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UNITAL l-PRIME LATTICE-ORDERED RINGS WITH POLYNOMIAL CONSTRAINTS ARE DOMAINS

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ABSTRACT. It is shown that a unital lattice-ordered ring in which the square of every element is positive must be a domain provided the product of two nonzero l-ideals is nonzero. More generally, the same conclusion follows if the condition $a^2 \geqslant 0$ is replaced by $p(a) \geqslant 0$ for suitable polynomials $p(x)$; and if it is replaced by $f(a, b) \geqslant 0$ for suitable polynomials $f(x, y)$ one gets an l-domain. It is also shown that if $a \wedge b = 0$ in a unital lattice-ordered algebra which satisfies these constraints, then the l-ideals generated by ab and ba are identical.

1. Introduction. In [**5**, p. 79] Diem has asked if an l-prime l-ring in which the square of every element is positive is an l-domain. In this paper we show that any such l-ring R is a domain provided the f-subring T of f-elements has zero annihilator in R or the T-T convex l-bimodule of R generated by $Ta + aT$ contains a for each nilpotent element a of index 2. Also, some polynomial constraints which generalize the condition that squares are positive are considered, and it is shown that an l-prime l-ring with such constraints is an l-domain, sometimes even a domain. Our original arguments were based on Lemmas 13 and (an earlier version of) 14. However, while this paper was being revised we realized that the simpler Lemma 2 was sufficient to get l-domains from l-prime l-rings.

A *lattice-ordered ring* (l-*ring*) is a ring R whose additive group is an l-group (that is, R is a lattice and each translation $x \to a + x$ is order preserving, and hence is an order automorphism) and in which the set of positive elements $R^+ = \{a \in R: a \geqslant 0\}$ is closed under multiplication. Some good references for background material on l-rings are [**4**; **2**; **3**, Chapters 13 and 17; **6**; **9**, Chapter I, pp. 164–176 and **14**, §2, pp. 192–202]. In particular, in Theorem 1 of [**14**] and Proposition 1.3 of [**9**] there is a list of many of the basic equations, inequalities and properties that result from the interaction of the lattice and ring structures in an l-ring.

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The right (left) module M over the l-ring R is called an *l-module* if M is an l-group and $M^+R^+ \subseteq M^+$ ($R^+M^+ \subseteq M^+$). A *convex l-subgroup* (*submodule*) of M is a subgroup (submodule) X that is a sublattice which is also convex: $x \leq m \leq y$ and $x, y \in X$ imply $m \in X$; that is, X is the kernel of an l-group (l-module) homomorphism. The element $r \in R^+$ is an *f-element on* M_R if for all $a, b \in M$

(1) $a \wedge b = 0$ implies $ar \wedge b = 0$.

If R^+ consists of f-elements on M, then M is called an *f-module over R*. An l-module over R is an f-module precisely when it is embeddable in a product of totally ordered R-modules [**13**, Theorem 1.1 or **1**, p. 54]. Note that when M_R is an f-module, the map $x \to xr$ is a lattice homomorphism of M for each $r \in R^+$ (see, for example [**4**, Lemma 1, p. 52 or **2**, Theorem 1.4.4, p. 25]). If R and S are l-rings, then M is an R-S *l-bimodule* (*f-bimodule*) if M is a left l-module (f-module) over R, a right l-module (f-module) over S and $r(xs) = (rx)s$ for all $r \in R$, $x \in M$, and $s \in S$. The R-S l-bimodule is an f-bimodule if and only if it is embeddable in a product of totally ordered R-S l-bimodules. In particular, R is an *f-ring* (that is, R is an R-R f-bimodule) precisely when it is embeddable in a product of totally ordered rings [**4**, Theorem 12, p. 57]. By an *f-element of the l-ring R* we mean an element $a \in R^+$ which is an f-element on both the l-modules R_R and ${}_RR$. An *l-algebra* over the commutative unital totally ordered domain F is a ring R which is a *torsion-free algebra* over F and which is also an f-module over F. Of course, any l-ring R is an l-algebra over the integers $\mathbf{Z}$; and if R is also an l-module and algebra over the totally ordered *field F*, then it is an l-algebra over F.

An (right, left) ideal of the l-ring R is an (*right*, *left*) *l-ideal* of R if it is also a convex l-subgroup of the additive l-group of R. R is called *l-prime* if the product of two nonzero l-ideals is nonzero, and R is an *l-domain* if the product of two nonzero positive elements is nonzero. R is called (*l-reduced*) *reduced* if it has no nonzero (positive) nilpotent elements, and *l-semiprime* if it has no nonzero nilpotent l-ideals. Recall that R is l-semiprime (l-prime) if and only if for all $a \in R^+$ ($a, b \in R^+$), $aRa = 0$ ($aRb = 0$) implies $a = 0$ ($a = 0$ or $b = 0$) [**5**, 2.5, p. 73 or **11**]. An l-ideal P is an *l-prime l-ideal* of R if R/P is an l-prime l-ring. By the *lower l-radical* of the l-ring R we mean $\beta(R)$ = the intersection of all the l-prime l-ideals of R. The lower l-radical is a nil l-ideal, and R is l-semiprime if and only if $\beta(R) = 0$ [**5**, 2.13 or **11**]. We also note that, just as for rings, an l-reduced l-prime l-ring is an l-domain. Birkhoff and Pierce [**4**, p. 63] have shown:

(2) If R is an f-ring, then $N_n = \{a \in R\colon a^n = 0\}$ is a nilpotent l-ideal of index at most n.

Let R be an l-algebra over F, and let I be an l-ideal of R. Then $I_1 = \{x \in R\colon |x| \leq \alpha i$ for some $\alpha \in F^+$ and $i \in I^+\}$ is the algebra l-ideal of R generated by I. Since $I_1^2 \subseteq I$, if I is an l-prime l-ideal, then it is an algebra ideal. So $\beta(R)$ is the lower l-radical of the *l-algebra R*.

If a is an element of the l-module M, then its *positive part*, *negative part* and *absolute value* are defined by $a^+ = a \vee 0$, $a^- = (-a) \vee 0$ and $|a| = a \vee (-a)$, respectively. Then $a = a^+ - a^-$, $|a| = a^+ + a^-$ and $a^+ \wedge a^- = 0$. Moreover, if $a \wedge b = 0$, then $a = x^+$ and $b = x^-$ for $x = a - b$. So for an l-ring R (1) is equivalent to the identity $x^+y^+ \wedge x^- = 0$. Since $y^+x^+ \wedge x^- = 0$ is the corresponding identity for ${}_RR$, the class of f-rings is a variety of l-rings. Also, each of the following conditions is equivalent to the corresponding parenthetical identity, and hence determines a variety of l-rings:

(3) $a \wedge b = 0$ implies $ab = 0$ ($x^+x^- = 0$).

(4) $a^2 \geqslant 0$ for each a in R ($(x^2)^- = 0$).

The variety of f-rings is contained in the variety determined by (3); and the latter is contained in that determined by (4): $a^2 = (a^+ - a^-)^2 = (a^+)^2 + (a^-)^2 \geqslant 0$ [**4**, p. 59]. Johnson [**9**, p. 174] has shown that an l-prime f-ring is a totally ordered domain (also see [**10**]), and Diem [**5**, p. 81] has shown that an l-prime l-ring which satisfies (3) is also a totally ordered domain (see Lemma 13 below).

Let $F[x, y]$ be a free noncommutative algebra over the totally ordered domain F. As a generalization of squares positive, a torsion-free l-algebra R over F is called a *PPI l-algebra* if there is a polynomial $f(x, y) \in F[x, y]$ such that $f(a, b) \geqslant 0$ for each $a, b \in R$ (we do not have any occasion to use more than two variables). Of course, we assume that $f(x, y) \notin F$, and if R is not unital, then the constant term of $f(x, y)$ is zero. If for each a in the l-algebra R there is a polynomial $p(x)$ in $F[x]$ (of positive degree) with $p(a) \in R^+$, then R will be called *p-positive*. A *PPI l*-algebra which satisfies $p(x) \geqslant 0$ is p-positive. In §3 we show that a unital l-prime p-positive l-algebra with properly conditioned polynomials is an l-domain, or even a domain.

In [**12**] Shyr and Viswanathan have called an l-ring R *square-archimedean* if for each $a, b \in R^+$ there is a positive integer n such that $ab + ba \leqslant n(a^2 + b^2)$. They showed that in a square archimedean l-ring R, $\beta(R)$ is the sum of the nilpotent l-ideals of R, and it is the largest nil l-ideal of R. In §3 we consider polynomials more general than $f(x, y) = -(xy + yx) + n(x^2 + y^2)$. We show that if R is an l-prime l-algebra with the property that for some $a, b \in R^+$ (or $a \in R$) there is a suitable polynomial $f(x, y)$ with $f(a, b) \geqslant 0$, then R is an l-domain if it is unital, or satisfies more general conditions.

In §4 we summarize the results of §§2 and 3 in terms of the lower l-radical $\beta(R)$ and strengthen the result of Shyr and Viswanathan. In §5 we show that in an l-algebra with the polynomial constraints considered previously, if $a \wedge b = 0$, then the l-ideals generated by ab and ba are identical. In §6 there are some examples and a remark connecting the general constraints with (3) and (4).

Finally, we fix some notation and give a few more useful facts. If X is a subset of the l-ring R, then $\langle X \rangle$ will denote the convex l-subgroup of R generated by X. Also,

$$M_2 = \{a \in R^+ : a^2 = 0\}.$$

(5) If R is a torsion free l-algebra over F and $0 < \beta \in F$ and $a \in R$ with $\beta a \geqslant 0$ ($\beta a \leqslant 0$), then $a \geqslant 0$ ($a \leqslant 0$).

(6) $\langle R^n \rangle = \{r \in R: |r| \leqslant s^n \text{ for some } s \in R^+\}$ is an l-ideal of R.

(7) If $a \wedge b = a \wedge c = 0$, then $a \wedge (b + c) = 0$.

(8) If $a, b \in R$ and $a_1 = a - a \wedge b$, $b_1 = b - a \wedge b$, then $a_1 \wedge b_1 = 0$.

