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# A GEOMETRIC INTERPRETATION OF THE CHERN CLASSES

BY
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ABSTRACT. Let $f_\xi\colon M \to BU$ be a classifying map of the stable complex bundle $\xi$ over the weakly complex manifold $M$. If $\tau$ is the stable right homotopical inverse of the infinite loop spaces map $\eta\colon QBU(1) \to BU$, we define $f'_\xi = \tau \cdot f_\xi$ and we prove that the Chern classes $c_k(\xi)$ are $f'^*_\xi(h^*_k(t_k))$, where $h_k$ is given by the stable splitting of $QBU(1)$ and $t_k$ is the Thom class of the bundle $\gamma^{(k)} = E\Sigma_k X_{\Sigma_k}\gamma^k$. Also, we associate to $f'$ an immersion $g\colon N \to M$ and we prove that $c_k(\xi)$ is the dual of the image of the fundamental class of the $k$-tuple points manifold of the immersion $g$, $g^*_k([N_k])$.

**1. Introduction.** In this paper, we give a geometric interpretation of the Chern classes of a stable complex vector bundle $\xi$ over a weakly complex manifold $M$. In the second section we define characteristic classes $\hat{c}_k$ in $H^{2k}(BU; \mathbf{Z})$ using the weak homotopy equivalence between $QBU(1)$ and $CBU(1)$ for an appropriate coefficient system $\mathcal{C}$ got in **[9, 4]**, the stable splitting of $CX$ given in **[4]** and the stable map $\tau\colon BU \to QBU(1)$ defined in **[1, 15 and 16]**. This section is finished by proving that $\hat{c}_k$ are the Chern classes using the computations of the homology of infinite loop spaces in **[3, 6 and 11]**. In the third section, if $f_\xi\colon M \to BU$ is a classifying map of the bundle $\xi$, we associate an immersion $f\colon N \to M$ to the composition $\tau f_\xi\colon M \to QBU(1)$ following **[7]**. Then, we prove that $\hat{c}_k(\xi)$ is the Poincaré dual of the fundamental class of the $k$-tuple points manifold of the immersion $f$. A crucial step is the extension of $f$ to an immersion in "good position" of the normal bundle. The Appendix contains the proof of the existence and essential uniqueness of such extensions.

The research was done during my stay at the University of Warwick and it is part of my Ph. D. Thesis.

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**2. A new definition of the Chern classes.** In this paragraph we define classes $\hat{c}_k \in H^{2k}(BU; \mathbf{Z})$ and we prove that they are the Chern classes.

To define $\hat{c}_k$ let us recall some constructions of infinite loop space theory.

Let $\mathcal{C}$ be a coefficient system as defined in **[4]**. The universal $\mathcal{C}$-space associated to some space $X$ is defined as the quotient space

$$CX = \frac{\coprod \mathcal{C}_n \times X^n}{\sim}$$

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with the topology induced by the filtration given by

$$F_k CX = \operatorname{Im}\left( \coprod_{n=0}^{\kappa} \mathcal{C}_n \times X^n \to CX \right).$$

Also in [4] it is proved that for any coefficient system for which $\mathcal{C}_n$ is $\Sigma_n$-contractible, $CX$ has the weak homotopy type of the infinite loop space generated by $X$, $QX$, for any connected space $X$. This is true, in particular, for the coefficient systems given by the isometries operad $\mathcal{L}_\infty$, the little cube operad $\mathcal{C}_\infty$ and the system of configurations of points in $\mathbf{R}^\infty$, $\mathcal{F}(\mathbf{R}^\infty)$, where

$$\mathcal{F}(\mathbf{R}^\infty)_n = F(\mathbf{R}^\infty, n) = \left\{ (x_1 \cdots x_n) \in (\mathbf{R}^\infty)^n \colon x_i \neq x_j \right\}.$$

Using the approximation of $QBU(1)$ given by $C_\infty(BU(1))$, Snaith in [15] constructed a stable map $\tau$: $BU \to QBU(1)$ that is the right homotopy inverse of the unique map of infinite loop spaces $\eta$: $QBU(1) \to BU$ that extends the inclusion $BU(1) \subset BU$ (see [14] for a detailed proof).

