ENDOMORPHISMS OF THE COHOMOLOGY
OF COMPLEX GRASSMANNIANS

BY

MICHAEL HOFFMAN

ABSTRACT. For any complex Grassmann manifold G, we classify all endomorphisms of the rational cohomology ring of G which are nonzero on dimension 2. Some applications of this result are given.

1. Introduction. Let $G_{k,n}$ denote the complex Grassmann manifold of k-dimensional subspaces of $\mathbf{C}^{n+k}$. Then $G_{k,n}$ is a kn-dimensional complex manifold. In this paper we give a partial classification of endomorphisms of the cohomology ring $H^*(G_{k,n}; \mathbf{Q})$. This is useful as a first step toward classifying self-maps of $G_{k,n}$, and has several other consequences (see §7).

The cohomology ring $H^*(G_{k,n}; \mathbf{Q})$ is generated by elements $c_1, c_2, \ldots, c_k$, with $c_i \in H^{2i}(G_{k,n}; \mathbf{Q})$. Since c_1 is the only generator in dimension 2, any endomorphism of $H^*(G_{k,n}; \mathbf{Q})$ sends c_1 to a multiple of itself. The main result of this paper is as follows.

THEOREM 1.1. *Let h be an endomorphism of $H^*(G_{k,n}; \mathbf{Q})$ with $h(c_1) = mc_1$, $m \neq 0$. Then if $k < n$,*

$$h(c_i) = m^i c_i, \qquad 1 \leq i \leq k.$$

If $k = n$, there is the additional possibility

$$h(c_i) = (-m)^i (c^{-1})_i, \qquad 1 \leq i \leq k,$$

where $(c^{-1})_i$ is the $2i$-dimensional part of the inverse of $c = 1 + c_1 + \cdots + c_k$ in $H^*(G_{k,n}; \mathbf{Q})$.

REMARK. A Hermitian inner product on $\mathbf{C}^{n+k}$ gives a homeomorphism of $G_{k,n}$ onto $G_{n,k}$ by orthogonal complementation, so there is no loss of generality in considering only $G_{k,n}$ with $k \leq n$.

Theorem 1.1 includes a classification of automorphisms of $H^*(G_{k,n}; \mathbf{Q})$. Such a classification, for $k < n$, was first given in the unpublished work of Brewster [3], and Brewster's result is used in [4] to prove Theorem 1.1 for $k < n$. The proof given here draws heavily on [3], but is shorter and includes the case $k = n$. The main innovation is the use of the hard Lefschetz theorem.

To complete the classification of endomorphisms of $H^*(G_{k,n}; \mathbf{Q})$, it suffices to describe those endomorphisms h with $h(c_1) = 0$. We make the following conjecture.

CONJECTURE 1.2. *The only endomorphism h of $H^*(G_{k,n}; \mathbf{Q})$ with $h(c_1) = 0$ is the zero endomorphism.*

If this conjecture is true, then the hypothesis "$m \neq 0$" in Theorem 1.1 can be omitted. The conjecture is true if $n \geq 2k^2 - k - 1$ or $k \leq 3$, as follows from [16 and 5] (except for the cases $n = k = 2$ and $n = k = 3$, which are easily checked).

Theorem 1.1 is proved in §§3–6, after a discussion of the cohomology ring $H^*(G_{k,n}; \mathbf{Q})$ in §2. Some applications are discussed briefly in §7.

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2. The cohomology ring.

The following characterization of $H^*(G_{k,n}; \mathbf{Q})$ comes from results in [2].

THEOREM 2.1. *Let γ be the canonical k-plane bundle over $G_{k,n}$. Then if $c = 1 + c_1 + \cdots + c_k$ is the total Chern class of γ,*

$$H^*(G_{k,n}; \mathbf{Q}) = \mathbf{Q}[c_1, \ldots, c_k]/I_{k,n},$$

where $I_{k,n}$ is the ideal generated by $(c^{-1})_{n+i}$, $1 \leq i \leq k$ (here $(c^{-1})_q$ is the part of the formal inverse of c in dimension $2q$).

REMARK. $I_{k,n}$ contains all the elements $(c^{-1})_q$, $q > n$, and in fact

$$c^{-1} = 1 + (c^{-1})_1 + \cdots + (c^{-1})_n \in H^*(G_{k,n}; \mathbf{Q})$$

is the Chern class of the orthogonal complement bundle of γ.

It is well known that the universal Chern classes $c_i \in H^*(BU_k; \mathbf{Q})$ can be identified with the elementary symmetric functions

$$e_i(t_1, \ldots, t_k) = \sum_{m_1 < \cdots < m_i} t_{m_1} \cdots t_{m_i}$$

in $\mathbf{Q}[t_1, \ldots, t_k]$, where each t_i is in dimension 2. Since $H^*(G_{k,n}; \mathbf{Q})$ is a quotient of $H^*(BU_k; \mathbf{Q}) = \mathbf{Q}[c_1, \ldots, c_k]$, we can regard $H^*(G_{k,n}; \mathbf{Q})$ as a quotient of the ring $\mathbf{Q}[t_1, \ldots, t_k]^{\Sigma_k}$ of symmetric functions in k variables. To describe the ideal $I_{k,n}$ in this setting, we shall require some facts about the symmetric functions.

We first establish notation for partitions. A partition λ of an integer $n \geq 0$ is a sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ of integers with $\lambda_1 + \cdots + \lambda_l = n$. We call λ_1 the greatest part of λ, and $l(\lambda) = l$ the length of λ. We write $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$ and $|\lambda| = n$. Sometimes we shall use superscripts to indicate repetition, so $3^2 21$ means 3321 and not 921. Associated to λ is its diagram, which is a left-justified array of boxes with λ_i boxes in the ith row. For example, $\lambda = 3321$ has diagram as shown below.
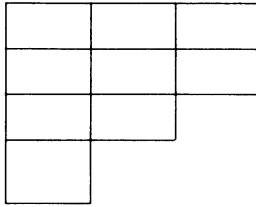
By reflecting the diagram of λ through its main diagonal, we get the diagram of another partition λ' with $|\lambda'|=|\lambda|$. Formally,

$$\lambda'_p = \text{order of } \left\{q \,|\, \lambda_q \geqslant p\right\}.$$

The partition λ' is called the conjugate partition of λ: conjugation is an involution exchanging greatest part and length.

Now we consider the ring $\Lambda_k = \mathbf{Q}[t_1,\ldots,t_k]^{\Sigma_k}$ of symmetric functions in k variables. As an algebra, Λ_k is generated by the elementary symmetric functions $e_1,\ldots,e_k$. Another set of symmetric functions $h_1, h_2,\ldots$ is defined by

$$h_i = \sum_{p_1+ \cdots +p_k=i} t_1^{p_1}t_2^{p_2} \cdots t_k^{p_k}.$$

The h_i are called symmetric functions. The following identity relates the complete and elementary symmetric functions [**15**, p. 14]:

(1) $$(1 + e_1 + \cdots + e_k)(1 - h_1 + h_2 - \cdots) = 1.$$

Another identity is proved in [**15**, p. 15].