(9) If $a^* \wedge b^* = 0$ in a homomorphic image R^* of R, then there exist a and b in R, mapping to a^* and b^*, respectively, and $a \wedge b = 0$.

2. Squares positive. Our first lemma is included for ease of reference, and is, for $F = \mathbf{Z}$ (except (d)), Example 15 of [**4**, p. 55]. The next two lemmas determine when an l-semiprime l-ring is l-reduced or reduced.

LEMMA 1. *Let R be a torsion-free l-algebra over the totally ordered domain F, and let*

$$T = \{c \in R : |c| \text{ is an } f\text{-element of } R\}.$$

Then:

(a) *T is a convex f-subalgebra of R.*
(b) *If R is unital and $1 > 0$, then $F \subseteq T$.*
(c) *If $0 \neq \beta \in F$ and $a \in R$ with $\beta a \in T$, then $a \in T$.*
(d) *R is a T-T f-bimodule.*

PROOF. We will only prove (c). If $x \wedge y = 0$ in R, then $|\beta a| x \wedge y = 0$ implies

$$|\beta|(|a|x \wedge y) = |\beta||a|x \wedge |\beta|y = |\beta a|x \wedge |\beta|y = 0.$$

So $|a|x \wedge y = 0$ since R is F-torsion-free; similarly, $x|a| \wedge y = 0$, so $a \in T$.

We will consistently denote the f-subring of f-elements of R by T, or $T(R)$, if necessary.

LEMMA 2. *Let R be an l-ring. If $a \in R^+$ is an f-element of R and $a^2 = 0$, then $aRa = 0$.*

PROOF. Let $z \in R^+$. Then $(az - za)^+ \wedge (az - za)^- = 0$ and hence $(az - za)^+ a \wedge a(az - za)^- = 0$. Since $(az - za)^+ a = (aza - za^2)^+ = aza$ and $a(az - za)^- = (a^2 z - aza)^- = aza$, we have $aza = aza \wedge aza = 0$.

Recall that $M_2 = \{a \in R^+ : a^2 = 0\}$ and $N_2 = \{a \in R : a^2 = 0\}$.

LEMMA 3. *Let R be an l-ring.*

(a) *R is l-reduced if and only if it is l-semiprime and $M_2 \subseteq T$.*
(b) *R is reduced if and only if it is l-semiprime and $N_2 \subseteq T$.*
(c) *R is an l-domain if and only if it is l-prime and $M_2 \subseteq T$.*
(d) *R is a reduced l-domain if and only if it is l-prime and $N_2 \subseteq T$.*

PROOF. (a) If R is l-semiprime and $M_2 \subseteq T$, then $M_2 = 0$ by Lemma 2; hence R is l-reduced.

(b) Suppose that R is l-semiprime and $N_2 \subseteq T$. If $a \in N_2$, then $|a| \in T$ and $|a|^2 = |a^2| = 0$ since T is an f-subring. So $|a| = 0$ by Lemma 2, and hence R is reduced.

(c) follows from (a), and (d) follows from (b).

In the following $T^0 = \langle T^0 \rangle$ is defined to be $\mathbf{Z}$ and $u^0 = 1$ (even if $1 \notin R$). The next result is a generalization of [**14**, Lemma 4(b), p. 203].

LEMMA 4. *Let R be an l-ring with squares positive. Suppose that $a \in R$ and $k, l, m, n \in \mathbf{Z}^+$ with $1 \leqslant l \leqslant m + k + 2$. If $\langle T^k \rangle a^{2^n} \langle T^m \rangle \subseteq \langle T^l \rangle$, then*

$$\langle T^k \rangle a \langle T^{n+m} \rangle + \langle T^{k+n} \rangle a \langle T^m \rangle \subseteq \langle T^l \rangle.$$

PROOF. We use induction on n. If $n = 0$ this is trivial. Suppose it is true for some integer n and $\langle T^k \rangle a^{2^{n+1}} \langle T^m \rangle \subseteq \langle T^l \rangle$. Then $\langle T^k \rangle a^2 \langle T^{n+m} \rangle + \langle T^{k+n} \rangle a^2 \langle T^m \rangle \subseteq \langle T^l \rangle$. If $t \in T^+$, then $0 \leqslant (a \pm t)^2$ yields $-(t^2 + a^2) \leqslant ta + at \leqslant t^2 + a^2$ and hence $|ta + at| \leqslant t^2 + a^2$. But R is a T-T f-bimodule, and $|at|$, $|ta| \leqslant |at + ta|$ holds in any totally ordered T-T bimodule which is a homomorphic image of R, since $t \geqslant 0$; so it also holds in R. Now $|at| \leqslant t^2 + a^2$ implies

$$|t^k a t^{n+m+1}| = t^k |at| t^{n+m} \leqslant t^{k+n+m+2} + t^k a^2 t^{n+m} \in \langle T^l \rangle;$$

so $t^k a t^{n+m+1} \in \langle T^l \rangle$. Thus $\langle T^k \rangle a \langle T^{n+m+1} \rangle \subseteq \langle T^l \rangle$ by (6), and, similarly, $\langle T^{k+n+1} \rangle a \langle T^m \rangle \subseteq \langle T^l \rangle$.

The subset X of the l-ring R is said to have *local bi-f-superunits* if for each $x \in X$ there is an element $e \in T^+$ with $|x| \leqslant |x| e + e|x| + e|x|e$ (that is, x is in the convex l-T-T-bimodule of R generated by $Tx + xT$). The following theorem implies that a unital l-prime l-ring with squares positive is a domain.

THEOREM 1. *Let R be an l-ring in which the square of every element is positive.*

(a) *R is l-reduced (an l-domain) if and only if it is l-semiprime (l-prime) and* $M_2 = \{a \in R^+ : a^2 = 0\}$ *has local bi-f-superunits.*

(b) *R is reduced (a domain) if and only if it is l-semiprime (l-prime) and* $N_2 = \{a \in R: a^2 = 0\}$ *has local bi-f-superunits.*

PROOF. (a) Suppose that R is l-semiprime and M_2 has local bi-f-superunits. If $a \in M_2$, then by Lemma 4, with $k = m = 0$ and $n = l = 1$, $aT + Ta \subseteq T$, and hence $aT + Ta + TaT \subseteq T$. If U is the convex l-subgroup of R generated by $aT + Ta + TaT$, then $U = \{u \in R: |u| \leqslant at + ta + tat$ for some $t \in T^+\} \subseteq T$, and $a \in U$ since a has a bi-f-superunit. So $M_2 \subseteq T$ and R is l-reduced by Lemma 3(a). If R is also l-prime, then it is an l-domain by Lemma 3(c).

(b) If R is l-semiprime and N_2 has local bi-f-superunits, then, as in the previous paragraph, $N_2 \subseteq T$. So R is reduced by Lemma 3(b). If R is also l-prime, then it is a reduced l-domain. But if $ab = 0$, then $a^2 b^2 = 0$ implies $a^2 = 0$ or $b^2 = 0$, and hence $a = 0$ or $b = 0$.

Another version of Theorem 1 is implied by the following two lemmas. The *left annihilator* of a subset X of R is $l_R(X) = \{a \in R: ax = 0$ for each $x \in X\}$; the *right annihilator* of X will be denoted by $r_R(X)$.

LEMMA 5. *Let R be an l-ring and suppose that* $X \subseteq T$ *with* $X \subseteq X_1 - X_1$ *where* $X_1 = (X \cap R^+) \cup \{0\}$. *Then* $r_R(X) = r_R(\langle X \rangle)$ *is a right l-ideal of R, and* $l_R(X) = l_R(\langle X \rangle)$ *is a left l-ideal of R.*

PROOF. Let $x \in X$ and $r \in r_R(X)$. Then $x = x_1 - x_2$ where $x_1, x_2 \geqslant 0$ and $x_1, x_2 \in X \cup \{0\}$. If $|s| \leqslant |r|$, then

$$\begin{aligned} |xs| &= |(x_1 - x_2)s| \leqslant |x_1 s| + |x_2 s| \\ &= x_1|s| + x_2|s| \leqslant x_1|r| + x_2|r| = |x_1 r| + |x_2 r| = 0. \end{aligned}$$

So $s \in r_R(X)$ and $r_R(X)$ is a right l-ideal of R. Since $X \subseteq \langle X \rangle$, $r_R(\langle X \rangle) \subseteq r_R(X)$. Since $\langle X \rangle = \{u \in R: |u| \leqslant x_1 + \cdots + x_n$ for some $0 \leqslant x_i \in X_1\}$, if $r \in r_R(X)$ and $u \in \langle X \rangle$ with $|u| \leqslant x_1 + \cdots + x_n$, then $|ur| \leqslant |u||r| \leqslant x_1|r| + \cdots + x_n|r| = 0$, since $|r| \in r_R(X)$. Thus $ur = 0$ and $r \in r_R(\langle X \rangle)$. So $r_R(X) \subseteq r_R(\langle X \rangle)$. Similarly, $l_R(X) = l_R(\langle X \rangle)$ is a left l-ideal of R.

LEMMA 6. *Let R be an l-ring with squares positive and suppose that* $a \in R$ *with* $a^{2^n} \in T$. *If* $u \wedge v = 0$ *in R, then* $|a|u \wedge v \in r_R(T^n)$ *and* $u|a| \wedge v \in l_R(T^n)$. (*If* $n = 0$, $r_R(T^n) = l_R(T^n) = 0$.)

PROOF. By Lemma 4 with $k = m = 0$ and $l = 1$, $aT^n + T^n a \subseteq T$. If $n = 0$ the result is obvious; so assume $n \geqslant 1$. If $0 \leqslant s \in \langle T^n \rangle$, then $s \leqslant t^n$ for some $t \in T^+$ by (6). So $s(|a|u \wedge v) \leqslant t^n(|a|u \wedge v) = |t^n a|u \wedge t^n v = 0$. Since $\langle T^n \rangle = \langle T^n \rangle^+ - \langle T^n \rangle^+$, $|a|u \wedge v \in r_R(\langle T^n \rangle) = r_R(T^n)$ by Lemma 5.