In [4] it is proved that, for any connected $X$, there is a splitting of $QX$ given by stable maps

$$h_k \colon QX \to D_k(X)$$

where

$$D_k(X) = \frac{F_k(F(\mathbf{R}^\infty)(X))}{F_{k-1}(F(\mathbf{R}^\infty)(X))} = F(\mathbf{R}^\infty, k) \ltimes_{\Sigma_k} X^k.$$

The following proposition is easily proved.

**Proposition 2.1.** *If $X = T(\zeta)$ is the Thom space of some bundle $\zeta$, then $D_k(X) \cong T(\zeta^{(k)})$, where*

$$\zeta^{(k)} = F(\mathbf{R}^\infty, k) \times_{\Sigma_k} \zeta^k.$$

As $BU(1)$ has the homotopy type of the Thom space of the universal line bundle $T(\gamma^1)$, by composing we have a stable map

$$h_k \colon QBU(1) \to T(\gamma^{(k)}),$$

$\gamma^{(k)}$ is a complex vector bundle, so it has a Thom class $t_k \in H^{2k}(T(\gamma^{(k)}); \mathbf{Z})$. We define the characteristic class

$$\hat{c}_k = \tau^* h_k^*(t_k).$$

As usual, if $\xi$ is any stable complex vector bundle we define $\hat{c}_k(\xi) = f_\xi^*(\hat{c}_k)$, where $f_\xi$: $M \to BU$ is the classifying map of $\xi$.

As $\gamma^{(k)}$ has also a Thom class in complex cobordism $t_k^U \in MU^{2k}(T(\gamma^k))$, we can define in the same way $\hat{c}_k^U$, $\hat{c}_k^U(\xi)$ and obviously they are mapped to $\hat{c}_k$, $\hat{c}_k(\xi)$ by the standard map of spectra $MU \to \mathbf{HZ}$.

Now, we want to identify $\hat{c}_k$ as a polynomial in the Chern classes $\{c_n\}$. It is convenient to reduce it mod $p$ to $\hat{c}_k^{(p)} \in H^{2k}(BU; \mathbf{Z}/p\mathbf{Z})$ since there we can use the computations of the homology of infinite loop spaces [3, 4].

**Proposition 2.2.** *$\hat{c}_k^{(p)}$ agrees with the reduction* mod *$p$ of the universal Chern class, $c_k^{(p)}$, for any prime $p$.*

PROOF. It is well known that $H_*(BU;\mathbf{Z})$ is a polynomial algebra with generators $\{a_n\}_{n\in\mathbf{N}}$, where $\{a_n\}_{n\in\mathbf{N}}$ is a basis of $H_*(BU(1);\mathbf{Z})$ as a $\mathbf{Z}$-module, and the same is true for homology with coefficients in $\mathbf{Z}_p$ and the reductions $a_n^{(p)}$ **[17]**.

Then, to compute $\hat{c}_k^{(p)}$ we only need to evaluate it on each monomial $a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}$ of dimension $2k$.

$$\left\langle \hat{c}_k^{(p)}, a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}\right\rangle = \left\langle \tau^* h_k^*\left(t_k^{(p)}\right), a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}\right\rangle = \left\langle t_k^{(p)}, h_{k*}\tau_*\left(a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}\right)\right\rangle.$$

To see the actions of $h_{k*}$ and $\tau_*$ let us recall that $H_*(QBU(1);\mathbf{Z}_p)$ is the Dyer-Lashof algebra with generators $\{a_n^{(p)}\}$ **[3]**. As $\tau_*|_{H_*(BU(1))} = 1$ and $\tau$ is an $H$-map **[14]**, we have

$$\tau_*\left(a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}\right) = a_{i_1}^{(p)}\cdots a_{i_n}^{(p)}.$$