PROPOSITION 2.2. *For any partition* $\lambda = \lambda_1\lambda_2 \cdots \lambda_l$ *with* $l \leqslant k$,

(2) $$\det\left(h_{\lambda_p-p+q}\right)_{1\leqslant p,q\leqslant l} = \det\left(e_{\lambda'_p-p+q}\right)_{1\leqslant p,q\leqslant \lambda_1}.$$

The common value of both sides of (2) is called the Schur function (or S-function) associated to λ and denoted s_λ. The s_λ are symmetric functions, and in fact $\{s_\lambda \,|\, l(\lambda) \leqslant k\}$ is a basis for Λ_k [**15**, p. 24].

We are now in a position to describe $I_{k,n}$, hence $H^*(G_{k,n};\mathbf{Q})$, in terms of symmetric functions. By (1), we see that identifying c_i and e_i with identifies $(c^{-1})_i$ with $(-1)^i h_i$. Hence $I_{k,n}$, considered as an ideal in Λ_k, is generated by $h_{n+1},\ldots,h_{n+k}$. This ideal contains all the h_i, $i > n$ (use (1)). So we have the following result.

THEOREM 2.3. *There is an isomorphism* $\Lambda_k/\{h_1 \,|\, i > n\} \xrightarrow{\sim} H^*(G_{k,n};\mathbf{Q})$ *taking* $e_i(t_1,\ldots,t_k)$ *to* c_i.

Now it is a classical result of algebraic geometry that $H^*(G_{k,n};\mathbf{Q})$ has basis $\{\sigma_\lambda \,|\, \lambda_1 \leqslant n, \; l(\lambda) \leqslant k\}$, where the elements $\sigma_\lambda \in H^{2|\lambda|}(G_{k,n};\mathbf{Q})$ are the Poincaré duals of the Schubert cycles (see e.g. [**8**]). The next result gives a simple relation between the σ_λ and the Schur functions s_λ.

THEOREM 2.4. *The isomorphism of 2.3 sends* s_λ *to* $(-1)^{|\lambda|}\sigma_\lambda$.

PROOF. From [8, p. 410], we have $c_i = (-1)^i \sigma_{11\cdots 1} = (-1)^i \sigma_{1^i}$ in $H^*(G_{k,n}; \mathbf{Q})$. But we have also $\sum_{r=0}^d (-1)^r \sigma_{1^r} \sigma_{d-r} = 0$ for $d > 0$ [8, p. 205]. Thus $\sum_{r=0}^d c_r \sigma_{d-r} = 0$ for $d > 0$: but then $\sigma_i = (c^{-1})_i$ is the image of $(-1)^i h_i$. Since

$$\sigma_\lambda = \det(\sigma_{\lambda_p - p + q})_{1 \leqslant p, q \leqslant l(\lambda)}$$

[8, p. 205], the result follows.

REMARK. The relationship between the s_λ and the σ_λ was first explained in [12], though the formal similarity between the multiplication rules for the σ_λ (i.e., the classical Schubert calculus) and those for the s_λ was noted earlier; see [17] for details.

In the sequel we shall write $c^\lambda = \det(c_{\lambda_p - p + q})_{1 \leqslant p, q \leqslant l}$ and $c_\lambda = c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_l}$ for a partition $\lambda = \lambda_1 \lambda_2 \cdots \lambda_l$. By 2.2 and 2.4 $c^\lambda = (-1)^{|\lambda|} \sigma_{\lambda'}$. Thus $H^{2i}(G_{k,n}; \mathbf{Q})$ has basis $\{c^\lambda | \lambda \in P(i)\}$, where $P(i)$ is the set of partitions λ with $|\lambda| = i$, $\lambda_1 \leqslant k$, and $l(\lambda) \leqslant n$. Now $P(i)$ has a total ordering defined by putting $\mu > \lambda$ if the first nonzero difference $\mu_r - \lambda_r$, $1 \leqslant r \leqslant n$, is positive. It is evident (from expanding the determinant) that

$$c^\lambda = c_\lambda + \sum_{\mu \in P(i),\, \mu > \lambda} D_\mu c_\mu$$

for some integers D_μ. Thus, we have the following fact.

PROPOSITION 2.5. $H^{2i}(G_{k,n}; \mathbf{Q})$ has basis $\{c_\lambda | \lambda \in P(i)\}$.

3. The hard Lefschetz theorem. In this section we use the hard Lefschetz theorem to establish a result (Theorem 3.3) which "almost" proves Theorem 1.1 by induction. This partial result is the basis for the remainder of the proof.

The hard Lefschetz theorem was first stated by Lefschetz [13] for projective varieties, though the first rigorous proof is due to Hodge [10]. Other references are [18 and 8].

THEOREM 3.1 (HARD LEFSCHETZ THEOREM). *Let M be a compact Kähler manifold of complex dimension d, with Kähler class $u \in H^2(M; \mathbf{Q})$. Then if $0 \leqslant r \leqslant d$, multiplication by u^{d-r} gives an isomorphism $H^r(M; \mathbf{Q}) \xrightarrow{\sim} H^{2d-r}(M; \mathbf{Q})$.*

Since $G_{k,n}$ is a Kähler manifold with Kähler class $\sigma_1 = -c_1$, we immediately have the following corollary.

COROLLARY 3.2. *Multiplication by c_1^{kn-2i}, $0 \leqslant 2i \leqslant kn$, gives an isomorphism*

$$H^{2i}(G_{k,n}; \mathbf{Q}) \xrightarrow{\sim} H^{2kn-2i}(G_{k,n}; \mathbf{Q}).$$

Thus, $\{c_1^{kn-2i} c_\mu | \mu \in P(i)\}$ is a basis for $H^{2kn-2i}(G_{k,n}; \mathbf{Q})$.

By Poincaré duality, cup product is a nonsingular dual pairing

$$H^{2i}(G_{k,n}; \mathbf{Q}) \otimes H^{2kn-2i}(G_{k,n}; \mathbf{Q}) \to \mathbf{Q}.$$

Thus, if we define a matrix $M = (M_{\mu\lambda})_{\mu, \lambda \in P(i)}$ by $c_1^{kn-2i} c_\mu c_\lambda = M_{\mu\lambda} c_k^n$, then 3.2 implies M is nonsingular. This observation is the key to proving the next result.

THEOREM 3.3. *Let h be an endomorphism of $H^*(G_{k,n}; \mathbf{Q})$ with $h(c_t) = m^t c_t$, $1 \leqslant t < i$, for some $m \neq 0$. Then $h(c_i) = m^i c_i$ or*

$$h(c_i) = -m^i c_i - 2m^i \sum_{\lambda \in P(i), \lambda \neq i} \frac{A_{\lambda i}}{A_{ii}} c_\lambda,$$

where the $A_{\lambda i}$ are appropriate (signed) minors of M.

