

AN EXTENSION OF THE THEORY
OF COMPLETELY DECOMPOSABLE
TORSION-FREE ABELIAN GROUPS

BY

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ABSTRACT. We construct a class of strongly indecomposable finite rank torsion-free groups that includes the rank-one groups, and develop a complete set of invariants for them and their direct sums.

0. Introduction. Perhaps the only class of countable torsion-free abelian groups, for which a satisfactory complete set of invariants has been developed, is the class of countable completely decomposable torsion-free groups [FCHS, §85]. The theory of completely decomposable groups begins by classifying rank-one torsion-free groups up to isomorphism by their type, then shows that direct sums of rank-one groups are isomorphic only if they are composed of the same number of groups of each type. As an added bonus it turns out that summands of completely decomposable groups are completely decomposable.

We can attempt to generalize this elegant theory by starting with a larger class than the rank-one groups. A natural place to look is among the Butler groups: those groups that are isomorphic to pure subgroups of a finite direct sum of rank-one torsion-free groups [BTLR, ARNL]. In this paper we describe a class **C** of Butler groups, that contains the rank-one groups and also contains strongly indecomposable groups of every finite rank, and exhibit a complete set of invariants for distinguishing among groups in **C**. Applying the global Azumaya theorems of [AHR] we also obtain a complete set of invariants for the class of direct sums of groups in **C**.

Let $n \geq 2$ be an integer. We shall be concerned with n -tuples $A = (A_1, \dots, A_n)$ of subgroups A_i of the rational numbers Q . For such an A we define the rank- $(n - 1)$ group $G(A)$ by

$$G(A) = \left\{ (a_1, \dots, a_n) \in A_1 \oplus \dots \oplus A_n : \sum_{i=1}^n a_i = 0 \right\}.$$

This construction is slightly more general than it might appear at first glance. Any pure subgroup K of $A_1 \oplus \dots \oplus A_n$ of corank one is the kernel of a nonzero

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homomorphism from $A_1 \oplus \cdots \oplus A_n$ to Q and so can be written as

$$K = \left\{ (a_1, \dots, a_n) \in A_1 \oplus \cdots \oplus A_n : \sum_{i=1}^n q_i a_i = 0 \right\}$$

for suitable integers q_i . Replacing A_i by $q_i A_i$ whenever $q_i \neq 0$ we see that K can be written as a direct sum of a group of the form $G(A')$, where A' is an m -tuple of groups each equal to some A_i , and the direct sum of the remaining $n - m$ groups A_i .

If the types τ_{ij}^A of $A_i \cap A_j$ are incomparable for distinct pairs $\{i, j\}$, we call A an *n-tuple of doubly incomparable subgroups of Q* . Let \mathbf{C} denote the class of groups that are isomorphic to $G(A)$ for such an A . Taking $n = 2$, and $A_1 = A_2$ an arbitrary subgroup of Q , we see that \mathbf{C} contains all rank-one torsion-free groups.

Given a doubly incomparable n -tuple A of subgroups of Q we construct two quantities. The first is a set of types T_A ; the second is a matrix E_A of integers and symbols $-\infty$, whose columns are indexed by T_A , and whose rows are indexed by the primes. The set T_A is a complete set of invariants for $G(A)$ under quasi-isomorphism (Theorem 1.3). The matrix E_A is determined by T_A up to a finite number of entries, and together with T_A forms a complete set of invariants for $G(A)$ under isomorphism.

We will have occasion to employ three operations on an n -tuple A of subgroups of Q that do not change the isomorphism class of $G(A)$. The first, *permutation*, replaces A_i by $A_{\sigma(i)}$ for some permutation σ of $\{1, \dots, n\}$. The second, *scalar multiplication*, replaces each A_i by qA_i for a fixed nonzero rational number q . We denote the resulting n -tuple by qA . Clearly, multiplication by q induces an isomorphism from $A_1 \oplus \cdots \oplus A_n$ to $qA_1 \oplus \cdots \oplus qA_n$ that takes $G(A)$ isomorphically onto $G(qA)$.

The third operation on A , called *trimming*, replaces each A_i by $A'_i = A_i \cap \sum_{j \neq i} A_j$. As A'_i is the image of the projection of $G(A)$ into A_i , it is clear that $G(A') = G(A)$. Moreover, $A'_i \cap A'_j = A_i \cap A_j$ so $\tau_{ij}^{A'} = \tau_{ij}^A$ whence A' is doubly incomparable if A is. We call A' the *trimmed version* of A and say that A is *trimmed* if $A = A'$. It is easy to show that A' is trimmed, and is, in fact, the largest trimmed n -tuple contained in A .

