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CONTRACTION OPERATORS QUASISIMILAR TO A UNILATERAL SHIFT[1]

BY

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ABSTRACT. Let U_n denote the unilaterial shift of finite multiplicity n. It is shown that a contraction operator T is quasisimilar to U_n if and only if T is of Class $C_{1\cdot}$, the canonical isometry V associated with T is pure and T is n-cyclic with analytically independent vectors. For this, the notions of operators of analytic type and analytic independence of vectors are introduced. A characterization of the cyclic vectors of the Backward Shift is also presented.

0. Introduction. In this paper, necessary and sufficient conditions are obtained for a contraction operator to be quasisimilar to a unilateral shift. Two Hilbert space operators T and S are quasisimilar if there exist operators X and Y which are one-to-one, have dense range and satisfy $XT = SX$ and $TY = YS$. Quasisimilarity was introduced by Sz.-Nagy and Foiaş and they gave a simple characterization of operators quasisimilar to a unitary operator [7]. The topic was further studied by W. S. Clary who obtained a characterization of cyclic subnormal operators quasisimilar to the unilateral shift of multiplicity one [1]. W. W. Hastings generalized the result to the case of quasisimilarity between subnormal operators and isometries [5]. Both authors develop standard representations of the subnormal operators in terms of measures supported by $\bar{D} = \{z\colon |z| \leq 1\}$. Thus these subnormal operators are contractions and so an appropriate general question would be what type of contractions are quasisimilar to U_n?

1. Notation and definitions. A Hilbert space operator is a bounded linear transformation $T\colon H \to K$ from a Hilbert space H into a Hilbert space K. If $K = H$, we say that T is an operator on H. T is said to be a contraction if $\|T\| \leq 1$. The set of analytic polynomials in one complex variable z is denoted by P. Let T be an operator on H. A subspace M of H is said to be invariant for T if $TM \subseteq M$. For any given $f \in H$ let $P(f; T) = \{p(T)f\colon p \in P\}$ and $M(f; T) =$ closure of $P(f; T)$ in H. Then $M(f; T)$ is invariant for T. If there is a vector f in H such that $M(f; T) = H$, we say that T is cyclic and f is a cyclic vector for T. For n vectors $f_1, \ldots, f_n$ in H let $M(f_1, \ldots, f_n; T)$ be the closure of $P(f_1, \ldots, f_n; T) = \{\sum_{i=1}^n p_i(T) f_i\colon p_i \in P\}$ in H. Then $M(f_1, \ldots, f_n; T)$ is invariant for T. T is said to be n-cyclic if there are n vectors

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$f_1,\dots,f_n$ in H with $M(f_1,\dots,f_n; T) = H$ while no smaller set of vectors has this property.

Let T on H and S on K be operators. An operator X: $H \to K$ is called a quasiaffinity if X is one-to-one and has dense range. T is said to be a quasiaffine transform of S if there is a quasiaffinity X: $H \to K$ satisfying $XT = SX$. T and S are said to be quasisimilar if each of them is a quasiaffine transform of the other. That is, if there exists quasiaffinities X: $H \to K$ and Y: $K \to H$ satisfying $XT = SX$ and $YS = TY$. It is known that the order of cyclicity is preserved under quasisimilarity. T is said to be of class $C_{1\cdot}$ if $T^n h \nrightarrow 0$ for all h in H except $h = 0$; of class C_{10} if T is of class $C_{1\cdot}$ and $T^{*n}h \to 0$ for all h in H.

Let T be a contraction on H. If A is the algebra of functions analytic in $D = \{z: |z| < 1\}$ and continuous on $\bar{D}$ and $f \in A$ with power series $\Sigma a_n z^n$, then the operator series $\Sigma a_n T^n$ is convergent in norm to a limit denoted by $f(T)$. For $f \in H^\infty$ and $0 < r < 1$, the function f_r defined by $f_r(z) = f(rz)$ belongs to A and so $f_r(T)$ is an operator on H. Let H_T^∞ denote the set of all functions f in H^∞ for which $f_r(T)$ has a strong limit as $r \to 1$ [7]. This limit is denoted by $f(T)$.

DEFINITION 1.1. An operator T on a space H is said to be of *analytic type* if and only if T is a contraction and $H_T^\infty = H^\infty$.

A completely nonunitary (c.n.u.) contraction is of analytic type [7, p. 111]. If T is unitary and the spectral measure of T is absolutely continuous with respect to m, normalized Lebesgue measure on the unit circle C, then T is of analytic type [7, p. 116]. In particular, the bilateral shifts and the unilateral shifts are of analytic type. In general, a contraction T is of analytic type if and only if the spectral measure of its unitary part is absolutely continuous.