As seen in **[11]** $h_{k*}$ sends to zero any monomial in $\{Q^I(a_i^{(p)})\}$ with height different from $k$ and preserves any monomial of height $k$. But the only one of height $k$ and dimension $2k$ is $(a_1^{(p)})^k$ so we have

$$\left\langle c_k^{(p)}, a_{a_1}^{(p)}\cdots a_{i_n}^{(p)}\right\rangle = 0 \quad \text{for } a_{i_1}^{(p)},\ldots,a_{i_n}^{(p)} \neq \left(a_1^{(p)}\right)^k.$$

On the other hand,

$$\left\langle t_k^{(p)}, \left(a_1^{(p)}\right)^k\right\rangle = 1,$$

since by **[6]** the cell representing $(a_1^{(p)})^k$ is the same that represents $t_k$ (i.e.: $\{*\}\times_{\Sigma_k} D(\gamma|_{S^2})^k$) so $\hat{c}_k^{(p)}$ is the dual of $(a_1^{(p)})^k$ with respect to the basis of monomials in $\{a_n^{(p)}\}$, thus $\hat{c}_k^{(p)} = c_k^{(p)}$ **[2]**.

COROLLARY 2.3. $\hat{c}_k = c_k \in H^{2k}(BU;\mathbf{Z})$.

PROOF. $\hat{c}_k$ is a homogeneous polynomial of dimension $2k$ in $\{c_n\}$, $P_{2k}$. $\hat{c}_k^{(p)}$ is the same polynomial in $\{c_n^{(p)}\}$ with coefficients reduced mod $p$. As the coefficient of any monomial different from $c_k$ in $P_{2k}$ is congruent to zero modulo any prime $p$, it has to be zero. Similarly, the coefficient of $c_k$ is one. So, $\hat{c}_k = c_k$.

**3. Geometric interpretation.** In this section we give a geometric interpretation of the classes $\hat{c}_k$ defined in §2. First we recall the geometric models for maps from a manifold $M$ to $QBU(1)$ and the action that $h_k$ has on them as described in [7].

Let $\zeta$ be a vector bundle over $B\zeta$ and let $M$ be a smooth manifold. We denote by $\mathfrak{T}(M,\zeta)$ the group of cobordism classes of triples $(N, g, \tilde{g})$ where $N$ is a manifold, $g$ is an embedding $g\colon N\to M\times\mathbf{R}^\infty$ projecting to an immersion $f\colon N\to M$ whose normal bundle

$$\nu = f^*(T(M))/TN$$

is classified by the map $\tilde{g}$, i.e. $\tilde{g}$ is a bundle map:

$$\begin{array}{ccc} E_\nu & \overset{\tilde{g}}{\rightarrow} & E_\zeta \\ \downarrow p_\nu & & \downarrow p_\zeta \\ N & \underset{\tilde{g}_1}{\rightarrow} & B_\zeta \end{array}$$

If $\hat{M}$ is the one point compactification of $M$, we define a homomorphism

$$\beta\colon \mathfrak{T}(M,\zeta) \to [\hat{M}, F(\mathbf{R}^\infty, T(\zeta))]$$

as follows.

For any $(N, g, \tilde{g})$ we extend $f$ to an immersion $\bar{f}\colon \nu \to M$ satisfying:

(i) $(\bar{f}, e\cdot p_\nu)\colon E_\nu \to M \times \mathbf{R}^\infty$ is an embedding, where $e$ is the composite map

$$N \xrightarrow{g} M \times \mathbf{R}^\infty \xrightarrow{\pi_2} \mathbf{R}^\infty.$$

(ii) There is an integer $n$ such that $\bar{f}^{-1}(m)$ has at most $n$ points for any $m \in M$.

