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ON THE PROPER HOLOMORPHIC EQUIVALENCE FOR A CLASS OF PSEUDOCONVEX DOMAINS

BY

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ABSTRACT. A complete and explicit description of the holomorphic proper mappings between weakly pseudoconvex domains of the class Δ_p (see ($\cdot$) below) is given.

The study of proper holomorphic mappings (i.e. holomorphic mappings which antitransform compact sets into compact sets),

$$(1) \qquad F\colon D_1 \to D_2, \qquad D_1, D_2 \subset C^n \ (n > 1),$$

between bounded pseudoconvex domains with smooth boundary has had, recently, a considerable impulse and has been applied, mainly, to investigate

(a) under which hypothesis there is no obstruction for the existence of such an F,

(b) the regularity up to the boundary of (1),

(c) for which domains proper means, in effect, biholomorphic.

The first class of problems has been taken into consideration, for instance, in **[8, 5, 4]** and a typical result is, for example, that a strictly pseudoconvex domain cannot be mapped properly onto a weakly pseudoconvex domain.

About (b), see, for instance, **[4, 2, 3]** and the bibliography there included.

The property enunciated in (c) has been proved in the case $D_1 = D_2 =$ unit ball (see **[1** and **9]**), when D_1 and D_2 are strictly pseudoconvex domains with, in addition, D_2 simply connected (see **[7** and **4]**), and in the case $D_1 = D_2$ and strictly pseudoconvex (see **[7]**). The aim of this note is to look at questions of type (a) and (c), above, posed for the following class of weakly pseudoconvex domains,

$$(\cdot) \quad \Delta_p = \left\{(z_1,\ldots,z_n)\colon |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1\right\}, \qquad (p_1,\ldots,p_n) \in (\mathbf{Z}^+)^n.$$

Precisely, we prove

THEOREM. *In order for a proper holomorphic mapping from Δ_p onto Δ_q to exist it is necessary and sufficient that*

$$(\cdot\cdot) \qquad p/q = (p_1/q_1,\ldots,p_n/q_n) \in (\mathbf{Z}^+)^n.$$

Furthermore, assuming ($\cdot\cdot$), the only proper holomorphic mapping between Δ_p and Δ_q is, up to biholomorphisms of Δ_q, $F(z_1,\ldots,z_n) = (z_1^{p_1/q_1},\ldots,z_n^{p_n/q_n})$.

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COROLLARY. *Every proper self-mapping of Δ_q is a biholomorphism.*

1. In this section we find some holomorphic automorphisms of Δ_p; in §2, as a corollary of the theorem there stated, it will be proved that they form the whole automorphism group of Δ_p.

Let us suppose Δ_p, in order to make easier the writing of what follows, to be of the form

$$\Delta_p = \left\{(z_1,\dots,z_n)\colon |z_1|^2 + \cdots + |z_k|^2 + |z_{k+1}|^{2p_{k+1}} + \cdots + |z_n|^{2p_n} < 1\right\}, \qquad k \geqslant 1,$$

where the p_i's, for $i = k+1,\dots,n$, are assumed bigger than 1. Furthermore, let us denote

$$\mathbf{B}_k(0,1) = \mathbf{B}_k = \left\{(z_1,\dots,z_k)\colon |z_1|^2 + \cdots + |z_k|^2 < 1\right\}$$

and $z^k = (z_1,\dots,z_k)$, $z = (z^k, z_{k+1},\dots,z_n)$. For a fixed point $z_0^k \in \mathbf{B}_k$, let us consider a holomorphic automorphism of $\mathbf{B}_k$, say $T_{z_0^k}$, which carries the point z_0^k into the origin of $\mathbf{C}^k$. Then we have

PROPOSITION 1.1. *Every holomorphic mapping $S\colon \Delta_p \to \mathbf{C}^n$ of the form*

$$S(z) = \left(T_{z_0^k}(z^k),\, c_{k+1}z_{k+1}\left[\frac{1-|z_0^k|^2}{\left(1-\langle z^k, z_0^k\rangle\right)^2}\right]^{1/2p_{k+1}},\dots,\right.$$
$$\left. c_n z_n\left[\frac{1-|z_0^k|^2}{\left(1-\langle z^k, z_0^k\rangle\right)^2}\right]^{1/2p_n}\right)$$

where $\langle\,,\rangle$ is the canonical hermitian scalar product in $\mathbf{C}^k$ and $|c_j| = 1$ for $j = k+1,\dots,n$, is for every choice of z_0^k an automorphism of Δ_p.

