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STABILITY OF THE TRAVELLING WAVE SOLUTION OF THE FITZHUGH-NAGUMO SYSTEM

BY

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ABSTRACT. Travelling wave solutions for the FitzHugh-Nagumo equations have been proved to exist, by various authors, close to a certain singular limit of the equations. In this paper it is proved that these waves are stable relative to the full system of partial differential equations; that is, initial values near (in the sup norm) to the travelling wave lead to solutions that decay to some translate of the wave in time. The technique used is the linearised stability criterion; the framework for its use in this context has been given by Evans [6–9]. The search for the spectrum leads to systems of linear ordinary differential equations. The proof uses dynamical systems arguments to analyse these close to the singular limit.

1. Introduction. Travelling waves play a central role in the theory of reaction-diffusion equations. Many techniques have been developed to find such waves, i.e., prove their existence; see Conley and Gardner [**4**], Gardner and Smoller [**16**], and Dunbar [**5**] for recent results. However, the equation of their stability relative to the PDE has remained fairly open. Scalar equations are now well understood; see Fife [**12**], Fife and McLeod [**13**], and Bramson [**1**]. For systems, the only fully established results involve assumptions on the nonlinearity that permit the application of a maximum principle type argument, i.e., some monotonicity; see Klaasen and Troy [**19**], and Gardner [**15**]. Feroe [**11**] has performed some numerical calculations on the stability problem for the FitzHugh-Nagumo equations with a special assumption of piecewise linearity on the nonlinear term.

In this paper I shall prove a stability result for the FitzHugh-Nagumo equations. These equations are a paradigm example of a system of equations to which the maximum principle is difficult to apply; see Terman [**22**].

The FitzHugh-Nagumo equations are the following system of reaction-diffusion equations:

$$u_t = u_{xx} + f(u) - w, \qquad w_t = \varepsilon(u - \gamma w). \tag{1.1}$$

The function

$$f(u) = u(u-a)(1-u) \tag{1.2}$$

is a cubic, where $a < 1/2$. The constants ε and γ are positive. I shall be interested in the case $\varepsilon \ll 1$ and $\gamma \ll 1$; γ is often assumed to be zero.

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These equations were originally formulated as a simplification to the Hodgkin-Huxley equations for nerve conduction; see FitzHugh **[14]** and Nagumo et al. **[21]**. They have since become a central example in reaction-diffusion equations.

A solution to (1.1) is determined by an initial value

$$(1.3) \qquad u(x,0) = u_0(x), \qquad w(x,0) = w_0(x),$$

where x ranges over $\mathbf{R}$. In the nerve conduction case the variable x is the distance along the nerve fiber.

The initial value problem (1.1), (1.3) can be solved (at least for small time) in many different function spaces; see Rauch and Smoller **[22]**. A natural one for our purposes is the space

$$\mathrm{BC}(\mathbf{R}, \mathbf{R}^2) = \{u\colon \mathbf{R} \to \mathbf{R}^2 \mid u \text{ is bounded and uniformly continuous}\}$$

supplied with the supremum norm.

A travelling wave for (1.1) is a solution that is a function of the single variable $\xi = x - ct$, i.e., $(u(\xi), w(\xi))$ satisfies

$$(1.4) \qquad -cu' = u'' + f(u) - w, \quad -cw' = \varepsilon(u - \gamma w) \qquad (' = d/d\xi).$$

A travelling pulse is a travelling wave that satisfies $(u, w) \to (0,0)$ as $\xi \to \pm\infty$.

For the nerve conduction problem, $(0,0)$ is the rest state and the nerve impulse is such a travelling wave.

The existence of a relevant travelling pulse, for some value of c, has been proved by many authors for $\varepsilon \ll 1$; see Carpenter **[2]**, Conley **[3]**, Hastings **[17]** and Langer **[20]**. Whether there exists such a pulse for ε not necessarily small is an open question. The significance of ε small is that (1.4) then becomes a singular perturbation and the pulse is constructed by piecing together solutions of certain reduced systems. The most explicit construction is given by Langer **[20]**.

Call this travelling pulse $(u_\varepsilon(\xi), w_\varepsilon(\xi))$. I shall be interested in its stability relative to the original PDE (1.1). If (1.1) is recast in a moving coordinate frame, i.e., in terms of variables $\xi = x - ct$ and t, it becomes

$$(1.5) \qquad u_t = u_{\xi\xi} + cu_\xi + f(u) - w, \qquad w_t = cw_\xi + \varepsilon(u - \gamma w).$$

The travelling wave is an equilibrium (time independent) solution of (1.5). The fact that any translate of a travelling wave is also a travelling wave must be taken into account when defining stability. In the following:

$$U = (u, w) \quad \text{and} \quad U_\varepsilon(\xi) = (u_\varepsilon(\xi), w_\varepsilon(\xi)).$$

DEFINITION. The travelling wave, for fixed $\varepsilon > 0$, is said to be stable if there exists $\delta > 0$ so that if $U(\xi, t)$ is a solution of (1.5) and there is a k_1 so that $\|U(\xi + k_1, 0) - U_\varepsilon(\xi)\|_\infty < \delta$, then there is a k_2 such that

$$(1.6) \qquad \|U(\xi + k_2, t) - U_\varepsilon(\xi)\|_\infty \to 0$$

as $t \to +\infty$.

This says that if a solution to (1.5) starts near some translate of the travelling wave, it tends to some other translate of it as $t \to +\infty$. A standard technique for determining stability is to use the linearised criterion. If the right-hand side of (1.5) is linearised about its equilibrium solution $U_\varepsilon(\xi)$, the resulting operator is

$$L\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} p_{\xi\xi} + cp_\xi + f'(u_\varepsilon)p - r \\ cr_\xi + \varepsilon(p - \gamma r) \end{pmatrix}, \tag{1.7}$$

where

$$\begin{pmatrix} p \\ r \end{pmatrix}(\xi) \in \mathrm{BC}(\mathbf{R}, \mathbf{R}^2).$$

The linearised criterion for stability of the travelling pulse is that the spectrum of L (except for 0) lies in a left half-plane $\{\lambda\colon \operatorname{Re}\lambda < a\}$ where $a < 0$, and 0 is a simple eigenvalue. Note that 0 must be in the spectrum because the translate of a travelling wave is another travelling wave. 0 being a simple eigenvalue means that this is the only neutral effect. This paper is devoted to proving

THEOREM. *Let L be given by* (1.7), $L\colon \mathrm{BC} \to \mathrm{BC}$. *Then*
(1) *there exists $a < 0$ so that* $\sigma(L)\setminus\{0\} \subset \{\lambda\colon \operatorname{Re}\lambda < a\}$;
(2) 0 *is a simple eigenvalue.*

Whether linearised stability implies stability relative to the full (nonlinear) equations, in the sense of the definition above, is a separate question. Henry [**18**] has some general theorems but these require a sectorial operator, and L is not sectorial as it has some spectrum that is asymptotically vertical; see §3.

In [**8**] Evans proved a "linearised stability implies stability" theorem for "nerve impulse equations". This is a class of equations that includes the FitzHugh-Nagumo system with the stated parameter values. The theorem in [**8**], in fact, states that the linear PDE is stable if the above described conditions on the spectrum hold. There is then a result in [**6**] which states that the travelling wave is stable for the full PDE. Using this, the following can be concluded from the theorem.

COROLLARY. *If $\varepsilon \ll 1$, $U_\varepsilon(\xi)$ is stable in the sense of the definition.*

In the next section the construction of the travelling pulse solution, found by the authors mentioned, is sketched. A theorem is then proved that gives an exact description of the fact that the pulse approaches the singular orbit as $\varepsilon \to 0$.

The spectrum of L falls naturally into two pieces: the normal spectrum, consisting of eigenvalues of finite multiplicity; and the essential spectrum, which is the rest. It is shown in §3 that the essential spectrum lies in a half-plane $\{\lambda\colon \operatorname{Re}\lambda < a\}$ for some $a < 0$. This essentially follows from proving that the system (1.1) is stable at $(0, 0)$, which is an assumption Evans makes for the theorem referenced above from [**8**].

In the set $\{\lambda\colon \operatorname{Re}\lambda > a\}$ an analytic function, due to Evans, $D(\lambda)$ can be defined. The zeroes of $D(\lambda)$ are eigenvalues of L. The description of $D(\lambda)$ is also given in §3.

$D(\lambda)$ is used to approximately locate the eigenvalues of L. They must lie close to the eigenvalues for a certain reduced system that is associated with some pieces in the singular travelling wave ($\varepsilon = 0$).

The reduced system is analysed in §4 and this approximate location of the eigenvalues of the full system is proved in §5.

It then follows that the only danger to stability comes from eigenvalues that lie near zero. In §6 I prove that there are at most two eigenvalues near zero. This is a computation of the winding number of D applied to a small circle about 0 (actually, it is not D, but an analytic continuation $\tilde{D}$). Since D is analytic, this winding number measures the number of zeroes inside the circle. It is proved that this winding number is exactly 2.

Zero is of necessity an eigenvalue, due to translation of the waves. Therefore, the other eigenvalue is real. In §7 the proof is completed by showing that this other eigenvalue is negative. Evans derived a very beautiful technique for determining this kind of information. He showed that the sign of the quantity $(d/d\lambda)D(\lambda)|_{\lambda=0}$ is determined by the direction in which the stable and unstable manifolds cross in the construction of the pulse. This is determined by using Langer's construction of the pulse.

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2. Description of the pulse. The travelling pulse satisfies (1.4), rewritten as a system

$$u' = v, \quad v' = -cv - f(u) + w, \quad w' = -(\varepsilon/c)(u - \gamma w). \tag{2.1}$$

The phase space of (2.1) is $\mathbf{R}^3$. The origin $(0, 0, 0)$ is a critical point of (2.1) and the pulse solution is a homoclinic orbit to the origin.

This homoclinic orbit is constructed for $\varepsilon \ll 1$. Langer describes the limiting behavior of this orbit, as $\varepsilon \to 0$, in some detail in his §2. I shall review this description, using his notation as much as possible.

When $\varepsilon = 0$ each plane $w =$ constant is invariant for (2.1). There exist values $w_{\max}$ and $w_{\min}$, with $w_{\min} < 0$, so that if $w_{\min} < w < w_{\max}$ then the reduced system

$$u' = v, \qquad v' = -cv - f(u) + w \tag{2.2}$$

has three critical points. When $w = 0$ there is a $c^* < 0$ for which there exists a heteroclinic orbit, called J_F, joining $(0, 0, 0)$ to the right-most critical point $(1, 0, 0)$. For c^* there is a w^* for which an orbit, called J_B, exists to (2.2) joining the right to the left critical point. F and B stand for front or back; an explanation for this will be given after the pulse is described further.

The singular limit of the homoclinic orbit ($\varepsilon \to 0$) consists of four pieces:

(1) J_F;

(2) $E_R^* = \{(u, v, w): v = 0, 0 \leqslant w \leqslant w^*$ and u is the largest root of $w = f(u)\}$;

(3) J_B;

(4) $E_L^* = \{(u, v, w): v = 0, 0 \leqslant w \leqslant w^*$ and u is the smallest root of $w = f(u)\}$;

see Figure 1.

Let $S_0 = J_F \cup E_R^* \cup J_B \cup E_L^* \subset \mathbf{R}^3$. S_0 is the singular orbit. It is called singular because E_R^* and E_L^* consist of critical points. The existence theorem says that, given any neighborhood N of S_0, there is an ε_0 so that (2.1) has a solution for some $c = c(\varepsilon)$ for all $\varepsilon \in [0, \varepsilon_0]$, which is homoclinic to $(0,0,0)$ and lies entirely in N. Moreover, $c(\varepsilon) \to c^*$ as $\varepsilon \to 0$. Call this orbit of (2.1), S_ε.

This picture is not new to Langer's proof but was already in the earlier proofs. Langer's contribution was to add that if N is a small enough neighborhood of S_0, there is a unique solution for each ε for unique c.

Langer uses a transversality argument. He shows that two certain manifolds intersect transversely in (u, v, w, c)-space for $\varepsilon = 0$; therefore, they still intersect for ε small. The uniqueness follows from the transversality. For the stability proof, some information about the nature of this transversality will play a central role; see §7.

If the pulse solution is graphed with U as a function of ξ, a profile is obtained that looks like a nerve impulse but with a long latent period in the middle. The part close to J_F is the front and that close to J_B is the back.

I shall need a more explicit description of S_ε. This is contained in the following theorem.

THEOREM 2.1. *If ε_0 is sufficiently small, there exists a homeomorphism h: $S^1 \times [0, \varepsilon_0] \to \bigcup S_\varepsilon$, where the union is taken over $\varepsilon \in [0, \varepsilon_0]$.*

PROOF. Firstly, parametrise S_0 in any way, i.e., choose a map h_0: $S^1 \to S_0$. I shall show this can be extended.

Let U_0, U_1, U_2, U_3 denote the four corners of S_0; see Figure 2. Define $B \subset \mathbf{R}^3$ by

$$B = [-\gamma_1, \gamma_1] \times [-\gamma_2, \gamma_2] \times [-\gamma_3, \gamma_3].$$

Let $B_i = U_i + B$; see Figure 2. Choose the γ_i linearly related so that J_F and J_B cross ∂B_i through faces parallel to the $u = 0$ plane and E_R^*, E_L^* cross through faces parallel to $w = 0$.

Let b_i, $i = 1, \ldots, 8$, be the successive intersection points of $\partial(B_0 \cup B_1 \cup B_2 \cup B_3)$ with S_0 starting at $J_F \cap B_0$ and proceeding in a counterclockwise direction. Set

$$M_F = \{(u, v, w): u = 0 \text{ and } v^2 + w^2 \leqslant \gamma_F\}.$$

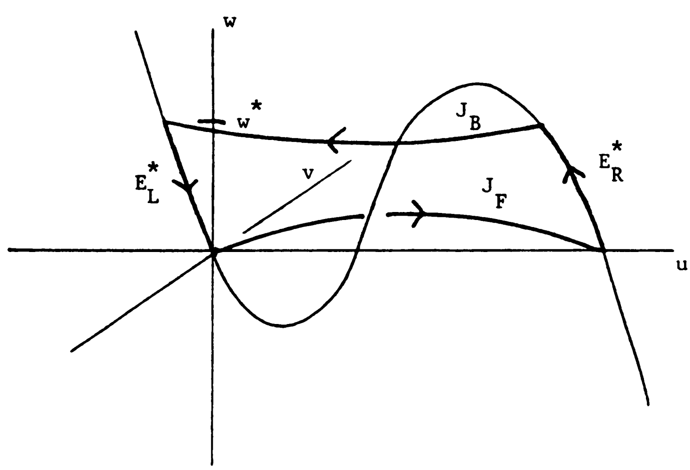


FIGURE 1

Now choose γ_F so that $b_1 + M_F \subset \partial B_0$ and $b_2 + M_F \subset \partial B_1$. Similarly, set

$$M_R = \{(u, v, w) : w = 0 \text{ and } u^2 + v^2 \leqslant \gamma_R\}.$$

Choose γ_R so that $b_3 + M_R \subset \partial B_1$ and $b_4 + M_R \subset \partial B_2$. Define M_B and M_L similarly to M_F and M_R, respectively; again choose γ_B and γ_L so that the obvious conditions are satisfied.

Let $\tilde{J}_F = J_F \setminus \{B_0 \cup B_1\}$; form a tube about $\tilde{J}_F$ by setting

$$N_F = \bigcup_{j \in \tilde{J}_F} (j + M_F).$$

Define N_R, N_B and N_L in the obvious fashion. Let

$$N = B_0 \cup N_F \cup B_1 \cup N_R \cup B_2 \cup N_B \cup B_3 \cup N_L.$$

N is a neighborhood of S_0 formed out of tubes joining boxes that cover each corner.

The size of the neighborhood is determined by γ_1, say, since each of the other γ's is related to it. Let $\kappa = \gamma_1$; then $N = N(\kappa)$ and, as $\kappa \to 0$, $N \to S_0$ as a set. Consequently, for fixed κ, there is an $\varepsilon_0 > 0$ so that $S_\varepsilon \subset N$ for all $\varepsilon \in [0, \varepsilon_0]$.

By the chosen parametrisation of S_0, $h \mid S^1 \times \{0\} = h_0$ is already defined. Now I shall extend h_0 to $S^1 \times [0, \varepsilon_0]$. Let $(\theta, \varepsilon) \in S^1 \times [0, \varepsilon_0]$; there are two cases to consider.

Case I. $h_0(\theta) \notin B_i$ for any i. Then $h_0(\theta) \in N_F \cup N_R \cup N_B \cup N_L$. Let $M_\theta = h_0(\theta) + M_F$ and set $h(\theta, \varepsilon) = S_\varepsilon \cap M_\theta$.