Let A and B be n -tuples of subgroups of Q . Call A and B *equivalent* if they can be made equal via a succession of trimmings, permutations, and scalar multiplications. An equivalence class of n -tuples may be thought of as a sort of multiple type. We shall show that the following three conditions are equivalent if A and B are doubly incomparable.

- (1) A is equivalent to B ,
- (2) $G(A)$ is isomorphic to $G(B)$,
- (3) $T_A = T_B$ and $E_A = E_B$.

We have already observed that (1) implies (2). Theorem 2.1 shows that (3) implies (1), and Theorems 1.3 and 2.3 show that (2) implies (3).

In §3 we consider the case where each type in T_A is the type of a subring of Q ; in this case we can replace E_A by an n -tuple of positive integers. In §4 we consider groups that are quasi-isomorphic to groups in \mathbf{C} and show that if, at each prime, K is isomorphic to some group in \mathbf{C} , then K is in \mathbf{C} (Theorem 4.4). This result, in conjunction with Theorem 1.5 which states that the endomorphism rings of groups

in \mathbf{C} are subrings of Q , implies that the class of direct sums of groups in \mathbf{C} is closed under the taking of summands. In §5 we apply the general theory of [AHR] to get a complete set of invariants for direct sums of groups in \mathbf{C} .

1. Types and quasi-isomorphism. Let A be an n -tuple of subgroups of Q . Let $\tau_i^A = \sup\{\tau_{ij}^A: j \neq i\} = \text{type}(A_i \cap \sum_{j \neq i} A_j)$, and let $T_A = \{\tau_i^A: i = 1, \dots, n\}$ where we think of T_A as a multiset if some of the τ_i are equal. If A is doubly incomparable we shall show how to compute T_A from $G(A)$, and show that T_A forms a complete set of invariants for $G(A)$ under quasi-isomorphism. For I a nonempty subset of $\{1, 2, \dots, n\}$, let τ_I^A be the infimum of $\{\tau_i^A: i \in I\}$. When there is only one n -tuple A under consideration, we will suppress the superscript A on τ_I^A . First we show how to retrieve the set of τ_{ij} from the type set of $G(A)$.

THEOREM 1.1. *If A is an n -tuple of subgroups of Q , then the type set of $G(A)$ consists of all τ_I with $\text{card } I \geq 2$. Thus, the maximal elements in the type set of $G(A)$ are precisely the τ_{ij} .*

PROOF. The type of an element x of K is the infimum of the types of those A_i in which x has a nonzero coordinate. As the sum of the coordinates of x is zero, x cannot have precisely one nonzero coordinate. As $\bigcap_{i \in I} A_i$ is a nonzero subgroup, there is an element x in K such that x has a nonzero coordinate in A_i if and only if $i \in I$. \square

If K is a group and τ is a type then $K(\tau) = \{x \in K: \text{type}(x) \geq \tau\}$.

LEMMA 1.2. *Let A be a doubly incomparable n -tuple of subgroups of Q , let $K = G(A)$ and let M be the set of maximal types of elements of K . Let τ and τ' be two distinct types in M . Then $K(\tau) + K(\tau')$ contains an element of type τ'' in M distinct from τ and τ' if and only if $\tau = \tau_{ij}$ and $\tau' = \tau_{jk}$ for some i, j and k .*

PROOF. On the one hand, $K(\tau_{ij}) + K(\tau_{jk})$ contains an element of type τ_{ik} . On the other hand, if the pairs $\{i, j\}$ and $\{k, l\}$ are disjoint, then the nonzero elements of $K(\tau_{ij}) + K(\tau_{kl})$ have type τ_{ij}, τ_{kl} or $\inf(\tau_{ij}, \tau_{kl})$. \square

THEOREM 1.3. *Let A and B be n -tuples of subgroups of Q . If $T_A = T_B$ then $G(A)$ and $G(B)$ are quasi-isomorphic. Conversely, if A and B are doubly incomparable, and $G(A)$ is quasi-isomorphic to $G(B)$, then $T_A = T_B$.*

PROOF. Suppose $T_A = T_B$. By trimming and permuting A and B we may assume that $B_i = q_i A_i$ for some rational numbers q_i , and we can then further assume that the q_i are integers. Then $G(B) \subset G(A)$ and $G(A)/G(B) \subset (A_1/q_1 A_1) \oplus \dots \oplus (A_n/q_n A_n)$ which is bounded by $q_1 q_2 \dots q_n$.