PROOF. To prove the statement it is sufficient, after having observed that S is well defined as $|z_0^k| < 1$, to show that

$$S(z) \in \Delta_p \quad \text{if } z \in \Delta_p, \qquad \text{and} \qquad S(z) \in \partial\Delta_p \quad \text{iff } z \in \partial\Delta_p.$$

Because of the special form of $S_{k+1},\dots,S_n$ (i.e. the last $n-k$ components of S) and the fact that $T_{z_0^k}$ is, by assumption, an automorphism of the ball in $\mathbf{C}^k$, we have

$$|S_1(z)|^2 + \cdots + |S_k(z)|^2 + |S_{k+1}(z)|^{2p_{k+1}} + \cdots + |S_n(z)|^{2p_n}$$
$$= |T_{z_0^k}|^2 + \frac{1-|z_0^k|^2}{\left|\left(1-\langle z^k, z_0^k\rangle\right)\right|^2}\left(|z_{k+1}|^{2p_{k+1}} + \cdots + |z_n|^{2p_n}\right) < 1$$

where the last inequality is deduced from (see [**10**, p. 26])

$$|T_{z_0^k}|^2 = 1 - \frac{\left(1-|z_0^k|^2\right)\left(1-|z^k|^2\right)}{\left|1-\langle z^k, z_0^k\rangle\right|^2}.$$

REMARK (a). For every point of Δ_p lying in the linear subspace $z_{k+1} = \cdots = z_n = 0$ there exists an automorphism of Δ_p which carries this point into the origin, and obviously vice-versa.

REMARK (b). The hypothesis that $p_1, \ldots, p_k = 1$ (that is, the first k are ones, see $(\cdot)$) is completely unessential. It is sufficient to make a formal change of indices to get a general statement of the proposition. In the case $k = 0$ the biholomorphisms taken into consideration will be, obviously, $S(z) = (c_1 z_1, \ldots, c_n z_n)$ with $|c_i| = 1$, $i = 1, \ldots, n$.

2. This section is mainly devoted to the proof of the theorem stated in the introduction. We shall start by observing a geometric property which has to be satisfied by a proper holomorphic mapping between domains of the class $(\cdot)$. Then, let $F = (F_1, \ldots, F_n)\colon \Delta_p \to \Delta_q$ and let us denote

$$A_p^* = \{ i \in \{p_1, \ldots, p_n\} \text{ s.t. } i > 1 \}$$

and A_q^* the analogous set for $q \in (\mathbf{Z}^+)^n$ we have

PROPOSITION 2.1. *There exists a function $j\colon A_q^* \to A_p^*$ such that*

$$F\left[\{ z_{j(i)} = 0 \} \cap \Delta_p \right] = \{ z_i = 0 \} \cap \Delta_q, \qquad i \in A_q^*.$$

Furthermore, such a function is 1-1.

PROOF. F has an analytic extension to a neighbourhood of $\overline{\Delta}_p$ [**3** Remark B, p. 112]; moreover [**3**, Theorem 2], calling $H_I(p) = \prod_{i \in I} \{ z_i = 0 \} \cap \Delta_p$, we have

$$\partial \left(F^{-1} \left(H_{A_q^*}(q) \right) \right) \subseteq \partial H_{A_p^*}(p)$$

where ∂ means "boundary of". This implies $F^{-1}(H_{A_q^*}(q)) \subseteq H_{A_p^*}(p)$ or, in other words,

$$\{ z \in \Delta_p\colon F_i(z) = 0 \} \subseteq H_{A_p^*}(p), \qquad i \in A_q^*. \tag{2.2}$$

The relation (2.2) implies (see for instance [**6**, p. 60]) that for every $i \in A_q^*$ there exists an index belonging to A_p^*, say k, such that on Δ_p we have $F_i|_{z_k = 0} \equiv 0$. Fix, once and for all, the correspondence $i \to k$ and call it j. The function j is injective, otherwise it might happen for $i_1 \neq i_2$,

$$F_{i_1}|_{z_k = 0} = F_{i_2}|_{z_k = 0} = 0$$

and hence, since F is a proper mapping between $\Delta_p \cap \{ z_k = 0 \}$ and its image, this is impossible because this last set is contained in a coordinate subspace of dimension $n - 2$.

REMARK. The injectivity of the function j says that a necessary condition for the proper equivalence between Δ_p and Δ_q is that $\#(A_p^*) \geqslant \#(A_q^*)$.