A priori, the right-hand side is just a set. But from the equation $u' = v$ in (2.1) it is clear that it contains just one point and so the map is well defined.

Case II. $h_0(\theta) \in B_i$. Consider B_1; the others are analogous. Form a rectangle M_θ in $\mathbf{R}^3$ as follows. Let $P_\theta =$ plane containing $h_0(\theta)$ and the line $u = u_1 - \gamma_1$, $w = \gamma_3$, where $U_1 = (u_1, v_1, w_1)$ is the corner point. Let $M_\theta = P_\theta \cap B_1$. Define $h(\theta, \varepsilon) = S_\varepsilon \cap M_\theta$.

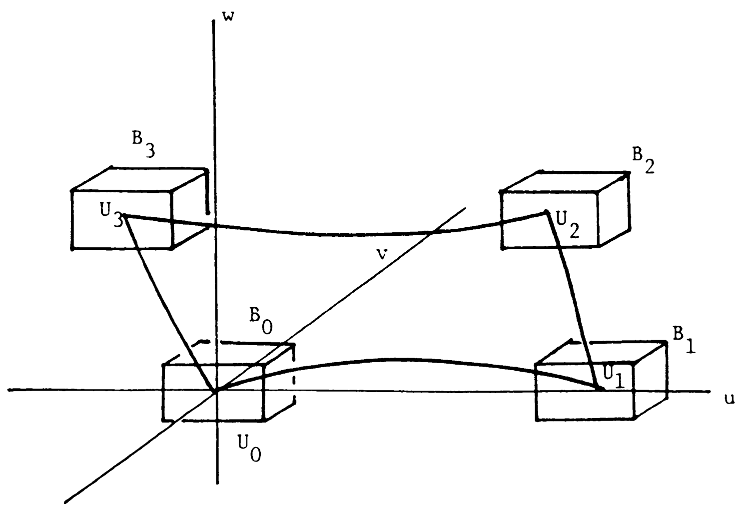


FIGURE 2

It is considerably harder to see that h is well defined in this case due to the subtlety of the behavior of the wave near the corners. B_0 is actually straightforward because of the approximation of the stable and unstable manifolds by the eigenspaces.

Let $h_0(\bar{\theta}_1) = b_2$ and $h_0(\bar{\theta}_2) = b_3$; these are the entrance and exit points of S_0 through B_1. I shall prove the following lemma.

LEMMA 2.1. *If κ is sufficiently small (and consequently ε_0), $S_\varepsilon \cap M_\theta$ contains a unique point for $\bar{\theta}_1 \leqslant \theta \leqslant \bar{\theta}_2$.*

PROOF. I shall divide this into two cases. Choose $\bar{\theta}$ so that $h_0(\bar{\theta}) \in J_F \cap B_1$ but $\bar{\theta} \neq \bar{\theta}_1$ and $h_0(\bar{\theta}) \neq U_1$. Let $m_\theta =$ slope of M_θ projected onto (u, w) space; see Figure 3. Reset γ_1 and γ_3, if necessary, so that $m_{\bar{\theta}} > f'(0) + \delta$ for some $\delta > 0$.

Let $n_\theta =$ normal to M_θ with a positive u component. It suffices to show that

$$n_\theta \cdot (u_\varepsilon'(\xi), v_\varepsilon'(\xi), w_\varepsilon'(\xi)) > 0 \tag{2.3}$$

for any $\theta \in [\bar{\theta}_1, \bar{\theta}_2]$ such that $(u_\varepsilon(\xi), v_\varepsilon(\xi), w_\varepsilon(\xi)) \in M_\theta$.

Case I. $\theta \in [\bar{\theta}_1, \bar{\theta}]$. Suppose (2.3) were violated for all $\kappa > 0$ with some θ in $[\bar{\theta}_1, \bar{\theta}]$; then there would be a sequence of points on S_ε as $\varepsilon \to 0$ for which (2.3) failed. These would converge to a point $h_0(\theta)$ on S_0. In fact, $h_0(\theta) \in J_F$ and, by continuity of the vector field (call it V),

$$n_\theta \cdot V(h_0(\theta)) \leqslant 0.$$

Since $h_0(\theta) \in J_F$ this is impossible unless $h_0(\theta) = U_1$, but it cannot be in Case I.

Case II. $\theta \in [\bar{\theta}, \bar{\theta}_2]$. To obtain information about the derivative along S_ε, consider the variational equations

$$\delta u' = \delta v, \quad \delta v' = -c\delta v - f'(u_\varepsilon)\delta u + \delta w, \quad \delta w' = -(\varepsilon/c)(\delta u - \gamma\delta w). \tag{2.4}$$

If ε is small and $U_\varepsilon \in B_1$, (2.4) is well approximated by the system linearised at U_1 with $\varepsilon = 0$;

$$\delta u' = \delta v, \quad \delta v' = -c\delta v - f'(u_1)\delta u + \delta w, \quad \delta w' = 0. \tag{2.5}$$

Because they are linear, both (2.4) and (2.5) induce flows on S^2 by equating two vectors in $\mathbf{R}^3 \setminus \{0\}$ if one is a positive multiple of the other. The flow of (2.5) is

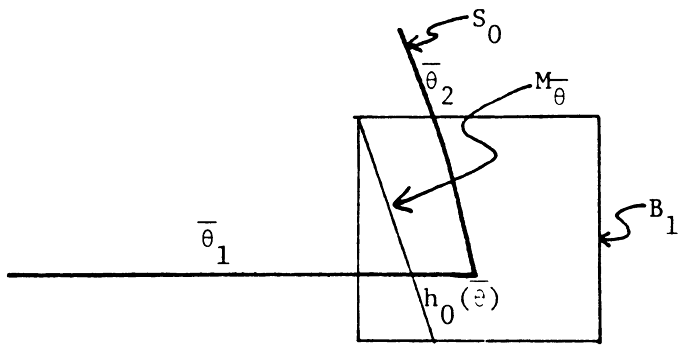


FIGURE 3

qualitatively the same as the linearisation at rest. It has one unstable subspace and two stable ones. Let these be spans of the eigenvectors X_1 (unstable), X_2 and X_3.

The associated flow on S^2 has two attracting critical points, two repelling ones, and two saddles; see Figure 4. These come from the eigenspaces. Let X_2 be the eigenvector that gives the saddle. Set $C = \text{span}\{X_2, X_3\} \cap S^2$ and let V be a given neighborhood of C in S^2.

If $\tilde{U}_\varepsilon(\xi) = U'_\varepsilon(\xi)/|U'_\varepsilon(\xi)|$, this satisfies the flow induced on S^2 from (2.4). If ε_0 is small enough, $\tilde{U}_\varepsilon(\xi) \in V$, while $U_\varepsilon(\xi) \in B_1$; otherwise, it would be driven to some neighborhood of $\text{span}\{X_1\} \cap S^2$, since these two points are attractors for the flow on S^2 derived from (2.5). If this happened, $U_\varepsilon(\xi)$ would leave B_1 other than through the top, which it does not.

Consider the vector $(u'_\varepsilon(\xi), w'_\varepsilon(\xi))$. Since $w'_\varepsilon(\xi) > 0$, if (2.3) were violated it is easy to check that $0 > w'_\varepsilon(\xi)/u'_\varepsilon(\xi) > f'(0) + \delta$ for $\theta \in [\bar{\theta}, \bar{\theta}_2]$. But this is impossible if ε_0 is small enough, because (2.5) is then well approximated by (2.4) and, inside C, $\text{span}\{X_1\} \cap S^2$ is a pair of attracting points. Moreover, $X_1 = (-1, 0, -f'(u_1))$. This completes the proof of the lemma.

Returning to the proof of Theorem 2.1, it is now known that $h(\theta, \varepsilon)$ is well defined. It is obviously one-to-one, since the M_θ's are all disjoint. Because $S^1 \times [0, \varepsilon_0]$ is compact, it remains to show that h is continuous. By Langer's proof, since it uses the implicit function theorem, h is continuous in ε for each θ. By continuity of the flow this is uniform in θ; full continuity therefore follows.

3. Essential spectrum and the definition of $D(\lambda)$. Firstly, I shall give the definitions used in splitting up the spectrum. Let B be a Banach space and $L: B \to B$ a linear operator.

DEFINITION. $\lambda \in \mathbf{C}$ is said to be in the normal spectrum, denoted $\sigma_n(L)$, if it is an isolated eigenvalue of finite multiplicity.

The essential spectrum, $\sigma_e(L)$, is the complement of this in $\sigma(L)$, i.e., $\sigma_e(L) = \sigma(L) \setminus \sigma_n(L)$.

Now let L be the linearised operator about the travelling wave given by (1.7) and let $B = \text{BC}(\mathbf{R}, \mathbf{R}^2)$. In this section I shall prove that $\sigma_e(L)$ is bounded away from the imaginary axis in the left half-plane. Also I shall define Evans' analytic function $D(\lambda)$, which is the tool for finding eigenvalues.

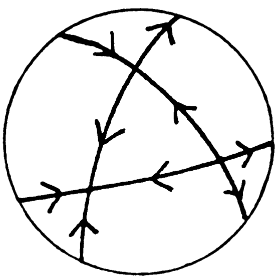


FIGURE 4

Consider the equation

$$(L - \lambda I)\binom{p}{r} = 0 \tag{3.1}$$

where $\binom{p}{r}(\xi) \in B_c$, complexified B. Rewrite (3.1) as a system

$$\begin{aligned} p' &= q, \\ q' &= -cq + (\lambda - f'(u))p + r, \\ r' &= -(\varepsilon/c)p + ((\lambda + \varepsilon\gamma)/c)r. \end{aligned} \tag{3.2}$$

I have dropped the ε on U_ε, so with a slight abuse of notation, $U(\xi) = (u(\xi), v(\xi), w(\xi))$ is the underlying travelling wave.

Let $z = (p, q, r) \in \mathbf{C}^3$ and write (3.2) as

$$z' = Az, \tag{3.3}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - f'(u) & -c & 1 \\ -\varepsilon/c & 0 & (\lambda + \varepsilon\gamma)/c \end{pmatrix}. \tag{3.4}$$

Equation (3.3) is a nonautonomous one on $\mathbf{C}^3$. As $\xi \to \pm\infty$, $u(\xi) \to 0$; therefore (3.3) is asymptotically autonomous and the asymptotic system is

$$z' = A_0 z, \tag{3.5}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ \lambda - f'(0) & -c & 1 \\ -\varepsilon/c & 0 & (\lambda + \varepsilon\gamma)/c \end{pmatrix}. \tag{3.6}$$

The set $S = \{\lambda \in \mathbf{C}: A_0 = A_0(\lambda)$ has an imaginary eigenvalue$\}$ will determine the necessary information about $\sigma_e(L)$.

Lemma 3.1. *If $\varepsilon > 0$, $\mathbf{C} \setminus S$ has a component G for which there exists an $a < 0$ such that $\{\lambda: \operatorname{Re} \lambda > a\} \subset G$.*

Proof. Let $P = P(\alpha, \varepsilon, \lambda) = \det(A_0 - \alpha I)$. Then

$$P = (\alpha^2 + c\alpha + f'(0) - \lambda)((\lambda + \varepsilon\gamma)/c - \alpha) - \varepsilon/c. \tag{3.7}$$

Fix $\varepsilon > 0$. S consists of those λ for which

$$P(\alpha, \varepsilon, \lambda) = 0 \tag{3.8}$$

for some $\alpha \in i\mathbf{R}$. If $\varepsilon = 0$, the set of λ's for which $P(i\tau, 0, \lambda) = 0$, for some $\tau \in \mathbf{R}$, is easily seen to be the imaginary axis union the parabola $\operatorname{Re} \lambda = -(\operatorname{Im} \lambda)^2/c^2 + f'(0)$.

For (3.8) the solutions λ will be near this curve and near the imaginary axis. The latter are the only ones to worry about. For fixed τ, at $\varepsilon = 0$,

$$\frac{d\lambda}{d\varepsilon} = -\frac{\partial P}{\partial \varepsilon} \Big/ \frac{\partial P}{\partial \lambda}.$$

Now

$$\partial P/\partial\lambda = (-\tau^2 + f'(0))/c,$$

since $f'(0) < 0$, $\partial P/\partial\lambda \neq 0$, and λ is a function of ε for fixed $\alpha = i\tau$ near $\varepsilon = 0$ such that $\lambda(0) \in i\mathbf{R}$. For each fixed τ this gives all λ's for which $A_0(\lambda)$ has an imaginary eigenvalue because (3.7) is quadratic in λ.

Since

$$d\lambda/d\varepsilon = -\left(\gamma + 1/\left(\tau^2 - f'(0)\right)\right) < 0,$$

the set of λ's near the imaginary axis lies in the left-hand plane. If $\gamma > 0$ is fixed, the curve thus defined is bounded uniformly away from the imaginary axis. This proves the lemma.

The point of this lemma is that there is no essential spectrum of L in G. Evans shows this for his more general class of problems in Theorem 3 of [**8**]. The idea is fairly standard and worth explaining briefly.

Set $L = L_0 + R$, where

$$L_0\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} p'' + cp' + f'(0)p + r \\ cp' + \varepsilon(p - \gamma r) \end{pmatrix} \tag{3.9}$$

and

$$R\begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} (f'(u) - f'(0))p \\ 0 \end{pmatrix}.$$

L_0 is the linearisation about the rest state and R is the perturbation due to the wave. $\sigma(L_0)$ has actually been found in Lemma 3.1. The equation $(L_0 - \lambda I)\binom{p}{r} = 0$ becomes $z' = A_0 z$ when rewritten as a system.

It is a standard computation to see that $\sigma(L_0) = S$. R is a relatively compact perturbation of L_0. It follows that any component of $\mathbf{C} \setminus S$ is entirely the essential spectrum, or the only spectrum in it is normal. Evans further shows that if $\lambda < 0$ and large, it is not an eigenvalue. It follows that the only spectrum in G is normal. This kind of argument establishes the following lemma.

LEMMA 3.2. $\sigma(L) \cap G \subset \sigma_n(L)$.

REMARKS. (1) $a = a(\varepsilon)$ and tends to 0 as $\varepsilon \to 0$, so there is not a right half-plane whose boundary is bounded to the left of the imaginary axis independently of ε.

(2) S contains a curve that is asymptotically vertical, thus preventing L and L_0 from being sectorial.

Lemmas 3.1 and 3.2 show that $\sigma_e(L)$ causes no problem for stability. Hence, I need only be concerned with locating eigenvalues. As stated earlier, this is done by defining an analytic function $D(\lambda)$ whose domain is G.

Consider again A_0, given by (3.6). I claim that if $\lambda \in G$, $A_0(\lambda)$ has only one eigenvalue of positive real part. It is easy to check this for $\varepsilon = 0$ from the definition of G. It therefore follows for $\varepsilon > 0$. Call this eigenvalue $\alpha^+ = \alpha^+(\lambda, \varepsilon)$. Its associated eigenvector can be written

$$X^+ = (1, \alpha^+, -\varepsilon/[c\alpha^+ - (\lambda + \varepsilon\gamma)]).$$

Since $P(\alpha, \varepsilon, \lambda) = 0$ simplifies as $\varepsilon \to 0$, α^+ can be given explicitly in the limit

$$\alpha^+(\lambda, 0) = \left(-c + \left(c^2 - 4(f'(0) - \lambda)\right)^{1/2}\right)/2.$$

In the following, assume $\varepsilon \neq 0$. I shall motivate the definition of $D(\lambda)$ by seeing what it means to look for an eigenvalue. An eigenvalue of L in G is a λ for which there is a solution of (3.2) that is bounded at $\pm\infty$. For it to be bounded at $-\infty$, it must be asymptotic to the unstable eigenspace.

By Evans **[9]** there is a unique solution $\zeta(\lambda, \xi)$ to (3.3) that satisfies

$$\zeta(\lambda, \xi) - X^+ e^{\alpha^+ \xi} \to 0$$

as $\xi \to -\infty$ faster than $e^{\operatorname{Re} \alpha^+ \xi}$. Furthermore, $\zeta(\lambda, \xi)$ is a $\mathbf{C}^3$-valued analytic function of $\lambda \in G$ for each fixed ξ.

This function $\zeta(\lambda, \xi)$ is therefore a candidate to be an eigenfunction and, up to a scalar multiple, it is the only one.