Then we define a map $H$ by

$$H(m) = \begin{cases} \{e(p_\nu(x))\colon x \in \bar{f}^{-1}(m)\} & \text{if } m \in \operatorname{Im} \bar{f}, \\ * & \text{otherwise,} \end{cases}$$

and $\beta([(N, g, \tilde{g})]) = [H]$.

In [7] the following is proved.

THEOREM 3.1. *$\beta$ is a group isomorphism.*

If $f\colon N \to M$ is a self-transverse immersion,

$$\tilde{N}_k = \{(x_1 \cdots x_k) \in N^k\colon x_i \neq x_j \text{ and } f(x_i) = f(x_j) \text{ for any } i, j\}$$

is a submanifold of $N^k$. Then, we define the manifold of $k$-tuple points of $f$, $N_k$, as the quotient of $\tilde{N}_k$ by the $\Sigma_k$-action given by permuting factors. The map induced by $f$, $f_k\colon N_k \to M$, is an immersion whose normal bundle $\nu_k$ is the quotient of $\nu^k|_{\tilde{N}_k}$ by the same $\Sigma_k$-action.

If we define the operation

$$\theta_k\colon \mathfrak{T}(M,\zeta) \to \mathfrak{T}(M,\zeta^{(k)})$$

by $\theta_k([(N, g, \tilde{g})]) = [(N_k, g', \tilde{g}')]$, where $g'$ and $\tilde{g}'$ are associated to $f_k$ then, the following is proved in [7].

THEOREM 3.2. *The diagram*

$$\begin{array}{ccc} \mathfrak{T}(M,\zeta) & \overset{\beta}{\rightleftarrows} & [\hat{M}, f(\mathbf{R}^\infty), (t(\zeta))] \\ \downarrow \theta_k & & \downarrow h_{k*} \\ \mathfrak{T}(M,\zeta^{(k)}) & \overset{\beta}{\rightleftarrows} & [\hat{M}, F(\mathbf{R}^\infty)(T(\zeta^{(k)}))] \end{array}$$

*commutes.*

Now we are going to define extensions of $f$ in good position and show some advantages. We define the manifold of based $k$-tuple points of $f$, $N'_k$, as the quotient of $\tilde{N}_k$ by the $\Sigma_{k-1}$-action given by permuting the first $(k-1)$ factors. The map $f'_k$: $N'_k \to N$ that sends each pointed $k$-tuple to the base point is an immersion whose normal bundle $\nu'_k$ is the quotient of $\nu^{k-1} \times \{0\}|_{\tilde{N}_k}$ by the same $\Sigma_{k-1}$-action. The projection $\pi_k\colon N'_k \to N_k$ is a $k$-cover and we call $\bar{\pi}_k\colon \nu'_k \to \nu_k$ the obvious monomorphism of bundles over it. (Notice that $\pi_k^*(\nu_k)$ is isomorphic to $\nu'_k \oplus f'^*_k(\nu)$.)

Then the map $\bar{f}: \nu \to M$ is said to be an extension of $f$ in good position if:

(i) $\bar{f}$ is an immersion extending $f$.

(ii) For any $k$, there are immersions

$$\bar{f}'_k: \nu'_k \to N$$

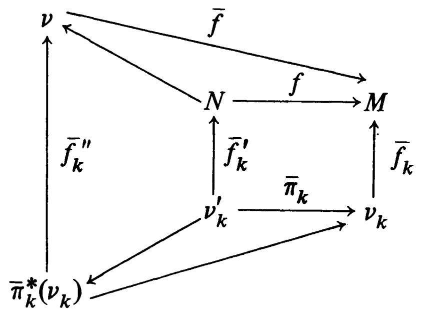
extending $f'_k$ and

$$\bar{f}_k: \nu_k \to M$$

extending $f_k$, and a map of vector bundles over $f'_k$,

$$\bar{f}''_k: \pi^*_k(\nu_k) \to \nu,$$

such that the diagram




commutes.

(iii) Im $\bar{f}_k$ is the set of multiple points of $\bar{f}$ of multiplicity greater than or equal to $k$, for any $k$.