PROOF. For convenience, let $Q(i) = P(i) - \{i\}$. In view of 2.5, we can write

$$(1) \qquad\qquad h(c_i) = \sum_{\lambda \in P(i)} a_\lambda c_\lambda.$$

For $\mu \in Q(i)$, applying h to the relation

$$c_1^{kn-2i} c_\mu c_i = M_{\mu i} c_k^n$$

gives

$$m^{kn-i} c_1^{kn-2i} c_\mu \left(\sum_{\lambda \in P(i)} a_\lambda c_\lambda \right) = M_{\mu i} m^{kn} c_k^n.$$

(Note c_k^n is a multiple of c_1^{kn} by 3.2.) Rewriting everything as a multiple of c_k^n, we get

$$(2) \qquad\qquad M_{\mu i}(a_i - m^i) + \sum_{\lambda \in Q(i)} M_{\mu \lambda} a_\lambda = 0.$$

Similarly, applying h to $c_1^{kn-2i} c_i^2 = M_{ii} c_k^n$ yields

$$(3) \qquad M_{ii}(a_i^2 - m^{2i}) + 2a^i \sum_{\mu \in Q(i)} M_{\mu i} a_\mu + \sum_{\mu, \lambda \in Q(i)} M_{\mu \lambda} a_\mu a_\lambda = 0.$$

Now multiply (2) by a_μ, sum over $\mu \in Q(i)$, and subtract the resulting equation from (3) to get

$$M_{ii}(a_i^2 - m^{2i}) + (a_i + m^i) \sum_{\mu \in Q(i)} M_{\mu i} a_\mu = 0.$$

Provided $a_i \neq -m^i$, we can divide this by $a_i + m^i$ to obtain

$$M_{ii}(a_i - m^i) + \sum_{\mu \in Q(i)} M_{\mu i} a_\mu = 0.$$

Together with (2) for $\mu \in Q(i)$, this gives a system

$$\sum_{\lambda \in P(i)} M_{\mu \lambda}(a_\lambda - \delta_{\lambda i} m^i) = 0, \qquad \mu \in P(i),$$

where δ denotes Kronecker delta. By the nonsingularity of M, we then have $a_\lambda = \delta_{\lambda i} m^i$, i.e. $h(c_i) = m^i c_i$.

Now suppose $a_i = -m^i$. Then (2) gives

$$(4) \qquad\qquad \sum_{\lambda \in P(i)} M_{\mu \lambda} a_\lambda = m^i M_{\mu i}$$

for $\mu \in Q(i)$. By cofactor expansion,

$$(5) \qquad\qquad \sum_{\mu \in P(i)} A_{\nu \mu} M_{\mu \lambda} = \delta_{\nu \lambda} \det M.$$

Multiply (4) by $A_{i\mu}$ and sum over $\mu \in Q(i)$ to get

$$\sum_{\mu \in Q(i)} \sum_{\lambda \in P(i)} A_{i\mu} M_{\mu\lambda} a_\lambda = m^i \sum_{\mu \in Q(i)} A_{i\mu} M_{\mu i}.$$

By the use of (5), this can be reduced to

$$(6) \qquad A_{ii} \sum_{\lambda \in P(i)} M_{i\lambda} a_\lambda = m^i A_{ii} M_{ii} - 2m^i \det M.$$

Since $\det M \neq 0$, evidently $A_{ii} \neq 0$. Equation (6) divided by A_{ii}, together with (4) for $\mu \in Q(i)$, gives a system

$$\sum_{\lambda \in P(i)} M_{\mu\lambda} a_\lambda = m^i M_{\mu i} - \delta_{i\mu} m^i \frac{2 \det M}{A_{ii}}, \qquad \mu \in P(i),$$

which can be solved using (5) to give

$$a_\lambda = m^i \delta_{i\lambda} - 2m^i \frac{A_{i\lambda}}{A_{ii}}.$$

In what follows, h will be an endomorphism of $H^*(G_{k,n}; \mathbf{Q})$ with $h(c_1) = mc_1$, $m \neq 0$. Coefficients a_μ, $\mu \in P(i)$, will be defined by (1) for $1 \le i \le k$. Then Theorem 3.3 has the following consequence.

COROLLARY 3.4. *For* $1 \le i \le k$, $a_i = \pm m^i$.

PROOF. By 3.3, we have $h^2(c_1) = m^{2i} c_i$ whenever $h(c_t) = m^t c_t$, $t < i$. Iterate to get $h^N(c_i) = m^{Ni} c_i$, $1 \le i \le k$, for some N. Then the conclusion follows.

We shall set $a_i = \varepsilon_i m^i$ for $1 \le i \le k$, so each ε_i is ± 1 (note $\varepsilon_1 = 1$). By 3.3, to prove $h(c_1) = m^i c_i$ it suffices to prove $\varepsilon_t = 1$ for $t \le i$.

4. Computations in top dimension. By the remarks following 3.4, to establish Theorem 1.1 for $n > k$ it suffices to show $\varepsilon_i = 1$, $1 \le i \le k$. In this section we prove $\varepsilon_2 = 1$ for $n > k \ge 2$, and compute $h(c_2)$ when $n = k$ and $\varepsilon_2 = -1$. The method is explicit computation based on a formula (Proposition 4.2) for coefficients in $H^{2kn}(G_{k,n}; \mathbf{Q})$.

If μ is a partition without ones, $|\mu| \le kn$, and $\mu_1 \le k$, we define M_μ by the relation

$$c_1^{kn - |\mu|} c_\mu = M_\mu c_k^n$$

in $H^{2kn}(G_{k,n}; \mathbf{Q})$. Now suppose $\varepsilon_2 = -1$. By 3.3,

$$(1) \qquad h(c_2) = -m^2 c_2 - 2m^2 \frac{A_{2.11}}{A_{2.2}} c_1^2 = -m^2 c_2 + 2m^2 \frac{M_2}{M_\varnothing} c_1^2$$

($\varnothing$ is the empty partition, so $c_1^{kn} = M_\varnothing c_k^n$). Assume $kn \ge 6$ (as when $n > k \ge 2$). Then applying h to $c_1^{kn-6} c_2^3 = M_{222} c_k^n$, using (1), expanding out the resulting equation and rewriting everything as a multiple of c_k^n, we get

$$(2) \qquad \frac{M_{222}}{M_2} - 3 \frac{M_{22}}{M_\varnothing} + 2 \left(\frac{M_2}{M_\varnothing} \right)^2 = 0.$$

So $\varepsilon_2 = -1$ implies (2) for $kn \ge 6$: we shall show (2) cannot hold for $n > k \ge 2$.

First we give a combinatorial description of Poincaré duality in $H^*(G_{k,n}; \mathbf{Q})$. Given a partition $\lambda \in P(i)$, the dual partition $\bar{\lambda} \in P(kn - i)$ is defined by $\bar{\lambda}_q = k - \lambda_{n+1-q}$. Then the following result holds.

PROPOSITION 4.1. *Let* $\lambda \in P(i)$, $\mu \in P(kn - i)$. *Then in* $H^*(G_{k,n}; \mathbf{Q})$, $c^\lambda c^\mu = \delta_{\bar{\lambda}\mu} c_k^n$.

PROOF. This follows from the corresponding rule in the Schubert calculus [**8**, p. 198], using $c^\lambda = (-1)^{|\lambda|}\sigma_{\lambda'}$.

Now let μ be a partition. For each box b in the diagram of μ, we define the hook length $H(b)$ by

$$H(b) = \text{number of boxes below } b + \text{number of boxes to right of } b + 1$$

(so $H(b)$ is the length of the "hook" containing b). We set

$$N(\mu) = \frac{|\mu|!}{\prod_{b \in \text{diagram of } \mu} H(b)}.$$

For example, if $\mu = 3321$, the hook lengths of boxes in the diagram of μ are

6	4	2
5	3	1
3	1	
1		

from which we compute

$$N(\mu) = \frac{9!}{2 \cdot 3^2 \cdot 4 \cdot 5 \cdot 6} = 168.$$

In fact, $N(\mu)$ is always an integer. The numbers $N(\mu)$ appear in the next result.