To see if it is bounded at $+\infty$, one uses the adjoint to (3.3),

$$z^{*\prime} = Bz^*, \tag{3.10}$$

where $B = -A^*$, so

$$B = \begin{pmatrix} 0 & f'(u) - \bar{\lambda} & \varepsilon/c \\ -1 & c & 0 \\ 0 & -1 & -(\bar{\lambda} + \varepsilon\gamma)/c \end{pmatrix}. \tag{3.11}$$

The asymptotic system for (3.10) is

$$z^{*\prime} = B_0 z^*, \tag{3.12}$$

where B_0 is the same as B but with u replaced by 0. Obviously $B_0 = -A_0^*$, and the eigenvalues of B_0 are the negatives of the complex conjugates of the eigenvalues of A_0. B_0 therefore has a unique eigenvalue of negative real part in G; call it $\beta^- = \beta^-(\lambda, \varepsilon) = -\bar{\alpha}^+$. Its associated eigenvector is

$$Y^- = \left(1, (c - \beta^-)^{-1}, \left[(\beta^- - c)\left(\beta^- + (\bar{\lambda} + \varepsilon\gamma)/c\right)\right]^{-1}\right).$$

(3.10) therefore has a unique solution $\eta(\lambda, \xi)$ satisfying

$$\eta(\lambda, \xi) - Y^- e^{\beta^- \xi} \to 0$$

as $\xi \to +\infty$ faster than $e^{\operatorname{Re} \beta^- \xi}$. Furthermore, $\eta(\lambda, \xi)$ is a $\mathbf{C}^3$-valued analytic function of λ for each fixed ξ.

DEFINITION. The function $D(\lambda) = \zeta(\lambda, \xi) \cdot \eta(\lambda, \xi)$.

One checks easily that this is well defined, i.e. independent of ξ:

$$\begin{aligned} \frac{\partial}{\partial \xi} D(\lambda) &= \frac{\partial}{\partial \xi} \zeta(\lambda, \xi) \cdot \eta(\lambda, \xi) + \zeta(\lambda, \xi) \cdot \frac{\partial}{\partial \xi} \eta(\lambda, \xi) \\ &= A\zeta \cdot \eta + \zeta \cdot B\eta = A\zeta \cdot \eta - \zeta \cdot A^* \eta \\ &= 0. \end{aligned}$$

I shall collect the important properties of $D(\lambda)$.

Properties of $D(\lambda)$. (1) $D\colon G \to \mathbf{C}$ is analytic.

(2) Zeroes of $D(\lambda)$ are eigenvalues of L.

(3) The order of a zero is equal to the algebraic multiplicity of the eigenvalue.

(1) follows from the fact that ζ and $\bar{\eta}$ are analytic functions of λ, into $\mathbf{C}^3$, for each fixed ξ. The reason for this can be seen from the proof of Lemma 3.3 below. (2) has a very pretty geometric interpretation. Since (3.3) is linear, its solution operator takes planes to planes (a plane being a two-dimensional complex subspace of $\mathbf{C}^3$). The information as to how this occurs is contained in the adjoint equation (3.10). In fact, the normal to a plane evolving under (3.3) will satisfy (3.10) if its complex amplitude is determined appropriately. The eigenvector Y^- is exactly the one that is normal to the stable subspace for (3.5). If $D(\lambda) = 0$ then, as $\xi \to +\infty$, $\zeta(\lambda, \xi)$ is perpendicular to Y^- and so is asymptotic to the stable subspace of (3.5). Therefore, $\zeta(\lambda, \xi) \to 0$ as $\xi \to +\infty$ and one has an eigenfunction; λ is therefore an eigenvalue. It is not hard to see that this is the only way a bounded, uniformly continuous solution of (3.3) can be found. (3) is somewhat more difficult to see and I refer to Evans **[9]**.

I shall need, in §5, an analytic continuation of $D(\lambda)$ to a right half-plane $\{\lambda\colon \operatorname{Re}\lambda > b\}$, where $b < 0$ and independent of ε. I shall prove this as a lemma which includes the proof of (1).

LEMMA 3.3. *There exists $b < 0$, independent of ε, and an analytic function $\tilde{D}(\lambda)$ on the set $\tilde{G} = \{\lambda\colon \operatorname{Re}\lambda > b\}$ so that $\tilde{D}|_G = D$.*

PROOF. It will be obvious from the construction that $\tilde{D}$ extends D. The problem with D is that the boundary of its domain G collapses onto the imaginary axis as $\varepsilon \to 0$. The proof is then to produce $\zeta(\lambda, \xi)$ and $\eta(\lambda, \xi)$, satisfying their respective defining conditions. This is possible on a set of the form $\tilde{G}$.

The eigenvalue $\alpha^+(\lambda, \varepsilon)$ can be extended to a set of the form $\tilde{G}$ for some $b < 0$ independent of ε. This cannot be done preserving the condition that α^+ is the unique eigenvalue of positive real part, but it can be done with α^+ the eigenvalue of largest real part.

In a strip $H = \{\lambda\colon b < \operatorname{Re}\lambda < 0\}$, if $\varepsilon = 0$ there are three distinguished eigenvalues:

$$\alpha^+ = \left(-c + \left(c^2 - 4(f'(0) - \lambda)\right)^{1/2}\right)/2, \quad \alpha^0 = \lambda/c,$$
$$\alpha^- = \left(-c - \left(c^2 - 4(f'(0) - \lambda)\right)^{1/2}\right)/2,$$

where a branch of the square root that is continuous near $\arg z = 0$ is being used. It is clear that b can be chosen so that $\operatorname{Re}\alpha^+ > \operatorname{Re}\alpha^0 > \operatorname{Re}\alpha^-$ for $\lambda \in H$. One checks easily that if $\varepsilon \ll 1$ and $\lambda \in H$ there are eigenvalues of A_0—$\alpha^+(\lambda, \varepsilon)$, $\alpha^0(\lambda, \varepsilon)$ and $\alpha^-(\lambda, \varepsilon)$—corresponding to each of these. Furthermore, $|\partial\alpha^+/\partial\varepsilon|$, $|\partial\alpha^0/\partial\varepsilon|$ and $|\partial\alpha^-/\partial\varepsilon|$ are bounded independently of $\lambda \in H$. It follows that $\varepsilon > 0$ can be chosen so that

$$\operatorname{Re}\alpha^+(\lambda, \varepsilon) > \max\left\{\operatorname{Re}\alpha^0(\lambda, \varepsilon), \operatorname{Re}\alpha^-(\lambda, \varepsilon)\right\}$$

for all $\lambda \in H$.

Under these conditions $\zeta(\lambda, \xi)$ can be constructed. The construction of $\eta(\lambda, \xi)$ is analogous with one added difficulty; see comment at end. The construction follows Evans.

Write (3.3) as

$$z' = A_0 z + P(\xi)z.$$

It is easy to check that there exists $C, k > 0$, so that

(3.13) $$\|P(\xi)\| < Ce^{k\xi} \quad \text{for } \xi < 0.$$

Define the iteration scheme:

(3.14) $$\zeta_0(\lambda, \xi) = X^+ e^{\alpha^+ \xi},$$

$$\zeta_n(\lambda, \xi) = X^+ e^{\alpha^+ \xi} + \int_{-\infty}^{\xi} \exp(A_0(\xi - s))\, P(s)\zeta_{n-1}(\lambda, s)\, ds.$$

That $\zeta_n(\lambda, \xi)$ is well defined for $\lambda \in \tilde{G}$ and an analytic function, for fixed ξ, is established inductively. The following estimates are shown to hold at the same time. Fix $\lambda_0 \in \tilde{G}$. There exists a neighborhood N of λ_0 and constants C_1, C_2 independent of n so that for $\xi \leqslant \xi^*$, some ξ^*,

$$(1) \quad |\zeta_n(\lambda, \xi)| \leqslant C_1 \exp(\tau^- \xi),$$

$$(2) \quad \left|\zeta_n(\lambda, \xi) - X^+ e^{\alpha^+ \xi}\right| \leqslant C_2 \exp(\tau^+ \xi),$$

where $\tau^- = \inf_{\lambda \in N} \{\operatorname{Re} \alpha^+(\lambda)\}$ and $\tau^+ = \sup_{\lambda \in N} \{\operatorname{Re} \alpha^+(\lambda)\}$. I shall drop mentioning the dependence of α^+ on ε.

The key point is that for $\lambda \in \tilde{G}$, $\alpha^+(\lambda)$ is the eigenvalue of largest real part. Consequently, $\|\exp(A_0(\lambda))\| \leqslant \exp(\operatorname{Re} \alpha^+(\lambda))$.

Choose N so that $\tau_- + k > \tau_+$. Suppose (1) hold up to $n - 1$; then

$$\left|\int_{-\infty}^{\xi} \exp(A_0(\xi - s))P(s)\zeta_{n-1}(\lambda, s)\, ds\right|$$
$$\leqslant CC_1 \int_{-\infty}^{\xi} \exp(\operatorname{Re} \alpha^+(\lambda)(\xi - s) + ks + \tau^- s)\, ds.$$

This integral converges uniformly for all $\lambda \in N$, which shows that $\zeta_n(\lambda, \xi)$ is well defined and analytic in $\lambda \in \tilde{G}$. By setting

$$C_2 = CC_1/(\tau^- + k - \tau^+),$$

(2) holds. To check (1):

$$|\delta_n(\lambda, \xi)| \leqslant |X^+| \exp(\operatorname{Re} \alpha^+ \xi) + C_2 \exp(\tau^+ \xi).$$

As long as C_2 is chosen larger than $|X^+|$, ξ^* can be picked so that

$$\left(|X^+| + C_2 \exp(\tau^+ - \tau^-)\xi\right) \leqslant C_2$$

for all $\xi \geqslant \xi^*$. It is trivial that (1) and (2) are satisfied for $n = 0$.

By a very similar inductive argument, N, ξ^* and C_3 can be found so that

$$\sup_{\lambda \in N} |\zeta_{n+1}(\lambda, \xi) - \zeta_n(\lambda, \xi)| \leqslant \frac{C_3}{2^n} \exp(\tau^+ \xi).$$

It follows that $\zeta_n(\lambda, \xi) \to \zeta(\lambda, \xi)$, for each fixed ξ, uniformly on compact subsets of $\tilde{G}$. $\zeta(\lambda, \xi)$ is therefore analytic and satisfies the integral equation

$$\zeta(\lambda, \xi) = X^+ \exp(\alpha^+ \xi) + \int_{-\infty}^{\xi} \exp(A_0(\xi - s))P(s)\zeta(\lambda, s)\, ds$$

and therefore satisfies (3.3).

By letting $n \to \infty$ in (2) above, $\zeta(\lambda, \xi)$ is seen to satisfy the defining condition of ζ in the set G.

The distinguished solution to the adjoint equation, $\eta(\lambda, \xi)$, is constructed in the same way. However, in this case $k \to 0$ as $\varepsilon \to 0$. The size of N will therefore depend on ε. For fixed $\varepsilon > 0$, the construction goes through and $\eta(\lambda, \xi)$ is analytic in $\tilde{G}$. Obviously η does not exist for $\varepsilon = 0$, although ζ does.

$\tilde{D}(\lambda)$ is then set as $\zeta(\lambda, \xi) \cdot \eta(\lambda, \xi)$.

4. Analysis of the reduced system. The zeroes of $D(\lambda)$ (or $\tilde{D}$) will be related to the eigenvalues of the reduced systems, that is, the linearisation of the PDE about the front or the back. I shall redevelop the theory of the preceding section for the reduced system. It is slightly different because the underlying wave is heteroclinic rather than homoclinic. The necessary information about the zeroes of the reduced D-function can then be given, as the stability is well understood in these cases; see Fife and McLeod [**13**].

I shall consider a system which is exactly the one for the front, but the analysis for the back only requires appropriate reinterpretation.

Consider the PDE (in a moving frame with speed c)

(4.1) $$u_t = u_{\xi\xi} + cu_\xi + f(u),$$

where $f(u)$ is given by (1.2). The travelling wave equation is

(4.2) $$u' = v, \qquad v' = -cv - f(u).$$

As mentioned in §2, there exists c^* so that (4.2) possesses a solution $(u_R(\xi), v_R(\xi))$ so that

$$(u_R, v_R) \to (0,0) \quad \text{as } \xi \to -\infty \qquad \text{and} \qquad (u_R, v_R) \to (1,0) \quad \text{as } \xi \to +\infty.$$

Linearise (4.1) about this wave:

(4.3) $$L_R p = p'' + cp' + f'(u_R)p.$$

Let $\sigma(L_R)$ be the spectrum of L_R relative to $B = \mathrm{BC}(\mathbf{R}, \mathbf{R})$. Write $(L_R - \lambda I)p = 0$ as a system

(4.4) $$p' = q, \qquad q' = -cq + (\lambda - f'(u_R))p.$$

This has an asymptotic system at $-\infty$,

(4.5) $$p' = q, \qquad q' = -cq + (\lambda - f'(0))p,$$

which I write as

$$z' = M_0 z.$$

Set $S_0 = \{\lambda: M_0(\lambda) \text{ has an imaginary eigenvalue}\}$.

There is an analogous picture at $+\infty$, where 0 is replaced by 1 throughout the above. Then $\mathbf{C} \setminus S_0 \cup S_1$ has a component G_{R}, containing the right half-plane, so that $\sigma(L_{\mathrm{R}}) \cap G_{\mathrm{R}} \subset \sigma_n(L_{\mathrm{R}})$.

The function that plays the $D(\lambda)$ role can now be formulated. M_0 has eigenvalues and associated eigenvectors:

$$\mu^+ \quad (\operatorname{Re}\mu^+ > 0) \quad X_{\mathrm{R}}^+,$$
$$\mu^- \quad (\operatorname{Re}\mu^- < 0) \quad X_{\mathrm{R}}^+,$$

where

$$\mu^{\pm} = \left\{-c \pm \left(c^2 + 4(f'(0) - \lambda)^{1/2}\right)\right\}/2, \qquad X_{\mathrm{R}}^{\pm} = (1, \mu^{\pm}).$$

Write (4.4) as $z' = Mz$. Then for $\lambda \in G_{\mathrm{R}}$, there is a unique solution of $z' = Mz$ so that

$$\zeta_R(\lambda, \xi) - X_{\mathrm{R}}^+ e^{\mu^+ \xi} \to 0 \quad \text{as } \xi \to -\infty$$

faster than $e^{\operatorname{Re}\mu^+ \xi}$, and $\zeta_{\mathrm{R}}(\lambda, \xi)$ is analytic in λ.

The adjoint equation is

$$z^{*\prime} = Nz^*,$$

where $N = -M^*$, and this has an asymptotic system

(4.6) $$z^{*\prime} = N_1 z^*,$$

where $N_1 = -M_1^*$. N_1 has eigenvalues and eigenvectors:

$$\nu^+ = -\bar{\mu}^+, \quad (\operatorname{Re}\nu^+ > 0) \quad Y_{\mathrm{R}}^+ = \left(1, (c - \nu^+)^{-1}\right),$$
$$\nu^- = -\bar{\mu}^+, \quad (\operatorname{Re}\nu^- < 0) \quad Y_{\mathrm{R}}^- = \left(1, (c - \nu^-)^{-1}\right).$$

Also there is a unique solution of (4.6), $\eta_{\mathrm{R}}(\lambda, \xi)$, so that

$$\eta_{\mathrm{R}}(\lambda, \xi) - Y_{\mathrm{R}}^- e^{\nu^- \xi} \to 0 \quad \text{as } \xi \to +\infty$$

faster than $e^{\operatorname{Re}\nu^- \xi}$, and it is analytic in λ. $D_{\mathrm{R}}(\lambda)$ is then defined as $\zeta_{\mathrm{R}}(\lambda, \xi) \cdot \eta_{\mathrm{R}}(\lambda, \xi)$. It has domain G_{R}.

The stability of the travelling wave is well understood; see Fife and McLeod [**13**]. I shall translate the known facts into properties of $D_{\mathrm{R}}(\lambda)$. This may seem to be backwards, but it is through $D_{\mathrm{R}}(\lambda)$ that these known results will be used.

Facts about $D_{\mathrm{R}}(\lambda)$. (1) $D_{\mathrm{R}}(0) = 0$.

(2) $D_{\mathrm{R}}(\lambda) \neq 0$ for $\lambda \in \{\lambda\colon \operatorname{Re}\lambda > d\}$, some $d < 0$, except at 0.

(3) $(d/d\lambda) D_{\mathrm{R}}(\lambda)|_{\lambda=0} > 0$.

(1) follows from the standard feature of translation of waves. By a maximum principle argument, 0 is the eigenvalue of largest real part and there are only finitely many eigenvalues in G_{R}; (2) therefore follows. I shall prove (3) directly in the following lemma.

LEMMA 4.1. $(d/d\lambda) D_{\mathrm{R}}(\lambda)|_{\lambda=0} > 0$.

