen [15] for the case $G = \text{SL}_n \mathbf{C}$.

CONNECTEDNESS THEOREM. *Let $f: X \rightarrow Z \times Z$ be a regular map of the compact irreducible variety X to $Z \times Z$ and let $\Delta \subset Z \times Z$ be the diagonal embedding of Z . Let $l = \min\{\text{rk}(g_i)\}$ where $\mathfrak{g} = \bigoplus g_i$ is the decomposition of the Lie algebra of G into*

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simple ideals. Then

- (i) $\dim f(X) \geq 2 \dim(Z) - l \Rightarrow f^{-1}(\Delta) \neq \emptyset$.
- (ii) $\dim f(X) > 2 \dim(Z) - l \Rightarrow f^{-1}(\Delta)$ is connected.

From Table 1, we see that $l \leq \text{ca}(Z)$. In [10A], Faltings explains that, in the above theorem, “ l ” may be replaced by the better bound “ $\text{ca}(Z)$ ”.

For Z other than projective space, it is not clear what are the higher homotopy results implied by the connectedness theorem; see Fulton and Lazarsfeld [12, §10.1].

I should mention, also, that part (i) of the connectedness theorem follows from work of Sommese [21, Proposition 1.1].

EXAMPLE 1. $G = \text{SL}(n + 1, \mathbb{C})$, $l = n = \text{rk}(G)$, and $\text{ca}(G/P) = n$ for each parabolic P . See Hansen [15, p. 3], for examples that the results (i), (ii) are sharp.

EXAMPLE 2. $G = O(2n, \mathbb{C})$, $l = n = \text{rk}(G)$, and $\text{ca}(G/P) = 2n - 3$ for each parabolic P . Let $Z \subset \mathbb{P}^{2n-1}$ be the $2n - 2$ dimensional quadric given by the equation $\sum_{i=1}^{2n} z_i^2 = 0$. The above results imply that $X \cap Y \neq \emptyset$ whenever X and Y are closed subvarieties of Z satisfying

$$\dim(X) + \dim(Y) \geq 2(2n - 2) - (2n - 3) = 2n - 1.$$

This result is sharp:

Let X and Y be the images, respectively, of the maps $\mathbb{P}^{n-1} \rightarrow Z$ given by $x \mapsto (x, ix)$ and $y \mapsto (y, -iy)$. Clearly, $X \cap Y = \emptyset$, while $\dim(X) + \dim(Y) = 2n - 2$.

EXAMPLE 3. $G = O(2n + 1, \mathbb{C})$, $l = n = \text{rk}(G)$, and

$$\text{ca}(G/P) = \begin{cases} 2n - 1 & \text{for one special } P, \\ 2n - 2 & \text{for every other parabolic } P. \end{cases}$$

Let Z be the $2n - 1$ dimensional quadric. The above results imply that

- (*) $X \cap Y \neq \emptyset$ whenever X and Y are closed subvarieties of Z satisfying $\dim(X) + \dim(Y) \geq 2n$.

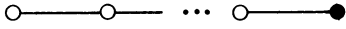
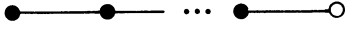
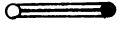
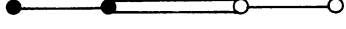
In fact, as pointed out to me by Sommese, one has a stronger (and sharper) result by replacing $2n$ by $2n - 1$ in (*). Sharpness follows from Example 2 since the $2n - 2$ dimensional quadric is contained in Z . The validity follows from the fact that $H^*(Z, \mathbb{C})$ is ring isomorphic to $H^*(\mathbb{P}^{2n-1}, \mathbb{C})$.

The computational part of the paper concerns a desingularization of the unipotent variety of G . This map has been studied by Springer [22] and Steinberg [25]. Their results on fibre dimensions deal largely with the case $P =$ a Borel subgroup of G . R. Elkik [8] has also noticed the connection between this desingularization and the cotangent bundle of G/P (see Remark 3.3 for more details). In [26], Steinberg explains that, following my letter to him, he did the computations for determining $\text{ca}(G/P)$. For the case $P = B$ a Borel subgroup, he gives the succinct formula $\text{ca}(G/B) = |\rho^*|$, where ρ is a highest root for the simple group G and ρ^* is its dual, or coroot. This formula is equivalent to Theorem 4.2.

The notion of k -ampleness is due to Andrew Sommese [20, §1], and I would like to thank him for suggesting that I consider doing these computations for G/P . The preprint of Faltings provided further motivation to compare ampleness and rank in the simple groups.

I would like to thank James Carrell, Hanspeter Kraft, George Maxwell, Peter Slodowy and Robert Steinberg for helpful letters and conversations. A special thanks goes to Bomshik Chang for discussions on Weyl groups.

TABLE 1

G (all roots long)		$ca(G/P) = \rho $ for every parabolic subgroup P .	
A_n		n	
D_n		$2n - 3$	
E_6		11	
E_7		17	
E_8		29	
G	Coxeter-Dynkin diagram (The short roots are darkened)	$ca(G/B)$	$ \rho $
B_n		$2n - 2$	$2n - 1$
C_n		n	$2n - 1$
G_2		3	5
F_4		8	11
$ca(G/P_\Theta) = ca(G/B) + \min\{d(\alpha): \alpha \notin \Theta\}$			
$ca(Z_1 \times Z_2) = \min\{ca(Z_1), ca(Z_2)\}$ (cf. (3.4))			
$ \rho $ is the height of the highest root			
$d(\alpha)$ = no. of nodes from α to the long roots			

2. Background material. In this section, I describe the notation that is used in the paper. The references Borel [3, 4, 5], Carter [6], Humphreys [16], Samelson [18], Serre [19] and Steinberg [24] contain proofs and elaborations.