PROPOSITION 4.2. *Let* $\lambda \in P(s)$. *Then in* $H^{2kn}(G_{k,n}; \mathbf{Q})$, $c_1^{kn-s} c^\lambda = N(\bar{\lambda}) c_k^n$.

PROOF. We start with the formula $e_1^r = \sum_{|\mu|=r} N(\mu) s_\mu$, which holds in Λ_k [**15**, p. 37]. Using the correspondence of § 2 (and the fact $N(\mu') = N(\mu)$), we then have $c_1^r = \sum_{\mu \in P(r)} N(\mu) c^\mu$ in $H^*(G_{k,n}; \mathbf{Q})$. Now put $r = kn - s$, multiply by c^λ, and use 4.1 to get the conclusion.

In what follows, we put $N_\mu = N(\bar{\mu})$. For small $|\mu|$, the ratio $N_\mu/N_\varnothing$ is fairly easy to compute. For example, we can show (setting $\xi = kn$ and $\eta = n - k$)

$$\frac{N_2}{N_\varnothing} = \frac{\xi - \eta - 1}{2(\xi - 1)}, \qquad \frac{N_3}{N_\varnothing} = \frac{(\xi - \eta - 1)(\xi - 2\eta - 4)}{6(\xi - 1)(\xi - 2)}$$

and

$$\frac{N_{22}}{N_\varnothing} = \frac{\xi(\xi - \eta - 1)(\xi + \eta - 1)}{12(\xi - 1)(\xi - 2)(\xi - 3)}.$$

From formulas like these, we obtain expressions for the coefficients M_μ. For example, since $c_2^2 = c^{22} + c_1 c^3$ we have $M_{22} = N_{22} + N_3$ and thus (noting $M_\varnothing = N_\varnothing$)

$$\frac{M_{22}}{M_\varnothing} = \frac{\xi(\xi - \eta - 1)(\xi + \eta - 1)}{12(\xi - 1)(\xi - 2)(\xi - 3)} + \frac{(\xi - \eta - 1)(\xi - 2\eta - 4)}{6(\xi - 1)(\xi - 2)}$$

$$= \frac{(\xi - \eta - 1)[\xi^2 - 5\xi + 8 - \eta(\xi - 4)]}{4(\xi - 1)(\xi - 2)(\xi - 3)}.$$

With these techniques, a lengthy but straightforward computation shows

$$\frac{M_{222}}{M_2} - 3\frac{M_{22}}{M_\varnothing} + 2\left(\frac{M_2}{M_\varnothing}\right)^2 = \frac{2\eta(\xi + 5)(\xi + \eta - 1)}{(\xi - 1)^2(\xi - 2)(\xi - 3)(\xi - 4)(\xi - 5)}.$$

If $n > k \geqslant 2$ (hence $\xi \geqslant 6$ and $\eta > 0$), the right-hand side is clearly positive. This contradicts (2), so $\varepsilon_2 \neq -1$ for $n > k \geqslant 2$. If $n = k$,

$$\frac{M_2}{M_\varnothing} = \frac{N_2}{N_\varnothing} = \frac{\xi - 1}{2(\xi - 1)} = \frac{1}{2}$$

and (1) is $h(c_2) = -m^2 c_2 + m^2 c_1^2$. We have established the following result.

THEOREM 4.3. *If $k \geqslant 2$, then $\varepsilon_2 = 1$ when $n > k$. If $n = k$ and $\varepsilon_2 = -1$, then $h(c_2) = m^2(-c_2 + c_1^2)$.*

Since Theorem 1.1 is trivial for $k = 1$ (the case of projective space), we immediately have the following corollary.

COROLLARY 4.4. *Theorem 1.1 holds if $k \leqslant 2$.*

5. Comparison of coefficients. In this section we obtain information on the ε_i, show some of the coefficients a_μ to be zero, and establish Theorem 1.1 in some special cases by examining the images under h of the relations

$$R_i = (c^{-1})_{n+i}, \qquad 1 \leqslant i \leqslant k,$$

in $\mathbf{Q}[c_1, \ldots, c_k]$. Most of the arguments of this section and the following one are adapted from [**3**].

If h is a well-defined endomorphism of $H^*(G_{k,n}; \mathbf{Q})$, we must have relations

$$(1) \qquad h(R_i) = K_i R_i + \sum_{1 \leqslant |\mu| < i} K_\mu^{(i)} c_\mu R_{i-|\mu|}$$

in $\mathbf{Q}[c_1, \ldots, c_k]$ for $1 \leqslant i \leqslant k$. On the other hand, we can write R_i explicitly in terms of the c_λ and then use

$$h(c_t) = \sum_{\mu \in P(t)} a_\mu c_\mu, \qquad 1 \leqslant t \leqslant k,$$

to write $h(R_i)$. It will be convenient to introduce some additional notation.

For any partition λ, set $m_i(\lambda) = $ order of $\{q \mid \lambda_q = i\}$. Then $m_1(\lambda), m_2(\lambda), \ldots$ uniquely determine λ, and $m_1(\lambda) + m_2(\lambda) + \cdots = l(\lambda)$. We set

$$C(\lambda) = (-1)^{l(\lambda)} \frac{l(\lambda)!}{m_1(\lambda)! m_2(\lambda)! \cdots m_k(\lambda)!}$$

for partitions λ with $\lambda_1 \leqslant k$. Then a formal computation using the multinomial theorem shows

$$R_i = (c^{-1})_{n+i} = \sum_{|\lambda| = n+i,\, \lambda_1 \leqslant k} C(\lambda) c_\lambda.$$

Thus

$$(2) \quad h(R_i) = \sum_{|\lambda| = n+i,\, \lambda_1 \leqslant k} C(\lambda) h(c_\lambda) = \sum_{|\lambda| = n+i,\, \lambda_1 \leqslant k} C(\lambda) \prod_{t=1}^{l(\lambda)} \left(\sum_{\mu \in P(\lambda_t)} a_\mu c_\mu \right).$$

By comparing coefficients of specific terms in (1) and (2), we shall obtain relations in the a_μ.

In what follows, we define p and r by $n = pk + r$, $0 \leqslant r < k$, and set $\alpha = \varepsilon_{k-1}\varepsilon_k$.

PROPOSITION 5.1. *If* $i \geqslant \min\{r+1, k-r+1\}$, *then* $\varepsilon_i = \alpha^{k-i}\varepsilon_k$.