DEFINITION 1.2. Let T be an operator of analytic type on the space H. Then n vectors $f_1,\dots,f_n$ in H are said to be *analytically independent* under T if a relation $F_1(T)f_1 + \cdots + F_n(T)f_n = 0$ with $F_1,\dots,F_n$ in H^∞ implies that $F_1 = \cdots = F_n = 0$.

An interesting result based on this notion of analytic independence is a simple characterization of the cyclic vectors of the backward shift U^* given below. Here U denotes the simple unilateral shift. The simple bilateral shift is denoted by W. The orthogonality of two vectors f and g is indicated by $f \perp g$. In the special case of U or W and for F in H^∞, the operator $F(U)$ or $F(W)$ is just multiplication by the function F and so we will write Ff instead of $F(U)f$ or $F(W)f$. We will also use the fact that any function in the classical spaces H^p, $p > 0$, is a quotient of two functions in H^∞ [4].

THEOREM 1.3. *A nonzero function in H^2 is a cyclic vector of U^* if and only if* 1 *and $\bar{h}$ are analytically independent under W ($\bar{h}$ is the complex conjugate of h).*

PROOF. Suppose h is not a cyclic vector. Then there is a nonzero function f in H^2 such that $f \perp U^{*n}h$ for all $n \geq 0$. That is, for $n = 0, 1, 2, \dots,$ $(f, U^{*n}h) = \int z^n f\bar{h}\, dm = 0$. Since $f\bar{h}$ is in $L^1(m)$ it follows that $f\bar{h}$ is in H^1 and so is a quotient of H^∞ functions, say, $f\bar{h} = G_1/G_2$. Also, f being in H^2 is a quotient of H^∞ functions: $f = F_1/F_2$. Thus $(F_1/F_2)\bar{h} = G_1/G_2$, or equivalently, $G_2F_1\bar{h} = F_2G_1$, showing that 1 and $\bar{h}$ are analytically dependent.

Conversely, suppose $F\bar{h} = G$, where F, G are nonzero functions in H^∞. Since G is in H^∞ we have for $n = 1, 2, \ldots$,

$$0 = \int z^n G\, dm = \int z^n F\bar{h}\, dm = (U^n F, h) = (zF, U^{*n-1}h).$$

This shows that zF is orthogonal to $U^{*k}h$ for $k = 0, 1, 2, \ldots$. Since zF is nonzero we conclude that h is not a cyclic vector.

REMARKS. Several characterizations of the cyclic vectors of U^* appear in [**3**]. Some of the other results in [**3**] can be obtained easily from the concept of analytic independence. The theorem also shows that polynomials, rational functions or inner functions cannot be cyclic vectors of U^*.

2. Quasisimilarity. For a contraction T on H, we recall two constructions from [**7**]. First, T has a canonical decomposition $T = T_0 \oplus T_1$ on $H = H_0 \oplus H_1$ such that T_0 on H_0 is unitary (called the unitary part) and T_1 on H_1 is completely nonunitary (called the c.n.u. part). Second, T has a unitary dilation U_T on a space K having the following properties:

(1) H is a subspace of K

(2) $T^n h = \Pr U_T^n h$, $n \geqslant 1$, $h \in H$,

where Pr denotes the orthogonal projection of K onto H. U_T is called *minimal* if the smallest subspace of K which reduces U_T and contains H is all of K. If T is c.n.u. then both T and U_T are of analytic type.

LEMMA 2.1. *Let T: $H \to H$ be a contraction of analytic type. For every outer function F in H^∞, $F(T)$ is a quasiaffinity on H.*

PROOF. By the canonical decomposition of T one reduces the lemma to Proposition 3.1 in [**7**] for the c.n.u. part and to the Riesz brothers' theorem for the unitary part.

PROPOSITION 2.2. *Let T: $H \to H$ be a contraction of analytic type. For all $n \geqslant 1$ the following are equivalent*:

(a) *There exists n vectors $f_1, \ldots, f_n$ in H which are analytically independent under T and satisfying $M(f_1, \ldots, f_n; T) = H$.*

(b) *There exists a quasiaffinity Y: $H_n^2 \to H$ such that $YU_n = TY$.*

PROOF. Here H_n^2 denotes the direct sum of n copies of H^2 on which U_n acts in the usual manner. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in the ith place, $i = 1, \ldots, n$.