Let G be a connected complex semisimple Lie group. The centralizer of a subset, A , of G is $Z(A) = \{g \in G: ga = ag \forall a \in A\}$. The normalizer of A is $N(A) = \{g \in G: gAg^{-1} \subset A\}$.

Let \mathfrak{g} denote the Lie algebra of G , with Lie bracket $[\cdot, \cdot]$. For each $b \in \mathfrak{g}$, $\text{ad}(b) \in \text{End}(\mathfrak{g})$ is defined by $\text{ad}(b)(c) = [b, c]$. The Killing form on \mathfrak{g} is given by $(b, c) = \text{Trace}(\text{ad}(b) \circ \text{ad}(c))$. This pairing is nondegenerate and induces an identification, \mathcal{K} , of \mathfrak{g}^* , the vector space dual of \mathfrak{g} , with \mathfrak{g} .

For $g \in G$, let $C(g)(x) = gxg^{-1}$, and denote by $\text{Ad}(g): \mathfrak{g} \rightarrow \mathfrak{g}$ the differential of $C(g)$ at the identity element of G . Let $\exp: \mathfrak{g} \rightarrow G$ be the exponential map. For each $a \in \mathfrak{g}$, $t \mapsto \exp(ta)$ is the 1-parameter subgroup of G whose tangent vector at $t = 0$ is a . One has

$$\text{Ad}(\exp(a)) = e^{\text{ad}(a)} = \sum_{m=0}^{\infty} (\text{ad}(a))^m / m!.$$

Fix a maximal torus H in a Borel subgroup B of G . Let \mathfrak{h} be the Lie algebra of H . Let $R \subset \mathfrak{h}^*$ be the root system with respect to H , and R^+ (resp. R^-) the positive (resp. negative) roots with respect to B . We write $\alpha > 0$ (resp. $\alpha < 0$) for $\alpha \in R^+$ (resp. $\alpha \in R^-$). Let $\Sigma = \{\alpha_1, \dots, \alpha_n\}$ denote the simple roots, i.e. a basis for R ; $n = \dim(H) = \text{rank}(G)$.

Each $\alpha \in R$ may be expressed as $\alpha = \sum n_i \alpha_i$ with either all $n_i \geq 0$ (i.e. $\alpha \in R^+$) or all $n_i \leq 0$ (i.e. $\alpha \in R^-$). The height of α is $|\alpha| = \sum n_i$. An element ρ of R is a highest root when $\rho + \alpha \notin R$ whenever $\alpha > 0$. For simple G there is a unique such root. Also, R is a reduced root system, i.e. for each $\alpha \in R$, the only multiples of α that again belong to R are $\pm\alpha$.

One has the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \{g^\alpha : \alpha \in R\}$$

where $g^\alpha = \{a \in \mathfrak{g} : \text{ad}(b)(a) = \alpha(b)a \forall b \in \mathfrak{h}\}$ is the 1-dimensional root space associated to α . Moreover, $[g^\alpha, g^\beta] \subset g^{\alpha+\beta}$, so that $[g^\rho, g^\alpha] = 0$ when ρ is a highest root and $\alpha > 0$. Also $(g^\alpha, g^\beta) = 0$ when $\alpha + \beta \neq 0$ while g^α and $g^{-\alpha}$ are perfectly paired by the Killing form. The homomorphism

$$x_\alpha = \exp | g^\alpha : g^\alpha \rightarrow X_\alpha \subset G$$

is an isomorphism onto X_α , the root subgroup associated to α . One has $X_\rho \subset Z(X_\alpha)$ when ρ is a highest root and $\alpha > 0$.

For each subset $\Theta \subset \Sigma$, one has the standard parabolic subgroup

$$(2.1) \quad P = P_\Theta = H \cdot \prod \{X_\alpha : \alpha \in \langle \Theta \rangle\} \cdot \prod \{X_\alpha : \alpha > 0, \alpha \notin \langle \Theta \rangle\}.$$

The products are taken in any fixed order, and $\langle \Theta \rangle$ denotes the span of Θ in R . The third factor is the unipotent radical, U_p of P , and $P \subset N(U_p)$. Every parabolic subgroup of G is conjugate to some standard parabolic.

Similarly, the Lie algebra of P_Θ is

$$\mathfrak{p} = \mathfrak{h} \oplus \{g^\alpha : \alpha \in \langle \Theta \rangle\} \oplus \{g^\alpha : \alpha > 0, \alpha \notin \langle \Theta \rangle\}.$$

The third summand is the nilpotent radical N_p of P and is also the annihilator of \mathfrak{p} in \mathfrak{g} with respect to the Killing form.

(2.2) We denote $U = U_B = \prod \{X_\alpha : \alpha > 0\}$. Then $X_\rho \subset Z(U)$ when ρ is a highest root. The map

$$H \times \mathbb{C}^N \rightarrow B \quad (N = \#(R^+)),$$

$$(t, c) \mapsto t \cdot \prod \{x_\alpha(c_\alpha) : \alpha > 0\}$$

is an isomorphism of varieties.