PROOF. First we examine the coefficient of $c_{r+u+1}c_{kv-u}c_k^{p-1}$ in $h(R_i)$ for $0 \leqslant u \leqslant k - r + 1$. From (1) the coefficient is

$$C(k^{p-1}k - ur + u + 1)K_1,$$

(recall that k^{p-1} means k repeated $p - 1$ times) while from (2) it is

$$C(k^{p-1}k - ur + u + 1)a_{r+u+1}a_{k-u}a_k^{p-1}.$$

Thus $K_1 = a_{r+u+1}a_{k-u}a_k^{p-1}$ for $0 \leqslant u \leqslant k - r - 1$. It follows that

$$(3) \qquad \varepsilon_{r+1}\varepsilon_k = \varepsilon_{r+2}\varepsilon_{k-1} = \cdots = \varepsilon_k\varepsilon_{r+1}.$$

A similar analysis of the coefficient of $c_{r+u+2}c_{k-u}c_k^{p-1}$ in $h(R_2)$, $0 \leqslant u \leqslant k - r - 2$, shows $K_2 = a_{r+u+2}a_{k-u}a_k^{p-1}$ and thus

$$(4) \qquad \varepsilon_{r+2}\varepsilon_k = \varepsilon_{r+3}\varepsilon_{k-1} = \cdots = \varepsilon_k\varepsilon_{r+2}.$$

Now work back and forth between (3) and (4) to obtain $\varepsilon_i = \alpha^{k-i}\varepsilon_k$ for $i \geqslant r + 1$.

To finish the proof, we need only consider the case $k < 2r$. Here we examine the coefficient of $c_{k-r+u+1}c_r c_{k-u}c_k^{p-1}$ in $h(R_{k-r+1})$, $0 \leqslant u \leqslant r - 1$, to find

$$a_{k-r+u+1}a_r a_{k-u}a_k^{p-1} = K_{k-r+1}.$$

Examining the coefficient of $c_{k-r+u+1}c_r c_{k-u}c_k^{p-1}$ in $h(R_{k-r+2})$, $0 \leqslant u \leqslant r - 2$, gives

$$a_{k-r+u+2}a_r a_{k-u}a_k^{p-1} = \begin{cases} K_{k-r+2}, & 2r > k+1, \\[2mm] K_{k-r+2} - \dfrac{1}{p+2}K_r^{(k-r+2)}, & 2r = k+1. \end{cases}$$

From these equations we get ones corresponding to (3) to (4), and we proceed as above to show $\varepsilon_i = \alpha^{k-i}\varepsilon_k$ for $i \geqslant k - r + 1$.

In the case $r = 0$, 5.1 is enough to prove Theorem 1.1.

COROLLARY 5.2. *If* $n = pk$, *then Theorem* 1.1 *holds.*

PROOF. By 5.1, $\varepsilon_i = \alpha^{k-i}\varepsilon_k$ for $1 \leqslant i \leqslant k$. Since $\varepsilon_i = 1$, $1 = \alpha^{k-1}\varepsilon_k$ and so $\varepsilon_i = \alpha^{i+1}$. Thus $\varepsilon_2 = \alpha$. If $\alpha = \varepsilon_2 = 1$, then $\varepsilon_i = 1$ for $1 \leqslant i \leqslant k$ and $h(c_i) = m^i c_i$, $1 \leqslant i \leqslant k$, by the remarks following 3.4.

Now suppose $\varepsilon_2 = -1$. Then $n = k$ and

$$h(c_2) = m^2(-c_2 + c_1^2) = m^2(c^{-1})_2$$

by 4.3. Let $\Theta: H^*(G_{k,k}; \mathbf{Q}) \to H^*(G_{k,k}; \mathbf{Q})$ be defined by

$$\Theta(c^\mu) = (-1)^{|\mu|}c^{\mu'} = \sigma_\mu.$$

Then Θ is an involution taking c_i to $(c^{-1})_i$. We have $\Theta h(c_1) = m(c^{-1})_1 = -mc_1$ and $\Theta h(c_2) = m^2 c_2 = (-m)^2 c_2$. But then (if we replace m by $-m$) Θh is an endomorphism with $\varepsilon_1 = \varepsilon_2 = 1$, so

$$\Theta h(c_i) = (-m)^i c_i, \qquad 1 \leq i \leq k,$$

by the argument of the preceding paragraph. Hence $h(c_i) = (-m)^i(c^{-1})_i$ for $1 \leq i \leq k$, and Theorem 1.1 holds.

Henceforth we assume $n > k$; by 4.4, we can assume $k \geq 3$. Let $f = [k/2]$. In the remainder of this section we prove $a_{k\,k-f} = 0$, $\varepsilon_f \varepsilon_{k-f} = \varepsilon_k$, and $\varepsilon_k = \alpha^k$. The arguments depend on the parity of k: we consider first the case where k is even.

PROPOSITION 5.3. *If $k = 2f$, then $\varepsilon_k = 1$ and $a_{ff} = 0$.*

PROOF. By the preceding result, we can assume $r \neq 0$. In this case,

$$(5) \qquad\qquad 2a_{ff} = (p + 2)(a_f^2 - a_k).$$

This can be established by comparing coefficients of $c_f^2 c_{f+1} c_k^{p-1}$ and $c_{f+1} c_k^p$ in $h(R_{f-r+1})$ if $0 < r \leq f$, or by comparing coefficients of $c_f^2 c_k^p$ and c_k^{p+1} in $h(R_{k-r})$ if $f \leq r < k$. Thus $\varepsilon_k = 1$ if and only if $a_{ff} = 0$. The remainder of the proof divides into three cases.

Case 1. $0 < r < f$. Compare coefficients of $c_f^3 c_k^{p-1}$ and $c_f c_k^p$ in $h(R_{f-r})$ to get $6a_{ff} = (p + 2)(a_f^2 - a_k)$, and compare this with (5) to obtain the conclusion.

Case 2. $r = f$. We suppose $f \geq 3$: the case $r = f = 2$ requires a special argument, which is omitted. Let $1 \leq t \leq f - 1$. Compare coefficients of $c_t c_f c_k^p$ and $c_{t+f} c_k^p$ in $h(R_t)$ to get

$$(6) \qquad\qquad a_{ft} = (p + 2)(a_t a_f - a_{t+f}).$$

Now compare coefficients of $c_t c_f^3 c_k^{p-1}$ and $c_{t+f} c_k^p$ to get

$$6a_{ft}a_{ff} - 6(p + 2)a_t a_f a_{ff} - 3(p + 2)a_f^2 a_{ft} = (p + 2)(p + 3)(a_{t+f} a_k - a_t a_f^3).$$

Use (5) and (6) to substitute for a_{ff} and a_{ft} respectively. The result is

$$(7) \qquad\qquad \varepsilon_{t+f} = \varepsilon_t \varepsilon_f \varepsilon_k, \qquad 1 \leq t \leq f - 1.$$

If $\varepsilon_k = -1$, then (7) together with $\varepsilon_2 = 1$ gives

$$(8) \qquad\qquad \varepsilon_{f+2} = -\varepsilon_f.$$

(This uses $f \geq 3$.) Compare coefficients of $c_{f+1}^3 c_k^{p-1}$ and $c_f c_{f+1} c_{f+2} c_k^{p-1}$ in $h(R_3)$ (again using $f \geq 3$) to get $a_{f+1}^2 = a_f a_{f+2}$: but then $\varepsilon_f \varepsilon_{f+2} = 1$, contradicting (8). So $\varepsilon_k = 1$.

Case 3. $f < r < k$. Compare coefficients of $c_f c_{f+1} c_k^p$ and $c_f^3 c_{f+1} c_k^{p-1}$ in $h(R_{k-r+1})$ to get $6a_{ff} = (p + 3)(a_f^2 - a_k)$. Together with (5), this gives the conclusion.

We note that the argument of Case 2 above can be improved to yield the following result.