PROOF OF THE THEOREM. The proof is divided into two parts: in the first, we prove the Theorem in the case $q_1 = q_2 = \cdots = q_n = 1$; in the second part, we prove the general statement exploiting the result in the particular case. Suppose then, that

$\Delta_q = \mathbf{B} =$ unit ball. In this situation, the only fact to prove is that the proper mappings F: $\Delta_p \to \mathbf{B}$ are, up to automorphisms of the ball, the mapping

$$(z_1,\ldots,z_n) \mapsto (z_1^{p_1},\ldots,z_n^{p_n}).$$

Consider a convenient open neighbourhood U of

$$z^0 = (z_1^0,\ldots,z_n^0) \in \partial\Delta_p, \qquad z_1^0,\ldots,z_n^0 \neq 0,$$

and the mapping

$$(z_1,\ldots,z_n) \mapsto (z_1^{p_1},\ldots,z_n^{p_n}). \tag{2.3}$$

For suitable U, (2.3) is invertible. Denote by Ψ: $V \to U$ a fixed local inverse of (2.3). On the other hand (shrinking U if necessary), F is invertible on U, z^0 being a strictly pseudoconvex point of $\partial\Delta_p$, so the composite mapping,

$$G = F \circ \Psi\colon V \to F(U)$$

is holomorphic, 1-1, and maps, by construction,

$$V \cap \partial\mathbf{B} \to F(U) \cap \partial\mathbf{B}.$$

Hence [**1**], G is the restriction to V of an automorphism Θ of the ball. This means

$$F(z_1,\ldots,z_n) = \Theta(z_1^{p_1}, z_2^{p_2},\ldots,z_n^{p_n}),$$

and the statement of the theorem then follows in this particular case.

Consider, now, the general case. First of all we can assume, without loss of generality and combining Propositions 2.1 and 1.1, that

$$F(0,\ldots,0) = (0,\ldots,0). \tag{2.4}$$

In fact, by Proposition 2.1, we have that

$$F_i(0,\ldots,0) = 0, \qquad i \in A_q^*,$$

and hence Proposition 1.1 allows us to consider (2.4) satisfied.

Now take into consideration the following holomorphic mapping:

$$E = (F_1^{q_1},\ldots,F_n^{q_n})\colon \Delta_p \to \mathbf{B}.$$

The first part of the statement, already proved, implies then, because of 2.4, that

$$E(z_1,\ldots,z_n) = A\begin{pmatrix} z_1^{p_1} \\ \vdots \\ z_n^{p_n} \end{pmatrix} \tag{2.5}$$

where A is a unitary matrix, $A = (a_{ij})$, and hence

$$\delta_{ik} = \sum_s a_{is}\bar{a}_{ks}. \tag{2.6}$$

Let $i \in A_q^*$: by Proposition 2.1 there exists $j(i) \in A_p^*$ such that $F_i|_{z_{j(i)}=0} \equiv 0$ which by (2.5) forces us to have

$$a_{ir} = 0, \qquad r \neq j(i),\ i \in A_q^*. \tag{2.7}$$

The relation (2.5) written down for $i \in A_q^*$, taking into account (2.7), actually gives

$$|a_{ij(i)}|^2 = 1, \quad a_{kj(i)} = 0, \qquad k \neq i, i \in A_q^*. \tag{2.8}$$

Let us now consider the square matrix (j is injective)

$$B = (a_{rs}), \qquad r \notin A_q^*, s \notin j(A_q^*).$$

B, by virtue of (2.8), is unitary as

$$\sum_{s \notin j(A_q^*)} a_{is}\bar{a}_{ks} = \sum_{s=1}^{n} a_{is}\bar{a}_{ks} = \delta_{ik}, \qquad i, k \notin A_q^*.$$

This completes the proof, observing that by composing F with the automorphism of Δ_q which operates on the z_i variables, $i \notin A_q^*$, as the linear unitary mapping carried by B^{-1}, and on the other variables by multiplication by $a_{ij(i)}^{-1}$, F can be seen to satisfy

$$(F_1^{q_1}, \ldots, F_n^{q_n}) = (z_1^{p_1}, \ldots, z_n^{p_n}).$$

A remark of some interest is that, having used in the proof only the automorphisms given by Proposition 1.1, we get

COROLLARY. *The automorphism group of Δ_p is the set given in Proposition* 1.1(*see also* **[11]**).

During the submission of this note, the author came to know about a generalisation (to any smooth pseudoconvex domain with real analytic boundary) of the corollary of the theorem which, therefore, he includes in the bibliography **[12]**.

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