(b) $\Rightarrow$ (a). Let $f_i = Ye_i$. Since $\{e_i: i = 1, \ldots, n\}$ generate H_n^2 under U_n it follows from the properties of Y that $H = M(f_1, \ldots, f_n; T)$. We assert that the f_i are analytically independent under T. Let us assume the contrary, so that there are H^∞ functions F_i, not all zero, satisfying $\Sigma F_i(T) f_i = 0$. That is, $0 = \Sigma F_i(T) Ye_i = Y(\Sigma F_i e_i)$. Since Y is one-to-one, we must have $\Sigma F_i e_i = 0$. This means $F_i = 0$ for all i, which is a contradiction.

(a) $\Rightarrow$ (b). Let U_T be the minimal unitary dilation of T and μ_i be the measue corresponding to f_i arising from the spectral measure of U_T. By hypothesis, μ_i is absolutely continuous with respect to m. Let $h_i = d\mu_i/dm$, so that $h_i \in L^1(m)$. Choose outer functions F_i in H^∞ satisfying $|F_i|^2 = (1 + h_i)^{-1}$ a.e. (m) [**6**, p. 53].

Now we define $Y\colon H_n^2 \to H$ by $Ye_i = F_i(T)f_i$ and extend to the set of vectors of the form $(p_1,\ldots,p_n)$, where p_i are polynomials, by $Y(p_1,\ldots,p_n) = \Sigma p_i(T)F_i(T)f_i$. So Y satisfies $YU_n = TY$. For polynomials $p_1,\ldots,p_n$,

$$\begin{aligned}\|Y(p_1,\ldots,p_n)\|_H^2 &= \left\|\sum p_i(T)F_i(T)f_i\right\|^2 \leqslant n \cdot \sum \|p_i(T)F_i(T)f_i\|^2 \\ &\leqslant n \cdot \sum \|p_i(U_T)F_i(U_T)f_i\|^2 = n \cdot \sum \int |p_iF_i|^2 \, d\mu_i \\ &= n \cdot \sum \int |p_i|^2 h_i(1 + h_i)^{-1} \, dm \\ &\leqslant n \cdot \sum \|p_i\|_{H^2}^2 = n \cdot \|(p_1,\ldots,p_n)\|_{H_n^2}^2.\end{aligned}$$

Thus Y is bounded on the set of vectors of the form $(p_1,\ldots,p_n)$. Since this set is dense in H_n^2, Y has a unique bounded extension to all of H_n^2.

To show that Y is one-to-one, suppose that $Y(g_1,\ldots,g_n) = 0$. Each g_i is in H^2 and so is a quotient of two H^∞ functions: $g_i = G_{1i}/G_{2i}$, where G_{2i} are outer. Let $G_2 = \prod_{i=1}^n G_{2i}$ and $G'_{2i} = G_2/G_{2i}$. Then we have

$$\begin{aligned}0 &= Y(g_1,\ldots,g_n) = G_2(T)Y(g_1,\ldots,g_n) = Y(G_2(g_1,\ldots,g_n)) \\ &= Y(G'_{21}G_{11},\ldots,G'_{2n}G_{1n}) = Y\left(\sum G'_{2i}G_{1i}e_i\right) = \sum (G'_{2i}G_{1i})(T)F_i(T)f_i.\end{aligned}$$

Since G'_{2i}, F_i are outer functions, the analytic independence of f_i under T implies that $G_{1i} = 0$ for all i. That is, $g_i = 0$ for all i.

Next, we show that Y has dense range. Suppose p is a polynomial. We have

$$\int |p/F_i|^2 \, dm = \int |p|^2(1 + h_i) \, dm < \infty.$$

Hence by [**4**, Theorem 2.11], p/F_i is in H^2. Write $g_i = p/F_i$. Then g_i is in H^2 and $F_ig_i = p$. Then

$$\begin{aligned}F_i(T)Y(g_ie_i) &= Y(F_ig_ie_i) = Y(pe_i) = p(T)Ye_i \\ &= p(T)F_i(T)f_i = F_i(T)p(T)f_i.\end{aligned}$$

Since F_i is outer, from Lemma 2.1 we get $Y(g_ie_i) = p(T)f_i$. It is now clear that ran Y contains all vectors of the form $\Sigma p_i(T)f_i$, where p_i are polynomials. Thus Y has dense range and the proof is complete.

We recall the following results from [**7**].

PROPOSITION 2.3 (WOLD DECOMPOSITION [**7**, p. 3]). *Every isometry is the direct sum of a unitary operator and a unilateral shift.*

If the isometry has no unitary part it is said to be *pure*. Thus a pure isometry is a unilateral shift.

PROPOSITION 2.4. *Every isometry has a minimal unitary extension* [**7**, *p.* 6].