Conversely, if $G(A)$ and $G(B)$ are quasi-isomorphic, then they have the same set M of maximal types. Call two distinct elements of M *overlapping in $G(A)$* if they satisfy the condition of Lemma 1.2. As $G(A)$ and $G(B)$ are quasi-isomorphic, two elements of M overlap in $G(A)$ if and only if they overlap in $G(B)$. The set $T_A = T_B$ consists of the suprema of maximal sets of pairwise overlapping elements of M . \square

The following example shows that the requirement that A and B be doubly incomparable in Theorem 1.3 cannot be removed, and suggests the problem of

finding conditions on T_A that allow T_A to be read from (the quasi-isomorphism class of) $G(A)$.

EXAMPLE 1.4. Let p_1, p_2, p_3 and p_4 be distinct primes, and let R_i denote the ring generated over the integers by $1/p_i$, and R_{ij} denote $R_i + R_j$. Let

$$A = (R_{12}, R_{34}, R_{14}, R_{23}) \quad \text{and} \quad B = (R_{12}, R_{34}, R_{13}, R_{24}).$$

Then A and B are trimmed 4-tuples of subgroups of Q such that $T_A \neq T_B$, but we shall show that $G(A)$ and $G(B)$ are isomorphic. It suffices to show that $G(A)$ is isomorphic to $R_1 \oplus \cdots \oplus R_4$ modulo the subgroup generated by $(1, 1, 1, 1)$. Map $R_1 \oplus \cdots \oplus R_4$ to $G(A)$ by taking (w, x, y, z) to $(w - x, y - z, -w + z, x - y)$. The only difficulty is showing that this map is onto, and this can easily be done by localizing at each prime. \square

If S is a set of types, we let $K(S) = \sum_{\tau \in S} K(\tau)$. If we let τ^+ denote the set of types greater than τ , then $K(\tau^+)$ is $K^*(\tau)$ in the usual notation. We denote the smallest type in a group K by 0 . The following theorem shows that the groups in \mathbf{C} are B_0 -groups, that is, $K(\tau^+)$ is pure for any type τ [ARNL, p. 14].

THEOREM 1.5. *Let A be a doubly incomparable n -tuple of subgroups of Q and $K = G(A)$. If τ is a type, then*

$$\begin{aligned} K(\tau^+) &= 0 \quad \text{if } \tau = \tau_{ij} \text{ for some } i \text{ and } j, \\ &= K(\tau) \quad \text{otherwise.} \end{aligned}$$

PROOF. As the τ_{ij} are the maximal types of elements in K , the first assertion is clear. If no element of K has type precisely τ then the second assertion is clear. Otherwise $K(\tau) = G(A')$, where $A' = \{A_i : i \in I\}$ for some subset I of $\{1, \dots, n\}$ of cardinality at least three, so it suffices to show that $K(0^+) = K$. We shall show that $K(0^+)_p = K_p$ for each prime p . Let (a_1, \dots, a_n) be an element of K_p and choose $i \neq j$ such that $(A_i)_p \subset (A_j)_p$. For notational convenience we assume $i = 1$ and $j = 2$. Then

$$(a_1, a_2, \dots, a_n) = (a_1, -a_1, 0, \dots, 0) + (0, a_1 + a_2, a_3, \dots, a_n).$$

The first term is in $K(\tau_{12})_p \subset K(0^+)_p$ and the second term is in $K(0^+)_p$ by induction n . \square

The requirement that A be doubly incomparable in Theorem 1.5 cannot be dropped in light of the following example.

EXAMPLE 1.6. Define the 3-tuple A of subgroups of Q as follows. Let A_1 be the ring generated by $\frac{1}{2}$, let A_2 be the ring generated by $\frac{1}{3}$, and let A_3 be 5 times the ring generated by $\frac{1}{6}$. Then $K = G(A)$ is not a B_0 -group because $K(0^+)_5$ is generated by $(5, 0, -5)$ and $(0, 5, -5)$ over the integers localized at 5, but K contains $(1, -1, 0)$. Note that A is trimmed and K is quasi-decomposable. \square

The next theorem shows that the groups in \mathbf{C} are candidates for the global Azumaya theory developed in [AHR].