THEOREM 2. *Let R be an l-ring in which the square of every element is positive and suppose that* $l_R(T) = r_R(T) = 0$. *Then*:

(a) *R is reduced if and only if it is l-semiprime.*

(b) *R is a domain if and only if it is l-prime.*

PROOF. By Lemma 6, $N_2 \subseteq T$, and hence (a) follows from Lemma 3(b). If R is l-prime, then it is a reduced l-domain by Lemma 3(d), and hence a domain (see the proof of Theorem 1).

3. Polynomial constraints which generalize squares positive. In this section we show that Theorems 1 and 2 are true for l-algebras which satisfy polynomial constraints more general than $x^2 \geqslant 0$. The types of constraints that we use are illustrated in the next two results which are generalizations of [**14**, Theorem 7, p. 200].

Let F be a totally ordered domain. A polynomial $f(x, y) \in F[x, y]$ will be called *nice* if it has at least one monomial of degree 1 in x and each of its monomials of degree 1 in x has a negative coefficient. So if $f(x, y)$ is nice, then $f(x, y) = -g(x, y) + p(y) + h(x, y)$ where $0 \neq g(x, y)$ is of degree 1 in x and all its coefficients are positive, and $h(x, y) = 0$ or each of its monomials is of degree at least 2 in x. For example, for each $\alpha \in F$, $f(x, y) = -(xy + yx) + \alpha(x^2 + y^2)$ is nice; so is $(y - x)^n$ and modifications obtained by putting in appropriate coefficients $\alpha \in F$ in the monomials of $(y - x)^n$. Note that y need not appear in the nice polynomial $f(x, y)$. *We will consistently denote the "parts" of a nice polynomial* $f(x, y)$ *by* $g(x, y)$, $p(y)$ *and* $h(x, y)$, *as in the definition.*

The derivative of $p(x) \in F[x]$ will be denoted by $p'(x)$. If $f(x, y)$ is a nice polynomial then $f(x, 1)'(0) < 0$.

LEMMA 7. *Let R be a unital torsion-free l-algebra over the totally ordered domain F. The following statements are equivalent for the nilpotent element a of R.*

(a) $|a| < 1$.

(b) *There is a polynomial* $p(x)$ *in* $F[x^2]$ *with* $p(a^n + 1) \geqslant 0$ *and* $p(a^n - 1) \geqslant 0$ *for each* $n \geqslant 1$, *and* $0 \neq p'(1) \cdot 1 \in R^+$.

(c) *For each integer* $n \geqslant 1$ *there are polynomials* $p_1(x)$ *and* $q_1(x)$ *in* $F[x]$ *with* $p_1(a^n + 1) \geqslant 0$, $q_1((a^n - 1)^2) \geqslant 0$ *and* $p_1'(1)q_1'(1) \cdot 1 > 0$ *in R.*

(d) *For each integer* $n \geqslant 1$ *there are polynomials* $p_2(x)$ *and* $q_2(x)$ *in* $F[x]$ *with* $p_2(a^n + 1) \geqslant 0$, $q_2(a^n - 1) \geqslant 0$ *and* $p_2'(1)q_2'(-1) \cdot 1 < 0$ *in R.*

(e) $1 \in R^+$ *and for each b in* $\{\pm a^n : n \geqslant 1\}$ *there is a polynomial* $f(x, y) \in F[x, y]$ *such that* $f(b, 1) \geqslant 0$ *and* $f(x, 1)'(0) < 0$.

(f) $1 \in R^+$, $|a|$ *is nilpotent and if* $u \wedge v = 0$ *with* $u \leqslant |a^m|$ *for some* $m \in \mathbf{Z}^+$ *and* $v \leqslant 1$, *then there is a nice polynomial* $f(x, y) \in F[x, y]$ *with* $f(u, v) \geqslant 0$.

(g) *For each integer* $n \geqslant 1$ *there are polynomials* $p_3(x)$ *and* $q_3(x) \in F[x]$, *with only odd terms, such that* $p_3(b)^+ p_3(b)^- = 0$ *if* $b = \pm(a^n + 1)$, *and* $q_3(b)^+ q_3(b)^- = 0$ *if* $b = \pm(a^n - 1)$; *and* $p_3(1)p_3'(1)q_3(1)q_3'(1) \cdot 1 > 0$ *in* R.

PROOF. For (a) → (b) let $p(x) = x^2$ and use the fact that T is an f-ring (Lemma 1(a)). For (b) → (c) let $p_1(x) = p(x)$ and $q_1(x) = h(x)$ where $p(x) = h(x^2)$ in (b). For (c) → (d) let $q_2(x) = q_1(x^2)$ and $p_2(x) = p_1(x)$.

(d) → (e). Let $b = a^n$ and take $p_2(x)$, $q_2(x) \in F[x]$ with $p_2(a^n + 1) \geqslant 0$, $q_2(a^n - 1) \geqslant 0$ and $p_2'(1)q_2'(-1) \cdot 1 < 0$. If $\beta = p_2'(1)q_2'(-1) > 0$, then $1 < 0$ in R by (5). So $\beta < 0$, $(-\beta) \cdot 1 > 0$ and $1 \in R^+$. Now

$$\begin{aligned} 0 \leqslant q_2(b-1) &= \alpha_0 + \alpha_1(b-1) + \alpha_2(b-1)^2 + \cdots + \alpha_m(b-1)^m \\ &= \left(\alpha_1 - 2\alpha_2 + \cdots + (-1)^{m-1} m\alpha_m\right)b + \alpha_0 + h(b) \\ &= q_2'(-1)b + \alpha_0 + h(b) \end{aligned}$$

where $h(x) \in x^2F[x]$. Similarly, there exists $h_1(x) \in x^2F[x]$ with

$$0 \leqslant p_2(b+1) = p_2'(1)b + \gamma_0 + h_1(b).$$

If $q_2'(-1) < 0$, then $f_+(x, y) = q_2'(-1)x + \alpha_0 + h(x)$ is a nice polynomial with $f_+(b, 1) \geqslant 0$. Also, $p_2'(1) > 0$ since $p_2'(1)q_2'(-1) < 0$, and $f_-(x, y) = -p_2'(1)x + \gamma_0 + h_2(x)$ is a nice polynomial with $f_-(-b, 1) \geqslant 0$; here, if $h_1(x) = \Sigma\gamma_i x^i$, then $h_2(x) = \Sigma(-1)^i\gamma_i x^i$.

If $q_2'(-1) > 0$, then again we get two nice polynomials $f_\pm(x, y)$ with $f_+(b, 1) \geqslant 0$ and $f_-(-b, 1) \geqslant 0$.

(e) → (a). By induction on the index of nilpotency of a we may assume that $a^k \in T$ if $k \geqslant 2$. Let $f(x, y) = g(x, y) + p(y) + h(x, y)$ be a polynomial with $f(x, 1)'(0) < 0$ and $f(a, 1) = g(a, 1) + p(1) + h(a, 1) \geqslant 0$, where the monomials of $g(x, y)$ (respectively, $h(x, y)$) are of degree 1 (respectively, 2) in x. Then, since $g(a, 1) = -\beta a$ where $\beta = -f(x, 1)'(0) > 0$ and $h(a, 1) \in a^2F[a] \subseteq T$, we have $\beta a \leqslant s$ for some $s \in T$. By using a similar polynomial for $-a$, we get $-\gamma a \leqslant t$ for some $t \in T$ and $0 < \gamma \in F$. So $-\beta t \leqslant \gamma\beta a \leqslant \gamma s$ and $a \in T$ by Lemma 1(a) and (c). Since (a) holds in any totally ordered ring, it must hold in any f-ring.

(f) → (a). By induction on the index of nilpotency of $b = |a|$, we may assume that $b^n = 0$, $n \geqslant 2$, and $b^k \in T$ if $k \geqslant 2$. Let $c = b \wedge 1$, and let $u = b - c$ and $v = 1 - c$. Then $c, v \in T$ and $u \wedge v = 0$ by (8). Let $f(x, y) = -g(x, y) + p(y) + h(x, y)$ be a nice polynomial with $f(u, v) \geqslant 0$. Then $0 \leqslant g(u, v) \leqslant p(v) + h(u, v)$. Each term of $h(u, v)$ is of the form $\alpha w = \alpha u^{n_1}v^{m_1}u^{n_2}v^{m_2} \cdots u^{n_t}v^{m_t}$ with $N = \Sigma n_i \geqslant 2$. Since $v \leqslant 1$, $0 \leqslant w \leqslant u^N \leqslant b^N \in T$; so $\alpha w \in T$ and hence $h(u, v) \in T$. Whence $g(u, v) \in T$ since $p(v) \in T$. Now $g(u, v)$ contains a term of the form αu, αuv^m, $\alpha v^m u$ or $\alpha v^m uv^k$, where $\alpha > 0$ and $m, k \geqslant 0$. Since $g(x, y)$ has positive coefficients, if d is this term, then $0 \leqslant d \leqslant g(u, v)$ and hence u, uv^m, $v^m u$ or $v^m uv^k \in T$ (Lemma 1(c)). But $v = 1 - c$ is an invertible element in T (since $c^n = 0$), hence $u \in T$ and $b = u + c \in T$.

(g) → (d). Since $p_3(x)$ has only odd terms $p_3(-b) = -p_3(b)$; and hence $p_3(-b)^+ = p_3(b)^-$ and $p_3(-b)^- = p_3(b)^+$. So if $b = a^n + 1$, then $p_3(b)^+ p_3(b)^- = 0$ and $p_3(b)^- p_3(b)^+ = 0$, and hence

$$p_3(b)^2 = [p_3(b)^+ - p_3(b)^-]^2 = [p_3(b)^+]^2 + [p_3(b)^-]^2 \geqslant 0.$$

Similarly, $q_3(b)^2 \geqslant 0$ if $b = a^n - 1$. Let $p_2(x) = p_3(x)^2$ and $q_2(x) = q_3(x)^2$. Then $p_2(a^n + 1) \geqslant 0$, $q_2(a^n - 1) \geqslant 0$ and $p_2'(1)q_2'(-1) \cdot 1 < 0$ in R.