PROPOSITION 5.4. *If $k = 2f$ and $r = f$, then Theorem 1.1 holds.*

PROOF. As in Case 2, we assume $f \geqslant 3$ and omit the special argument for $r = f = 2$. Now $\varepsilon_k = 1$ by 5.3, so (7) is

$$(9) \qquad \varepsilon_{t+f} = \varepsilon_t \varepsilon_f, \qquad 1 \leqslant t \leqslant f - 1.$$

By 5.1,

$$(10) \qquad \varepsilon_{t+f} = \alpha^{k-f-t} \varepsilon_k = \alpha^{f+t}$$

for $1 \leqslant t \leqslant f$. Thus $\varepsilon_t \varepsilon_f = \alpha^{f+t}$, $1 \leqslant t \leqslant f - 1$. Since $\varepsilon_1 = \varepsilon_2 = 1$, this means $\alpha^{f+1} = \varepsilon_f = \alpha^{f+2}$, i.e. $\alpha = 1$. But then $\varepsilon_f = 1$, and by (10) $\varepsilon_{f+t} = 1$ for $1 \leqslant t \leqslant f$. Hence (9) becomes $\varepsilon_t = 1$, $1 \leqslant t \leqslant f - 1$, and we have $\varepsilon_i = 1$ for $1 \leqslant i \leqslant k$. Theorem 1.1 then follows in this case.

Now we turn to the case where k is odd.

PROPOSITION 5.5. *If $k = 2f + 1$, then $\varepsilon_k = \alpha$.*

PROOF. We can assume $r \neq 0$. If $1 \leqslant r \leqslant f + 1$, we compare coefficients of $c_{f+1}^3 c_k^{p-1}$ in $c_{f+2} c_k^p$ in $h(R_{f-r+2})$ to get

$$a_{f+1}^3 a_k^{p-1} = a_{f+2} a_k^p.$$

If $f + 2 \leqslant r \leqslant k - 1$, we compare coefficients of $c_{f+1}^4 c_k^{p-1}$ and $c_{f+1} c_{f+2} c_k^p$ in $h(R_{k-r+2})$ to get

$$a_{f+1}^4 a_k^{p-1} = a_{f+1} a_{f+2} a_k^p.$$

Either equation implies $\varepsilon_k = \varepsilon_{f+1} \varepsilon_{f+2}$, and $\varepsilon_{f+1} \varepsilon_{f+2} = \alpha$ by 5.1.

PROPOSITION 5.6. *If $k = 2f + 1$, then $\varepsilon_f \varepsilon_{f+1} = \alpha$ and $a_{f+1\,f} = 0$.*

PROOF. Unless $r = f$ or $r = f + 1$. $\varepsilon_f \varepsilon_{f+1} = \alpha$ by 5.1. We consider two cases.
Case 1. $0 < r \leqslant f$. Compare coefficients of $c_{f+1} c_k^p$ and $c_f c_{f+1}^2 c_k^{p-1}$ in $h(R_{f-r+1})$ to get

$$(11) \qquad 2a_{f+1\,f} = (p + 2)(a_f a_{f+1} - a_k).$$

Except for $r = f$, we have $\varepsilon_f \varepsilon_{f+1} = \alpha = \varepsilon_k$ and the conclusion follows. Suppose now $r = f \geqslant 2$ ($r = f = 1$ requires a special argument). Then compare coefficients of $c_{f+2} c_k^p$ and $c_f c_{f+1} c_{f+2} c_k^{p-1}$ in $h(R_2)$ to obtain

$$a_{f+1\,f} = (p + 2)(a_f a_{f+1} - a_k).$$

Together with (11), this gives the conclusion.
Case 2. $f + 1 \leqslant r < k$. Comparing coefficients of c_k^{p+1} and $c_{f+1} c_k^p$ in $h(R_{k-r})$ yields

$$(12) \qquad a_{f+1\,f} = (p + 2)(a_f a_{f+1} - a_k),$$

which proves the conclusion unless $r = f + 1$. Suppose $r = f + 1$, $f \geqslant 2$ (again $f = 1$ requires a special argument). Then compare coefficients of $c_{k-1}c_k^p$ and $c_f^2 c_k^p$ in $h(R_{f-1})$ to get

$$2a_{ff} = (p + 2)\left(a_f^2 - a_{k-1}\right).$$

Now $\varepsilon_{k-1} = 1$ by 5.5, so $a_{ff} = 0$. Using this fact, analyze the coefficient of $c_f^3 c_{f+1} c_k^{p-1}$ to obtain $3a_{f+1f} = (p + 3)(a_f a_{f+1} - a_k)$. Along with (12), this implies the conclusion.

We close this section with the following improvement of 5.1.

THEOREM 5.7. *If* $s \geqslant \min\{k - r + 1, r + 1, f\}$, *then* $\varepsilon_s = \alpha^s$.

PROOF. Suppose first that $s \geqslant \min\{k - r + 1, r + 1\}$. By 5.1, $\varepsilon_s = \alpha^{k-s}\varepsilon_k$. If k is even, $\varepsilon_k = 1$ by 5.3 and $\alpha^{k-s} = \alpha^s$, so the conclusion follows. If k is odd, $\varepsilon_k = \alpha$ by 5.6 and $\alpha^{k-s} = \alpha^{s+1}$, so $\varepsilon_s = \alpha^{s+1}\alpha = \alpha^s$.

Now $\min\{k - r + 1, r + 1\} > f$ only if $r = f$, or if $r = k + 1$ and k is odd; we finish the proof by checking these cases. If $r = f$ and k is even, use 5.4. If k is odd and $r = f$ or $r = f + 1$, 5.6 gives $\varepsilon_f \varepsilon_{f+1} = \alpha$, from which the conclusion follows.

6. Completion of the proof. In this section we complete the proof of Theorem 1.1, using the comparison-of-coefficients technique. We first establish relations among some of the coefficients a_μ with $l(\mu) = 2$, and then use them to show $\varepsilon_i = 1$, $1 \leqslant i \leqslant k$, when $n > k$.

PROPOSITION 6.1. *If* $\min\{k - r + 1, r + 1\} < v \leqslant u < k$ *and* $u + v > k + 1$, *then*

$$a_{v\,k-u}a_u + a_{u\,k-u}a_v = a_{v-1\,k-u}a_{u+1}.$$

PROOF. We consider four cases.

Case 1. $v \geqslant r + 1$. Comparing coefficients of $c_{k-u}c_v c_u c_k^{p-1}$ and $c_{k-u}c_{v-1}c_{u+1}c_k^{p-1}$ in $h(R_{v-r})$ yields

(1) $-a_{v\,k-u}a_u - a_{u\,k-u}a_v + (p + 2)a_{k-u}a_v a_u$

$$= -a_{v-1\,k-u}a_{u+1} + (p + 2)a_{k-u}a_{v-1}a_{u+1}.$$

Because of the restrictions on u and v, we can apply 5.7 to get $a_{v-1}a_{u+1} = \alpha^{u+v}m^{u+v} = a_v a_u$ and thus (1) reduces to the conclusion.