Proof. To compute $(d/d\lambda)D_R(\lambda)$ at $\lambda = 0$, it suffices to consider λ real. So $\zeta_R(\lambda, \xi) \in \mathbf{R}^2$ and $\eta_R(\lambda, \xi) \in \mathbf{R}^2$. Let $\zeta_R = (r_R, \theta_R)$ and $\eta_R = (r_R^*, \theta_R^*)$ in polar coordinates on the plane. Then

$$D_R(\lambda) = r_R r_R^* \cos(\theta_R - \theta_R^*).$$

Since $D_R(0) = 0$, one computes

$$\frac{d}{d\lambda} D_R(\lambda) = -r_R r_R^* \sin(\theta_R - \theta_R^*)\left\{\frac{\partial}{\partial\lambda}\theta_R - \frac{\partial}{\partial\lambda}\theta_R^*\right\}$$

where the right-hand side can be evaluated at any ξ. Since $\theta_R - \theta_R^* = -\pi/2$, at $\lambda = 0$

$$\frac{d}{d\lambda} D_R(\lambda) = -r_R r_R^* \left\{\frac{\partial}{\partial\lambda}\theta_R - \frac{\partial}{\partial\lambda}\theta_R^*\right\}.$$

From (4.6)

$$\theta_R^* \to \arctan\left(\frac{1}{c - \nu_-}\right) \quad \text{as } \xi \to +\infty$$

and

$$\frac{\partial}{\partial\lambda}\left\{\arctan\frac{1}{c - \nu_-}\right\} = \frac{\partial}{\partial\lambda}\left(\frac{\nu}{(c - \nu)^2 + 1}\right) < 0$$

since $\partial\nu_-/\partial\lambda < 0$.

It follows that

$$\lim_{\xi \to +\infty} \frac{\partial}{\partial\lambda}\theta_R^* \leqslant 0.$$

To check the term involving $\partial\theta_R/\partial\lambda$, one computes

(4.7) $$\theta_R' = -c\sin\theta_R\cos\theta_R + (\lambda - f'(u_R))\cos^2\theta_R - \sin^2\theta_R$$

for θ_R as a function of ξ. From (4.7), if $\partial\theta_R/\partial\lambda = 0$,

(4.8) $$\{\partial\theta_R/\partial\lambda\}' = \cos^2\theta_R > 0.$$

By the same kind of argument as above,

$$\lim_{\xi \to -\infty} \frac{\partial}{\partial\lambda}\theta_R \geqslant 0,$$

and so, from (4.7) and (4.8),

$$\frac{\partial}{\partial\lambda}\theta_R > 0 \quad \text{for all } \xi, \qquad \lim_{\xi \to +\infty} \inf \frac{\partial}{\partial\lambda}\theta_R > 0.$$

It follows that $\partial\theta_R/\partial\lambda > 0$, as desired.

If $\varepsilon = 0$ and $w = 0$ in (2.1), one obtains the system (4.2) coupled with the equation $w' = 0$. J_F is a trajectory of (2.1) in this invariant plane. The system restricted to this plane fits exactly into the form described in this section.

If $\varepsilon = 0$ but $w = w^*$ (see §2), the equation for the back trajectory is obtained; it is

(4.9) $$u' = v, \qquad v' = -cv - f(u) + w_*, \qquad w' = 0.$$

If the third coordinate is dropped, one obtains a system that is analogous to the above reduced system. The nonlinearity $f(u) - w^*$ has the graph given in Figure 5. The situation has merely been reversed; the right and left critical points have their roles interchanged. An analytic function is then defined for which the properties given for $D_R(\lambda)$ still hold.

5. Approximate location of eigenvalues. In this section I shall prove that any points in $\sigma(L) \cap G$ must lie close to eigenvalues of one of the reduced systems. It then follows that the only dangerous eigenvalues are close to zero.

The trajectories J_F and J_B described in §2 are, respectively, travelling waves for the PDEs

$$u_t = u_{\xi\xi} + c^* u_\xi + f(u), \quad u_t = u_{\xi\xi} + c^* u_\xi + f(u) - w^*.$$

As solutions to these equations call them $u_F(\xi)$ and $u_B(\xi)$, fixing some point at $\xi = 0$, say $u_F(0) = a$ and $u_B(0) = 0$. Let L_F and L_B be the linearised operators about these solutions. Let $\sigma_F = \sigma(L_F)$ and $\sigma_B = \sigma(L_B)$ relative to $B = \mathrm{BC}(\mathbf{R}, \mathbf{R})$.

Recall that $\tilde{G} = \{\lambda\colon \operatorname{Re}\lambda > b\}$. It is obvious that b can be chosen so that $\tilde{G} \subset G_F \cap G_B$, where G_F = domain of D_F, G_B = domain of D_B, and D_F and D_B are the analytic functions for the front and back as given in §4.

Let $V = V_\delta$ = union of open balls of radius δ about each point in $(\sigma_F \cup \sigma_B) \cap \tilde{G}$. This section is devoted to proving the following theorem.

THEOREM 5.1. *Given $\delta > 0$ there exists $\varepsilon_0 > 0$ so that if $\varepsilon \in (0, \varepsilon_0]$, $D(\lambda) \neq 0$ for $\lambda \in G \setminus V_\delta$.*

COROLLARY 5.1. *If $\lambda \in G \setminus V$ then λ is not an eigenvalue of L.*

The idea of the proof of Theorem 5.1 is to follow $\zeta(\lambda, \xi)$ until ξ is very large and then evaluate $D(\lambda)$. If ξ is large enough at the evaluation point, $\eta(\lambda, \xi)$ is essentially determined there.

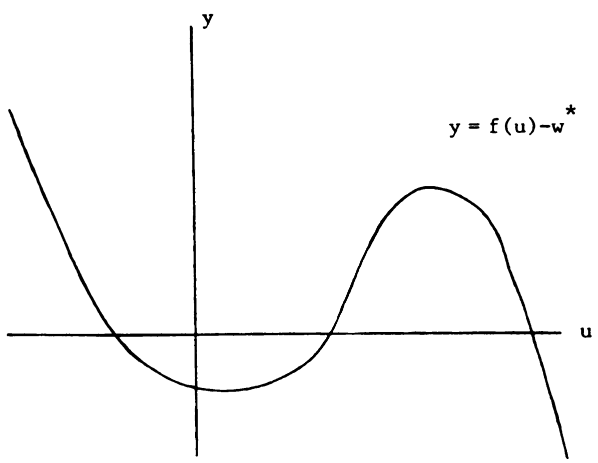


FIGURE 5

Following $\zeta(\lambda, \xi)$ as ξ varies can be thought of as following it "around" the travelling wave. $\zeta(\lambda, \xi)$ satisfies (3.2), its dependence on ξ is through $u(\xi)$, the first component of the travelling wave $U(\xi)$. If a copy of $\mathbf{C}^3$ is attached to each point of the orbit (S_ε), $\zeta(\lambda, \xi)$ lies in that copy if $U(\xi)$ is the underlying point.

To make this more precise, couple the travelling wave system (2.1) with the eigenvalue system (3.2),

$$(5.1) \qquad \begin{aligned} &u' = v, \quad v' = -cv - f(u) + w, \quad w' -(\varepsilon/c)(u - \gamma w), \quad p' = q, \\ &q' = -cq + (\lambda - f'(u))p + r, \quad r' = -(\varepsilon/c)p + [(\lambda + \varepsilon\gamma)/c]r, \end{aligned}$$

where $U = (u, v, w) \in \mathbf{R}^3$ and $z = (p, q, r) \in \mathbf{C}^3$. The natural setting for (5.1) is the complexified tangent bundle to $\mathbf{R}^3$, denoted $T_c\mathbf{R}^3$. This is isomorphic to $\mathbf{R}^3 \times \mathbf{C}^3$. (5.1) induces a flow on $T_c\mathbf{R}^3$ that depends continuously on $(\lambda, c, \varepsilon) \in \mathbf{C} \times \mathbf{R} \times \mathbf{R}$.

The travelling wave for $\varepsilon \neq 0$ is denoted $S_\varepsilon \subset \mathbf{R}^3$, with $c = \bar{c}(\varepsilon)$. If the flow above is restricted to S_ε, we obtain a flow on $S_\varepsilon \times \mathbf{C}^3$, the component on $\mathbf{C}^3$ coming from (3.2). This flow depends on $\lambda \in \mathbf{C}$ and is defined for $\varepsilon \in [0, \varepsilon_0]$.

$\zeta(\lambda, \xi)$ will be followed around $S_\varepsilon \times \mathbf{C}^3$ as a trajectory for this flow, i.e. $(u(\xi), \zeta(\lambda, \xi))$ will be followed as $U(\xi)$ goes around S_ε.

Not all of the information in $\zeta(\lambda, \xi)$ will be necessary to deduce $D(\lambda) \neq 0$. In fact, only its "direction" is important; the appropriate context is the projectivised space.

Since (5.1) is linear in $z \in \mathbf{C}^3$, the flow can be projectivised in the second component. Using coordinates $(U, z) \in T_c\mathbf{R}^3$, $PT_c\mathbf{R}^3 = \mathbf{R}^3 \times \mathbf{C}^3 \setminus \{0\}/\sim$, where $(U_1, z_1) \sim (U_2, z_2)$ if $U_1 = U_2$ and there exists an $\alpha \in \mathbf{C}$ so that $z_1 = \alpha z_2$.

Clearly $PT_c\mathbf{R}^3 \simeq \mathbf{R}^3 \times \mathbf{CP}^2$, where $\mathbf{CP}^2$ is two-dimensional complex projective space. Let π: $\mathbf{C}^3 \to \mathbf{CP}^2$ be the natural map; $\pi(z)$ is the equivalence class determined by z, which is $\operatorname{span}_{\mathbf{C}}\{z\} \setminus \{0\}$. I shall use the notation $\hat{z} = \pi(z)$. Extend π to $PT_c\mathbf{R}^3$:

$$\pi: T_c\mathbf{R}^3 \to PT_c\mathbf{R}^3, \qquad (U, z) \to (U, \hat{z}).$$

Since (5.1) is linear, it induces a flow on $PT_c\mathbf{R}^3$. If $F(\xi)$ is the time-ξ map of (5.1), then $\hat{F}(\xi)$, the time-ξ map of the projectivized flow is the unique map for which the diagram

$$\begin{array}{ccc} T_c\mathbf{R}^3 & \xrightarrow{F(\xi)} & T_c\mathbf{R}^3 \\ \pi \downarrow & & \downarrow \pi \\ PT_c\mathbf{R}^3 & \xrightarrow{\hat{F}(\xi)} & PT_c\mathbf{R}^3 \end{array}$$

commutes. All continuous dependence on parameters is inherited by $\hat{F}(\xi)$.

$S_\varepsilon \times \mathbf{CP}^2$ is an invariant subspace of $PT_c\mathbf{R}^3$ for fixed ε. There is therefore a (global, since the space is compact) flow on $S_\varepsilon \times \mathbf{CP}^2$ depending on $\lambda \in \mathbf{C}$. Using Theorem 2.1 the flow on $\bigcup S_\varepsilon \times \mathbf{CP}^2$, where the union is over $\varepsilon \in [0, \varepsilon_0]$, can be considered to lie on $S^1 \times [0, \varepsilon_0] \times \mathbf{CP}^2$.

Local coordinates can be put on $\mathbf{CP}^2$ in the following way. Let $z = (p, q, r) \in \mathbf{C}^3$ and $\pi(z) \in \mathbf{CP}^2$. If $p \neq 0$, then $\pi(z)$ is given by the coordinates $(q/p, r/p)$. This is obviously independent of which point in $\pi^{-1}(\pi(z))$ is used. Each of the other components can be used to get other local coordinate systems, but I shall always use the above.

For each $\hat{z} \in \mathbf{CP}^2$, so that $\hat{z} = (q/p, r/p)$, there is a distinguished vector in $\mathbf{C}^3$, call it $\tilde{z} = (1, q/p, r/p)$, so that $\pi(\tilde{z}) = \pi(z)$. In other words $\tilde{z}$ is a normalised version of z.

The adjoint system (3.10) can be dealt with similarly. The natural phase space here is the projectivised, complexified cotangent bundle $PT_c^*\mathbf{R}^3$! I shall not use this at all, however.

Now let $\zeta(\lambda, \xi)$ and $\eta(\lambda, \xi)$ have their usual meanings. Suppose at a certain value of ξ, $\zeta(\lambda, \xi)$ and $\eta(\lambda, \xi)$ are well defined. It is clear that if $\tilde{\zeta}(\lambda, \xi) \cdot \tilde{\eta}(\lambda, \xi) \neq 0$ then $\zeta(\lambda, \xi) \cdot \eta(\lambda, \xi) \neq 0$. This proves the following lemma.

LEMMA 5.1. *If $\lambda \in \tilde{G}$ and there exists a $\xi \in \mathbf{R}$, so that $\tilde{\zeta}(\lambda, \xi) \cdot \tilde{\eta}(\lambda, \xi) \neq 0$, then $\tilde{D}(\lambda) \neq 0$.*

The asymptotic systems are constant coefficient linear systems on $\mathbf{C}^3$. By projectivising such an autonomous linear flow, one obtains a flow on $\mathbf{CP}^2$. The following special considerations apply.

Let $A\colon \mathbf{C}^3 \to \mathbf{C}^3$ be linear. Then A induces a vector field on $\mathbf{CP}^2$ as follows:

$$
\begin{array}{ccc}
\mathbf{C}^3 & \overset{(\mathrm{id}, A)}{\longrightarrow} & T\mathbf{C}^3 \\
\pi \downarrow & & \downarrow D\pi \\
\mathbf{CP}^2 & \overset{g}{\longrightarrow} & T\mathbf{CP}^2
\end{array}
\tag{5.2}
$$

I shall call it g.

Let C_α be a one-dimensional eigenspace for A associated with an eigenvalue α. $\pi(C_\alpha)$ is then a critical point for the flow on $\mathbf{CP}^2$. If one linearises g at $\pi(C_\alpha)$, $Dg(\pi(C_\alpha))$ can be considered as a linear map on $\mathbf{C}^2$. I shall prove the following lemma.

LEMMA 5.2. *If A has eigenvalues α, α_1, α_2, and α is simple, then $Dg(\pi(C_\alpha))\colon \mathbf{C}^2 \to \mathbf{C}^2$ has eigenvalues $\alpha_1 - \alpha$ and $\alpha_2 - \alpha$.*

PROOF. Suppose $\alpha_1 \neq \alpha_2$. Then choose a basis for $\mathbf{C}^3$ in which A becomes

$$
\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}.
$$

Let (p, q, r) be the coordinates in this basis. Then, near $\pi(C_\alpha)$, $(q/p, r/p)$ are local coordinates on $\mathbf{CP}^2$. Consider $D\pi\colon T\mathbf{C}^3 \to T\mathbf{CP}^2$; let $((p, q, r), (z_1, z_2, z_3))$ be a generic point in $T\mathbf{C}^3$. Then

$$
D\pi((p, q, r), (z_1, z_2, z_3)) = \left(\left(\frac{q}{p}, \frac{r}{p} \right), \left(\frac{z_2}{p} - \left(\frac{q}{p} \right) \frac{z_1}{p}, \frac{z_3}{p} - \left(\frac{r}{p} \right) \frac{z_1}{p} \right) \right)
$$

from the quotient rule. Chasing a point around (5.2):

$$\begin{array}{ccc} (p,q,r) & \to & ((p,q,r),(\alpha p,\alpha_1 q,\alpha_2 r)) \\ \downarrow & & \downarrow \\ \left(\frac{q}{p},\frac{r}{p}\right) & \to & \left(\left(\frac{q}{p},\frac{r}{p}\right),\left((\alpha_1-\alpha)\frac{q}{p},(\alpha_2-\alpha)\frac{r}{p}\right)\right) \end{array}$$

In these coordinates the second component is linear and is therefore $Dg(\pi(C_\alpha))$. As a matrix it has the form

$$\begin{pmatrix} \alpha_1-\alpha & 0 \\ 0 & \alpha_2-\alpha \end{pmatrix}$$

and the lemma holds for this case. If $\alpha_1 = \alpha_2$ and the geometric multiplicity is only one, $Dg(\pi(C_\alpha))$ would be

$$\begin{pmatrix} \alpha_1-\alpha & 0 \\ 1 & \alpha_2-\alpha \end{pmatrix}$$

and the result still holds.

As an application of this lemma, consider the linear equation with constant coefficients $z' = Az$. If α is the eigenvalue of A of largest real part, then $C_\alpha \in \mathbf{CP}^2$ is an attracting critical point. Similarly, if α were of smallest real part, C_α would be a repelling critical point. Paraphrasing this, one can say that unstable subspaces become stable critical points and stable subspaces become unstable critical points.