Since T is a convex f-subring of R (Lemma 1(a)) and hence satisfies (3) and (4), for the implication (a) → (f) we may let $f(x, y) = -(xy + yx) + x^2 + y^2$, and for (a) → (g) we may let $p_3(x) = q_3(x) = x$. The proof is complete.

The next lemma shows that polynomials also determine when the idempotents are in T.

LEMMA 8. *The following statements are equivalent for the unital torsion-free l-algebra R over the totally ordered domain F.*

(a) *The idempotents of R are contained in the interval* $[0, 1]$ (*and are central*).

(b) *There is a polynomial $p(x)$ in $F[x]$ with $p(f) \geqslant 0$ for each idempotent f, and* $[p(1) - p(0)] \cdot 1 > 0$ *in R.*

(c) *For each idempotent f there are polynomials $p(x)$ and $q(x)$ in $F[x]$ with $p(f) \geqslant 0$, $q(1 - f) \geqslant 0$ and $[p(1) - p(0)][q(1) - q(0)] \cdot 1 > 0$ in R.*

(d) *For each idempotent f there are polynomials $p(x)$ and $q(x)$ in $F[x]$, with zero constant terms, such that $p(f)^+ p(f)^- = q(f)^- q(f)^+ = 0$ and $p(1)q(1) > 0$.*

PROOF. Since T is an f-ring (Lemma 1(a)) squares are positive in T and T satisfies $x^+ x^- = 0$; so (a) implies (b) and (d), and clearly (b) implies (c). Also, for (d) implies (a) we can simply note that for f idempotent $p(f) = p(1)f$ and $q(f) = q(1)f$, and so $f^+ f^- = f^- f^+ = 0$. Hence $f = f^2 \geqslant 0$ and $1 - f \geqslant 0$. Now we show that (c) → (a).

By (5) $1 \in R^+$, since $[p(1) - p(0)][q(1) - q(0)] \cdot 1 > 0$. Also $0 \leqslant p(f) = p(0) + [p(1) - p(0)]f$ and $0 \leqslant q(1 - f) = q(1) - [q(1) - q(0)]f$ yield

$$-p(0) \leqslant [p(1) - p(0)]f \quad \text{and} \quad [q(1) - q(0)]f \leqslant q(1).$$

So, as in the proof of (e) → (a) of Lemma 7, $f \in T$. But (a) is satisfied in any unital f-algebra [7, p. 539]. For if $f = f^2$ in a unital totally ordered algebra, then $0 \leqslant f \leqslant 1 - f$ or $0 \leqslant 1 - f \leqslant f$, and hence $f = 0$ or 1. Thus, a unital f-algebra satisfies (a), since it is a subdirect product of totally ordered algebras. Consequently, by Lemma 1(a), the idempotents of R are contained in $[0, 1]$ and commute, and hence are central.

Note that the conditions on the coefficients of the polynomials are important. For any algebraic l-algebra R will satisfy the constraint $p(a) \in R^+$, but it need not satisfy (a) of Lemmas 7 and 8.

Results analogous to Theorem 1 follow from Lemmas 7 and 3. We state one such result which uses (d) of Lemma 7.

THEOREM 3. *Let R be a unital torsion-free l-algebra over the totally ordered domain F.*

(a) *R is l-reduced (an l-domain) with $1 \in R^+$ if and only if R is l-semiprime (l-prime) and for each element a in $M_2 = \{a \in R^+ : a^2 = 0\}$ there is a polynomial $q_2(x)$ in $F[x]$ with $q_2(a-1) \geqslant 0$ and $q_2'(-1) \cdot 1 < 0$ in R.*

(b) *R is reduced (a reduced l-domain) with $1 \in R^+$ if and only if R is l-semiprime (l-prime) and for each element a in $N_2 = \{a \in R: a^2 = 0\}$ there are polynomials $p_2(x)$ and $q_2(x)$ in $F[x]$ with $p_2(a+1) \geqslant 0$, $q_2(a-1) \geqslant 0$ and $p_2'(1)q_2'(-1) \cdot 1 < 0$ in R.*

Next, we determine, in terms of polynomial constraints, when a unital l-domain is a domain. Let $\bar{F}$ be the totally ordered field of quotients of the totally ordered domain F, and let R be a torsion-free l-algebra over F. Then $\bar{R} = R \otimes_F \bar{F} = \{r/\alpha: r \in R$ and $0 \neq \alpha \in F\}$ is the F-divisible hull of R. If $\bar{R}$ is given the positive cone $\bar{R}^+ = \{r/\alpha: r \in R^+$ and $\alpha \in F^+\}$, then $\bar{R}$ is an l-algebra over $\bar{F}$ which contains R.

The F-l-algebra R will be called *normal* (*i-normal*) if for each a in R which is a zero divisor there is a polynomial $0 \neq p(x)$ in $F[x]$, with zero constant term, such that $p(a) \geqslant 0$ (and $p(1) \neq 0$).

LEMMA 9. *Let R be a unital, reduced, normal l-algebra over the totally ordered domain F, and suppose that R is an l-domain. Then the following statements are equivalent.*

(a) *R is a domain and $1 \in R^+$.*

(b) *If $c^2 = \alpha c$ with $c \in R$ and $0 < \alpha \in F$, then there is a polynomial $p(x)$ in $F[x]$ such that $p(c) \in R^+$ and $[p(\alpha) - p(0)] \cdot 1 > 0$ in R.*

(c) *The idempotents of $\bar{R} = R \otimes_F \bar{F}$ are positive.*

(d) *$\bar{R}$ is i-normal over $\bar{F}$ and $1 \in R^+$.*

PROOF. (a) → (b). If $c^2 = \alpha c$ with $\alpha > 0$, then $f = c/\alpha$ is an idempotent of $\bar{R}$, and since $\bar{R}$ is a domain, $f = 0$ or 1. So $c = 0$ or α and we can let $p(x) = x$.

(b) → (c). First note that $1 \in R^+$ by (5). Let $f = c/\alpha$ be an idempotent in $\bar{R}$ with $\alpha > 0$. Then $1 - f = (\alpha - c)/\alpha$ is idempotent and $c^2 = \alpha c$ and $(\alpha - c)^2 = \alpha(\alpha - c)$. Let $p(x), q(x) \in F[x]$ be such that $p(c) \geqslant 0$, $q(\alpha - c) \geqslant 0$ and $p(\alpha) - p(0) > 0$, $q(\alpha) - q(0) > 0$. Then $p(c) = p(\alpha f) = p(0) + [p(\alpha) - p(0)]f \geqslant 0$ and $q(\alpha - c) = q(\alpha(1-f)) = q(0) + [q(\alpha) - q(0)](1-f) \geqslant 0$. So $-p(0) \leqslant [p(\alpha) - p(0)]f$ and $[q(\alpha) - q(0)]f \leqslant q(\alpha)$, and hence $f \in T(\bar{R})$ since $F \subseteq T$ by Lemma 1.

(c) → (a). Since $\bar{T} = T \otimes_F \bar{F}$ is the set of f-elements of the l-domain $\bar{R}$, $\bar{T}$ is an f-ring (Lemma 1(a)) and hence is a domain. But the idempotents of $\bar{R}$, being positive, are contained in $\bar{T}$; and hence 0 and 1 are the only idempotents of $\bar{R}$. Let $ab = 0$ in R; then, since R is a normal l-algebra, there are nonzero polynomials $p(x)$ and $q(x)$ in $xF[x]$ with $p(a) \geqslant 0$ and $q(b) \geqslant 0$. Since R is an l-domain and $p(a)q(b) = 0$, either $p(a) = 0$ or $q(b) = 0$; suppose $p(a) = 0$ and $a \neq 0$. Then, since $\bar{R}$ is reduced, the algebraic element a is strongly regular in $\bar{F}[a]$; that is, $a = a^2h(a)$ for some polynomial $h(x)$ in $\bar{F}[x]$. For, since $\bar{F}[a]$ is reduced, $\bar{F}[a] \simeq \bar{F}[x]/(g(x))$ with $g(x)$ square free; so that $\bar{F}[a]$, as a ring, is a direct sum of fields (or see [**8**, p. 165]). Since $f = ah(a)$ is an idempotent of $\bar{R}$, $f = 0$ or $f = 1$; thus $f = 1$ and $b = 0$.

(d) → (c). Let $f \neq 0, 1$ be an idempotent of $\bar{R}$. Since $\bar{R}$ is i-normal there exists $p(x) \in x\bar{F}[x]$ with $0 \leqslant p(f) = p(1)f$ and $p(1) \neq 0$. Then $p(1)^2 f \geqslant 0$ and hence $f \geqslant 0$ by (5).

Since (a) trivially implies (d) the proof is complete.

Note that the equivalence of (b) and (c) in Lemma 9 holds for any unital l-algebra. From Theorem 3 and Lemmas 7 and 9 we get the following two corollaries.

COROLLARY 1. *Let R be a unital torsion-free l-algebra over the totally ordered domain F. Then R is a domain with $1 \in R^+$ if and only if it is a normal l-prime l-algebra which satisfies* (i) *and* (ii).

(i) *If $a \in R$ with $a^2 = 0$, then there are polynomials $p_2(x)$ and $q_2(x) \in F[x]$ with $p_2(a + 1) \geqslant 0$, $q_2(a - 1) \geqslant 0$ and $p_2'(1)q_2'(-1) \cdot 1 < 0$ in R.*

(ii) *If $c^2 = \alpha c$ where $0 < \alpha \in F$ and $c \in R$, then there exists $p(x) \in F[x]$ with $p(c) \in R^+$ and $[p(\alpha) - p(0)] > 0$.*

The F-l-algebra is *weakly p-positive* if for each a in R there is a polynomial $p(x) \in F[x]$ (of degree $\geqslant 1$) with $p(a) \geqslant 0$ and $p'(1) > 0$ in F; it is *strongly p-positive* if for each a in R, $p(x)$ exists with positive coefficients with $p(a) \geqslant 0$.