THEOREM 1.7. *If A is a doubly incomparable n -tuple of subgroups of Q and $K = G(A)$, then the endomorphism ring of K is $\{x \in Q : xA_i \subset A_i \text{ for } i = 1, \dots, n\}$. Hence, K is strongly indecomposable.*

PROOF. Let φ be an endomorphism of K . As $K(\tau_{ij}) \cong A_i \cap A_j$ is fully invariant, φ is multiplication by x_{ij} on $K(\tau_{ij})$ where x_{ij} is a rational number such that $x_{ij}A_i \subset A_i$ and $x_{ij}A_j \subset A_j$. If i, j and k are distinct, then $K(\tau_{ij})$ has nontrivial intersection with $K(\tau_{ij}) \oplus K(\tau_{jk})$. Thus $x_{ij} = x_{jk} = x_{ik}$, so φ is multiplication by a fixed element x such that $xA_i \subset A_i$ for $i = 1, \dots, n$, on $K(0^+) = K$. \square

As T_A determines the quasi-isomorphism class of $G(A)$, we should be able to find necessary and sufficient conditions on T_A for $G(A)$ to be strongly indecomposable. More specifically, these conditions should be on the structure of the lattice generated by T_A .

2. The isomorphism invariants. For x a nonzero rational number, and p a prime, we let $v_p x$ denote the unique integer m such that $x = p^m a/b$ where a and b are integers that are not divisible by p . If $A = (A_1, \dots, A_n)$ is an n -tuple of subgroups of Q , and p is a prime, we set

$$e_A(i, p) = \inf\{v_p x : x \in A_i\}.$$

Note that if $1 \in A_i$, then $-e_A(i, p)$ is the p -height of 1 in A_i . Also observe that x is in $(A_i)_p$ if and only if $v_p x \geq e_A(i, p)$. Each $e_A(i, p)$ is either an integer or $-\infty$, and all but finitely many of them are nonpositive. The matrix e_A completely determines A , and any such matrix is e_A for some A .

We may think of the columns of the matrix e_A as being indexed by T_A rather than by $\{1, \dots, n\}$ so that e_A becomes independent of the particular ordering of the n -tuple A . If T_A is a multiset, a case we are not particularly interested in, then there are minor technical problems in indexing e_A by T_A which we shall let the reader resolve. To construct the invariant E_A we let A' be the trimmed version of A and set

$$E_A(\tau, p) = e_{A'}(\tau, p) - \min\{e_{A'}(\sigma, p) : \sigma \in T_A \text{ and } e_{A'}(\sigma, p) \neq -\infty\}.$$

Thus the p th row of E_A specifies the relative positions of the groups $(A_i)_p$ in Q .

Recall from the introduction that two n -tuples of subgroups of Q are called *equivalent* if they can be made equal by a series of permutations, trimmings, and scalar multiplications.

THEOREM 2.1. *Let A and B be n -tuples of subgroups of Q . If $T_A = T_B$ and $E_A = E_B$, then A is equivalent to B .*

PROOF. We may assume that A and B are trimmed. As $T_A = T_B$ we can, by permuting B , assume there are rational numbers q_i so that $B_i = q_i A_i$. Multiplying A by a suitable rational number we may assume that the q_i are integers, that $v_p q_i = 0$ if $E_A(\tau_i, p) = -\infty$, and that $v_p q_i = 0$ for some i such that $E_A(\tau_i, p)$ is finite, unless $E_A(\tau_i, p) = -\infty$ for all i . If after all that we have $v_p q_i > 0$ for some i , then we can find j such that $E_A(\tau_j, p)$ is finite and $v_p q_j \neq v_p q_i$. But then

$$\begin{aligned} E_A(\tau_i, p) - E_A(\tau_j, p) &= e_A(i, p) - e_A(j, p) \\ &\neq e_B(i, p) - e_B(j, p) = E_B(\tau_i, p) - E_B(\tau_j, p), \end{aligned}$$

contradicting $E_A = E_B$. Thus we may assume that $A = B$. \square

LEMMA 2.2. *Let A be a doubly incomparable n -tuple of subgroups of Q , let p be a prime, and let $K = G(A)$. If $e_A(2, p)$, and at least one of $e_A(1, p)$ and $e_A(3, p)$, are finite, then the cyclic group*

$$K(\tau_{12}, \tau_{23}, \tau_{13})_p / K(\tau_{12}, \tau_{23})_p$$

has order p^m where

$$m = \sup\{e_A(1, p), e_A(2, p), e_A(3, p)\} - \sup\{e_A(1, p), e_A(3, p)\}.$$

PROOF. The group in question is isomorphic to $K(\tau_{13})_p / (K(\tau_{13})_p \cap K(\tau_{12}, \tau_{23})_p)$. The $K(\tau_{ij})_p$ are cyclic Z_p -submodules of $(A_1 \oplus A_2 \oplus A_3)_p$. If $v_p q_{ij} = \sup\{e_A(i, p), e_A(j, p)\}$, then we have

$$\begin{aligned} K(\tau_{13})_p &= Z_p(q_{13}, 0, -q_{13}), & K(\tau_{12})_p &= Z_p(q_{12}, -q_{11}, 0), \\ K(\tau_{23})_p &= Z_p(0, q_{23}, -q_{23}). \end{aligned}$$