It remains to consider the cases where $v \leqslant r$. In view of the hypothesis, $v \leqslant r$ implies $\min\{k - r + 1, r + 1\} = k - r + 1$, and thus $k - r + 1 < v \leqslant r$. In all the remaining cases, the argument is based on comparison of the coefficients of $c_{k-u}c_v c_u c_{r-1}c_k^{p-1}$ and $c_{k-u}c_{v-1}c_{u+1}c_{r-1}c_k^{p-1}$ in $h(R_{v-1})$.

Case 2. $v = k - r + 2 \leqslant r$. Here $u + v > k + 1$ implies $u > r - 1$. Comparison of the coefficients indicated above yields

$$-a_{v\,k-u}a_{r-1}a_u - a_{u\,k-u}a_v a_{r-1} + (p + 3)a_{k-u}a_v a_{r-1}a_u$$

$$= -a_{v-1\,k-u}a_{r-1}a_{u+1} - a_{r-1\,v-1}a_{k-u}a_{u+1} + (p + 3)a_{k-u}a_{v-1}a_{r-1}a_{u+1},$$

from which follows

$$\left(a_{v\,k-u}a_u + a_{u\,k-u}a_v\right)a_{r-1} = a_{v-1\,k-u}a_{u+1}a_{r-1} + a_{r-1\,v-1}a_{k-u}a_{u+1}.$$

Thus, to get the conclusion we must show $a_{r-1\,v-1} = a_{r-1\,k-r+1}$ is zero. Compare coefficients of c_k^{p+1} and $c_{k-r+1}c_{r-1}c_k^p$ in $h(R_{k-r})$ to obtain

$$a_{r-1\,k-r+1} = (p+2)(a_k - a_{r-1}a_{k-r+1}),$$

and note the right-hand side is zero by 5.7.

Case 3. $k - r + 2 < v \le u < r - 1$. Compare the indicated coefficients to get

$$-a_{v\,k-u}a_u a_{r-1} - a_{u\,k-u}a_v a_{r-1} + (p+3)a_{k-u}a_v a_u a_{r-1}$$
$$= -a_{v-1\,k-u}a_{u+1}a_{r-1} + (p+3)a_{k-u}a_{v-1}a_{u+1}a_{r-1},$$

from which the conclusion follows as in Case 1.

Case 4. $k - r + 2 < v \le r$, $u \ge r - 1$. Comparison of the indicated coefficients yields

$$-a_{v\,k-u}a_{r-1}a_u - a_{u\,k-u}a_v a_{r-1} - a_{r-1\,k-u}a_v a_u + (p+3)a_{k-u}a_v a_{r-1}a_u$$
$$= -a_{v-1\,k-u}a_{r-1}a_{u+1} - a_{r-1\,k-u}a_{v-1}a_{u+1} + (p+3)a_{k-u}a_{v-1}a_{r-1}a_{u+1}.$$

from which the conclusion follows as above.

PROPOSITION 6.2. *If* $f + 1 < u < k$, *then* $a_{f+1\,k-u}a_u + a_{u\,k-u}a_{f+1} = a_{f\,k-u}a_{u+1}$.

PROOF. This follows from 6.1 unless $\min\{k - r + 1, r + 1\} > f$, which happens only if $r = f$ or if $r = f + 1$ and k is odd. The case $r = f$, k even, was disposed of in 5.4. If $r = f$ and k is odd, the argument of Case 1 above applies (with $v = f + 1$). Finally, suppose $r = f + 1$ and k is odd. Then comparing coefficients of $c_{k-u}c_f c_{f+1}c_u c_k^{p-1}$ and $c_{k-u}c_f^2 c_{u+1}c_k^{p-1}$ in $h(R_f)$, using $a_{f+1\,f} = a_{ff} = 0$ (see 5.6, Case 2), gives

$$-a_{u\,k-u}a_f a_{f+1} - a_{f\,k-u}a_{f+1}a_u - a_{f+1\,k-u}a_f a_u + (p+3)a_{k-u}a_f a_{f+1}a_u$$
$$= -2a_{f\,k-u}a_f a_{u+1} + (p+3)a_{k-u}a_f^2 a_{u+1}.$$

Using 5.7, we then obtain the conclusion as in the proof of 6.1.

We use Propositions 6.1 and 6.2 to deduce the following result.

THEOREM 6.3. *If* $\min\{k - r + 1, r + 1, f\} \le w \le u < k$ *and* $w + u > k$, *then*

$$\alpha^{u-w}m^{u-w}a_{w\,k-u} = (u - w + 1)a_{u\,k-u}.$$

PROOF. We proceed by induction on $u - w$. For $u = w$ the conclusion is trivial. Suppose the result holds for $u - w < i$, and take u, w with $u - w = i$ and which satisfy the hypothesis. Then by 6.1 with $v = w + 1$ (or by 6.2 if $w = f$),

$$a_{w\,k-u}a_{u+1} = a_{w+1\,k-u}a_u + a_{u\,k-u}a_{w+1}.$$

Divide by a_{w+1}, using 5.7, to get

$$(2) \qquad \alpha^{u-w}m^{u-w}a_{w\,k-u} = \alpha^{u-w+1}m^{u-w+1}a_{w+1\,k-u} + a_{u\,k-u}.$$

By the induction hypothesis,

$$\alpha^{u-w+1}m^{u-w+1}a_{w+1\,k-u} = (u - w)a_{u\,k-u}.$$

Substitute this in (2) to obtain the conclusion.

Now we complete the proof of Theorem 1.1.

LEMMA 6.4. *For* $i \geq 2$, $\varepsilon_i = \alpha^i$.

PROOF. We can assume $\min\{k - r, r, f - 1\} \geq 2$, since otherwise the result follows from 5.7. Let $2 \leq i \leq \min\{k - r, r, f - 1\}$, and compare coefficients of $c_i c_{k-i} c_k^p$ and $c_i c_{k-i+1} c_{k-1} c_k^{p-1}$ in $h(R_{k-r})$ to get

$$a_{k-i\,i} a_k = (p + 2)(a_{k-i} a_k - a_{k-i+1} a_{k-1}) a_i.$$

The right-hand side is zero by 5.7, so $a_{k-i\,i} = 0$. Then 6.3 implies $a_{r+1\,i} = 0$, so comparing coefficients of $c_{r+1} c_k^p$ and $c_i c_{r+1} c_{k-1} c_k^p$ in $h(R_i)$ yields $a_{r+i} a_k = a_i a_{r+1} a_{k-1}$. Apply 5.7 to this to get $\varepsilon_i = \alpha^i$ for $2 \leq i \leq \min\{k - r, r, f - 1\}$. Then the conclusion follows (again using 5.7).

THEOREM 6.5. *If* $k < n$, *then* $h(c_i) = m^i c_i$, $1 \leq i \leq k$.

PROOF. We must prove $\varepsilon_i = 1$ for $1 \leq i \leq k$: in view of the preceding lemma, it suffices to show $\alpha = 1$. By 5.2, we can assume $r \neq 0$. The case $r = f$, k even, was disposed of in 5.4. This leaves three cases.

Case 1. $0 < r < f$, k even. Examine the coefficients of $c_{f+1} c_k^p$ and $c_1 c_f c_k^p$ in $h(R_{f-r+1})$ to obtain $a_{f+1} a_k^p = K_{f-r+1}$ and $(p + 2)(a_{f+1} - a_1 a_f) a_k^p + a_{f\,1} a_k^p = K_1^{(f-r+1)}$. Using these facts, together with $a_{ff} = 0$ (by 5.3), analyze the coefficient of $c_1 c_f^3 c_k^{p-1}$ to get

(3) $2a_{f\,1} = a_1 a_f - a_{f+1}$.