COROLLARY 2. *Let R be a unital, weakly p-positve, torsion-free l-algebra over the totally ordered domain F.*

(a) *If $1 \in R^+$, then R is a reduced l-domain if and only if it is l-prime.*

(b) *If F is a field and $1 \in R^+$, then R is a domain if and only if it is an i-normal l-prime l-algebra.*

(c) *If R is strongly p-positive, then $1 \in R^+$ and R is a domain if and only if it is a normal l-prime l-algebra.*

PROOF. (a) follows from Lemmas 7(c) and 3(d), and then (b) follows from Lemma 9(d). If R is a strongly p-positive normal l-prime l-algebra, then $p(1) \cdot 1 \in R^+$ with $p(x) \in F^+[x]$ implies $1 \in R^+$, and hence $F^+ \subseteq R^+$. Thus R is a domain by Corollary 1.

Example 1 in §6 shows that (b) is false if $F = \mathbf{Z}$, even if R is commutative and the idempotents of R are positive. It also shows that a weakly p-positive l-algebra need not be strongly p-positive. We also note that [**16**, Example 2] shows that a commutative unital l-domain with all idempotents positive, which is a p-positive l-algebra over a totally ordered field F, need not be reduced. In this example each element a satisfies an inequality $(x - \alpha)^2 \geqslant 0$. In fact, if R is any l-algebra with squares positive and $R_1 = R + F$ is the l-algebra obtained from R by freely adjoining F in the usual manner (so $R_1^+ = \{(r, \alpha)\colon r \in R^+$ and $\alpha \in F^+\}$), then R_1 is a p-positive l-algebra with $1 > 0$. Each element of R_1 satisfies $(x - \alpha)^2 \geqslant 0$ for some $\alpha \in F$. R_1 will be an l-domain if R is an l-domain. Analogous statements are true for any p-positive l-algebra.

If A is a finite subset of a strongly p-positive l-algebra R, then there is a polynomial $p(x) \in F^+[x]$ with $p(a) \geqslant 0$ for each a in A. For if a_1 and a_2 are in R and if $p_1(x), p_2(x) \in F^+[x]$ with $p_2(a_2) \in R^+$ and $p_1(p_2(a_1)) \in R^+$, then $p(a_i) \in R^+$ for $i = 1,2$ where $p(x) = p_1(p_2(x))$. Similarly, the direct sum of a family of strongly p-positive l-algebras is strongly p-positive. Since the direct sum need not be unital, we note that throughout this paper, the condition "$1 \in R^+$" may be replaced by "R has *central f-units*"; that is, for each $a \in R$ there is an idempotent e in T which is central in R and $a = ae$.

We turn next to two-variable polynomials and give the following generalization of Lemma 4.

Lemma 10. *Let R be a torsion-free l-algebra over the totally ordered domain F. Suppose that $a \in R$ and $1 \leqslant k \in \mathbf{Z}$. Assume that for each $t \in T^+$ and each integer $m \geqslant 0$ there are two nice polynomials $f_i(x, y) = -g_i(x, y) + p_i(y) + h_i(x, y) \in F[x, y]$, $i = 1, 2$, with $f_1(a^{k^m}, t) \geqslant 0$, $f_2(-a^{k^m}, t) \geqslant 0$ and such that*:

(i) *$g_1(x, y)$ or $g_2(x, y)$ has a monomial ending in x and $g_2(a^{k^m}, t) \leqslant g_1(a^{k^m}, t)$.*

(ii) *$h_i(x, y) \in F[x^k, y]$; so $h_i(x, y) = q_i(x^k, y)$ for $i = 1, 2$.*

If $a^{k^n} \in T$ for some $n \geqslant 0$, then for each $s \in T \cup \{1\}$ and for each $t \in T$ there is an integer $N \geqslant 0$ with $t^N sa \in T$.

Moreover, if the degree in y of each monomial of $g_i(x, y)$ which ends in x (for all $t \in T^+$ and $m \geqslant 0$) is bounded by M_1, and the degree of each $q_i(x, y)$ in x is bounded by M_2, then we may take $N \leqslant M_1(M_2^n + M_2^{n-1} + \cdots + 1)$.

Proof. Let $t \in T$ and $s \in T \cup \{1\}$. We may assume that $s \geqslant 0$ and $t \geqslant 0$. For if $|t|^N |s| a \in T$, then

$$|t^N sa| \leqslant |t|^N |s| |a| = ||t|^N |s| a| \in T$$

implies that $t^N sa \in T$ by Lemma 1. Let $t_1 = t \vee s$ if $s \neq 1$ and let $t_1 = t$ if $s = 1$. We argue by induction on n. If $n = 0$, then $a \in T$ and we can let $N = 0$. Assume the result is true for the integer n and $a^{k^{n+1}} \in T$, and let $b = a^k$. Then $b^{k^n} \in T$ and hence for each $s_1 \in T \cup \{1\}$ there is an integer N_1 with $t_1^{N_1} s_1 b \in T$ (and $N_1 \leqslant M_1(M_2^n + M_2^{n-1} + \cdots + 1)$ if M_1 and M_2 exist). Now for each integer $r \geqslant 1$ there is an integer N_r with $t_1^{N_r} s_1 b^r \in T$ (and $N_r \leqslant r M_1(M_2^n + M_2^{n-1} + \cdots + 1)$). For if $s_2 = t_1^{N_r} s_1 b^r \in T$, then there exists an integer M with $t_1^M s_2 b \in T$ (and $M \leqslant M_1(M_2^n + M_2^{n-1} + \cdots + 1)$); but $t_1^M s_2 b = t_1^M t_1^{N_r} s_1 b^{r+1}$ and hence $N_{r+1} = M + N_r$ (and $N_{r+1} \leqslant (r+1) M_1(M_2^n + M_2^{n-1} + \cdots + 1)$).

Let $f_1(x, y) = -g_1(x, y) + p_1(y) + h_1(x, y)$ be a nice polynomial which satisfies (ii) and such that $f_1(a, t_1) \geqslant 0$. If u is a term of $h_1(a, t_1) = q_1(a^k, t_1) = q_1(b, t_1)$, then

$$u = \alpha t_1^{i_1} b^{j_1} t_1^{i_2} b^{j_2} \cdots t_1^{i_l} b^{j_l}$$

with $0 \neq \alpha \in F$, $l \geqslant 1$, $i_1 \geqslant 0$, $j_l \geqslant 0$ and $j_1 \geqslant 1$. We claim that $t_1^L u \in T$ for some L (and $L \leqslant (\sum_{\nu=1}^{l} j_\nu) M_1(M_2^n + M_2^{n-1} + \cdots + 1)$). If $l = 1$ this follows from the previous paragraph. Assume that $l \geqslant 2$ and $t_1^{L_1}(\alpha t_1^{i_1} b^{j_1} \cdots t_1^{i_{l-1}} b^{j_{l-1}}) = s_3 \in T$ (and $L_1 \leqslant (\sum_{\nu=1}^{l-1} j_\nu) M_1(M_2^n + M_2^{n-1} + \cdots + 1)$). Then, again, there is an integer L_2 with

$$t_1^{L_1 + L_2} u = t_1^{L_2}(s_3 t_1^{i_l}) b^{j_l} \in T$$

and so

$$L = L_1 + L_2$$

(and

$$L \leqslant \left(\left(\sum_{\nu=1}^{l} j_\nu \right) M_1\left(M_2^n + M_2^{n-1} + \cdots + 1\right) \leqslant M_1\left(M_2^{n+1} + M_2^n + \cdots + M_2\right) \right).$$

Thus, there exists L_3 with $t_1^{L_3}h_1(a, t_1) \in T$ (and $L_3 \leqslant M_1(M_2^{n+1} + \cdots + M_2)$).

Similarly, if $f_2(x, y) = -g_2(x, y) + p_2(y) + h_2(x, y)$ is a nice polynomial which satisfies (i) and (ii) and $f_2(-a, t_1) \geqslant 0$, then there is an integer L_4 with $t_1^{L_4}h_2(-a, t_1) \in T$ (and $L_4 \leqslant M_1(M_2^{n+1} + M_2^n + \cdots + M_2)$). Let L_5 be the larger of L_3 and L_4 ($L_5 \leqslant M_1(M_2^{n+1} + M_2^n + \cdots + M_2)$). Then $t_1^{L_5}g_i(a, t_1) \in T$. For $g_1(a, t_1) \leqslant p_1(t_1) + h_1(a, t_1)$ and $g_2(-a, t_1) \leqslant p_2(t_1) + h_2(-a, t_1)$. But $g_2(-a, t_1) = -g_2(a, t_1)$, so

$$-\left(p_2(t_1) + h_2(-a, t_1)\right) \leqslant g_2(a, t_1) \leqslant g_1(a, t_1) \leqslant p_1(t_1) + h_1(a, t_1).$$

Thus

$$\begin{aligned} -t_1^{L_5}\left(p_2(t_1) + h_2(-a, t_1)\right) \leqslant t_1^{L_5}g_2(a, t_1) &\leqslant t_1^{L_5}g_1(a, t_1) \\ &\leqslant t_1^{L_5}\left(p_1(t_1) + h_1(a, t_1)\right) \end{aligned}$$

and $t_1^{L_5}g_i(a, t_1) \in T$ by Lemma 1(a).

Now suppose $g_1(a, t_1)$ has a term of the form $\beta t_1^{L_6}a$. But $t_1 \geqslant 0$ and all the coefficients of $g_1(x, y)$ are in F^+, so $|\beta t_1^{L_6}a| \leqslant |g_1(a, t_1)|$, since this inequality holds in any totally ordered F-T-T bimodule which is a homomorphic image of R, and R is a subdirect product of these modules. Thus $\beta |t_1^{L_5}t_1^{L_6}a| \leqslant t_1^{L_5}|g_1(a, t_1)| = |t_1^5 g_1(a, t_1)|$, and if $N = L_5 + L_6$ then $t_1^N a \in T$ by Lemma 1(c) (and

$$N \leqslant M_1\left(M_2^{n+1} + M_2^n + \cdots + M_2\right) + M_1 = M_1\left(M_2^{n+1} + M_2^n + \cdots + 1\right)).$$

If $N = 0$, then $a \in T$ and $t^N sa \in T$. If $N \geqslant 1$, then $0 \leqslant t^{N-1}s \leqslant t_1^N$ and hence $|t^{N-1}sa| = t^{N-1}s|a| \leqslant t_1^N|a| = |t_1^N a|$; so $t^{N-1}sa \in T$ by Lemma 1(a).