To prove Theorem 5.1, I shall divide $G \setminus V$ into two sets:

(1) $G_1 = \{\lambda : \lambda \in G \setminus V \text{ and } |\lambda| > k\}$ for some fixed $k > 0$.

(2) $G_2 = \{G \setminus V\} \setminus G_1$.

Evans [9] proves an asymptotic estimate for $|\lambda| \to +\infty$ that shows if $\lambda \in G_1$ for some $k > 0$, then λ is not an eigenvalue, i.e., $D(\lambda) \neq 0$.

The main task then is to prove that for $\lambda \in G_2$, $D(\lambda) \neq 0$. This will be proved for any k. $\hat{\zeta}(\lambda, \xi)$ will be followed around S_ε, and then at large ξ, $\hat{\zeta}(\lambda, \xi)$ will be used to determine $\tilde{\zeta}(\lambda, \xi)$. $\tilde{\zeta} \cdot \tilde{\eta}$ will then be proved to be nonzero so, by Lemma 5.1, λ could not be an eigenvalue.

I shall actually restrict λ to a larger set than G_2. Let

$$\Omega = \{\lambda \in \mathrm{cl}(\tilde{G}) : \lambda \notin V \text{ and } |\lambda| \leqslant k\}.$$

Then $G_2 \subset \Omega$ and Ω is compact. It follows from Lemma 3.3 that $\zeta(\lambda, \xi)$, $\eta(\lambda, \xi)$ and $D(\lambda)$ are all defined in Ω and analytic in $\mathrm{int}(\Omega)$.

There are various flows I shall want to consider, depending on how many parameters are fixed. As stated earlier, the full equations (5.1) induce a flow on $\bigcup_{0 \leqslant \varepsilon \leqslant \varepsilon_0} S_\varepsilon \times \mathbf{CP}^2$, where ε_0 satisfies all the requirements collected to date. Using h: $S^1 \times [0, \varepsilon_0] \to \bigcup S_\varepsilon$, from Theorem 2.1, there is a flow on $S^1 \times [0, \varepsilon_0] \times \mathbf{CP}^2$.

With the parameter λ set by the flow, there is a flow on $S^1 \times [0, \varepsilon_0] \times \mathbf{CP}^2 \times \Omega$. Call this flow $H(t)$. If λ is fixed, let $H^\lambda(t)$ be the flow on $S^1 \times [0, \varepsilon_0] \times \mathbf{CP}^2$. $H_\varepsilon(t)$ and $H_\varepsilon^\lambda(t)$ then have the obvious meaning.

Control on $\hat{\zeta}(\lambda, \xi)$ will be afforded by proving certain properties for the flow $H_0(t)$ on $S^1 \times \mathrm{CP}^2$ and then perturbing this information to $H_\varepsilon(t)$.

Recall the construction of a tubular neighborhood about the singular orbit S_0 given in the proof of Theorem 2.1. The corner boxes are B_0, B_1, B_2, B_3. Let the following points b_i, $i = 0, 1, 2, 3$, be the indicated crossing points of S_0 with the respective boxes:

$$b_0 \in S_0 \cap \partial B_0 \cap \{u = \gamma_1\}, \qquad b_1 \in S_0 \cap \partial B_1 \cap \{u = u_1 - \gamma_1\},$$
$$b_2 \in S_0 \cap \partial B_2 \cap \{u = u_2 - \gamma_1\}, \quad b_3 \in S_0 \cap \partial B_3 \cap \{u = u_3 + \gamma_1\}.$$

All notation is defined in §2; see the proof of Theorem 2.1. Recall that $h_0 = h|S^1 \times \{0\}$. Let θ_i be given by the condition

$$h_0(\theta_i) = b_i, \qquad i = 0, 1, 2, 3.$$

Recall further from §3 that U_0, U_1, U_2 and U_3 denote the four corners of S_0; see Figure 2. Let $\bar{\theta}_i$ be determined by the conditions

$$h_0(\theta_i) = U_i, \qquad i = 0, 1, 2, 3.$$

The first property of the H_0 flow is that it possesses a certain attractor that sits over the right-hand slow manifold in the construction of S_0.

The attractor will be a set of points of the form $(\theta, \hat{X}(\theta, \lambda), \lambda) \in S^1 \times \mathrm{CP}^2 \times \Omega$, where $\theta \in [\bar{\theta}_1, \theta_2]$. It will then be

$$K = \bigcup_{\bar{\theta}_1 \leqslant \theta \leqslant \theta_2} \bigcup_{\lambda \in \Omega} (\theta, \hat{X}(\theta, \lambda), \lambda). \tag{5.3}$$

In order to describe $\hat{X}(\theta, \lambda)$, let $u_\theta = u$-component of $h_0(\theta) \in S_0 \subset \mathbf{R}^3$. $\{\theta\} \times \mathrm{CP}^2$ is an invariant subset of $S^1 \times \mathrm{CP}^2$ under $H_0(t)$ if $\bar{\theta}_1 \leqslant \theta \leqslant \bar{\theta}_2$. The flow on CP^2 is the projectivised version of

$$p' = q, \qquad q' = -cq + (\lambda - f'(u_\theta))_p, \qquad r' = 0. \tag{5.4}$$

b can be chosen, to set $\tilde{G}$, so that (5.4) has a unique simple eigenvalue of largest real part for each (θ, λ); call it $\alpha^+(\theta, \lambda)$. Call some associated eigenvector $X^+(\theta, \lambda)$. $\hat{X}(\theta, \lambda)$ is then set as $\hat{X}^+(\theta, \lambda)$.

The set

$$K_1 = \bigcup_{\bar{\theta}_1 \leqslant \theta \leqslant \bar{\theta}_2} \bigcup_{\lambda \in \Omega} (\theta, \hat{X}(\theta, \lambda), \lambda)$$

will form part of the attractor. It needs to be extended to θ_2, i.e., away from the corner to the edge of B_2.

Let $\theta \in [\bar{\theta}_2, \theta_2]$. Then $h_0(\theta) \in J_{\mathrm{B}}$. J_{B} is parametrised by $\xi \in \mathbf{R}$ and given by $(u_{\mathrm{B}}(\xi), u'_{\mathrm{B}}(\xi), w^*)$. From §4 there exists a uniquely determined, up to normalisation, solution for the linearised eigenvalue equations over the back; call this $\zeta_{\mathrm{B}}(\lambda, \xi)$. Now set $\hat{X}(\theta, \lambda) = \hat{\zeta}_{\mathrm{B}}(\lambda, \xi)$, where θ and ξ are related by the condition $\theta_{\mathrm{B}}(\xi) = \theta$.

Now $\hat{X}(\theta, \lambda)$ is defined for all $\theta \in [\bar{\theta}_1, \theta_2]$. The attractor is then the set K given by (5.3). I need to show that K is an attractor in some suitable sense.

LEMMA 5.3. *K is an attractor for the flow $H_0(t)$ relative to the set*

$$[0, \theta_2] \times \mathrm{CP}^2 \times \Omega = F. \tag{5.5}$$

In other words, there is a neighborhood Q of K in F so that $\omega(Q) \cap F = K$.

REMARK. θ_2 depends on the size of the tubular neighborhood, i.e. κ. Lemma 5.3 holds for all sufficiently small κ.

PROOF. First consider the set K_1. Since $X^+(\theta, \lambda)$ is an eigenvector for the eigenvalue of largest real part, it follows from Lemma 5.2 that $(\theta, \hat{X}^+(\theta, \lambda))$ is an attractor in $\{\theta\} \times \mathbf{CP}^2$ for each fixed λ.

To show that K_1 is an attractor it suffices to show that the rate of convergence to $(\theta, \hat{X}(\theta, \lambda))$ is bounded away from zero uniformly in $\theta \in [\bar{\theta}_1, \bar{\theta}_2]$ and $\lambda \in \Omega$. $\alpha^+(\theta, \lambda)$ depends continuously on θ and λ; moreover, the rate of convergence to the attracting point $(\theta, \hat{X}^+(\theta, \lambda))$ in $\{\theta\} \times \mathbf{CP}^2$ is determined by the quantity

$$\operatorname{Re}\big(\alpha^+(\theta, \lambda) - \alpha^0(\theta, \lambda)\big), \tag{5.6}$$

where $\alpha^0(\theta, \lambda)$ is the eigenvalue of next largest real part. In the proof of Lemma 3.3, it is shown that (5.6) is bounded away from zero and positive if $\lambda \in \tilde{G}$. (5.4) is the same as (3.5) except that 0 is replaced by u_θ. But there is an $a < 0$ so that $f'(u_\theta) < a < 0$ for all $\theta \in [\bar{\theta}_1, \bar{\theta}_2]$, so it is clear that this also holds here. By compactness of $[\bar{\theta}_1, \bar{\theta}_2] \times \Omega$, (5.6) can be uniformly bounded away from zero.

It follows that K_1 is an attractor relative to the set

$$\big[\bar{\theta}_1, \bar{\theta}_2\big] \times \mathbf{CP}^2 \times \Omega. \tag{5.7}$$

I claim that it is, in fact, an attractor in the set

$$\big[0, \bar{\theta}_2\big] \times \mathbf{CP}^2 \times \Omega. \tag{5.8}$$

Relative to $[0, \bar{\theta}_1] \times \mathbf{CP}^2 \times \Omega$, the invariant set $\bar{\theta}_1 \times \mathbf{CP}^2 \times \Omega$ is itself an attractor. This is trivial because the underlying flow on $(0, \bar{\theta}_1)$ is just the front solution (see Figure 6) and increases to $\bar{\theta}_1$. It can then also be said that (5.7) is an attractor relative to the set of (5.8). K_1 is therefore an attractor within an attracting invariant set and so is an attractor in (5.8).

The full attractor K is K_1 with a piece put on the tail. It suffices to show that the tail

$$K_2 = \bigcup_{\bar{\theta}_2 \leqslant \lambda \leqslant \theta_2} \bigcup_{\lambda \in \Omega} (\theta, \hat{X}(\theta, \lambda), \lambda)$$

is an attractor relative to the set

$$\big[\bar{\theta}_2, \theta_2\big] \times \mathbf{CP}^2 \times \Omega. \tag{5.9}$$

To this end, consider the flow induced on $\mathbf{R}^3 \times \mathbf{CP}^2$ from (5.1). Appending λ, there is a flow on $\mathbf{R}^3 \times \mathbf{CP}^2 \times \Omega$. The point $k_2(\lambda) = (U_2, \hat{X}^+(\bar{\theta}_2, \lambda), \lambda)$ is a critical point for each fixed λ. The linearisation in $\mathbf{R}^3 \times \mathbf{CP}^2 \times \Omega$ has one eigenvalue of positive real part, three of zero real part and the rest of negative real part. λ and w determine the ones of zero real part.

The point k_2, therefore, has a (real) four-dimensional center-unstable manifold $W^{\mathrm{cu}}(k_2)$, which is attracting relative to a compact neighborhood of $k_2(\lambda)$, say $V(\lambda)$. Set $V = \bigcup_{\lambda \in \Omega} V(\lambda)$ and $W = \bigcup_{\lambda \in \Omega} (W^{\mathrm{cu}}(k_2(\lambda)) \cap V(\lambda))$. Then $W \subset \mathbf{R}^3 \times \mathbf{CP}^2 \times \Omega$ and is attracting relative to V. Let $K_3 = W \cap (J_{\mathrm{B}} \times \mathbf{CP}^2 \times \Omega)$. Fix $\lambda_0 \in \Omega$ and define $K_3(\lambda_0) = K_3 \cap \{\lambda = \lambda_0\}$. Notice that the critical point $k_2(\lambda) \in K_3(\lambda)$. It is easy to check that $K_3(\lambda)$ contains none of the center directions in $W^{\mathrm{cu}}(k_2)$.

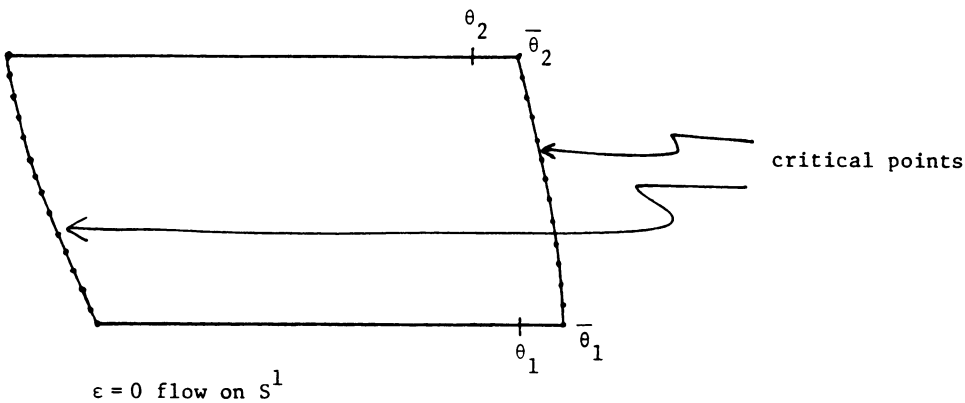


FIGURE 6

Therefore $K_3(\lambda)$ is one-dimensional and must lie in $W^u(k_2)$. It is therefore a curve of the form $(U_B(\xi), \hat{X}(\lambda, \xi))$, where $U_B(\xi)$ is the solution corresponding to $J_B \subset \mathbf{R}^3$, the back. The above is a trajectory in the (λ fixed) flow on $S_0 \times \mathbf{CP}^2$. Furthermore, it must satisfy, as $\xi \to -\infty$, $(U_B(\xi), \hat{X}(\lambda, \xi)) \to (U_2, \hat{X}^+(\bar{\theta}_2, \lambda))$.

But there is a unique curve that does this, namely $(U_B(\xi), \hat{\zeta}_B(\lambda, \xi))$.

Now extend h_0: $S^1 \to S_0$ to $\tilde{h}_0$: $S^1 \times \mathbf{CP}^2 \times \Omega \to S_0 \times \mathbf{CP}^2 \times \Omega$ by the identity.

Define K_2 by the condition $h_0(K_2) = K_3$. K_2 is of the form

$$\bigcup_{\bar{\theta}_2 \leqslant \theta \leqslant \eta} \bigcup_{\lambda \in \Omega} (\theta, \hat{X}(\theta, \lambda), \lambda)$$

for some $\eta > \bar{\theta}_2$. Choose K and, therefore, set θ_2 so that $\theta_2 < \eta$. Finally, reset V and, hence, K_2 so that $\eta = \theta_2$.

Since K_3 is an attractor in $V \cap (J_B \times \mathbf{CP}^2 \times \Omega)$, the same is true of K_2 in $[\bar{\theta}_2, \theta_2] \times \mathbf{CP}^2 \times \Omega$. Also, the only exit set in the boundary of the neighborhood lies in $\{\theta_2\} \times \mathbf{CP}^2 \times \Omega$. It follows that, if this neighborhood is called Q_2,

$$\omega(Q_2) \cap [\bar{\theta}_2, \theta_2] \times \mathbf{CP}^2 \times \Omega = K_2.$$

Now choose a neighborhood Q_1 of K_1 in $[0, \bar{\theta}_2] \times \mathbf{CP}^2 \times \Omega$. Since $K_1 \cap \{\bar{\theta}_2\} \times \mathbf{CP}^2 \times \Omega = K_2 \cap \{\bar{\theta}_2\} \times \mathbf{CP}^2 \times \Omega$, one can choose a Q_1 so that $Q_1 \cap \{\theta = \bar{\theta}_2\} = Q_2 \cap \{\theta = \bar{\theta}_2\}$. Let $Q = Q_1 \cup Q_2$. It is then not hard to see that Q is an attracting neighborhood of K relative to $[0, \theta_2] \times \mathbf{CP}^2 \times \Omega$.

The proof of Theorem 5.1 also needs an attractor that sits over the left-hand manifold. This would be a set of the form

$$K_L = \bigcup_{\bar{\theta}_3 \leqslant \theta \leqslant 2\pi} \bigcup_{\lambda \in \Omega} (\theta, \hat{X}(\theta, \lambda), \lambda),$$

where $\hat{X}(\theta, \lambda)$ is again the projectivised unstable eigenvector of the appropriate system.

The fact that this is an attractor is significantly easier to prove than for K because it lacks the tail of K. I shall only give the proof for K.

The following lemma about the behavior of the reduced system on the projectivised level will play a central role.

LEMMA 5.4. *Let $C \subset \Omega$ be a compact set so that $C \subset \operatorname{dom}(D_R) = G_R$ and $D_R(\lambda) \neq 0$ for all $\lambda \in C$. Then*

$$\lim_{\xi \to +\infty} \hat{\zeta}_R(\lambda, \xi) = \hat{X}_1^+(\lambda)$$

uniformly for $\lambda \in C$. $X_1^+(\lambda)$ is the eigenvector whose eigenvalue has positive real part for the system $z' = M_1 z$ (see §4).