Now $K(\tau_{13})_p \cap K(\tau_{12}, \tau_{23})_p = Z_p(r, 0, -r)$ where $v_p r = \sup\{e_A(i, p) : i = 1, 2, 3\}$ so $K(\tau_{13})_p \cap K(\tau_{12}, \tau_{23})_p = p^m K(\tau_{13})_p$ where m is as described. \square

We want to compute E_A from $G(A)$. A prime p is called *relevant* for A if τ_i is finite at p for at least two values of i . If p is not relevant for A , then $E_A(\tau_i, p) = 0$ if τ_i is finite at p . If τ_i is infinite at any prime p , then $E_A(\tau_i, p) = -\infty$. If $n = 2$, then all finite entries of E_A are 0. Thus it suffices to compute $E_A(\tau_i, p)$ when $n \geq 3$ and p is relevant and τ_i is finite at p .

THEOREM 2.3. *Let A be a doubly incomparable n -tuple of subgroups of Q with $n \geq 3$, let p be a relevant prime, and let $K = G(A)$. If τ_i is finite at p , then $E_A(\tau_i, p)$ is the supremum of the exponents of the cyclic groups*

$$K(\tau_{ij}, \tau_{ik}, \tau_{jk})_p / K(\tau_{ij}, \tau_{ik})_p$$

where j and k range over elements of $\{1, \dots, n\}$, different from i and from each other, such that either τ_j or τ_k is finite at p .

PROOF. Let μ_i denote the supremum in question. We may assume that A is trimmed and, by multiplying A by a suitable power of p , that $e_A(l, p) = E_A(\tau_l, p)$ for all l . If the pair j, k is used in forming the supremum, then either $e_A(j, p)$ or $e_A(k, p)$ is nonnegative, so by Lemma 2.2 we see that $\mu_i \leq e_A(i, p)$. Moreover, if $e_A(i, p) > 0$, then there exist j and k different from i and from each other such that $e_A(j, p)$ and $e_A(k, p)$ are either 0 or $-\infty$, and at least one is 0, so $\mu_i = e_A(i, p)$. \square

The following theorem characterizes those pairs T and E that are the invariants for some group in C .

THEOREM 2.4. *Let $n \geq 2$ be an integer, T a set of n types, and E a matrix of integers and symbols $-\infty$ whose columns are indexed by T and rows are indexed by the primes. Then there is a doubly incomparable n -tuple A of subgroups of Q such that $E = E_A$ and $T = T_A$ if and only if the following conditions hold.*

- (1) *The pairwise infima of types in T are incomparable.*
- (2) *$\tau \leq \sup\{\tau' \in T : \tau' \neq \tau\}$ for each τ in T .*
- (3) *$E(\tau, p) = -\infty$ if and only if τ is infinite at p .*

(4) Each row of E has at least two elements, and the least finite element, if any, is zero.

(5) If B is an n -tuple of subgroups of Q such that $T_B = T$, then for all but a finite number of primes p there is an integer k such that $E(\cdot, p) = e_B(\cdot, p) + k$.

PROOF. Suppose $E = E_A$ and $T = T_A$. We may assume that A is trimmed. The first condition holds because A is doubly incomparable; the second follows from the definition of the τ_i ; the third, fourth and fifth from the definition of E_A . Conversely, suppose these five conditions hold. Because of condition (2) we can find B satisfying the hypotheses of condition (5). We may assume that B is trimmed, so every row of e_B has at least two least elements. Let S be the finite set of exceptional primes of condition (5). Replacing each B_i by $q_i B_i$ where each q_i is divisible only by those primes in S we may assume that $e_B(\cdot, p) = E(\cdot, p)$ by condition (3). Set A equal to the resulting B . Then condition (4) assures that $E = E_A$ and condition (1) makes A doubly incomparable. \square

3. A special case. Let A be a doubly incomparable n -tuple of subgroups of Q . If the types in T_A are the types of subrings of Q , then these subrings are canonical representatives for their types and we can get a simpler looking set of invariants for $G(A)$.

THEOREM 3.1. *Let A and B be trimmed doubly incomparable n -tuples of subgroups of Q with $T_A = T_B$. If A_i is a subring of Q for $i = 1, \dots, n$, then there exist unique positive integers q_1, \dots, q_n such that:*

- (1) $G(q_1 A_1, \dots, q_n A_n) \cong G(B)$.
- (2) For any prime p , at least two of the q_i are not divisible by p .
- (3) If q_i is divisible by the prime p , then $pA_i \neq A_i$ and $pA_j \neq A_j$ for some $j \neq i$.