The minimal unitary extension of U_n is W_n, the bilateral shift of multiplicity n [**7**, p. 5].

PROPOSITION 2.5. *Suppose T is a contraction of class $C_{1\cdot}$ on the space H. There is a positive quasiaffinity X on H of norm one and an isometry V on H such that $XT = VX$. Further, for all h in H, $\|Xh\| = \inf_k \|T^k h\|$* [7, p. 79].

We shall call this isometry V the canonical isometry associated with T. If T has a unitary part then it is easy to see that V has a unitary part. Therefore V pure implies that T is c.n.u.

PROPOSITION 2.6. *Let $F_i = (f_{i1},\ldots,f_{in})$, $i = 1,\ldots,n$, be any n vectors in H_n^2. Then F_i are analytically dependent if and only if* $\det(f_{ij}) = 0$.

PROOF. Let A denote the matrix (f_{ij}). Since f_{ij} are in H^2, $\det A$ is a function of class N^+. Assume that $\det A = 0$. Hence for all z in D we have $\det A(z) = \det(f_{ij}(z)) = 0$. Let $k(z)$ be the row rank of $A(z)$ and put $k = \max_{z\in D} k(z)$. Then $0 < k < n$. Thus there exists z in D and a $k \times k$ minor of $A(z)$ whose determinant is not zero, while every minor of $A(z)$ of order greater than k has determinant zero for all z in D. Without loss of generality, we may assume that the principal $k \times k$ minor $B = (f_{ij})$, $1 \leqslant i, j \leqslant k$, has the nonzero determinant at some point z in D. Thus $\det B$ is a nonzero function of class N^+. Let

$$(h_1,\ldots,h_k) = (f_{k+1,1},\ldots,f_{k+1,k}) \cdot \det B \cdot B^{-1}.$$

Then each h_i is of class N^+. Hence there exists a nonzero function F in H^∞ such that $Fh_1,\ldots,Fh_k$ and $F \cdot \det B$ are all H^∞ functions. This follows from the fact that every function of class N^+ is a quotient of two functions in H^∞ [4]. We claim that

$$(*) \qquad (F \cdot \det B) \cdot F_{k+1} = F \cdot \sum_{i=1}^{k} h_i F_i.$$

To see this, let us set $G_i = (f_{i1},\ldots,f_{ik})$, $i = 1,\ldots,k+1$. Then by our choice of $h_1,\ldots,h_k$, we have $F \cdot \det B \cdot G_{k+1} = F \cdot \Sigma h_i G_i$. Hence, if $(*)$ were not true at some point z_0 in D, then $A(z_0)$ has a minor of order $k+1$ whose determinant is not zero, contradicting our choice of k. From $(*)$ we conclude that $F_1,\ldots,F_{k+1}$ and hence $F_1,\ldots,F_n$ are analytically dependent.

Conversely, suppose that $F_1,\ldots,F_n$ are analytically dependent: $\Sigma h_i F_i = 0$, where h_i are H^∞ functions, not all zero. Thus for all z in D we have $\Sigma h_i(z)F_i(z) = 0$. That is, for all z in D,

$$\big(h_1(z),\ldots,h_n(z)\big) \cdot \big(f_{ij}(z)\big) = 0.$$

Hence $\det(f_{ij}(z)) = 0$ for all points z in D at which at least one of the numbers $h_1(z),\ldots,h_n(z)$ is different from zero. But the common zeros of $h_1,\ldots,h_n$ are at most countable and so we must have $\det(f_{ij}) = 0$.

COROLLARY 2.7. *Any $n+1$ vectors in H_n^2 are analytically dependent.*

PROOF. Let $F_i = (f_{i1},\ldots,f_{in})$, $i = 1,\ldots,(n+1)$, be $n+1$ vectors in H_n^2. We may assume that the first n vectors $F_1,\ldots,F_n$ are analytically independent. By Proposition 2.6 we have

$$\det(f_{ij})_{1\leqslant i,j\leqslant n} \neq 0.$$

Let $(h_1,\dots,h_n) = F_{n+1}\cdot\det(f_{ij})\cdot(f_{ij})^{-1}$. Then there exists a function F in H^∞ such that $Fh_1,\dots,Fh_n$ and $F\cdot\det(f_{ij})$ are all in H^∞. We obtain

$$F\cdot\det(f_{ij})\cdot F_{n+1} = F\cdot(h_1,\dots,h_n)\cdot(f_{ij}) = \sum_{i=1}^{n} Fh_iF_i,$$

showing that $F_1,\dots,F_{n+1}$ are analytically dependent.