REMARK. The projectivising here is restricted to $\mathbf{C}^2$. So if $X \in \mathbf{C}^2$, $\hat{X} \in \mathbf{CP}^1$.

PROOF. $\zeta_R(\lambda, \xi)$ satisfies (4.4), which, when coupled with the travelling wave equations (4.2), gives a system on $\mathbf{R}^2 \times \mathbf{C}^3$. When projectivised this leaves a system on $\mathbf{R}^2 \times \mathbf{CP}^1$. Let $J_R \subset \mathbf{R}^2$ be the closure of the wave trajectory; then $J_R \times \mathbf{CP}^1$ is invariant.

The asymptotic system lies at $(1,0) \times \mathbf{CP}^1$ and is given by (4.5) with 0 replaced by 1. M_1 has two eigenvalues $\mu_1^\pm$ and associated eigenvectors $X_1^\pm$. The projectivised flow on $\mathbf{CP}^1$, therefore, has two critical points $\hat{X}_1^\pm$. $\hat{\zeta}_R(\lambda, \xi)$ must tend to one of these as $\xi \to +\infty$. The set $\omega(u_R(\bar{\xi}), \hat{\zeta}_R(\lambda, \bar{\xi}))$, some $\bar{\xi}$, is an invariant subset of $\{(1,0)\} \times \mathbf{CP}^1$ and is therefore one of the critical points.

If $\hat{\zeta}_R(\lambda, \xi) \to \hat{X}_1^-(\lambda)$, λ would be an eigenvalue. By assumption it is not. Therefore $\hat{\zeta}_R(\lambda, \xi) \to \hat{X}_1^+(\lambda)$.

That the convergence is uniform follows from the fact that the rate of convergence to $\hat{X}_1^+(\lambda)$ in $(1,0) \times \mathbf{CP}^2$ is determined by $\operatorname{Re}(\mu^-(\lambda) - \mu^+(\lambda))$, which can be seen to be bounded uniformly from zero, as it is continuous.

This is all the machinery I need to establish the central estimates in the proof of Theorem 5.1.

Let $\theta_\varepsilon(\xi)$ be the parameterisation induced on S_ε by the travelling wave equations. Define $T_i = T_i(\varepsilon)$ by

$$\theta_\varepsilon(T_i) = \theta_i, \qquad i = 0, 1, 2, 3.$$

I shall evaluate $\hat{\zeta}(\lambda, T_i)$ for $i = 0, 1, 2, 3$ and $\hat{\zeta}(\lambda, T_4)$ where T_4 is very large.

Now let j: $\mathbf{C}^2 \to \mathbf{C}^3$ be the inclusion map $j(p, q) = (p, q, 0)$. Let $\zeta_R(\lambda, \xi)$ be the eigenvalue solution for (4.4), i.e., the reduced system. Set $\zeta_F(\lambda, \xi) = j(\zeta_R(\lambda, \xi))$. Notice that this coincides with $\zeta(\lambda, \xi)$ provided in §3 for $\varepsilon = 0$.

Formulate the reduced system appropriate for the back; see comment at end of §4. Let $\zeta'_R(\lambda, \xi)$ be the eigenvalue solution. Set $\zeta_B(\lambda, \xi) = j(\zeta'_R(\lambda, \xi))$.

Let $\theta_F(\xi)$ and $\theta_B(\xi)$ be the parametrisations induced on $S^1 \times \{0\}$, from S_0, which correspond, respectively, to the front and the back. They should each be normalised in some fashion.

Set T_0^F, T_1^F, T_2^B and T_3^B by

$$\theta_F(T_0^F) = \theta_0, \quad \theta_F(T_1^F) = \theta_1, \quad \theta_B(T_2^B) = \theta_2, \quad \theta_B(T_3^B) = \theta_3.$$

In other words, these are the times at which the singular orbit S_0 hits the box edges.

I shall prove the theorem by five estimates of the following form:

(1) $|\tilde{\zeta}(\lambda, T_0) - \tilde{\zeta}_F(\lambda, T_0^F)| < \delta_0$;
(2) $|\tilde{\zeta}(\lambda, T_1) - \tilde{\zeta}_F(\lambda, T_1^F)| < \delta_1$;
(3) $|\tilde{\zeta}(\lambda, T_2) - \tilde{\zeta}_B(\lambda, T_2^B)| < \delta_2$;
(4) $|\tilde{\zeta}(\lambda, T_3) - \tilde{\zeta}_B(\lambda, T_3^B)| < \delta_3$;
(5) $|\tilde{\zeta}(\lambda, T_4) - X^+(\lambda)| < \delta_4$.

Part of the proofs of these estimates will be to show that $\tilde{\zeta}$ is well defined in each case. To get each estimate will require setting ε small and the truth of the preceding estimate. The way δ_i depends on ε is not the same in each case. Each estimate will require a lemma; the proof will then be completed by checking that the estimates can be followed iteratively to reach (5).

There are two different types of lemmas. In the following, κ, δ_1 and δ_3 will be fixed independently of ε. The following two are the first type.

LEMMA 5.5. *Given $\delta_2 > 0$, there exists $\varepsilon_2 > 0$ such that if $\varepsilon < \varepsilon_2$ then* (2) *implies* (3) *for all $\lambda \in \Omega$.*

LEMMA 5.6. *Given $\delta_4 > 0$, there exists $\varepsilon_3 > 0$ such that if $\varepsilon < \varepsilon_3$ then* (4) *implies* (5) *for all $\lambda \in \Omega$.*

Estimates (2) and (4) are understood to hold for these fixed δ_1 and δ_3. I shall only prove Lemma 5.5, as 5.6 is similar and easier.

PROOF OF LEMMA 5.5. I shall first set κ and δ_1. Fix κ_0 so that Lemma 5.3 holds with $\kappa = \kappa_0$. According to that lemma there is a neighborhood Q of K in $[0, \theta_2] \times \mathrm{CP}^2 \times \Omega = F$ so that $\omega(Q) \cap F = K$. θ_2 here depends on $\kappa = \kappa_0$; rename it θ_2^0.

I shall now reset κ to be some number smaller than κ_0. Let π_0: $S^1 \times \mathrm{CP}^2 \times \Omega \to S^1$ be the natural projection. Choose θ so that $\pi_0^{-1}(\bar{\theta}) \cap \mathrm{int}(Q) \neq \varnothing$. The first requirement of κ is that $J_\mathrm{F} \cap B_1 \subset h_0([\bar{\theta}, \bar{\theta}_1])$ and $J_\mathrm{B} \cap B_2 \subset h_0([\bar{\theta}_2, \theta_2^0])$ both be true.

Recalling that $\zeta_\mathrm{F}(\lambda, \xi) = j(\zeta_\mathrm{R}(\lambda, \xi))$ and noticing that $j(X_1^+(\lambda)) = X(\bar{\theta}_1, \lambda)$, it follows from Lemma 5.4 that there exists a $\bar{\xi}$ so that

$$(\theta_\mathrm{F}(\xi), \hat{\zeta}_\mathrm{F}(\lambda, \xi), \lambda) \in Q \tag{5.10}$$

for all $\lambda \in \Omega$ and $\xi \geqslant \bar{\xi}$, where Q remains the attractor neighborhood for the setting κ_0. The time T_1^F depends on κ. Set κ so that $T_1^\mathrm{F} \geqslant \bar{\xi}$. Then (5.10) holds if $\xi = T_1^\mathrm{F}$ for all $\lambda \in \Omega$. This completes the setting of κ

To set δ_1 I shall need to describe open subsets of Q in local coordinates on CP^2. Recall that if $p \neq 0$ in $z = (p, q, r) \in \mathbf{C}^3$, $(q/p, r/p)$ form local coordinates on CP^2. It is easy to check that if ε is sufficiently small $\hat{X}(\theta, \lambda)$ lies in such a coordinate patch for all $\theta \in [\bar{\theta}_1, \theta_2]$ and $\lambda \in \Omega$. In these coordinates, estimates (1)–(5) could be written with $\hat{}$ instead of $\tilde{}$. It is obvious that the two are equivalent.

Let N_δ be a ball of radius δ about $\hat{\zeta}_F(\lambda, T_1^\mathrm{F})$ in CP^2 with the above coordinates. Since Q is open and (5.10) is satisfied, there exists δ so that $\{\theta_F(T_1^\mathrm{F})\} \times N_\delta \times \{\lambda\} \subset Q$ if $\delta = \delta_1$ for $\lambda \in \Omega$. Set this to be δ_1 for estimate (2).

The fact that K is an attractor, relative to F, permits various statements that are true for $\varepsilon = 0$ to be perturbed to $\varepsilon > 0$. Recall that $H_0(t)$ is a flow on $S^1 \times \mathrm{CP}^2 \times \Omega$, as is $H_\varepsilon(t)$ and, further, the dependence on ε is continuous.

For $\varepsilon = 0$ it is true that, given a neighborhood R of K in F and $Q_0 \subset Q$, with Q_0 compact, there exists a $T > 0$ so that

$$H_0(T) Q_0 \subset \mathrm{int}(R). \tag{5.11}$$

(5.11) will perturb to $H_\varepsilon(t)$, $\varepsilon > 0$ and sufficiently small. So, for the same Q_0 and R,

$$H_\varepsilon(T) Q_0 \subset \mathrm{int}(R). \tag{5.12}$$

Further, if $R \subset Q$ is a neighborhood of K in F, there exists a $T > 0$ so that

$$H_0([T, 2T])R \cap F \subset \operatorname{int}(R). \tag{5.13}$$

This is equivalent to saying that R is an attractor neighborhood. But then this perturbs to $\varepsilon > 0$ also and

$$H_\varepsilon([T, 2T])R \cap F \subset \operatorname{int}(R). \tag{5.14}$$

Now determine R by the given δ_2. From the proof of Lemma 5.3 the point $(\theta_B(T_2^F), \hat{\zeta}_B(\lambda, T_2^F), \lambda) \in K$ for all $\lambda \in \Omega$. Let $M_\delta(\lambda)$ be a ball of radius δ about $\hat{\zeta}_B(\lambda, T_2^F)$ in local coordinates on CP^2. Given δ_2 choose R so that

$$R \cap \{\theta = \theta_2\} \subset \bigcup_{\lambda \in \Omega} \{\theta_B(T_2^F)\} \times M_\delta \times \{\lambda\}$$

for all $\lambda \in \Omega$ if $\delta = \delta_2$. δ_2 can be assumed to be small enough so that the above holds.

Estimate (3) will then clearly follow if it can be shown that $(\theta_\varepsilon(T_2), \hat{\zeta}(\lambda, T_2), \lambda) \in R$. This in turn follows if it can be shown that

$$(\theta_\varepsilon(\xi), \hat{\zeta}(\lambda, \xi), \lambda) \in R \cup F^c \tag{5.15}$$

for all sufficiently large ξ, where F^c = complement of F.

Now set Q_0 as $\bigcup_{\lambda \in \Omega}\{\theta_F(T_1^F)\} \times \mathrm{cl}(N_\delta) \times \{\lambda\}$, where $\delta = \delta_1/2$. This is a compact subset of Q. Since $\theta_\varepsilon(T_1) = \theta_F(T_1^F)$, if $(\theta_\varepsilon(T_1), \hat{\zeta}(\lambda, T_1), \lambda) \in Q_0$, then clearly (2) is satisfied.

But then there is an $\hat{\varepsilon}$ so that if $\varepsilon < \hat{\varepsilon}$, (5.12) is true. Further, there is an $\tilde{\varepsilon}$ so that (5.14) holds if $\varepsilon < \tilde{\varepsilon}$. So if $\varepsilon_2 = \min\{\hat{\varepsilon}, \tilde{\varepsilon}\}$, then, when $\varepsilon < \varepsilon_2$, (5.15) holds for sufficiently large ξ and the lemma is proved.

To prove Lemma 5.6 one uses the attractor over the left-hand slow manifold, as remarked before Lemma 5.3. The proof is almost identical and would require setting δ_3 and resetting κ. The following lemmas give the steps from (1) to (2) and (3) to (4).

LEMMA 5.7. *Given $\delta_1 > 0$ there exist $\varepsilon_1 > 0$ and $\delta_0 > 0$ so that if $\varepsilon < \varepsilon_1$ and* (1) *is satisfied for all $\lambda \in \Omega$, then* (2) *also holds for all $\lambda \in \Omega$.*

LEMMA 5.8. *Given $\delta_3 > 0$ there exist $\varepsilon_3 > 0$ and $\delta_2 > 0$ so that if $\varepsilon < \varepsilon_3$ and* (3) *is satisfied for all $\lambda \in \Omega$, then* (4) *also holds for all $\lambda \in \Omega$.*

Again I shall only prove Lemma 5.7. Lemma 5.8 is the same with the appropriate modification to replace the front with the back.

PROOF OF LEMMA 5.8. The flow $H_0(t)$ on $S^1 \times \mathrm{CP}^2 \times \Omega$ takes the curve $(\theta_F(T_0^F), \hat{\zeta}_F(\lambda, T_0^F), \lambda)$, where $\lambda \in \Omega$, in time $T_1^F - T_0^F$ to the curve of points $(\theta_F(T_1^F), \hat{\zeta}_F(\lambda, T_1^F), \lambda)$.

If κ is small enough, $\hat{\zeta}_F(\lambda, T_i^F)$, $i = 0, 1$, are both in the usual coordinate patch $(p \neq 0)$. If (1) is satisfied, $\{\hat{\zeta}(\lambda, T_0): \lambda \in \Omega\}$ lies in a neighborhood of the curve $\{\hat{\zeta}_F(\lambda, T_0^F): \lambda \in \Omega\}$ of radius δ_0.

T_0 is set so that $\theta_\varepsilon(T_0) = \theta_F(T_0^F)$, T_1 gives $\theta_\varepsilon(T_1) = \theta_F(T_1^F)$. Also, from the construction of the pulse, it is not hard to see that $T_1(\varepsilon) - T_0(\varepsilon) \to T_1^F - T_0^F$ as $\varepsilon \to 0$.

Therefore by continuity of the flow in ε, if ε_1, δ_0 are small enough and $\varepsilon < \varepsilon_1$, $\hat{\zeta}(\lambda, T_1)$ lies in a prescribed (δ_1) neighborhood of $\hat{\zeta}_F(\lambda, T_1^F)$. Estimate (2) then easily follows.

One more ingredient is needed for the proof of Theorem 5.1, that is, that (5) suffices.

LEMMA 5.9. *There exists $\delta_4 > 0$ so that if* (5) *is true for T_4 sufficiently large, uniformly in $\lambda \in \Omega$, then*

$$\operatorname{Re}(\tilde{\zeta}(\lambda, T_4) \cdot \tilde{\eta}(\lambda, T_4)) > 0 \tag{5.16}$$

for $\lambda \in \Omega$. In particular, such a λ is not an eigenvalue.

PROOF. First, compute

$$X^+ \cdot Y^- = 1 + \alpha^+/(c - \bar{\beta}^-) - \varepsilon\left[(c - \bar{\beta}^-)(\bar{\beta}^- + (\lambda + \varepsilon\alpha)/c)\right]^{-1}.$$

If $\varepsilon = 0$ this simplifies to

$$X^+ \cdot Y^- = 1 + \alpha^+/(c - \bar{\beta}^-).$$

One then checks that $\operatorname{Re}(\alpha^+/(c - \bar{\beta}^-)) > 0$ and so $\operatorname{Re}(X^+ \cdot Y^-) > 1$.

For the case $\varepsilon > 0$ one checks again that $(c - \bar{\beta}^-)(\bar{\beta}^- + (\lambda + \varepsilon\alpha)/c)$ is bounded away from zero, uniformly in λ as $\varepsilon \to 0$. It follows that if ε_0 is small enough, then b can be chosen so that there exists $a > 0$ for which $\operatorname{Re}(X^+ \cdot Y^-) > a$ for all $\lambda \in \tilde{G}$ and $\varepsilon \in [0, \varepsilon_0)$.

But $\tilde{\eta}(\lambda, \xi) \to Y^-(\lambda)$ as $\xi \to +\infty$, and by continuity in λ and compactness of Ω, if T_4 is large enough, $|\tilde{\eta}(\lambda, T_4) - Y^-|$ can be made as small as desired uniformly in $\lambda \in \Omega$. The fact that $\operatorname{Re}(\tilde{\zeta}(\lambda, T_4) \cdot \tilde{\eta}(\lambda, T_4)) < 0$ then follows for $\varepsilon \in [0, \varepsilon_0]$ with ε_0 sufficiently small, from (5).