(2.3) The finite group $\mathcal{W} = N(H)/H$ is the Weyl group of G with respect to H . For each $\omega \in \mathcal{W}$, let n_ω be a fixed representative in $N(H)$. The group \mathcal{W} embeds as a group of linear transformations on \mathfrak{h}^* , leaving R invariant. One has $n_\omega X_\alpha n_\omega^{-1} = X_{\omega(\alpha)}$ for each root subgroup X_α and $\omega \in \mathcal{W}$.

For each $\alpha \in R$, let $\sigma_\alpha \in \mathcal{W}$ be the reflection through α . Let $\sigma_i = \sigma_{\alpha_i}$ for each of the simple roots α_i . Then, \mathcal{W} is generated by the simple reflections $\{\sigma_1, \dots, \sigma_n\}$. The length, $l = l(\omega)$, of an element $\omega \in \mathcal{W}$ is defined as the least number of simple

reflections by which one may write $\omega = \sigma_{i_1} \cdots \sigma_{i_2} \sigma_{i_1}$. One has also that $l(\omega) = \#\{\alpha > 0: \omega(\alpha) < 0\}$. The simple reflection σ_i permutes the elements of $R^+ \setminus \{\alpha_i\}$. It follows that

$$l(\omega\sigma_i) = \begin{cases} l(\omega) + 1, & \omega(\alpha_i) > 0, \\ l(\omega) - 1, & \omega(\alpha_i) < 0. \end{cases}$$

There is a unique element, ω_0 , of \mathcal{W} taking R^+ to R^- . One has $l(\omega\omega_0) = \#(R^+) - l(\omega)$ for each $\omega \in \mathcal{W}$.

Let $(,)$ denote a \mathcal{W} -invariant positive definite pairing on \mathfrak{h}^* . When G is simple, at most two root lengths occur in R (long and short). If only one length occurs, it is called long.

G	Squared length ratio (long : short)
A_n, D_n, E_n	all roots long
B_n, C_n, F_4	2 : 1
G_2	3 : 1

The roots of a given length form an orbit of the Weyl group in R . The reflection σ_α is given by the formula $\sigma_\alpha(\beta) = \beta - \alpha^*(\beta)\alpha$ where $\alpha^*(\beta) = 2(\alpha, \beta)/(\alpha, \alpha) \in \mathbf{Z}$.

LEMMA 2.4. Assume that \mathfrak{g} is simple. Let $\alpha, \beta \in R$ such that $\alpha \neq \pm\beta$, $(\alpha, \beta) \neq 0$ and $(\alpha, \alpha) \leq (\beta, \beta)$. Then

- (i) $\alpha^*(\beta) \cdot \beta^*(\alpha) = 1, 2$ or 3 ,
- (ii) if $(\alpha, \beta) > 0$ then $\alpha^*(\beta) = (\beta, \beta)/(\alpha, \alpha)$ and $\beta^*(\alpha) = 1$, and
- (iii) if α and β are simple then $(\alpha, \beta) < 0$ and $\alpha^*(\beta) = -(\beta, \beta)/(\alpha, \alpha)$ and $\beta^*(\alpha) = -1$.

PROOF. (i) $\alpha^*(\beta) \cdot \beta^*(\alpha) = 4(\alpha, \beta)^2/(\alpha, \alpha)(\beta, \beta)$, so $\alpha^*(\beta) \cdot \beta^*(\alpha) = 0, 1, 2, 3$ or 4 by Schwarz's inequality, but the hypotheses preclude the values 0 and 4 .

(ii) $\alpha^*(\beta)/\beta^*(\alpha) = (\beta, \beta)/(\alpha, \alpha)$, so the result follows from (i).

(iii) Serre [19, V8, Lemme 3], for example, shows that $(\alpha, \beta) < 0$, and the result now follows from (ii).

3. Ampleness of the tangent bundle. With the notation of §2, $Z = G/P$ is a compact complex homogeneous space with group G . Sommese [20, §1] defines the k -ampleness of a vector bundle over a compact complex space. This notion is discussed, in particular, for the tangent bundle, TZ , in Goldstein [14, §2] which notation is reviewed below.

The global sections S_0, \dots, S_N of TZ determine the "ampleness map"

$$\phi: T^*Z \rightarrow \mathfrak{g}^*$$

where \mathfrak{g}^* is the vector space dual of the Lie algebra, \mathfrak{g} , of G . Explicitly,

$$\phi(\alpha) = z^{**}(\alpha) \quad \text{where } \alpha \in T_z^*Z, z^\#: G \rightarrow Z,$$

$$g \mapsto gz \quad \text{and } z^{**}: T_z^*Z \rightarrow T_g^*G = \mathfrak{g}^*.$$

The bundle TZ is k -ample when

$$\text{amp}(TZ) := \sup\{\dim(\phi^{-1}(\alpha)) : \alpha \in \mathfrak{g}^* \setminus \{0\}\}$$

has value at most k . The coampleness is $\text{ca}(Z) := \dim(Z) - \text{amp}(TZ)$.