PROOF. As $T_A = T_B$, and A is an n -tuple of rings, e_B has only finitely many entries different from 0 or $-\infty$. There exist unique positive integers q_1, \dots, q_n such that q_i is not divisible by p if $pA_i = A_i$ and setting $B' = (q_1 A_1, \dots, q_n A_n)$ we have $e_{B'} = E_B$. That the q_i satisfy the required conditions follows from the properties of E_B ; and since $E_B = E_{B'}$ we have $G(B') \cong G(B)$. \square

If we further restrict T_A to consist of *finitely generated* subrings of Q , then each A_i is specified by the finite set P_i of primes that are invertible in the ring R_i of the same type, and an integer q_i such that $A_i = q_i R_i$. Thus $G(A)$ is presented by specifying the finite sets P_1, \dots, P_n and the integers q_1, \dots, q_n . It then makes sense to ask if there is an algorithm for deciding whether $G(A)$ is isomorphic to $G(A')$. In fact, Theorem 3.1 provides a simple algorithm for making that decision: we reduce the n -tuple q_1, \dots, q_n to the normal form defined in that theorem. Now an arbitrary pure subgroup H of $R_1 \oplus \dots \oplus R_n$ may be specified by a matrix of integers $\{q_{ij}\}$ by setting $H = \{(a_1, \dots, a_n) : \sum q_{ij} a_j = 0 \text{ for all } i\}$. Does there exist a (good) algorithm for deciding when two such subgroups are isomorphic?

4. Local closure. Two torsion-free groups G and H are *isomorphic at a prime p* if G can be imbedded as a subgroup G' of H such that H/G' is bounded by an integer not divisible by p . Let C be a class of groups whose endomorphism rings are

subrings of Q , and let K be a group. Then the following are equivalent [AHR, Lemma 3.1].

- (1) For each prime p there is a group in C that is isomorphic to K at p .
- (2) The endomorphism ring of K is a subring of Q and K is isomorphic to a summand of a finite direct sum of groups in C .

If either of these two conditions holds we say that K is *locally in C* . We say that C is *locally closed* if, whenever K is locally in C , then K is in C . If C is locally closed, then summands of direct sums of groups in C are direct sums of groups in C [AHR, Theorem B].

We shall show that the class C of groups isomorphic to $G(A)$ for some doubly incomparable n -tuple A of subgroups of Q , is locally closed. Given K locally in C we may restrict our attention to groups that are quasi-isomorphic to K . Every rank-one group is in C , so we need only consider groups of rank two or more. As $K(\tau)$ is an additive functor of K , the local closure of the rank-two groups in C follows from the following theorem.

THEOREM 4.1. *Let A be a doubly incomparable triple of subgroups of Q and let K be a subgroup of finite index in $G(A)$. The $K = K(0^+)$ if and only if $K = G(A')$ for some A' .*

PROOF. Let m be the index of K in $G = G(A)$. We may assume that A is trimmed, and that the p th rows of e_A and E_A are the same for each prime p dividing m . If p is a prime dividing m then, by replacing A_i by $p^{-1}A_i$ if necessary, thereby possibly changing m but not the set of primes dividing m , we can arrange for $e_A(i, p)$ to be 0 or $-\infty$ for each prime p dividing m and all i .

If H is a subgroup of G such that $G/H \cong Z_p \oplus Z_p$ then $H = pG$. Thus there is an integer n so that $K \subset nG$ and nG/K is cyclic. Hence we may assume G/K is cyclic. If some irrelevant prime p divides m , then the order of G/pG does not exceed p , so $K \subset pG$. Thus we may assume that no irrelevant prime divides m . So we can write K as

$$K = \{(a_1, a_2, a_3) \in G : m | s_1 a_1 + s_2 a_2 + s_3 a_3\}$$

where $0 \leq s_i < m$. For i, j, k , a permutation of 1, 2, 3, let q_k be the GCD of $s_i - s_j$ and m . Then $K(\tau_{ij}) = (m/q_k)G(\tau_{ij})$. Suppose the s_i are all congruent modulo some prime p dividing m . As $a_1 + a_2 + a_3 = 0$ we may assume that the s_i are all congruent to 0 modulo p . But then K would contain $(m/p)G$ and so G/K would not be cyclic of order m . Thus the q_k are pairwise relatively prime and $K(0^+) \subset (m/q_1 q_2 q_3)G$.