PROOF OF THEOREM 5.1. First set δ_1, δ_3 and κ as required in Lemmas 5.5 and 5.6. Set $\varepsilon_0 < \varepsilon_i$, $i = 1, 2, 3, 4$. Fix $\delta_4 > 0$ so that the conclusion of Lemma 5.9 holds. Proceeding through the estimates, one sees that if $0 < \varepsilon < \varepsilon_0$, then (1) implies (5), which implies the theorem. It remains to show that (1) holds.

Recall that $\bar{\theta} = 0$ at the origin and $X^+(\lambda, \varepsilon)$ is the unstable eigenvector for the system (3.5). The point

$$(0, \hat{X}^+(\lambda, \varepsilon), \lambda) \tag{5.17}$$

is then an equilibrium point for all $\lambda \in \Omega$. Arguing in the same fashion as in the proof of Lemma 5.3, the curve $(\theta_\varepsilon(\xi), \hat{\zeta}(\lambda, \xi), \lambda)$, $\xi \in R$, is the unstable manifold W^u of (5.17). Since they depend continuously on parameters, as does (5.17), estimate (1) is easily seen to hold, again resetting κ if necessary.

6. Winding number computation. From the last section it is known that the only eigenvalues that offer any threat to stability are those near either the front or the back. Since, for both the front and the back, 0 is the eigenvalue of largest real part, any such dangerous eigenvalue must lie close to 0. In this section I shall prove that there are exactly two eigenvalues near 0.

Let B be a closed ball of radius δ about 0. Set $K = \partial B$. Choose δ small enough so that

(1) $B \cap \{\sigma_F \cup \sigma_B\} = \{0\}$, and

(2) $B \subset \tilde{G}$.

From (2) $\tilde{D}$ is well defined on K, for all $\varepsilon \in [0, \varepsilon_0]$, even if D is not.

If $C \subset \mathbf{C} \setminus \{0\}$ is a curve, let $W(C)$ be the usual winding number; i.e., C is given by a function ϕ: $S^1 \to \mathbf{C} \setminus \{0\}$. ϕ determines an element of $\pi_1(\mathbf{C} \setminus \{0\})$, the fundamental group; call it $\pi_1(\phi)$. Then $W(C) = \pi_1(\phi)$.

Let δ be as above and choose ε_0 so that Theorem 5.1 holds with this δ if $\varepsilon < \varepsilon_0$. In fact, the conclusion of Lemma 5.9, i.e., (5.16), holds for $\lambda \in \Omega$, not just $\lambda \in G_2$. In particular, $\tilde{D}(\lambda) \neq 0$ for $\lambda \in K$ so $W(\tilde{D}(K))$ is well defined. The result of this section is the following.

THEOREM 6.1. *With K given as above, if ε_0 is small enough,*

$$W(\tilde{D}(K)) = 2. \tag{6.1}$$

Since $\tilde{D}$ is an analytic function, the winding number counts the number of zeroes of $\tilde{D}$ (by multiplicity) inside B. It follows from Theorem 6.1 that there are exactly two zeroes. These zeroes may not correspond to eigenvalues which are zeroes of D. However, if there is an unstable eigenvalue, it must be a zero of D and hence a zero of $\tilde{D}$. It would therefore be counted by (6.1).

It is known from the previous section that $\hat{\zeta}(\lambda, \xi)$ can be followed around S_ε and used to show, by its value at large ξ, that $\tilde{D}(\lambda) \neq 0$ for all $\lambda \in K$. Information is, however, lost in projectivising and this is insufficient to determine (6.1). The extra information about complex amplitude must be recovered.

Set $\zeta(\lambda, \xi) = (p(\lambda, \xi), q(\lambda, \xi), r(\lambda, \xi))$. As mentioned in §5, if $\xi = T_i$, $p(\lambda, T_i) \neq 0$, $i = 0, \ldots, 4$, for all $\lambda \in K$ (since $K \subset \Omega$). This means $\tilde{\zeta}(\lambda, T_i)$ is defined for all such i. Define $\gamma_i(\lambda)$ for $\lambda \in K$, $i = 0, \ldots, 4$,

$$\zeta(\lambda, T_i) = \gamma_i(\lambda)\tilde{\zeta}(\lambda, T_i). \tag{6.2}$$

In fact, it is obvious that $\gamma_i(\lambda) = p(\lambda, T_i)$.

Recall that $\tilde{D}(\lambda)$ is independent of ξ and so can be evaluated at $\xi = T_4$. Now,

$$\tilde{D}(\lambda) = \zeta(\lambda, T_4) \cdot \eta(\lambda, T_4) = \gamma_4(\lambda)\{\tilde{\zeta}(\lambda, T_4) \cdot \eta(\lambda, T_4)\}.$$

Also, if T_4 is large enough,

$$\exp(\beta^- T_4)\eta(\lambda, T_4) = \tilde{\eta}(\lambda, T_4) + \varepsilon(\lambda, T_4),$$

where

$$|\varepsilon(\lambda, T_4)| \to 0 \quad \text{as } T_4 \to +\infty \tag{6.3}$$

uniformly for $\lambda \in K$. This follows from the defining condition for η. Putting this into the expression for $\tilde{D}(\lambda)$,

$$\tilde{D}(\lambda) = \gamma_4(\lambda)\exp(\beta^- T_4)\{\tilde{\zeta}(\lambda, T_4) \cdot \tilde{\eta}(\lambda, T_4) + \tilde{\zeta}(\lambda, T_4) \cdot \varepsilon(\lambda, T_4)\}.$$

From Lemma 5.9 and (6.3), the term in parentheses has winding number zero. Also if K is small enough, $\bar{\beta}^-(\lambda)$ is approximated by $\bar{\beta}^-(0)$ for all $\lambda \in K$, so $W(\exp(\beta^-(K)T_4)) = 0$. It follows that

$$W(\tilde{D}(K)) = W(\gamma_4(K)).$$

The proof will follow the same style as that of §5. I shall iteratively establish the following winding numbers:

(1) $W(\gamma_0(K)) = 0$;
(2) $W(\gamma_1(K)) = 1$;
(3) $W(\gamma_2(K)) = 2$;
(4) $W(\gamma_3(K)) = 2$;
(5) $W(\gamma_4(K)) = 2$.

The tube parameter κ may be reset in the following lemmas, but it will again not depend on ε.

LEMMA 6.1. *There exists ε_1 so that if $\varepsilon < \varepsilon_1$ then $W(\gamma_0(K)) = 0$.*

PROOF. From its definition,

$$\zeta(\lambda, \xi) = e^{\alpha^+\xi}X^+ + g(\xi),$$

where $|g(\xi)| \to 0$ faster than $e^{(\mathrm{Re}\,\alpha^+)\xi}$. From the proof of Lemma 3.3, this can be made uniform in $\lambda \in K$ and $\varepsilon \in [0, \varepsilon_0]$, i.e., there exists $\nu > 0$, k and ξ^* so that

$$|g(\xi)| \leqslant ke^{(\mathrm{Re}\,\alpha^+ + \nu)\xi}$$

for all $\xi \leqslant \xi^*$ and $\lambda \in K$, $\varepsilon \in [0, \varepsilon_0]$. Then if $\xi = T_0$ is negative enough,

$$\mathrm{Re}\,p(\lambda, T_0) > 0 \tag{6.4}$$

for all $\lambda \in K$ and $\varepsilon \in [0, \varepsilon_0]$. It is clear that κ can be reset so that the T_0 satisfying $\theta_\varepsilon(T_0) = b_0$ is negative enough for all $\varepsilon \in [0, \varepsilon_0]$.

From (6.4), since $\gamma_0(\lambda) = p(\lambda, T_0)$, (1) easily follows.

LEMMA 6.2. *There exists ε_2 so that if $\varepsilon < \varepsilon_2$ then $W(\gamma_1(K)) = 1$.*

PROOF. First consider the behavior of the reduced system, i.e., the front. Recall the definitions of T_0^{F} and T_1^{F} and set

$$\zeta_{\mathrm{F}}(\lambda, T^{\mathrm{F}}) = \gamma_1^{\mathrm{F}}(\lambda)\tilde{\zeta}_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}});$$

as usual, this is well defined. Now

$$D_{\mathrm{F}}(\lambda) = \zeta_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}}) \cdot \eta_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}}) = \gamma_1^{\mathrm{F}}(\lambda)\{\tilde{\zeta}_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}}) \cdot \eta(\lambda, T_1^{\mathrm{F}})\}.$$

From the same kind of argument as given earlier, and using Lemma 5.4,

$$W(\tilde{\zeta}_{\mathrm{F}}(K, T_1^{\mathrm{F}}) \cdot \eta(K, T_1^{\mathrm{F}})) = 0$$

if K is small enough (K determines T_1^{F}). But then

$$W(D_{\mathrm{F}}(K)) = W(\gamma_1^{\mathrm{F}}(K)).$$

From Lemma 4.1, $(d/d\lambda)D_{\mathrm{F}}(\lambda)|_{\lambda=0} > 0$. If δ is small enough (resetting ε_0 if necessary), K will be a small circle about 0. It follows that

$$W(D_{\mathrm{F}}(K)) = 1 \quad \text{and} \quad W(\gamma_1^{\mathrm{F}}(K)) = 1.$$

This is where the stability of the front is used.

I shall translate this information into a map. Let

$$C_0^{\mathrm{F}}(\lambda) = \mathrm{span}_{\mathbf{C}}\{\zeta_{\mathrm{F}}(\lambda, T_0^{\mathrm{F}})\}$$

and

$$E_0^{\mathrm{F}} = \{(w, \lambda) : w \in C_0^{\mathrm{F}}(\lambda), \lambda \in K\}.$$

Put coordinates on E_0^{F} by using the map

$$\mathbf{C} \times K \to E_0^{\mathrm{F}}, \qquad (z, \lambda) \mapsto (z\tilde{\zeta}_{\mathrm{F}}(\lambda, T_0^{\mathrm{F}}), \lambda).$$

If $(z, \lambda) \in E_0^{\mathrm{F}}$, take $z\tilde{\zeta}_{\mathrm{F}}(\lambda, T_0^{\mathrm{F}})$ as an initial condition for the eigenvalue flow, determined by (4.4), at time T_0^{F}. Following this, for each λ up until time T_1^{F}, one obtains a multiple of $\tilde{\zeta}_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}})$; call this $\phi^{\mathrm{F}}(z, \lambda)\tilde{\zeta}_{\mathrm{F}}(\lambda, T_1)$. So ϕ is a map ϕ^{F}: $E_0^{\mathrm{F}} \to \mathbf{C}$. Since the flow is linear, ϕ is linear in z. Set $\phi^{\mathrm{F}}(z, \lambda) = \Phi^{\mathrm{F}}(\lambda)z$. Φ^{F} is now a map Φ^{F}: $K \to \mathbf{C} \setminus \{0\}$. It is easy to check that Φ^{F} is continuous.

Now evaluate ϕ^{F} on $\gamma_1^{\mathrm{F}}(\lambda)$:

$$\phi^{\mathrm{F}}(\gamma_1^{\mathrm{F}}(\lambda), \lambda) = \Phi^{\mathrm{F}}(\lambda)\gamma_1^{\mathrm{F}}(\lambda).$$

But from the definition of ϕ^{F} it is easy to see that

$$\phi^{\mathrm{F}}(\gamma_1^{\mathrm{F}}(\lambda), \lambda) = \gamma_1^{\mathrm{F}}(\lambda).$$

Consequently,

$$\Phi^{\mathrm{F}}(\lambda) = \gamma_2^{\mathrm{F}}(\lambda)/\gamma_1^{\mathrm{F}}(\lambda).$$

Now approximate the $\varepsilon \neq 0$ case by the reduced system. Consider the eigenvalue system (3.3) again. Let $y(\lambda, \xi)$ be the solution of (3.3) satisfying the condition $y(\lambda, T_0) = \tilde{\zeta}(\lambda, T_0)$, which obviously depends on ε. Let $y_{\mathrm{F}}(\lambda, \xi)$ be the solution of the reduced system (4.4) (with $r' = 0$ appended) satisfying

$$y_{\mathrm{F}}(\lambda, T_0^{\mathrm{F}}) = \tilde{\zeta}_{\mathrm{F}}(\lambda, T_0^{\mathrm{F}}).$$

Because $\theta_\varepsilon(T_0) = b_0 = \theta_{\mathrm{F}}(T_0^{\mathrm{F}})$ and (5.1) is continuous, $|y(\lambda, T) - y_{\mathrm{F}}(\lambda, T)|$ can be made as small as desired for fixed T (if ε is small enough). But also $T_1 - T_0 \to T_1^{\mathrm{F}} - T_0^{\mathrm{F}}$ as $\varepsilon \to 0$. It follows that if ν is prescribed, there is an ε, so that $\varepsilon < \varepsilon_1$ implies

$$|y(\lambda, T_1) - y_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}})| < \nu. \tag{6.5}$$

Let $C_0(\lambda) = \mathrm{span}_{\mathbf{C}}\{\zeta(\lambda, T_0)\}$ and $E_0 = \{(\omega, \lambda)$: $\omega \in C_0(\lambda)$, $\lambda \in K\}$. Just as before let (z, λ) be coordinates on E_0 where $w = z\tilde{\zeta}(\lambda, T_0)$. By taking $z\tilde{\zeta}(\lambda, T_0)$ as the initial condition again, define $\phi(z, \lambda)$ by requiring that $\phi(z, \lambda)\tilde{\zeta}(\lambda, T_1)$ be the solution at time T_1. Again ϕ is linear in z. Set $\phi(z, \lambda) = \Phi(\lambda)z$. Also as above,

$$\Phi(\lambda) = \gamma_2(\lambda)/\gamma_1(\lambda).$$

$\Phi(\lambda)1 = \phi(1, \lambda) =$ the first component of $y(\lambda, T_1)$. Also $\Phi^{\mathrm{F}}(\lambda) =$ the first component of $y_{\mathrm{F}}(\lambda, T_1^{\mathrm{F}})$. But then, by (6.4), $|\gamma_1(\lambda)/\gamma_0(\lambda) - \gamma_1^{\mathrm{F}}(\lambda)/\gamma_0^{\mathrm{F}}(\lambda)|$ can be made as small as desired uniformly in $\lambda \in K$. Therefore, $\gamma_1(\lambda) - \{\gamma_1^{\mathrm{F}}(\lambda)/\gamma_0^{\mathrm{F}}(\lambda)\}\gamma_0(\lambda)$ can also be made small. But then

$$W(\gamma_1(K)) = W(\gamma_1^{\mathrm{F}}(K)/\gamma_0^{\mathrm{F}}(K)) + W(\gamma_0(K)).$$

From Lemma 6.1 and the above arguments it follows that $W(\gamma_1(K)) = 1 + 0 = 1$.

LEMMA 6.3. *If* $\varepsilon < \varepsilon_0$ *(as determined in* §5*) then* $W(\gamma_2(K)) = 1$.

PROOF. If $\varepsilon < \varepsilon_0$, from the proof of Theorem 5.1, if $\zeta = (p, q, r)$, then $p(\lambda, \xi) \neq 0$ for all $\lambda \in K$ and $T_1 \leqslant \xi \leqslant T_2$. Since $\gamma_1(\lambda) = p(\lambda, T_1)$ and $\gamma_2(\lambda) = p(\lambda, T_2)$, $p(\lambda, \xi)$ defines a homotopy of $\gamma_1: K \to \mathbf{C} \setminus \{0\}$ to $\gamma_2: K \to \mathbf{C} \setminus \{0\}$. Therefore $W(\gamma_2(K)) = W(\gamma_1(K))$ and the lemma is proved.

LEMMA 6.4. *There exists* ε_2 *so that if* $\varepsilon < \varepsilon_2$ *then* $W(\gamma_3(K)) = 2$.

PROOF. The analysis closely follows that of Lemma 6.2, but the back is used to approximate instead of the front. The conclusion is that

$$W(\gamma_3(K)) = W(\gamma_2(K)) + 1 = 2.$$

LEMMA 6.5. *If* $\varepsilon < \varepsilon_0$ *then* $W(\gamma_4(K)) = 2$.

PROOF. $p(\lambda, \xi)$ gives a homotopy, just as in the case of Lemma 6.3. Therefore, $W(\gamma_4(K)) = W(\gamma_3(K)) = 2$.

This completes the proof of Theorem 6.1.