This section reduces the determination of $\text{ca}(G/P)$ to a calculation with Weyl groups (Proposition 3.6). First, the ampleness map is translated into Lie group data (Lemma 3.2).

Notation. Recall that $\mathfrak{p} \subset \mathfrak{g}$ is the Lie algebra of the parabolic subgroup P . Let $\mathfrak{p}^\perp = \{\alpha \in \mathfrak{g}^* : \mathfrak{p} \subset \ker(\alpha)\}$.

Let $\tilde{\Phi} : G \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ be the map

$$(g, \alpha) \mapsto (\text{Ad}(g))^*{}^{-1}(\alpha).$$

Let $\Phi : G \times \mathfrak{p}^\perp \rightarrow \mathfrak{g}^*$ be the restriction of $\tilde{\Phi}$ to $G \times \mathfrak{p}^\perp$.

Define an action of P on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (gh^{-1}, (\text{Ad}(h))^*{}^{-1}(\alpha)).$$

The map Φ is constant on the orbits of P , and so induces a map

$$\tilde{\phi} : (G \times \mathfrak{p}^\perp)/P \rightarrow \mathfrak{g}^*.$$

The group G acts on $G \times \mathfrak{p}^\perp$ by

$$h \cdot (g, \alpha) = (hg, \alpha).$$

This action commutes with the action of P just described, thus defining a G -action on the space $(G \times \mathfrak{p}^\perp)/P$. The map $\tilde{\phi}$ is G -equivariant, where the action of G on \mathfrak{g}^* is the dual adjoint action.

LEMMA 3.1. *There is a commutative diagram (3.1.1), where Γ is a G -equivariant isomorphism of vector bundles over Z .*

$$(3.1.1) \quad \begin{array}{ccccc} G & \xleftarrow{a} & G \times \mathfrak{p}^\perp & & \\ & & & & \Phi \\ \pi \downarrow & & b \downarrow & & \searrow \\ G/P & \xleftarrow{\tilde{\rho}} & (G \times \mathfrak{p}^\perp)/P & \xrightarrow{\tilde{\phi}} & \mathfrak{g}^* \\ & & \Gamma \downarrow & & \nearrow \phi \\ Z & \xleftarrow{\rho} & T^*Z & & \end{array}$$

The maps $a, b, \pi, \tilde{\rho}$ and ρ are the natural projections.

PROOF. It remains to construct Γ . Define, first, a map

$$\begin{aligned} \tilde{\Gamma} : G \times \mathfrak{p}^\perp &\rightarrow T^*Z, \\ (g, \alpha) &\rightarrow \mathfrak{g}^{*-1}(\pi^{*-1}(\alpha)). \end{aligned}$$

Here, let z_0 denote the point $\pi(e)$ of G/P represented by the coset P ,

$$\pi^* : T^*_{z_0}Z \rightarrow T^*_eG = \mathfrak{g}^*$$

and

$$g^*: T_{g z_0}^* \rightarrow T_{z_0}^* Z.$$

Note that π^* is 1-1 and has image equal to \mathfrak{p}^\perp . It is elementary to verify that $\tilde{\Gamma}$ is G -equivariant, is constant on the orbits of P , that the induced map

$$\Gamma: (G \times \mathfrak{p}^\perp)/P \rightarrow T^*Z$$

is an isomorphism, and that $\phi \circ \Gamma = \tilde{\phi}$. \square

LEMMA 3.2. *There is a commutative diagram (3.2.1).*

$$(3.2.1) \quad \begin{array}{ccccc} T^*Z & \xrightarrow[\sim]{\Lambda_1} & (G \times N_p)/P & \xrightarrow[\sim]{\Lambda_2} & (G \times U_p)/P \\ \phi \downarrow & & \tilde{\psi} \downarrow & & \psi \downarrow \\ \mathfrak{g}^* & \xrightarrow[\sim]{\mathfrak{K}} \mathfrak{g} \supset & \mathfrak{U} & \xrightarrow[\sim]{\exp} & V \end{array}$$

The map Λ_1 is a vector bundle isomorphism, and Λ_2 is an isomorphism of varieties.

Here, U_p and N_p are the unipotent and nilpotent radicals, respectively, of P . The action of P on U_p is given by

$$h \cdot (g, x) = (gh^{-1}, h x h^{-1}).$$

The action of P on N_p is given by

$$h \cdot (g, v) = (gh^{-1}, \text{Ad}(h)(v)).$$

The map \mathfrak{K} is the Killing identification, and the exponential map, \exp , takes the nilpotent elements \mathfrak{U} of \mathfrak{g} isomorphically onto the unipotent variety V of G (cf. Springer [22, §3]). We have $\psi(g, x) = g x g^{-1}$, and $\tilde{\psi}(g, v) = \text{Ad}(g)(v)$.

PROOF. One verifies that $\mathfrak{K}(\text{ad}(c)^*) = -\text{ad}(c)$ for each $c \in \mathfrak{g}$. Thus, $\mathfrak{K}(\text{Ad}(g)^*) = \text{Ad}(g)^{-1}$. Also, $\mathfrak{K}(\mathfrak{p}^\perp) = N_p$ and $\exp(N_p) = U_p$. Using these facts, together with Lemma 3.1, diagram (3.2.1) may be constructed. \square

The goal in the remainder of this paper is to calculate the largest fibre dimension of the ampleness map ϕ (or $\tilde{\psi}$ or ψ).