We may assume that $s_k = 0$. If $(a_1, a_2, a_3) \in K$, then $s_i a_i + s_j a_j$ is divisible by m . As q_i divides s_j and m , and q_i and s_i are relatively prime, we have q_i divides a_i . Therefore,

$$K \subset K' = \{(a_1, a_2, a_3) \in G : a_i \in q_i A_i \text{ for } i = 1, 2, 3\}.$$

As q_1 and q_2 are relatively prime we can find integers x and y such that $xq_1 + yq_2 = 1$. Hence $(xq_1, yq_2, -1)$ is in G at all primes dividing m , and has order q_3 modulo K' . Repeating this argument we see that G/K' has order at least $q_1 q_2 q_3$. In particular, if

$K(0^+) = K$, then $q_1q_2q_3 = m$, so $K = K'$. The “only if” follows from Theorem 1.5. \square

Theorem 4.1 does not extend to the class $n = 4$ as the following example shows.

EXAMPLE 4.2. Let A be a doubly incomparable 4-tuple of subgroups of Q and let p be a prime such that $e_A(i, p) = 0$ for $i = 1, \dots, 4$. Then there is a subgroup $K \subset G(A)$ of index p such that $K = K(0^+)$ but K is not isomorphic to $G(A')$ for any A' .

PROOF. Define K to be

$$K = \{(a_1, a_2, a_3, a_4) \in G(A) : p \mid a_1 + a_2\}.$$

The p th row of E_A , computed from K by Theorem 2.3, is $(1, 1, 1, 1)$ which is not possible, so K cannot be isomorphic to $G(A')$ for any A' . To see that $K(0^+) = K$, it suffices to show that $K(0^+)_p = K_p$ as $K_q = G$ for $q \neq p$. But $(1, -1, 0, 0)$, $(0, 0, 1, -1)$ and $(0, p, -p, 0)$ are in $K(0^+)$ and generate K_p over Z_p . \square

Despite the fact that there are groups K with $K = K(0^+)$ that are quasi-isomorphic to a group in \mathbf{C} but not isomorphic to such a group, it is still true that \mathbf{C} is locally closed. To see this we first establish the following

LEMMA 4.3. *Let A and B be doubly incomparable n -tuples of subgroups of Q . Then $G(A)$ is isomorphic to $G(B)$ at the prime p if and only if $T_A = T_B$ and the p th rows of E_A and E_B are the same.*

PROOF. If $K = G(A)$ and $L = G(B)$ are isomorphic at the prime p , then $T_A = T_B$ by Theorem 1.3. Moreover, $K(S)_p \cong L(S)_p$ for any set S of types, so the p th rows of E_A and E_B are the same by Theorem 2.3. Conversely, suppose that the p th rows of E_A and E_B are the same. Then, upon trimming A and B and multiplying A by an appropriate rational number, we may assume that the p th rows of e_A and e_B are the same, and that $A \subset B$. Then $G(A) \subset G(B)$ and $G(B)/G(A)$ has order not divisible by p . \square

THEOREM 4.4. *Let \mathbf{C} be the class of groups $G(A)$ where A is a doubly incomparable n -tuple of subgroups of Q . Then \mathbf{C} is locally closed.*

PROOF. Suppose that, for each prime p , there is a group in \mathbf{C} that is isomorphic to K at p . By Lemma 4.3, there is a fixed group G in \mathbf{C} that is isomorphic to K at each p . By [AHR, Lemma 3.5] K is isomorphic to G . \square

COROLLARY 4.5. *Let \mathbf{C} be the class of groups $G(A)$ where A is a doubly incomparable n -tuple of subgroups of Q . Then summands of direct sums of groups in \mathbf{C} are isomorphic to direct sums of groups in \mathbf{C} .*

PROOF. By Theorem 4.4 the class \mathbf{C} is locally closed. The result then follows from [AHR, Theorem B]. \square

5. The class of direct sums. As we have a complete set of invariants for groups in \mathbf{C} , and the endomorphism rings of these groups are principal ideal domains, we can construct a complete set of invariants for direct sums of groups in \mathbf{C} by utilizing [AHR, Theorem D]. A direct sum of groups in \mathbf{C} may be specified by a family \mathcal{Q} of