In [7], as part of their characterization of those f-rings that can be embedded in unital f-rings, Henriksen and Isbell defined an f-ring to be *infinitesimal* if it satisfies the identity $x^2 \leqslant |x|$ (equivalently $nx^2 \leqslant |x|$ for each $n \in \mathbf{Z}^+$). In [**15**, Remark, p. 367] we have called an l-ring which satisfies the "dual" identities $n|x| \leqslant |x^2|$ *supertesimal.* Since the essential use of the nice polynomials $f(x, y)$ in Lemmas 7 and 10 is that "$x \leqslant$ higher powers of x", we make the following definitions.

A (p-) *pseudosupertesimal l-algebra* over F is an l-algebra R such that for all a, $r \in R$, with $r \geqslant 0$ (and $a \geqslant 0$), there is a nice polynomial $f(x, y) = -g(x, y) + p(y) + h(x, y)$ in $F[x, y]$ with $f(a, r) \geqslant 0$. A nice polynomial $f(x, y)$ is called *k-restricted* if $h(x, y) \in F[x^k, y]$. R is a (right) *k-restricted pseudosupertesimal l-algebra* if for all $a, r \in R$ with $r \geqslant 0$ there are two k-restricted nice polynomials $f_1(x, y)$ and $f_2(x, y)$ with $f_1(a, r) \geqslant 0$, $f_2(-a, r) \geqslant 0$, $g_2(a, r) \leqslant g_1(a, r)$ and $g_1(x, y) + g_2(x, y)$ has monomials which begin and end in x ($g_1(x, y) + g_2(x, y)$ has a monomial which ends in x). R is a (right) *p-k-restricted pseudosupertesimal l-algebra* if for all $a, r \in R^+$ there is a k-restricted polynomial $f(x, y)$ with $f(a, r) \geqslant 0$ and $g(x, y)$ has monomials which begin and end with x (which end in x). Finally, a *bounded pseudosupertesimal l-algebra* (*etc.*) is an l-algebra R for which there is an integer K such that for all $a, r \in R$ with $r \geqslant 0$ there is a nice polynomial $f(x, y)$ with $f(a, r) \geqslant 0$ and the degree of y in $g(x, y)$ is $\leqslant K$ and the degree of $h(x, y)$ in x

is $\leqslant K$. For example, a square archimedean l-ring is a bounded p-2-restricted pseudosupertesimal l-algebra over $\mathbf{Z}$. And a strongly p-positive l-algebra R is pseudosupertesimal, since if $p(x) \in F^+[x]$, then $f(x, y) = p(y - x)$ is a nice polynomial; and if R is unital, then for each element a of R there is a nice polynomial $f(x) = f(x, 1)$ with $f(a) \geqslant 0$; so R is p-2-restricted. Also, a commutative p-pseudosupertesimal l-algebra is p-2-restricted. If R is a *PPI* l-algebra with a nice k-restricted polynomial $f(x, y) = -g(x, y) + p(y) + h(x, y)$ and $g(x, y)$ has monomials which end in x, then R is right k-restricted; if R just satisfies $f(x^+, y^+)^- = 0$ then it is right p-k-restricted.

We can now give other generalizations of Theorems 1 and 2. The subset X of the l-ring R is said to have *local* (*left*) *f-superunits* if for each $x \in X$ there is an $e \in T^+$ with $|x| \leqslant e|x|$ and $|x| \leqslant |x|e$ ($|x| \leqslant e|x|$). The element $a \in R$ is *regular* if $l_R(a) = r_R(a) = 0$.

THEOREM 4. *Let R be a pseudosupertesimal torsion-free l-algebra over the totally ordered domain F, and suppose that $2 \leqslant k \in \mathbf{Z}$.*

(a) *If R is right p-k-restricted, then R is l-reduced (an l-domain) if and only if it is l-semiprime (l-prime) and $M_2 = \{a \in R^+ : a^2 = 0\}$ has local left f-superunits.*

(b) *If R is right k-restricted, then R is reduced if and only if it is l-semiprime and $N_2 = \{a \in R: a^2 = 0\}$ has local left f-superunits.*

PROOF. (a) Suppose that R is l-semiprime and $a \in M_2$ and $e \in T^+$ with $a \leqslant ea$. Since $a^k \in T$ and $a \geqslant 0$ we may use Lemma 10 with $f_2(x, y) = -g_1(x, y)$. Then $a \leqslant e^N a \in T$; hence $a \in T$ by Lemma 1(a) and R is l-reduced by Lemma 3(a).

The proof of (b) is similar.

THEOREM 5. *Let R be a pseudosupertesimal torsion-free l-algebra over the totally ordered domain F, and suppose that $k \geqslant 2$. Suppose that $l_R(T) = 0 = r_R(T)$ and R is bounded; or T contains a regular element of R.*

(a) *If R is p-k-restricted and l-semiprime (l-prime), then it is l-reduced (an l-domain).*

(b) *If R is k-restricted and l-semiprime, then it is reduced.*

PROOF. (a) If $a \in M_2$ and $t \in T^+$, then by Lemma 10 and its right counterpart $t^N a$ and at^N are in T for some integer N. So if $u \wedge v = 0$ in R, then $t^N(au \wedge v) = 0$ and $(ua \wedge v)t^N = 0$. If $s \in T$ is regular in R, then so is $t = s^2 \geqslant 0$; so $a \in T$. If R is bounded, then N is independent of t (Lemma 10), so $au \wedge v \in r_R(\langle T^N \rangle) = r_R(T^N)$ by Lemma 5, and $ua \wedge v \in l_R(T^N)$. If we also have $l_R(T) = r_R(T) = 0$, then again $a \in T$. Thus by Lemma 3(a) R is l-reduced.

The proof of (b) is similar to that of (a).

From Theorem 4 and Lemma 9(d) we get

COROLLARY 3. *Let R be a right k-restricted ($k \geqslant 2$) pseudosupertesimal l-algebra over the totally ordered field F, and suppose that R is unital with $1 \in R^+$. If R is an l-prime i-normal l-algebra, then R is a domain.*

4. The lower l-radical. If $\beta(R)$ is the lower l-radical of R, then since $R/\beta(R)$ is l-semiprime, Lemma 3 translates to

LEMMA 11. *Let R be an l-ring.*

(a) $\beta(R) = \{a \in R\colon |a|$ *is nilpotent*$\} = M$ *if and only if for* $0 \leq a \in M$ *and* $u \wedge v = 0$ *in R,* $au \wedge v \in \beta(R)$ *and* $ua \wedge v \in \beta(R)$. *This is true if* $M^+ \subseteq T$.

(b) $\beta(R) = \{a \in R\colon a$ *is nilpotent*$\} = N$ *if and only if for* $a \in N$ *and* $u \wedge v = 0$ *in* R, $|a|u \wedge v \in \beta(R)$ *and* $u|a| \wedge v \in \beta(R)$. *This is true if* $N \subseteq T$.

Lemmas 4, 7 and 10 (and the conditions in Theorems 4 and 5) offer a variety of polynomial characterizations of when $\beta(R) = M$ or $\beta(R) = N$. We record some of these explicitly (as implications). As usual, R is a torsion-free l-algebra over the totally ordered domain F.

THEOREM 6. *Each of the following conditions implies that* $\beta(R) = \{a \in R\colon |a|$ *is nilpotent*$\} = M \subseteq T$.

(a) *R is a right p-k-restricted pseudosupertesimal l-algebra for some integer $k \geq 2$ and R has local left f-superunits.*

(b) *R is a p-k-restricted pseudosupertesimal l-algebra, with* $l_R(T) = r_R(T) = 0$ *and R is bounded; or T contains a regular element of R* $(k \geq 2)$.

(c) *Here, we assume* $1 \in R^+$. *If* $u \wedge v = 0$ *with u nilpotent and* $v \leq 1$, *then there is a nice polynomial* $f(x, y) \in F[x, y]$ *with* $f(u, v) \geq 0$.

PROOF. By Lemma 11(a) we only need that $M^+ \subseteq T$. For (a) this follows from the argument in Theorem 4(a), and for (b) it follows from the argument in Theorem 5(a). For (c) use Lemma 7(f).

THEOREM 7. *Let R be a torsion-free l-algebra over the totally ordered domain F. Each of the following conditions implies that* $\beta(R) = \{a \in R\colon a$ *is nilpotent*$\} = N \subseteq T$.

(a) *The square of each element in R is positive; and R has local bi-f-superunits, or* $l_R(T) = r_R(T) = 0$.

(b) *R is a bounded k-restricted pseudosupertesimal l-algebra and* $l_R(T) = r_R(T) = 0$ $(k \geq 2)$.

(c) *R is a k-restricted pseudosupertesimal l-algebra and T contains a regular element of R* $(k \geq 2)$.

(d) $1 \in R^+$ *and R is weakly p-positive.*

PROOF. By Lemma 11(b) it suffices to show that each nilpotent element is in T. For (a) this follows from Lemmas 4 and 6. For (b) and (c) this follows from Lemma 10 (as in the proof of Theorem 5). For (d) it follows from Lemma 7(c).

Since $\beta(R)$ is an f-ring (in Theorems 6 and 7) it is the sum of the nilpotent l-ideals of R [5, Theorem 3.1]. Let $Z_n = \{a \in R\colon |a|^n = 0\}$ and $N_n = \{a \in R\colon a^n = 0\}$. If $M_2 = \{a \in R^+\colon a^2 = 0\} \subseteq T$, then $Z_2(R) = N_2(T)$ is an l-ideal of R. For if $a \in Z_2(R)$, then $|a| \in T$ implies that $a \in T$ since T is a convex l-subring. Since T is an f-ring (Lemma 1(a)), $|a^2| = |a|^2$, and hence $a \in N_2(T)$ and $Z_2(R) = N_2(T)$. By (2), $N_2(T)$ is a convex l-subgroup of R, and then by Lemma 3 $Z_2(R) = N_2(T)$ is an l-ideal of R. If $M_2(R/Z_2) \subseteq T(R/Z_2)$, then $Z_4(R)$ is an l-ideal of R. In particular, if R satisfies the hypotheses of (a) or (c) of Theorem 6, then each Z_{2^n} is a nilpotent l-ideal of index at most 2^n, and $\beta(R)$ is the union of $\{Z_{2^n}\}$.