REMARK 3.3.1. The map ψ has been studied by Springer [22] and Steinberg [25] when $P = B$. Let

$$W := \{(gP, u) \in G/P \times V: g^{-1} u g \in U_p\}.$$

The projection $\pi: W \rightarrow V$ is equivalent to the map ψ of Lemma 3.2. Springer shows, in the case $P = B$, that π is generically 1-1 and each fibre of π is connected. Steinberg extends these results to general parabolics. In the case $P = B$, he shows that

$$\dim(\pi^{-1}(u)) = \frac{1}{2} \times (\dim(Z(u)) - \text{rk}(G))$$

and obtains some results on the number of irreducible components in each fibre of π .

REMARK 3.3.2. The space $(G \times N_p)/P$ has also been studied by Elkik [8, §1], the identification with $T^*(G/P)$ being done in the language of schemes.

We next reduce the computation of $\text{ca}(G/P)$ to the case where G is simple. Without loss of generality, assume that G is simply connected, so $G = \times_{i=1}^m G_i$, where the G_i are simple normal subgroups of G . From the description of standard parabolics, one has $P = \times_{i=1}^m P_i$, so that $G/P \simeq \times_{i=1}^m (G/P_i)$. Let $Z_i = G/P_i$ and let $Z = G/P \simeq \times_{i=1}^m Z_i$. Let

$$\phi_i: T^*Z_i \rightarrow \mathfrak{g}_i^*$$

be the ampleness maps. Then

$$\phi = \bigoplus_{i=1}^m \phi_i: \bigoplus_{i=1}^m (T^*Z_i) \rightarrow \bigoplus_{i=1}^m \mathfrak{g}_i^*$$

is the ampleness map of Z . It follows that

$$(3.4) \quad \text{ca}(Z) = \min\{\text{ca}(Z_i): i = 1, \dots, m\}.$$

For the remainder of the paper, we assume that G is simple.

PROPOSITION 3.5. *Let G be a (simple) complex Lie group. Let ρ be the highest root of G with respect to some ordering, and let x_ρ be any nonidentity element of X_ρ , the root subgroup of G associated to ρ . Let P be a parabolic subgroup of G , with unipotent radical U_ρ . Then*

$$\text{ca}(G/P) = \text{cod}_G\{g \in G: g^{-1}x_\rho g \in U_\rho\}.$$

PROOF. We use the notation of Lemma 3.2. The projectivization of the ampleness map

$$\mathbf{P}(\psi): (G \times \mathbf{P}(N_\rho))/P \rightarrow \mathbf{P}(\mathcal{U})$$

is proper and G -equivariant. The space $\mathbf{P}(\mathcal{U})$ possesses a unique closed G -orbit \emptyset , which is the orbit of a highest root-vector line.² Thus, any point of $\mathbf{P}(\mathcal{U})$ may be specialized, within its orbit, to a point of \emptyset and we have that the maximum $\tilde{\psi}$ -fibre dimension (over $\mathcal{U} \setminus \{0\}$) occurs at $v_\rho \in \mathcal{U}$, where v_ρ is a highest root vector.

Viewing the ampleness map now as

$$\psi: (G \times U_\rho)/P \rightarrow V,$$

the maximum fibre dimension of ψ is

$$\begin{aligned} \dim(\psi^{-1}(x_\rho)) &= \dim\{(g, u) \in G \times U_\rho: gug^{-1} = x_\rho\} - \dim(P) \\ &= \dim\{g \in G: g^{-1}x_\rho g \in U_\rho\} - \dim(P) \end{aligned}$$

where $x_\rho = \exp(v_\rho)$. Thus,

$$\begin{aligned} \text{ca}(G/P) &= \dim(G/P) - \dim(\psi^{-1}(x_\rho)) \\ &= \dim(G) - \dim\{g \in G: g^{-1}x_\rho g \in U_\rho\}. \quad \square \end{aligned}$$

²This fact, as related to me by H. Kraft and P. Slodowy, is well known: The adjoint action of G on \mathfrak{g} is irreducible. So, there is a unique line in \mathfrak{g} which is invariant under the Borel group viz. the highest root-vector line (see e.g. [24, §3, 4a]). Using the fact that any two Borel subgroups of G are conjugate, one concludes that the orbit of this line in $\mathbf{P}(\mathcal{U})$ is the unique closed orbit.

The last result of this section expresses $ca(G/P)$ in terms of the length function of the Weyl group of G .

PROPOSITION 3.6. *Let P be the standard parabolic subgroup of the simple group G associated to the subset $\Theta \subset \Sigma$. Then*

$$(3.6.1) \quad ca(G/P) = \min\{l(\omega) : \omega \in \mathcal{W}, \omega(\rho) < 0 \text{ and } \omega(\rho) \notin \langle \Theta \rangle\}.$$

Moreover, the irreducible components of a largest fibre of the ampleness map are in 1-1 correspondence with those ω 's which minimize (3.6.1).