THEOREM 2.8. *Let T be a contraction on the space H. Then T is quasisimilar to U_n if and only if the following conditions hold*:

(1) *T is of class $C_{1\cdot}$.*

(2) *The canonical isometry V associated with T is pure.*

(3) *There exists n vectors $f_1,\dots,f_n$ in H which are analytically independent under T and satisfying $M(f_1,\dots,f_n;T) = H$.*

PROOF. *The conditions are sufficient.* Condition (2) implies that T is c.n.u. and so is of analytic type. By condition (3) and Proposition 2.2 there is a quasiaffinity Y: $H_n^2 \to H$ satisfying $YU_n = TY$. By condition (1) we have the quasiaffinity X on H satisfying $XT = VX$, where V is the canonical isometry associated with T. From (2), V is a unilateral shift. From $M(f_1,\dots,f_n;T) = H$ and $XT = VX$ it follows that $M(Xf_1,\dots,Xf_n;V) = H$. This shows that V has multiplicity at most n.

Suppose that the multiplicity of V is k where $k < n$. Indentifying V with U_k and applying Corollary 2.7 we see that $Xf_1,\dots,Xf_n$ are analytically dependent under V. Hence there exists H^∞ functions $F_1,\dots,F_n$, not all zero, satisfying $\Sigma F_i(V)Xf_i = 0$. This means $X(\Sigma F_i(T)f_i) = 0$ and hence $\Sigma F_i(T)f_i = 0$. This contradicts (3) and so V must have multiplicity n.

The conditions are necessary. We are assuming that T is quasisimilar to U_n. Let Y: $H \to H_n^2$ and Z: $H_n^2 \to H$ be quasiaffinities satisfying $YT = U_nY$ and $ZU_n = TZ$.

(1) T is of class $C_{1\cdot}$: Let h be any vector in H such that $T^kh \to 0$. Then $YT^kh \to 0$. This implies $U_n^kYh \to 0$. But, clearly, $\|U_n^kYh\| = \|Yh\|$ for all k and so we have $h = 0$.

Now $(U_n^*)^k \to 0$ strongly as $k \to \infty$ implies that $T^{*k} \to 0$ strongly. Thus T is of class C_{10} and so is c.n.u. and of analytic type.

(2) The canonical isometry V is pure: By Proposition 2.5 we have the quasiaffinity X satisfying $XT = VX$. From Y: $H \to H_n^2$ we construct an operator Y_0: $H \to H_n^2$ with the properties $Y_0V = U_nY_0$ and $Y_0X = Y$ as follows.

Just define $Y_0Xh = Yh$ for all h in H. Then Y_0 is densely defined and has dense range. For h in H we have for $k = 1,2,\dots,$

$$\|Y_0Xh\| = \|Yh\| = \|U_n^kYh\| = \|YT^kh\| \leq \|Y\|\cdot\|T^kh\|.$$

It follows that

$$\|Y_0Xh\| \leq \|Y\|\cdot\inf_k\|T^kh\| = \|Y\|\cdot\|Xh\|,$$

by Proposition 2.5. Thus Y_0 is bounded by $\|Y\|$ on the dense set of ran X. So it has a unique extension to all of H satisfying $\|Y_0\| \leq \|Y\|$. Now

$$Y_0VXh = Y_0XTh = YTh = U_nYh = U_nY_0Xh$$

shows that Y_0V and U_nY_0 agree on the dense set ran X. Hence we must have $Y_0V = U_nY_0$. The construction of Y_0 is complete.

Now $Y_0V = U_nY_0$ implies that $VXZY_0 = XZY_0V$. Let V_1 denote the minimal unitary extension of V. Since $VXZ = XZU_n$ one infers that V_1 has finite multiplicity and so cannot be unitarily equivalent to any proper part of it. Now let Y_1 be the unique bounded lifting of XZY_0 commuting with V_1 [**2**, Corollary 5.1]; obviously Y_1 has dense range. By [**2**, Lemma 4.1], V_1 is unitarily equivalent to $V_1 \mid (\ker Y_1)^{\perp}$ and it follows from $\ker Y_0 \subset \ker Y_1 = (0)$ that Y_0 is a quasiaffinity. Therefore, from $V^{*k}Y_0^* = Y_0^*U_n^{*k} \to 0$ strongly as $k \to \infty$, we infer that $V^{*k} \to 0$ strongly as $k \to \infty$. This means that V is pure.

(3) Take $f_i = Ze_i$. Then $f_1, \ldots, f_n$ satisfy the requirements. For, Z is a quasiaffinity implies $M(f_1, \ldots, f_n; T) = H$. The analytic independence of $f_1, \ldots, f_n$ under T also follows easily.

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