Similarly, if $N_2 \subseteq T$, then $N_2(R) = N_2(T)$ is an l-ideal of R; and if R satisfies the hypotheses of (d) or the first part of (a) of Theorem 7, then each N_{2^n} is a nilpotent l-ideal of index at most 2^n, and $\beta(R)$ is the union of $\{N_{2^n}\}$.

5. Disjoint elements almost commute. Recall that two elements a and b in an l-ring R are called disjoint if $a \wedge b = 0$.

It is well known that if a and b are two elements in an l-group with $a \wedge b = 1$, then $ab = ba$ [**3**, Theorem 6, p. 295]. Trivially, if a and b are disjoint elements of an l-ring which satisfies (3), then $ab = ba$. Examples in §6 show that a unital l-ring with squares positive need not have this property. However, Theorem 8 gives the appropriate analogue. We first present two lemmas.

An l-ring is *l-simple* if it has exactly two l-ideals. A unital totally ordered ring is l-simple if and only if whenever $a, b > 0$ there exist $c, d \geqslant 0$ with $a \leqslant cbd$. Some examples of commutative unital l-simple totally ordered rings F are subrings of the reals, totally ordered fields and (commutative) polynomial rings with coefficients in F, ordered appropriately. If R is an l-algebra over the totally ordered domain F, then an algebra l-ideal I is *closed* if R/I is F-torsion-free. For an arbitrary algebra l-ideal I, $\hat{I} = \{r \in R: \alpha r \in I$ for some $0 \neq \alpha \in F\}$ is the *closure* of I, and I is closed if and only if $I = \hat{I}$.

LEMMA 12. *Let R be an l-algebra over the totally ordered domain F.*

(a) *If for each $a \in R^+$ there exists $e \in R^+$ with $a \leqslant ea + ae + eae$, then each l-ideal of R is an algebra l-ideal.*

(b) *If F is l-simple, then each algebra l-ideal of R is closed.*

PROOF. (a) If I is an l-ideal of R, $a \in I^+$ and $\alpha \in F^+$, then $\alpha a \leqslant \alpha e a + a \alpha e + \alpha e a e$ implies $\alpha a \in I$.

(b) Let I be an algebra l-ideal of R. If $0 < \alpha \in F$ there exists $\beta \in F^+$ with $1 \leqslant \beta\alpha$. So if $r \in R$ with $\alpha r \in I$, then $|r| \leqslant \beta\alpha |r| = \beta |\alpha r| \in I$; hence $r \in I$.

Diem stated the next lemma for the case that R has squares positive, but, in fact, proved the more general result given here (a proof is also given in [**14**, p. 199]). It is the motivation for the somewhat surprising lemma which follows it.

LEMMA 13 [**5**, p. 78]. *An l-prime l-ring R is an l-domain if and only if it satisfies the two conditions*:

(a) *If $a, b \in R^+$ and $a^2 = b^2 = 0$, then $ab = 0$.*

(b) *If $a \wedge b = 0$ and $ab = 0$, then $ba = 0$.*

The element $a \in R^+$ is a *positive zero-divisor* if there is $0 \neq b \in R^+$ with $ab = 0$ or $ba = 0$.

LEMMA 14. *Let R be a torsion-free l-algebra over the totally ordered domain F. Suppose that*:

(a) *If $a \in R^+$ and $a^2 = 0$, then a is an f-element of R.*

(b) *If $u \wedge v = 0$, with u a positive zero divisor and $v \in T$, then there exists a polynomial $p(x) \in F^+[x]$ (of degree $\geqslant 1$) such that $p(v - u) \geqslant 0$; or there is a nice* *polynomial* $f(x, y) \in F[x, y]$ *with* $f(u, v) \geqslant 0$ *and* $f(x, y)$ *has a monomial of degree* 1 *in* x *which ends in* x.

Then if $a, b \in R$ *with* $a \wedge b = ab = 0$, *and* $e \in T^+$, *there exists* $N \in \mathbf{Z}^+$ *with* $ebe^N ae = 0$.

PROOF. We will repeatedly use the fact that T is an f-ring (Lemma 1(a)) and hence it satisfies (4).

Let $e \in T^+$ and let $a_1 = a \wedge e$ and $b_1 = b \wedge e$. We first show that $be^m a_1 e = 0$ for each $m \in \mathbf{Z}^+$. Let $b_2 = b - b_1$ and $e_2 = e - b_1$; and let $a_2 = a - a_1$ and $f_2 = e - a_1$. Then by (8) we get

$$b_2 \wedge e_2 = 0 \tag{10}$$

and

$$a_2 \wedge f_2 = 0. \tag{11}$$

Let $b_0 = b$ and $a_0 = a$; then since $a_1 b_i = 0$ we have

$$f_2 b_i = eb_i \quad \text{for } 0 \leqslant i \leqslant 2. \tag{12}$$

Also, since $a_i b_1 = 0$ we get

$$a_i e_2 = a_i e \quad \text{for } 0 \leqslant i \leqslant 2. \tag{13}$$

Now $a_1 \wedge b_1 e^m = 0$ and $a_1, b_1 e^m \in T$; so $b_1 e^m a_1 = 0$. Also (10) implies $b_2 e^m a_1^l \wedge e_2 = 0$, for any $l, m \in \mathbf{Z}^+$. But $e_2 \in T$, and $(b_2 e^m a_1^l)^2 = 0$ (if $l \geqslant 1$) implies $b_2 e^m a_1^l \in M_2 \subseteq T$; so

$$b_2 e^m a_1^l e = 0 \quad \text{for all } m \in \mathbf{Z}^+ \text{ and } l \geqslant 1, \tag{14}$$

since $b_2 e^m a_1^l e = b_2 e^m a_1^l e_2 = 0$, by (13). But then

$$be^m a_1 e = (b_2 + b_1) e^m a_1 e = b_2 e^m a_1 e + b_1 e^m a_1 e = 0.$$

By (11) $b_1 e^m a_2 \wedge f_2 = 0$, and therefore by (12) $eb_1 e^m a_2 = f_2 b_1 e^m a_2 = 0$. So

$$ebe^m ae = eb_2 e^m a_2 e \quad \text{for all } m \in \mathbf{Z}^+, \tag{15}$$

since $eb_1 e^m a_2 = be^m a_1 e = 0$ and

$$eb_2 e^m a_2 e = e(b - b_1) e^m (a - a_1) e = ebe^m ae - ebe^m a_1 e - eb_1 e^m a_2 e.$$

Since $(b_2 (f_2 e)^m a_2)(f_2 e)^s \in M_2 T^+ \subseteq T^+$ we get

$$b_2 (f_2 e)^m a_2 (f_2 e)^s a_2 \wedge f_2 = 0$$

by (11); and hence (12) implies

$$eb_2 (f_2 e)^m a_2 (f_2 e)^s a_2 = 0 \quad \text{for all } m, s \in \mathbf{Z}^+. \tag{16}$$

Let $p(x)$ be a polynomial in $F[x]$ of degree $\geqslant 1$ and with positive coefficients such that $p(f_2 e - a_2) \geqslant 0$. Then

$$0 \leqslant \alpha_0 + \alpha_1 (f_2 e - a_2) + \cdots + \alpha_n (f_2 e - a_2)^n = p(f_2 e - a_2) \tag{17}$$

and so ($\alpha_0 = 0$ if $1 \notin R^+$)

$$0 \leqslant g(a_2, f_2 e) \leqslant \alpha_0 + \sum_{k \geqslant 1} \alpha_k (f_2 e)^k + h(a_2, f_2 e) \tag{18}$$

where $-g(a_2, f_2e)$ is the sum of all those monomials in a_2 and f_2e in (17) which contain just one a_2, and $h(a_2, f_2e)$ is the sum of all those monomials which contain more than one a_2. A typical term in $h(a_2, f_2e)$ is of the form $\alpha w = \alpha(f_2e)^{m_1}a_2(f_2e)^{m_2}a_2\cdots(f_2e)^{m_t}$ with $m_i \in \mathbf{Z}^+$, $t \geqslant 3$ and $\alpha \in F$. By (16) $eb_2w = 0$ and hence $eb_2h(a_2, f_2e) = 0$. From (18) we get
From (18) we get

(19) $$0 \leqslant eb_2g(a_2, f_2e) \leqslant \sum \alpha_k eb_2(f_2e)^k.$$

A typical term in $g(a_2, f_2e)$ is $\alpha(f_2e)^m a_2(f_2e)^s$. But

(20) $$b_2(f_2e)^m a_2(f_2e)^s e \wedge b_2 = 0 \quad \text{for all } m, s \in \mathbf{Z}^+,$$

since $f_2 \leqslant e$ and

$$0 \leqslant b_2(f_2e)^m a_2(f_2e)^s e \wedge b_2 \leqslant b_2(f_2e)^m a_2(e^2)^s e \wedge b_2$$
$$= b_2(f_2e)^m a_2e_2e^{2s} \wedge b_2 = 0,$$

by (13) and (10); and (20) implies

(21) $$eb_2(f_2e)^m a_2(f_2e)^s e \wedge eb_2(f_2e)^k e = 0 \quad \text{for all } m, s, k \in \mathbf{Z}^+.$$

Now (19), (21) and (7) imply that

$$0 \leqslant eb_2g(a_2, f_2e)e = eb_2g(a_2, f_2e)e \wedge \sum \alpha_k eb_2(f_2e)^k e = 0,$$

and hence

(22) $$eb_2g(a_2, f_2e)e = 0.$$

However, one term in $g(a_2, f_2e)$ is $\alpha(f_2e)^m a_2$ with $0 < \alpha \in F$ and $m \geqslant 0$; since $g(x, y) \in F^+[x, y]$, (22) implies

(23) $$eb_2(f_2e)^m a_2e = 0.$$

Now for any $k \in \mathbf{Z}^+$

(24) $$b_2(f_2e)^k a_2 = b_2(e - a_1)e(e - a_1)e \cdots (e - a_1)ea_2 = b_2e^{2k}a_2,$$

since all other terms contain a factor $b_2e^r a_1^l e$ with $l \geqslant 1$, and $b_2e^r a_1^l e = 0$ by (14). Thus

(25) $$ebe^{2m}ae = eb_2e^{2m}a_2e = eb_2(f_2e)^m a_2e = 0$$

by (15), (24) and (23).