PROOF. By the Bruhat decomposition, each $g \in G$ may be expressed uniquely in the form

$$g = un_\omega b \quad \text{with } u \in U, \omega \in \mathcal{W} \text{ and } b \in B.$$

By Proposition 3.5, we have that

$$ca(G/P) = \text{cod}_G\{g \in G : g^{-1}x_\rho g \in U_P\}.$$

Now,

$$\begin{aligned} g^{-1}x_\rho g &= b^{-1}n_\omega^{-1}u^{-1}x_\rho un_\omega b \in U_P \Leftrightarrow n_\omega^{-1}x_\rho n_\omega \in U_P \\ &\Leftrightarrow \omega^{-1}(\rho) \in N_P \Leftrightarrow \omega^{-1}(\rho) > 0 \text{ and } \omega^{-1}(\rho) \notin \langle \Theta \rangle. \end{aligned}$$

The first equivalence follows from $B \subset N(U_P)$ and $x_\rho \in Z(U)$. The last two follow from 2.2, 2.3 and 2.1.

We conclude that the maximum fibre dimension for the ampleness map is

$$\begin{aligned} M &= \max\{\dim(Un_\omega B) : \omega^{-1}(\rho) > 0, \omega^{-1}(\rho) \notin \langle \Theta \rangle\} - \dim(P) \\ &= \dim(B) - \dim(P) + \max\{l(\omega) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle\}. \end{aligned}$$

Here we have used the formula

$$\dim(Un_\omega B) = \dim(B) + l(\omega)$$

and then made the substitution ω for ω^{-1} . Hence

$$\begin{aligned} ca(G/P) &= \dim(G/P) - M \\ &= \#(R^+) - \max\{l(\omega) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle\} \\ &= \min\{l(\omega\omega_0) : \omega(\rho) > 0, \omega(\rho) \notin \langle \Theta \rangle\}. \end{aligned}$$

(Recall that $\dim(G/B) = \#(R^+)$ and that $l(\omega\omega_0) = \#(R^+) - l(\omega)$ where ω_0 is the Weyl group element taking R^+ to R^- .) The proposition now follows by making the substitution $\omega \rightarrow \omega\omega_0$ and noting that $\omega_0(\rho) = -\rho$. \square

4. Computing $ca(G/P)$. The calculation is divided into three parts (recall that G is simple):

(i) Determine $ca(G/B)$ from the expression of the highest root $\rho = \sum n_i \alpha_i$ in terms of the simple roots $\Sigma = \{\alpha_1, \dots, \alpha_n\}$, as in Theorem 4.2.

(ii) For each standard maximal parabolic $P_i := P_{\Sigma \setminus \{\alpha_i\}}$, determine $ca(G/P)$.

Let $d(i)$ be the least number of steps, in the Coxeter-Dynkin diagram of G , from α_i to the long roots. Then, by Theorem 4.7

$$ca(G/P_i) = d(i) + ca(G/B).$$

(iii) Now, let $P = P_\Theta$ be any standard parabolic. By formula (3.6.1), we have

$$\begin{aligned} \text{ca}(G/P) &= \min\{l(\omega) : \omega\rho < 0 \text{ and } \omega\rho \text{ involves some } \alpha_i, \text{ with } \alpha_i \notin \Theta\} \\ &= \min\{\text{ca}(G/P_i) : \alpha_i \notin \Theta\} \\ &= \min\{d(i) : \alpha_i \notin \Theta\} + \text{ca}(G/B). \end{aligned}$$

The results are summarized in Table 1 in §1.

From formula (3.6.1), we see that, in particular,

$$(4.1.1) \quad \text{ca}(G/B) = \min\{l(\omega) : \omega(\rho) < 0\}.$$

To each simple root α_i associate the integer

$$(4.1.2) \quad v_i = \begin{cases} 1, & \alpha_i \text{ long,} \\ 2, & \alpha_i \text{ short, } G \neq G_2, \\ 3, & \alpha_i \text{ short, } G = G_2. \end{cases}$$

Suppose that β is a long root of height at least 2, and that $(\alpha_i, \beta) > 0$ for some i . Then

$$(4.1.3) \quad |\sigma_i(\beta)| = |\beta| - v_i$$

since $\alpha_i^*(\beta) = v_i$, as in Lemma 2.4.

THEOREM 4.2. *Let $\rho = \sum n_i \alpha_i$ be the expression of the highest root ρ in terms of simple roots. Then*

$$\text{ca}(G/B) = \sum n_i / v_i.$$

(The v_i are defined in (4.1.2).)

PROOF. It is well known that for each $\beta > 0$ there is some simple root α_i with $(\alpha_i, \beta) > 0$. This, together with (4.1.3) and formula (4.1.1), proves the theorem. \square

Table 1 contains the results of calculating $\text{ca}(G/B)$ for each of the simple Lie groups.

We turn next to calculating $\text{ca}(G/P_i)$ where P_i is the maximal parabolic $P_{\Sigma \setminus \{\alpha_i\}}$.

LEMMA 4.3. *Let $\omega \in \mathcal{U}$ minimize $l(\omega)$ in formula (4.1.1). Then*

- (i) $\omega(\rho) = -\alpha_i$ for some (long) simple root α_i , and
- (ii) $\omega^{-1}(\alpha_j) > 0$ for every other simple root α_j .

Let α_k be any long simple root. Then there exists a unique ω_k minimizing $l(\omega)$ in formula (4.1.1) and satisfying $\omega_k(\rho) = -\alpha_k$.