In the Appendix we prove the existence and uniqueness of extensions in good position, but now we point out some advantages.

Let us define $\underline{M}_k$ as the closure of (Im $\bar{f}_k$ − Im $\bar{f}_{k+1}$) in Im $\bar{f}_k$ and $\underline{N}_k$ as the closure of (Im $f_k$ − Im $f_{k+1}$) in Im $f_k$. Then the obvious map $\underline{M}_k \to \underline{N}_k$ is a cube bundle classified by the map $\underline{N}_k \to F(\mathbf{R}^\infty, k) \times_{\Sigma_k} B\zeta^k$. So, the next proposition follows directly from the definition of $\beta$.

PROPOSITION 3.3. *The map $h$: $M \to QT(\zeta)$ associated with the triple $(N, g, \tilde{g})$ restricted to $M$ −* Im $\bar{f}_{k+1}$ *factors through $F_k(F(\mathbf{R}^\infty)(T(\zeta)))$ and the composite map*

$$M - \text{Im}\, \bar{f}_{k+1} \xrightarrow{h} F_k(F(\mathbf{R}^\infty)(T(\zeta))) \xrightarrow{h_k} T(\zeta^{(k)})$$

*is the Thom-Pontrjagin construction on the bundle $\underline{M}_k \to \underline{N}_k$.*

Now, let $\xi$ be a stable complex vector bundle over a weakly complex manifold $M$, classified by the homotopy class of a map $f_\xi$: $M \to BU$. The composite map $f' = \tau f_\xi$: $M \to QBU(1)$ gives a cobordism class $[(N, g, \tilde{g})]$ in $\mathfrak{T}(M, \gamma^1)$. If the immersion $f$: $N \to M$ is the projection of $g$, we can choose an extension of $f$ in good position $\bar{f}$: $\nu \to M$ and get, using the next lemma, a geometric interpretation of $\hat{c}_k(\xi)$.

LEMMA 3.4. *Let $M'$ be a codimension zero submanifold of $M$. Then, the diagram*

$$\begin{array}{ccc} MU_{n-q}(M) & \overset{PD}{\simeq} & MU^q(M) \\ \downarrow i_* & & \\ MU_{n-q}(M, M') & & \downarrow j^* \\ \wr\uparrow e_* & & \\ MU_{n-q}(M - \operatorname{int} M', \partial M') & \overset{LD}{\simeq} & MU^q(M - \operatorname{int} M') \end{array}$$

*commutes for any $q$, where the vertical maps are induced by the corresponding inclusions, $e_*$ being an isomorphism by the excision axiom and the horizontal maps are given by the Lefschetz duality ($LD$) and Poincaré duality ($PD$).*

The proof is an immediate application of transversality to the geometric models of $MU_*(X)$ and $MU^*(X)$ given in **[12]**.

Now, in the case $M' = \operatorname{Im} \bar{f}_{k+1}$, we have

THEOREM 3.5. $i_*(\mathrm{PD}^{-1}(\hat{c}_k^U(\xi))) = i_*([(N_k, f_k, \nu_k)])$ *as elements of* $MU_{n-2k}(M, M')$, *where* $[(N_k, f_k, \nu_k)]$ *is the element of* $MU_{n-2k}(M)$ *given by the map* $f_k\colon N_k \to M$ *and the obvious complex orientation of* $\nu_k$.

PROOF. The diagram

$$\begin{array}{ccccc} MU^{2k}(M) & \xleftarrow{(f'_\xi)^*} & MU^{2k}(F(\mathbf{R}^\infty)(BU(1))) & & \\ & & & \nwarrow h_k^* & \\ \downarrow j^* & & \downarrow i_k^* & & MU^{2k}(D_k(F(\mathbf{R}^\infty)(BU(1)))) \\ & & & \swarrow p_k^* & \\ MU^{2k}(M - \operatorname{int} M') & \xleftarrow{(f'_\xi|)^*} & MU^{2k}(F_kF(\mathbf{R}^\infty)(BU(1))) & & \end{array}$$

commutes since the left-hand square is induced by inclusions and restrictions and the right-hand triangle commutes by definition of $h_k$.