Now compare coefficients of $c_{f+1} c_{f+r} c_k^{p-1}$ and $c_1 c_f c_{f+r} c_k^{p-1}$ in $h(R_1)$ to get

$$a_{f\,1} a_{f+r} + a_{f+r\,1} a_f = (p + 2)(a_1 a_f - a_{f+1}) a_{f+r}.$$

We can use (3) to reduce this to

(4) $a_{f+r\,1} a_f = (2p + 3) a_{f\,1} a_{f+r}$.

Now use 6.3 to rewrite $a_{f+r\,1}$ and $a_{f\,1}$ in terms of $a_{k-1\,1}$, and substitute into (4) to obtain $(f - r) a_{k-1\,1} = (2p + 3) f a_{k-1\,1}$ and thus $a_{k-1\,1} = 0$. Then $a_{f\,1} = 0$, and since $2a_{f\,1} = a_1 a_f - a_{f+1} = (1 - \alpha) \alpha^f m^{f+1}$ by (3), we have $\alpha = 1$.

Case 2. $0 < r \leq f$, k odd. Examine the coefficients of $c_{f+2} c_k^p$ and $c_1 c_{f+1} c_k^p$ in $h(R_{f-r+2})$ to get $a_{f+2} a_k^p = K_{f-r+2}$ and

$$(p + 2)(a_{f+2} - a_1 a_{f+1}) a_k^p + a_{f+1\,1} a_k^p = K_1^{(f-r+2)}.$$

(if $r = 1$, it is also necessary to examine the coefficient of $c_f c_{f+1} c_{f+2} c_k^{p-1}$ to show $K_f^{(f+1)} = 0$). Then analyze the coefficient of $c_1 c_f c_{f+1}^2 c_k^{p-1}$, using $a_{f+1\,f} = 0$ (by 5.6), to get

$$a_{f\,1} a_{f+1} + a_{f+1\,1} a_f = (a_1 a_{f+1} - a_{f+2}) a_f.$$

Now use 6.3 to rewrite $a_{f+1\,1}$ and $a_{f\,1}$ in terms of $a_{k-1\,1}$: substitute in the above equation to get

(5) $k a_{k-1\,1} = (a_1 a_{f+1} - a_{f+2}) \alpha^{f-1} m^{f-1}$.

Compare coefficients of $c_1 c_{f+1} c_{f+r} c_k^{p-1}$ and $c_{f+2} c_{f+r} c_k^{p-1}$ in $h(R_1)$ to obtain

$$(1 + \delta_{1r}) a_{f+1\,1} = (p + 2)(a_1 a_{f+1} - a_{f+2}),$$

where δ denotes Kronecker delta. Multiply by $\alpha^{f-1}m^{f-1}$ and use 6.3 to get

$$(1 + \delta_{1r})a_{k-1\,1} = (p + 2)(a_1 a_{f+1} - a_{f+2})\alpha^{f-1}m^{f-1}.$$

Comparison with (5) gives $(1 + \delta_{1r})a_{k-1\,1} = (p + 2)ka_{k-1\,1}$. Since $p \geqslant 1$, this implies $a_{k-1\,1} = 0$. But then by (5),

$$a_1 a_{f+1} - a_{f+2} = (1 - \alpha)\alpha^{f+1}m^{f+2} = 0,$$

so $\alpha = 1$.

Case 3. $f < r < k$. Compare the coefficients of $c_1 c_r c_k^p$ and $c_1 c_f c_{k-f} c_r c_k^{p-1}$ in $h(R_1)$, using $a_{k-ff} = 0$ and $\varepsilon_f \varepsilon_{k-f} = \varepsilon_k$ (by 5.3 or 5.6), to get

$$a_{r\,1}a_k = (p + 2)(a_{f\,1}a_{k-f} + a_{k-f\,1}a_f)a_r.$$

We can use 6.3 to write $a_{r\,1}$, $a_{f\,1}$, and $a_{k-f\,1}$ in terms of $a_{k-1\,1}$: substitute into the above equation to get $(k - r)a_{k-1\,1} = (p + 2)ka_{k-1\,1}$ and thus $a_{k-1\,1} = 0$. Hence $a_{r\,1} = 0$, and comparison of coefficients of $c_{r+1}c_k^p$ and $c_1 c_r c_k^p$ in $h(R_1)$ using this fact yields $a_{r+1} = a_1 a_r$. Then $\varepsilon_{r+1} = \varepsilon_r$: by 6.4, this means $\alpha = 1$.

7. Applications. Theorem 1.1 classifies all automorphisms of $H^*(G_{k,n}; \mathbf{Q})$. Since $H^*(G_{k,n}; \mathbf{Z})$ has no torsion, Theorem 1.1 remains true if $\mathbf{Q}$ is replaced by $\mathbf{Z}$. Either version establishes homotopy rigidity (in the sense of Liulevicius) for all complex Grassmann manifolds, by the main results of [**14** and **1**] respectively.

Let $(G_{k,n})_0$ denote the rationalization of $G_{k,n}$ (for definition see [**9**]). Then the first theorem of [**6**] implies the following result.

THEOREM 7.1. *The monoid of homotopy classes* $[(G_{k,n})_0, (G_{k,n})_0]$ *is anti-isomorphic to the monoid of endomorphisms of* $H^*(G_{k,n}; \mathbf{Q})$ *via the canonical map.*

Thus, Theorem 1.1 can be interpreted as a classification of homotopy classes of self-homotopy equivalences of $(G_{k,n})_0$. Using results in [**7**], it then follows that $G_{k,n}$ is generically rigid (i.e., any space in the genus of $G_{k,n}$ is homotopy equivalent to $G_{k,n}$; see [**7**] for definitions).

Conjecture 1.2, if true, would completely classify the endomorphisms of $H^*(G_{k,n}; \mathbf{Q})$. By 7.1, this would give a homotopy classification of self-maps of $(G_{k,n})_0$. We see another consequence in the next result.

PROPOSITION 7.2. *Suppose Conjecture* 1.2 *holds for* $G_{k,n}$. *If* $k < n$ *and* kn *is even, then* $G_{k,n}$ *has the fixed-point property.*

PROOF. Let f be a self-map of $G_{k,n}$. Assuming Conjecture 1.2,

$$f^*(c_i) = m^i c_i, \qquad 1 \leqslant i \leqslant k,$$

for some m. A straightforward computation (see [**5**]) shows that the Lefschetz number of f is nonzero for any m. Thus f has a fixed point by the Lefschetz fixed-point theorem.

REMARKS. (1) As noted in §1, Conjecture 1.2 is known to hold if $k \leqslant 3$ or $n \geqslant 2k^2 - k - 1$.

(2) If $k = n$ or kn is odd, then $G_{k,n}$ admits fixed-point-free maps.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, MEMORIAL UNIVERSITY OF NEWFOUNDLAND, ST. JOHN'S, NEWFOUNDLAND, CANADA A1B 3X7

Current address: Department of Mathematics, The Ohio State University, Columbus, Ohio 43210