PROOF. Suppose that $\omega^{-1}(\alpha_j) < 0$ and $\omega(\rho) \neq -\alpha_j$. Then $l(\sigma_j\omega) < l(\omega)$. But σ_j permutes $R^+ \setminus \{\alpha_j\}$, so that $\sigma_j\omega(\rho) < 0$. This contradicts the minimality of ω , and proves (i) and (ii).

The subdiagram of the Coxeter-Dynkin diagram for G consisting of the long roots is connected, so we may assume that $(\alpha_i, \alpha_k) \neq 0$. Then, as in Lemma 2.4,

$$\alpha_i^*(\alpha_k) = \alpha_k^*(\alpha_i) = -1,$$

so

$$\sigma_i \sigma_k(\alpha_i) = \sigma_i(\alpha_i + \alpha_k) = -\alpha_i + \alpha_k + \alpha_i = \alpha_k.$$

To see that $\omega_k = \sigma_i \sigma_k \omega$ is the required Weyl group element, it remains to see that $l(\omega_k) = l(\omega)$. This follows from properties (i) and (ii) of ω :

$$\begin{aligned} \omega^{-1}(\alpha_k) > 0 &\Rightarrow l(\sigma_k \omega) = l(\omega) + 1 \quad \text{and} \\ \omega^{-1} \sigma_k(\alpha_i) = \omega^{-1}(\alpha_i + \alpha_k) &= -\rho + \omega^{-1}(\alpha_k) < 0 \\ &\Rightarrow l(\sigma_i \sigma_k \omega) = l(\sigma_k \omega) - 1 = l(\omega). \end{aligned}$$

The uniqueness of ω_k is a standard result on parabolic subgroups of ${}^{\mathcal{O}}\mathcal{U}$, since ω_k is the *unique* element for which $\min\{l(\omega): \omega(\rho) = -\alpha_k\}$ is attained, e.g. see Carter [6, §2.5]. \square

For the maximal parabolics, formula (3.6.1) reads

$$(4.4.1) \quad \text{ca}(G/P_i) = \min\{l(\omega): \omega(\rho) < 0 \text{ and } \omega(\rho) \text{ involves } \alpha_i\}.$$

For each simple root α_i let

$$(4.4.2) \quad k(i) := \min\{l(\omega): \omega(\alpha_j) \text{ involves } \alpha_i \text{ for some long simple root } \alpha_j\}.$$

REMARK 4.5. $k(i) = 0 \Leftrightarrow \alpha_i$ is long, and $k(i) = 1 \Leftrightarrow \alpha_i$ is short and is adjacent to a long root in the Coxeter-Dynkin diagram for G .

For each simple root α_i , let $d(i)$ = the number of nodes, in the Coxeter-Dynkin diagram for G , from α_i to the long roots.

LEMMA 4.6. For each i , we have $k(i) = d(i)$.

PROOF. The only diagrams having more than one short root are those for C_n and F_4 . By Remark 4.5, it remains only to verify the lemma for these two cases. Samelson [18, pp. 79–86] contains the descriptions that I will be using of the root systems. The short roots are underlined. In each case, $\Sigma = \{\alpha_1, \dots, \alpha_n\}$ is the implicit labeling of the simple roots, and σ_i is the reflection through α_i .

I. $G = C_n$.

$$\begin{aligned} R^+ &= \{2a_i, \underline{a_i \pm a_j}, i < j\}, \\ \Sigma &= \{\underline{a_1 - a_2}, \dots, \underline{a_{n-1} - a_n}, 2a_n\}. \end{aligned}$$

Figure (4.6.1) shows the only paths (without backtracking) between the long positive roots:

$$(4.6.1) \quad 2a_n \xrightarrow{\sigma_{n-1}} 2a_{n-1} \xrightarrow{\sigma_{n-2}} \dots \xrightarrow{\sigma_2} 2a_2 \xrightarrow{\sigma_1} 2a_1.$$

It follows that $k(i) = n - i = d(i)$.

II. $G = F_4$.

$$\begin{aligned} R^+ &= \{a_i, a_i \pm a_j, i < j, \underline{\frac{1}{2} \times (a_1 \pm a_2 \pm a_3 \pm a_4)}\}, \\ \Sigma &= \{\underline{\frac{1}{2} \times (a_1 - a_2 - a_3 - a_4)}, \underline{a_4}, a_3 - a_4, a_2 - a_3\}, \\ \sigma_1 \sigma_2(a_3) &= \sigma_1(a_3 + a_4) = \underline{\frac{1}{2} \times (a_1 - a_2 + a_3 + a_4)}. \end{aligned}$$

Thus, $k(1) = 2 = d(1)$. \square

THEOREM 4.7. *Let $P_i = P_{\Sigma \setminus \{\alpha_i\}}$ be a maximal parabolic subgroup of the simple group G . Then*

$$\text{ca}(G/P_i) = d(i) + \text{ca}(G/B).$$

PROOF. By Lemma 4.6, it suffices to prove the result with $k(i)$ in place of $d(i)$. Let $\omega = \sigma_{i_m} \cdots \sigma_{i_2} \sigma_{i_1}$ be a reduced expression for some ω minimizing $l(\omega)$ in formula (4.4.1). By considering the sequence,

$$\beta_r = \sigma_{i_r} \cdots \sigma_{i_2} \sigma_{i_1}(\rho), \quad r = 0, \dots, m,$$

one sees that ω may be written as $\omega = \omega_2 \omega_1$ where ω_1 minimizes $l(\omega)$ in formula (4.1.1), ω_2 minimizes $l(\omega)$ in formula (4.4.2), and $l(\omega) = l(\omega_1) + l(\omega_2)$. \square