So, $j^*(\hat{c}_k^U(\xi)) = j^*(f'_\xi)^* h_k^*(t_k) = (f'_\xi|)^* p_k^*(t_k)$ and, as $f'_\xi|$ is the Thom-Pontrjagin construction of $\underline{M}_k$ over $\underline{N}_k$,

$$i_*\big(\mathrm{PD}^{-1}\big(\hat{c}_k^U(\xi)\big)\big) = e_*\mathrm{LD}^{-1}j^*\big(\hat{c}_k^U(\xi)\big) = e_*\mathrm{LD}^{-1}\big(f'_\xi|\big)^* p_k^*(t_k)$$

is represented by the triple $[(N_k, f_k, \nu'_k)]$, the same as $i_*([(N_k, f_k, \nu_k)])$.

COROLLARY 3.6. $\hat{c}_k(\xi) = \mathrm{PD} f_{k*}([N_k]) \in H^{2k}(M; \mathbf{Z})$ *where* $[N_k]$ *is the fundamental class of* $N_k$.

PROOF. Using the standard map $MU \to H\mathbf{Z}$ we get

$$i_*\big(\mathrm{PD}^{-1}\big(\hat{c}_k(\xi)\big)\big) = i_*\big(f_{k*}([N_k])\big) \in H_{n-2k}(M, M').$$

As $M'$ has the homotopy type of an $(n - 2(k + 1))$ dimensional complex, $i_*$ is a monomorphism in the given dimension, thus the result holds.

**4. Appendix.** The goal of this Appendix is to prove the existence and uniqueness up to regular homotopy of extensions in good position of a self-transverse immersion $f$. First, we recall the existence of some special kind of charts.

LEMMA 4.1 [5]. *Let $f\colon N \to M$ be a self-transverse immersion of codimension $a = \dim M - \dim N$. For any $y \in M$ such that $f^{-1}(y) = \{x_1,\dots,x_r\}$ there are charts at $y$, $(W, \psi)$ and at $x_i$, $(U_i, \phi_i)$ such that $f^{-1}(W)$ is the disjoint union of the $U_i$'s and the composition*

$$\left(\mathbf{R}^t \times (\mathbf{R}^a)^{r-1}, \mathbf{R}^t\right) \overset{\phi_i^{-1}}{\to} (U_i, \operatorname{Im} f_r') \overset{f}{\to} (W, \operatorname{Im} f_r) \overset{\psi}{\to} \left(\mathbf{R}^t \times (\mathbf{R}^a)^r, \mathbf{R}^t\right)$$

*is defined ($t = \dim M - ra$) and it is the inclusion*

$$\mathbf{R}^t \times (\mathbf{R}^a)^{r-1} \overset{\sim}{\to} \mathbf{R}^t \times H_i \to \mathbf{R}^t \times (\mathbf{R}^a)^r$$

*where $H_i = (\mathbf{R}^a)^{i-1} \times \{0\} \times (\mathbf{R}^a)^{r-i}$.*

Obviously if $M$ and $N$ are compact, there are finite coverings given by such charts and we assume that $W_j$ meets $\operatorname{Im} f_1,\dots,\operatorname{Im} f_{r_j}$ but does not meet $\operatorname{Im} f_{r_i-1}$.

From now on we use the following notation: $(\xi, \xi^1,\dots,\xi^{(r-1)}, P)$ always denotes an $r$-tuple of bundles over the manifold $P$ with fibre

$$\left((\mathbf{R}^a)^r, \bigcup_{i=1}^r H_i, \bigcup_{i,j=1}^r H_i \cap H_j,\dots, \bigcup_{i=1}^r L_i, \{0\}\right)$$

where $L_i = \{0\}^{i-1} \times \mathbf{R}^a \times \{0\}^{r-1}$.