doubly incomparable tuples of subgroups of Q ; we denote the direct sum of the groups $G(A)$ for A in \mathcal{Q} by $G(\mathcal{Q})$. We associate with \mathcal{Q} the multiset $\mathfrak{T}_{\mathcal{Q}} = \{T_A: A \in \mathcal{Q}\}$ and for each T in $\mathfrak{T}_{\mathcal{Q}}$ and prime p the multiset $\mathfrak{E}_{\mathcal{Q}}(T, p) = \{E_A(\cdot, p): A \in \mathcal{Q} \text{ and } T_A = T\}$ of functions from T to $Z \cup \{-\infty\}$. Notice that the multiset $\mathfrak{E}_{\mathcal{Q}}$ does not record the fact that the functions $E_A(\cdot, p)$ and $E_A(\cdot, p')$ come from the same A in \mathcal{Q} . Hence, we cannot recover the family $\{G(A): A \in \mathcal{Q}\}$, up to isomorphism, from $\mathfrak{T}_{\mathcal{Q}}$ and $\mathfrak{E}_{\mathcal{Q}}$. This reflects the fact that nonisomorphic multisets of groups in \mathbf{C} can give rise to isomorphic direct sums.

THEOREM 5.1. *Let \mathcal{Q} and \mathcal{Q}' be families of doubly incomparable tuples of subgroups of Q . Then $G(\mathcal{Q})$ is isomorphic to $G(\mathcal{Q}')$ if and only if $\mathfrak{T}_{\mathcal{Q}} = \mathfrak{T}_{\mathcal{Q}'}$ and $\mathfrak{E}_{\mathcal{Q}} = \mathfrak{E}_{\mathcal{Q}'}$.*

PROOF. By [AHR, Theorem D] the groups $G(\mathcal{Q})$ and $G(\mathcal{Q}')$ are isomorphic if and only if, for each C in \mathbf{C} and prime p , there is a bijection between $\{A \in \mathcal{Q}: C \text{ is isomorphic to } G(A) \text{ at } p\}$ and $\{A' \in \mathcal{Q}': C \text{ is isomorphic to } G(A') \text{ at } p\}$. If C is isomorphic to $G(A'')$, then, by Lemma 4.3, C is isomorphic to $G(A)$ at p if and only if $T_A = T_{A''}$ and $E_A(\cdot, p) = E_{A''}(\cdot, p)$. \square

The next theorem characterizes those pairs \mathfrak{T} and \mathfrak{E} that are the invariants of a direct sum of groups in \mathbf{C} .

THEOREM 5.2. *Let \mathfrak{T} be a multiset of finite sets of types, and for each T in \mathfrak{T} and prime p let $\mathfrak{E}(T, p)$ be a multiset of functions from T to $Z \cup \{-\infty\}$. Then there is a multiset \mathcal{Q} of doubly incomparable tuples of subgroups of Q such that $\mathfrak{T} = \mathfrak{T}_{\mathcal{Q}}$ and $\mathfrak{E} = \mathfrak{E}_{\mathcal{Q}}$ if and only if the following conditions hold.*

(1) *Each T in \mathfrak{T} has cardinality at least 2 and consists of types whose pairwise infima are incomparable.*

(2) *If $T \in \mathfrak{T}$ and $\tau \in T$, then $\tau \leq \sup\{\sigma: \tau \neq \sigma \in T\}$.*

(3) *If $T \in \mathfrak{T}$ with multiplicity $m \neq 0$, then $\mathfrak{E}(T, p)$ has cardinality m , and if $E(\cdot, p)$ is in $\mathfrak{E}(T, p)$, then*

(i) *$E(\tau, p) = -\infty$ if and only if τ is infinite at p , and*

(ii) *$E(\cdot, p)$ takes on its least value at least twice, and its least finite value is 0.*

(4) *If B is an n -tuple of subgroups of Q such that $T_B = T$, and k is a cardinal less than the multiplicity m of T in \mathfrak{T} , then for all but a finite number of primes p the function $E_B(\cdot, p)$ appears more than k times in $\mathfrak{E}(T, p)$.*

PROOF. The necessity of the first three conditions follows immediately from Theorem 2.4. To see that condition (4) is necessary, suppose there are infinitely many primes such that $E_B(\cdot, p)$ appears no more than k times in $\mathfrak{E}(T, p)$. Then there are m elements A of \mathcal{Q} such that $T_A = T$ and $E_B(\cdot, p) \neq E_A(\cdot, p)$, contradicting Theorem 2.4, condition (5).

To show sufficiency we may assume that \mathfrak{T} consists of a single type T with multiplicity m . Because of conditions (1) and (2) we can choose a doubly incomparable n -tuple A of subgroups of Q such that $T_A = T$, and by Theorem 2.4 we can arrange for $e_A(\cdot, p)$ to be a specified element of $\mathfrak{E}(T, p)$ for a finite number of primes p . Using condition (4) we can collect such A into the required family \mathcal{Q} . \square

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