If there is a nice polynomial $f(x, y) = -g(x, y) + p(y) + h(x, y)$ with $f(a_2, f_2e) \geqslant 0$, then we again get (18) (some α_k may be negative); and if $g(x, y)$ has a monomial which ends in x, the calculation from (18) through (25) is still valid.

COROLLARY 4. *Suppose that R satisfies the hypotheses of Lemma* 14, *and it has local left* (*right*) *f-superunits and $l_T(T) = 0$ ($r_T(T) = 0$). Then $a \wedge b = ab = 0$ implies $ba = 0$.*

PROOF. If $e \in T^+$ is a left superunit for $\{a, b\}$, then by Lemma 14 $0 \leqslant bae \leqslant ebe^N ae = 0$ for some N. If $t \in T^+$, then $e + t$ is also a left superunit for $\{a, b\}$; so $ba(e + t) = 0$ and hence $bat = 0$. Since $l_T(T) = 0$, $ba = 0$.

The F-l-algebra R is called (*right*) *weakly p-pseudosupertesimal* if whenever $u \wedge v = 0$ in R there exists a nice polynomial $f(x, y) = -g(x, y) + p(y) + h(x, y) \in F[x, y]$ (such that $g(x, y)$ has a monomial ending in x) and $f(u, v) \geq 0$. Note that this is a one variable constraint since $u = a^+$ and $v = a^-$ for $a = u - v$.

THEOREM 8. *Let R be a torsion-free l-algebra over the totally ordered domain F, and suppose that R has local f-superunits. Each of the following statements implies that the closed l-ideals of R generated by ab and ba are identical whenever $a \wedge b = 0$ in R.*

(a) *R has square positive.*

(b) *R is unital and strongly p-positive.*

(c) *R is unital and right weakly p-pseudosupertesimal.*

(d) *R is right p-k-restricted pseudosupertesimal with $k \geq 2$.*

PROOF. We first note that the hypotheses are satisfied by each homomorphic image R^* of R (for (c) use (9)). Let I be the l-ideal of R generated by ab; I is an algebra l-ideal by Lemma 12(a), with closure $\hat{I}$. If $R^* = R/\hat{I}$, then, in each case, we have seen that $M_2^* = M_2(R^*) \subseteq T^* = T(R^*)$. For (a) use Lemma 4; for (b) use Lemma 7(d) (or the fact that (b) implies (c)); for (c) use Lemma 7(f); for (d) use Lemma 10. Since $a^* \wedge b^* = a^*b^* = 0$, $b^*a^* = 0$ by Corollary 4. So $ba \in \hat{I}$, and similarly, ab is in the closed l-ideal of R generated by ba.

It is possible to strengthen Theorem 8(b) by assuming weakly p-positive and the following. Let $p(x) = p_1(x) - p_2(x)$ where $p_1(x)$ (respectively, $-p_2(x)$) is the sum of the terms of $p(x)$ with a positive (respectively, negative) coefficient. Then for each $a \in R$ we require $p(x) = p_1(x) - p_2(x) \in F[x]$ with $p(a) \geq 0$, $p(1) - p(0) > 0$ in R, and for each $i \geq 0$, $\gamma_i = \Sigma_{k \geq i+1} (\alpha_k - \beta_k) \geq 0$ (α_k and β_k are the coefficients of x^k in $p_1(x)$ and $p_2(x)$). Now the proof of Lemma 14 goes through with $e = 1$. For $b_2 f_2 = b_2(1 - a_1) = b_2$ by (14), and hence in (19) $b_2 g(a_2, f_2) = \Sigma_{i \geq 0} \gamma_i b_2 a_2 f_2^i$; so the argument after (19) is still valid.

6. Examples and a remark. Let R be a torsion-free l-algebra over the totally ordered domain F. In [**14**, Theorem 8] it is shown that the following statements are equivalent if R has a left f-superunit e:

(i) R satisfies $x^+ x^- = 0$.

(ii) If $a \wedge e = 0$, then $a = 0$.

(iii) If $a \geq 0$ and $a \wedge e$ is nilpotent, then $a \in T$.

(iv) If $a \geq 0$ and $(a \wedge e)^2 = 0$, then $a \in T$.

(v) R has squares positive and

(26) If $a \in R^+$ and $(a \wedge e)^2 = 0$, then $a^2 = 0$.

(vi) Assume $e = 1$. R is a *PPI* l-algebra with a polynomial $p(x)$ which satisfies (26).

In fact, it is easily seen that (iv) is equivalent to

(vii) $M_2 = \{a \in R^+ : a^2 = 0\} \subseteq T$ and R satisfies (26).

Thus, to get other equivalences, each of the polynomial constraints which generalize squares positive or $x^+ x^- = 0$ and implies $M_2 \subseteq T$ can be substituted for "squares positive" in (v). Hence, these constraints are not that far removed from their squares positive origin.

EXAMPLE 1. A commutative, unital, reduced, i-normal, weakly p-positive l-domain in which all the idempotents are positive, but which is not a domain (see [**4**, Example 9f (II), p. 48]).

Let $\bar{R} = \mathbf{Q} \oplus \mathbf{Q}$ be the (ring) direct sum of two copies of the rationals with positive cone $\bar{R}^+ = \{(u, v): 0 \leqslant v \leqslant u\}$ and let

$$R = \{(2n, 2m) + (k, k): n, m, k \in \mathbf{Z}\}.$$

Then $\bar{R}$ is an l-domain and if $a = (u, v) \in \bar{R}$, then either $p(a) \geqslant 0$ or $p(a) \leqslant 0$, where $p(x) = vx - x^2$; so R is an i-normal p-positive l-algebra over $\mathbf{Z}$.

The following table shows that R is weakly p-positive.

TABLE 1

$a = (u, v) \in \mathbf{Z} \times \mathbf{Z}$	$p(x)$ with $p(a) \in \bar{R}^+$ and $p'(1) > 0$
$a \in \bar{R}^+ \cup -\bar{R}^+ \cup \{(u, 1): u < 0\}$	$p(x) = x^2$
$u < 0$ and $v \geqslant 2$	$p(x) = vx^2 - x^3$
$u < 0$ and $v < u$	$p(x) = -vx^2 + x^3$
$u = 0$ and $v < 2$	$p(x) = x^2 - vx$
$u = 0$ and $v = 2$	$p(x) = 2x + x^2 - x^3$
$u = 0$ and $v > 2$	$p(x) = vx - x^2$
$u > 0$ and $v < 0$	$p(x) = x^2 - vx$
$u > 0$ and $v > u$	$p(x) = v^3x - x^4$

EXAMPLE 2. A unital l-ring with squares positive in which disjoint elements do not commute.

An example is given by the free algebra generated by the set X. Let Δ be the free semigroup (with identity e) generated by X, and let Y be the set X together with a total order. If $s = x_1 x_2 \cdots x_p \in \Delta$, then s is said to have *length* p: $l(s) = p$. We make Δ into a partially ordered semigroup by defining, for $s, t \in \Delta$, $s < t$ if

(i) $1 \leqslant l(s) < l(t)$ or

(ii) $s = x_1 \cdots x_m x_{m+1} \cdots x_p$, $t = x_1 \cdots x_m y_{m+1} \cdots y_p$, $p \geqslant 2$, and $x_{m+1} < y_{m+1}$ in Y for some $m \geqslant 0$.

In this ordering the set $X \cup \{e\}$ is trivially ordered and is at the "bottom" of Δ, whereas the elements of length $\geqslant 2$ form a chain above X. Let $R = A[\Delta] = \{f = \Sigma a_s s: s \in \Delta,\ a_s \in A\}$ be the semigroup ring of Δ over the totally ordered domain A. By the *support* of an element $f = \Sigma a_s s$ in R we mean $\{s \in \Delta: a_s \neq 0\}$. If R is given the positive cone $R^+ = \{f = \Sigma a_s s: a_s > 0$ if s is a maximal element in the support of $f\}$, then R is a unital l-ring with squares positive (this may be verified directly or it follows from [**16**, Theorem I(b) and Lemma 2]). If X has at least two elements and if x and y are distinct in X, then $x \wedge y = 0$ in R, but $xy \neq yx$. Another such example is obtained by strengthening the order of Δ slightly by adding

(iii) $e < t$ if $l(t) \geqslant 2$.

We also note that, if in (i) and (iii) we stipulate that $l(t) \geqslant 2n$, and if we require that $p \geqslant 2n$ in (ii), for a fixed positive integer n, then R will satisfy $(x^{2n})^- = 0$ but not $(x^m)^- = 0$ for $m < 2n$.

The referee has supplied the following simpler example (any example must take into account [**15**, Theorem 1] and the equivalence of (i) and (ii) in the first paragraph of this section).

EXAMPLE 3. Let θ be a nontrivial order preserving automorphism of the totally ordered field F. Let $F[x;\theta]$ be the twisted polynomial ring determined by θ. So the elements of $F[x;\theta]$ are polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$ where $a_i \in F$. The elements of $F[x;\theta]$ are added as usual and multiplied like polynomials subject to the commutation rule $xa = (a\theta)x$ for any $a \in F$. Let $p(x) > 0$ if $n \geqslant 2$ and $a_n > 0$, and let $a_0 + a_1x \geqslant 0$ if $a_0 \geqslant 0$ and $a_1 \geqslant 0$. Then squares in $F[x;\theta]$ are positive; $a \wedge x = 0$ for any $a \in F$, and $ax \neq xa$ if $a\theta \neq a$.

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