5. Let $L\Sigma$ denote the long roots in the basis Σ of the root system R of G , a simple complex Lie group. Let $n(P)$ be the number of irreducible components in a largest fibre of the ampleness map ψ (cf. Remark 3.3.1). By Proposition 3.6 and Lemma 4.3, we have that $n(B) = \#(L\Sigma)$. (In fact, when R contains only long roots, then $n(P_\Theta) = \#(\Sigma \setminus \Theta)$.) For the maximal parabolics, $n(P_i) = 1$. To see this, one need only verify the uniqueness of minimizing elements for formula (4.4.2). More generally,

$$n(P_\Theta) = \max\{1, \#(L\Sigma \setminus \Theta)\}.$$

BIBLIOGRAPHY

1. P. Bala and R. W. Carter, *Classes of unipotent elements in simple algebraic groups*, Math. Proc. Cambridge Philos. Soc. **79** (1976), 401–425; **80** (1976), 1–18.
2. I. N. Bernstein, I. M. Gel'fand and S. I. Gel'fand, *Schubert cells and cohomology of the spaces G/P* , Russian Math. Surveys **28** (1973), 1–26.
3. A. Borel, *Linear algebraic groups*, Algebraic Groups and Discontinuous Groups (Boulder, Colorado, 1965), Proc. Sympos. Pure Math., vol. 10, Amer. Math. Soc., Providence, R.I., 1966, pp. 3–19.
4. ———, *Linear algebraic groups*, Benjamin, New York, 1969.
5. A. Borel et al., *Seminar on algebraic groups and related finite groups*, Lecture Notes in Math., vol. 131, Springer-Verlag, Berlin and New York, 1970.
- 5A. W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Comment. Math. Helv. **54** (1979), 61–104.
6. W. C. Carter, *Simple groups of Lie type*, Pure and Appl. Math., vol. 28, Wiley, New York, 1972.
7. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Mat. Sb. (N.S.) **30** (1952), 349–462; English transl. in Amer. Math. Soc. Transl. (2) **6** (1957), 111–245.
8. R. Elkik, *Désingularisation des adhérences d'orbites polarisables et des nappes dans les algèbres de Lie réductives*, preprint.
9. G. B. Elkington, *Centralizers of unipotent elements in semisimple algebraic groups*, J. Algebra **23** (1972), 137–163.
10. G. Faltings, *Formale Geometrie und homogene Räume*, Invent. Math. **64** (1981), 123–165.
- 10A. ———, letter, June 1981.
11. W. E. Fulton and J. Hansen, *A connectedness theorem for projective varieties with applications to intersection and singularities of mappings*, Ann. of Math. (2) **110** (1979), 159–166.
12. W. E. Fulton and R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Proceedings, University of Illinois at Chicago Circle, 1980; Lecture Notes in Math., vol. 862, Springer-Verlag, Berlin and New York, 1981, pp. 26–93.
13. M. Gerstenhaber, *Dominance over the classical groups*, Ann. of Math. (2) **74** (1961), 532–569.
14. N. Goldstein, *Ampleness in complex homogeneous spaces and a Second Lefschetz Theorem*, preprint.
15. J. Hansen, *A connectedness theorem for flagmanifolds and Grassmannians*, Aarhus Universitet Preprint Series, No. 9, 1980/81.
16. J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Math., no. 9, Springer-Verlag, Berlin and New York, 1972.

- 16A. S. Kleiman, *The transversality of a general translate*, *Composito Math.* **28** (1974), 287–297.
17. K. Mizuno, *The conjugate classes of unipotent elements of the Chevalley groups E_7 and E_8* , *Tokyo J. Math.* **3** (1980), 391–459.
18. H. Samelson, *Notes on Lie algebras*, *Math. Studies*, no. 23, Van Nostrand Reinhold, Princeton, N.J., 1969.
19. J. P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966.
20. A. Sommese, *Submanifolds of Abelian varieties*, *Math. Ann.* **233** (1978), 229–256.
21. _____, *Complex subspaces of homogeneous complex manifolds. I*, *Duke Math. J.* **46** (1979), 527–548.
- 21A. _____, *Complex subspaces of homogeneous complex manifolds. II*, *Nagoya Math. J.* (to appear).
- 21B. _____, *Concavity theorems*, *Math. Ann.* **235** (1978), 37–53.
- 21C. _____, *Concavity theorems. II*, preprint.
22. T. A. Springer, *The unipotent variety of a semisimple group*, *Algebraic Geometry (Bombay Colloquium, 1968)*, Tata Institute, Bombay, 1969, pp. 373–391.
23. R. Steinberg, *Regular elements of semisimple algebraic groups*, *Inst. Hautes Étude Sci. Publ. Math.* **25** (1965), 49–80.
24. _____, *Conjugacy classes in algebraic groups* (notes by V. Deodhar), *Lecture Notes in Math.*, vol. 366, Springer-Verlag, Berlin and New York, 1974.
25. _____, *On the desingularization of the unipotent variety*, *Invent. Math.* **36** (1976), 209–224.
26. _____, letter, March 1981.

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