PROPOSITION 4.2. *Let $N_d$ be the manifold of the deepest intersection points. There are immersions $\bar f_d, \bar f_d'$ such that*

$$\begin{array}{ccc} \left(\nu_d', \nu_d'^{(1)},\dots,\nu_d'^{(d-1)}, N_d'\right) & \overset{\bar f_d'}{\to} & (N, \operatorname{Im} f_2',\dots,\operatorname{Im} f_d') \\ \downarrow & & \downarrow \\ \left(\nu_d^{(1)},\dots,\nu_d^{(d)}, N_d\right) & \overset{\bar f_d}{\to} & (\operatorname{Im} f, \operatorname{Im} f_2,\dots,\operatorname{Im} f_d) \end{array}$$

*commutes where*

$$\nu_d'^{(i-1)} = \frac{\nu^{d-1} \times \{0\}}{\Sigma_{i-1}} \quad \text{and} \quad \nu_d^{(i)} = \frac{\nu^d}{\Sigma_i}.$$

PROOF. By 4.1, there are embeddings of the trivial bundle

$$e_j\colon (W_j \cap \operatorname{Im} f_d) \times \left((\mathbf{R}^a)^d, \bigcup_{i=1}^d H_i,\dots\right) \to (M, \operatorname{Im} f_1,\dots,\operatorname{Im} f_d) \cap W_j$$

and we can glue them by inductive use of the isotopy lemma of tubular neighbourhoods of Mather [8].

Now, choosing an isomorphism $\theta\colon \pi_d^*(\nu_d) \to \bar f_d'^*(\nu)$, we define $\bar f_d''$ as the composite map of $\theta$ and the map $\bar f_d'^*(\nu) \to \nu$. Thus, $\bar f|_{\nu|_{\operatorname{Im} \bar f_d}}$ is the only map commuting the diagram given in the definition of good position.

THEOREM 4.3. *Given a self-transverse immersion* $f\colon N \to M$ *there is an extension in good position and any two given extensions are regularly homotopic* mod Im $f$.

PROOF. The uniqueness is an obvious consequence of the uniqueness of tubular neighbourhoods.

For the existence part, we construct $\bar{f}_k, \bar{f}'_k, \bar{f}''_k$ and $\bar{f}|_{\nu|_{\operatorname{Im}\bar{f}_k}}$ by downward induction. Let us sketch how the construction goes for $k = d - 1$.

We define

$$\left(\overline{N}_{d-1}, \overline{\overline{N}}_{d-1}\right) = \bar{f}_{d-1}^{-1}\left(\operatorname{Im}\bar{f}_d, \bar{f}_d(D(\nu_d))\right),$$

$$\left(\overline{N}'_{d-1}, \overline{\overline{N}}'_{d-1}\right) = \bar{f}'^{-1}_{d-1}\left(\operatorname{Im}\bar{f}'_d, \bar{f}'_d(D(\nu'_d))\right)$$

and the maps

$$g\colon \nu_{d-1}|_{\overline{N}_{d-1}} \to \nu_d \stackrel{\sim}{\to} D(\nu_d) \stackrel{\bar{f}_d}{\to} M,$$

$$g'\colon \nu'_{d-1}|_{\overline{N}'_{d-1}} \to \nu'_d \stackrel{\sim}{\to} D(\nu'_d) \stackrel{\bar{f}'_d}{\to} N.$$

Now, as before, we extend these maps glueing charts but leaving fixed $g|_{\overline{\overline{N}}_{d-1}}$ and $g'|_{\overline{\overline{N}}'_{d-1}}$. Thus, we get $\bar{f}_{d-1}$ and $\bar{f}'_{d-1}$ commuting the appropriate diagram and, as above, they give $\bar{f}''_{d-1}$ and $\bar{f}|_{\nu|_{\operatorname{Im}\bar{f}_{d-1}}}$.

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