7. Completion of proof. From Lemma 3.2, if ε is small, the essential spectrum $\sigma_e(L)$ lies entirely in a set $\{\lambda: \operatorname{Re}\lambda < a\}$, where $a < 0$, albeit dependent on ε. Therefore it is only eigenvalues that can cause instability. From Theorem 5.1 these eigenvalues must lie in a δ-neighborhood of 0, where $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Theorem 6.1 says that there are two zeroes of D in such a δ-neighborhood. Therefore there are at most two zeroes of D, and so there are at most two eigenvalues. Since 0 is definitely an eigenvalue (due to translation) the other zero of D must be real.

If $\lambda > 0$ and large it dominates the system (3.3)–(3.4). It is not hard to check that $D(\lambda) > 0$ in this situation; see Evans [9] for details. It would follow that the other zero of $D(\lambda)$ near zero is negative if it could be established that

$$(d/d\lambda)D(\lambda)|_{\lambda=0} > 0. \tag{7.1}$$

The main theorem will follow by proving (7.1).

Evans devised a beautiful technique for computing the sign of $(d/d\lambda)D(\lambda)|_{\lambda=0}$:

$$\frac{d}{d\lambda}D(\lambda) = \left\{\frac{\partial}{\partial\lambda}\zeta(\lambda, \xi)\right\} \cdot \eta(\lambda, \xi) + \zeta(\lambda, \xi) \cdot \left\{\frac{\partial}{\partial\lambda}\eta(\lambda, \xi)\right\}.$$

Furthermore, the right-hand side can be evaluated at any ξ. As $\xi \to +\infty$, $|\eta_\lambda(0, \xi)| \to 0$ and $|\zeta(0, \xi)| \to 0$ since η is determined at $+\infty$ and $\zeta(0, \xi)$ is the derivative of the travelling wave. Therefore

$$\left.\frac{d}{d\lambda}D(\lambda)\right|_{\lambda=0} = \lim_{\xi \to +\infty} \zeta_\lambda(\lambda, \xi) \cdot \eta(\lambda, \xi)|_{\lambda=0}.$$

$\zeta(\lambda, \xi) = (p(\lambda, \xi), q(\lambda, \xi), r(\lambda, \xi))$ satisfies system (3.2). Differentiating with respect to λ and evaluating at $\lambda = 0$, $\zeta_\lambda(0, \xi)$ satisfies

$$x' = y, \qquad y' = -cy - f'(u)x + z + p(0, \xi),$$
$$z' = -(\varepsilon/c)x + (\varepsilon\gamma/c)z + (1/c)r(0, \xi). \tag{7.2}$$

Fix $\varepsilon \in (0, \varepsilon_0]$. Let $\bar{c}(\varepsilon)$ be the speed at which the pulse exists. For each c in a neighborhood of $\bar{c}(\varepsilon)$, there exists a solution (unique up to parametrisation) that tends to 0 as $\xi \to -\infty$. Call this $U(c, \xi) = (u(c, \xi), v(c, \xi), w(c, \xi))$, with parametrisation set by $u(c, 0) = a$ with no $\xi < 0$ satisfying $u(c, \xi) = a$ (recall a is the middle zero of $f(u)$). This condition uniquely determines $U(c, \xi)$, if ε_0 is small enough. Note that $U(c, \xi)$ is the unstable manifold of the critical point $(0, 0, 0)$.

Differentiating the travelling wave system (2.1) with respect to c and evaluating at $c = \bar{c}(\varepsilon)$, U_c satisfies

$$x' = y, \qquad y' = -cy - f'(u)x + z - u_\xi, \qquad z' = -(\varepsilon/c)x + (\varepsilon\gamma/c)z - (1/c)w_\xi. \tag{7.3}$$

Now by definition, $(p(0, \xi), q(0, \xi), r(0, \xi))$ satisfies system (3.2) with $\lambda = 0$, which is

$$p' = q, \quad q' = -cq - f'(u)p + r, \quad r' = -(\varepsilon/c)p + (\varepsilon\gamma/c)r. \tag{7.4}$$

It is easily seen that if $(u(\xi), v(\xi), w(\xi))$ is the travelling wave then (u_ξ, v_ξ, w_ξ) also satisfies (7.4) with $c = \bar{c}(\varepsilon)$. The solution of (7.4) that decays to 0 as $\xi \to -\infty$ is unique up to a scalar multiple. So there is a scalar α for which $\alpha p(0, \xi) = u_\xi(\xi)$ and $\alpha r(0, \xi) = w_\xi(\xi)$. From (7.2) and (7.3), $\alpha\zeta_\lambda + U_c$ must then satisfy (7.4), but the only solution of (7.4) which decays to 0 as $\xi \to -\infty$ is $\zeta(0, \xi)$ itself. Since $\alpha\zeta_\lambda + U_c$ clearly does, there is a b so that $\alpha\zeta_\lambda + U_c = b\zeta$.

Moreover, α must be greater than 0, since $p(0, \xi) > 0$ for large negative ξ; it is asymptotic to X^+, whose first component is 1; and $u_\xi > 0$ for large negative ξ.

Substituting into the above expression,

$$\left.\frac{d}{d\lambda} D(\lambda)\right|_{\lambda=0} = \lim_{\xi \to +\infty} \left(\frac{b}{\alpha}\zeta - \frac{1}{\alpha}U_c\right) \cdot \eta(0, \xi) = \lim_{\xi \to +\infty} \left(\frac{b}{\alpha}\zeta \cdot \eta - \frac{1}{\alpha}U_c \cdot \eta\right) = -\frac{1}{\alpha} \lim_{\zeta \to +\infty} U_c \cdot \eta. \tag{7.5}$$

The last equality holds because $\zeta \cdot \eta = 0$, which is true at $\lambda = 0$ since it is an eigenvalue.

As noted in §3, $\eta(0, \xi)$ is normal to the stable subspace of $(0, 0, 0)$. Therefore, the limit on the right-hand side contains information about how the solution U crosses this subspace with respect to c. In other words, its sign is determined by the direction in which the unstable manifold crosses the stable manifold, with respect to c, at the value of c for which the wave exists.

Unfortunately the quantity $U_c \cdot \eta$ is not independent of ξ and so the limit cannot be dropped in (7.5). I shall determine quantities $P(\xi)$ and $N(\xi)$ for which

$$\lim_{\xi \to +\infty} P(\xi) \cdot N(\xi) = \lim_{\xi \to +\infty} U_c \cdot \eta,$$

but $P(\xi) \cdot N(\xi)$ will be independent of ξ and so can be evaluated anywhere along the pulse solution.

Append $c' = 0$ to the travelling wave system to obtain the system in $\mathbf{R}^4$:

$$u' = v, \quad v' = -cv - f(u) + w, \quad w' = -(\varepsilon/c)(u - \gamma w), \quad c' = 0. \tag{7.6}$$

The point $(0,0,0,\bar{c}(\varepsilon))$ is a critical point of (7.6). Let $W^{cu}(\varepsilon)$ be the center-unstable manifold of this point for (7.6). This is obtained locally and then iterated in forward time. Let $W^{cs}(\varepsilon)$ be the center-stable manifold obtained in a similar fashion.

If c is close to $\bar{c}(\varepsilon)$, $(U(c,\xi),c)$ lies in $W^{cu}(\varepsilon)$. Set

$$P(\xi)=\left(\frac{\partial U}{\partial c}(\bar{c}(\varepsilon),\xi),1\right).$$

This is tangent to the above curve at the point $(U(\bar{c}(\varepsilon),\xi),\bar{c}(\varepsilon))$. As such, it satisfies the equation of variations for (7.6):

$$\begin{aligned} p'&=q, \qquad q'=-cq-f'(u)p+r-vs,\\ r'&=-(\varepsilon/c)(p-\gamma w)s, \qquad s'=0, \end{aligned} \tag{7.7}$$

where $u=u(\bar{c}(\varepsilon),\xi)$.

To set $N(\xi)$, rewrite (7.7) as

$$x'=Ax \tag{7.8}$$

and let

$$y'=By \tag{7.9}$$

be the adjoint system ($B=-A^*$). $N(\xi)$ will be a solution of (7.9).

A and B both depend on ξ. Let $A_0=\lim_{\xi\to+\infty}A$ and $B_0=\lim_{\xi\to+\infty}B$. With $\varepsilon\neq 0$ there is only one eigenvalue of A_0 with positive real part. Therefore B_0 has only one of negative real part. The usual argument shows that there is a unique solution of (7.9), call it $N(\xi)$, up to a scalar multiple, that decays at $+\infty$. It is not hard to convince oneself that this solution is normal to $W^{cs}(\varepsilon)$ at $(U(\bar{c}(\varepsilon),\xi),\bar{c}(\varepsilon))$ for each ξ. Writing out A and taking the adjoint,

$$B=\begin{pmatrix} 0 & f'(u) & \varepsilon/c & 0\\ -1 & c & 0 & 0\\ 0 & -1 & -\varepsilon\gamma/c & 0\\ 0 & v & -(\varepsilon/c^2)(u-\gamma w) & 0 \end{pmatrix}. \tag{7.10}$$

From the form of B, the first three equations of (7.9) uncouple from the fourth. Therefore these first three equations are the same as those satisfied by $\eta(0,\xi)$. $N(\xi)$ must be a scalar multiple of $(\eta(0,\xi),k(\xi))$ for some function $k(\xi)$ found by solving the fourth equation of (7.10); this is because $\eta(0,\xi)$ decays to 0 as to $\xi\to+\infty$. Set $N(\xi)=(\eta(0,\xi),k(\xi))$.

$P(\xi)\cdot N(\xi)$ can now be computed:

$$P(\xi)\cdot N(\xi)=\frac{\partial U}{\partial c}(\bar{c}(\varepsilon),\xi)\cdot\eta(0,\xi)+k(\xi).$$

Since $|N(\xi)|\to 0$ as $\xi\to+\infty$, $k(\xi)\to 0$ as $\xi\to+\infty$. Therefore,

$$\lim_{\xi\to+\infty}P(\xi)\cdot N(\xi)=\lim_{\xi\to+\infty}\frac{\partial U}{\partial c}(\bar{c}(\varepsilon),\xi)\cdot\eta(0,\xi).$$

Since $P(\xi)$ satisfies (7.8) and $N(\xi)$ satisfies (7.9), $P(\xi)\cdot N(\xi)$ is actually independent of ξ, using the same argument as the one which shows $D(\lambda)$ is independent of

ξ. Therefore,

(7.11) $$P(\tau)\cdot N(\tau)=\lim_{\xi\to+\infty}U_c\cdot\eta$$

for any $\tau\in\mathbf{R}$.

The theorem will then be proved by finding a T for which

(7.12) $$P(T)\cdot N(T)<0.$$

From (7.11) and (7.5), it then follows that $(d/d\lambda)D(\lambda)|_{\lambda=0}>0$, as desired.

The proof of (7.12) will require the transversality in Langer's proof [20]. To explain and summarize what is needed from Langer's work, I shall first give some notation.

Consider again the travelling wave system (7.6) in $\mathbf{R}^4$. Now set $\varepsilon=0$ and $c=\bar{c}(0)=c^*$ (in the notation of §2). Recall from §2 (see Figure 1) that E_R^* and E_L^* are the parts of the right and left slow manifolds that partake in the singular solutions. Let E_R and E_L be extensions of these in the w directions with $c=c^*$, i.e.,

$$E_R=\{(u,v,w,c): v=0, w_1\leqslant w\leqslant w_2, c=c^* \text{ is the largest root of } w=f(u)\},$$

where $w_1<0$ and $w_2>w^*$ are suitably chosen; similarly for E_L. Let R^u be the center-unstable manifold of this curve of critical points; see Fenichel [10]. This is a three-dimensional object. Let L^s be the stable manifold of the left-hand slow manifold E_L, lying in the slice $c=c^*$. As usual, each of these is obtained locally and then iterated in the appropriate time direction.

Let $\omega=\omega(0)$ be the point in $\mathbf{R}^3$ where J_B (the back) intersects $\{u=a\}$; recall that a is the middle zero of f. Set K_0 to be the unit normal to L^s at $(\omega(0),\bar{c}(0))$ with positive v component. The following argument shows that this is well defined. As L^s is carried in backward time along J_B, the sign of the second component of the normal cannot change. With $\varepsilon=0$, one can find two vectors, $v_1(\xi)$ and $v_2(\xi)$, tangent to L^s at a given point. $v_1=(p_1,q_1,r_1)$ and $v_2=(p_2,q_2,r_2)$. v_1 is tangent to J_B with $p_1>0$ and $r_1=0$. v_2 is not tangent to J_B but has $r_2>0$; this can be found because $w=$ constant are invariant planes. But then any normal to L^s at a point on J_B must be a multiple of $v_1\times v_2$ and, from the above properties, could not have a zero second component.

The set $R^u\cap\{c=\bar{c}(0)\}\cap\{u=a\}$ is a curve near $(\omega(0),\bar{c}(0))$. Set Q_0 to be the unit tangent vector to this curve at $(\omega(0),\bar{c}(0))$ with negative w component. The argument that this is well defined is very similar to that for K_0; w replaces v because this is a tangent not a normal vector.

To define K_ε and Q_ε, $\varepsilon\neq0$, let $\omega(\varepsilon)$ be the point on the back of the pulse S_ε in $\{u=a\}$ for small ε. K_ε is then the unit normal to $W^{cs}(\varepsilon)$ at $(\omega(\varepsilon),\bar{c}(\varepsilon))$, again with positive v-component. Let Q_ε be the unit tangent vector to the curve $W^{cu}(\varepsilon)\cap\{u=a\}$ with negative w component. These are both well defined because they converge to K_0 and Q_0, respectively; see below.

With these definitions it is not hard to see that $K_\varepsilon\to K_0$ as $\varepsilon\to0$. Append yet another equation to (7.6), namely $\varepsilon'=0$. Embed E_L into the $\varepsilon=0$ subspace of $\mathbf{R}^5$ and consider W^{cs}, the center-stable manifold of E_L, now with ε varying. $W^{cs}\cap\{\varepsilon=0\}\cap\{c=c^*\}$ and $W^{cs}\cap\{\varepsilon=\bar{\varepsilon}\}=W^{cs}(\bar{\varepsilon})$, the center-stable manifold of the curve of critical points $(0,0,0,c,\bar{\varepsilon})$, c varying. These then vary smoothly in ε since they are slices of a smooth manifold. It follows that appropriately oriented normals vary continuously and so $K_\varepsilon \to K_0$ as $\varepsilon \to 0$.

It is considerably harder to see that $Q_\varepsilon \to Q_0$ as $\varepsilon \to 0$, since their definitions are very different. Indeed, that this is true is the hardest part of Langer's proof. The fact that Q_ε lies close to Q_0 carries information to the back about how the front is constructed.

From Langer's work I shall need, then, the following two facts, which I state as lemmas.

LEMMA 7.1. $Q_0 \cdot K_0 < 0$.

LEMMA 7.2. $Q_\varepsilon \to Q_0$.

Lemma 7.1 is the heart of the matter. Everything else is just designed to see that this is the correct quantity to compute. I shall leave the proof of Lemma 7.1 and its geometric explanation to the end.

I shall proceed by showing how to deduce Lemma 7.2 from Langer's construction and then prove (Lemma 7.3) that the above is what is needed.

PROOF OF LEMMA 7.2. Langer constructs a box B_ε about the right-hand slow manifold in $\mathbf{R}^4$ and then considers the intersection of various unstable manifolds with the face F on the boundary of the box that is near the exit point on J_B. These are

$$\alpha_0 = R^u \cap \{c = \bar{c}(0)\} \cap F, \qquad \alpha_\varepsilon = W^{cu}(\varepsilon) \cap F.$$

He then proves that α_ε is close ($O(\varepsilon)$) in the C^1 topology to α_0. Let π_0 be the point in $J_B \times \{\bar{c}(0)\} \cap F$. The travelling wave system (7.6) in $\mathbf{R}^4$ with $\varepsilon' = 0$ appended induces a smooth flow on $\mathbf{R}^5$, and, on some small neighborhood V of $\pi_0 \times [0, \varepsilon_0)$, it induces a C^1 diffeomorphism into the set $\{u = a\}$. Call this map ψ: $V \to \{u = a\}$. It obviously takes $\alpha_\varepsilon \times \{\varepsilon\}$ into $(W^{cu}(\varepsilon) \cap \{u = a\}) \times \{\varepsilon\}$ and $\alpha_0 \times \{0\}$ into $(R^u \cap \{c = \bar{c}(0)\} \cap \{u = a\}) \times \{0\}$. These curves therefore remain C^1-close. Since Q_ε and Q_0 are unit tangent vectors to these curves at $(\omega(\varepsilon), \bar{c}(\varepsilon))$ and $(\omega(0), \bar{c}(0))$, respectively, $Q_\varepsilon \to Q_0$ as $\varepsilon \to 0$. This proves the lemma.

REMARK. Langer's proof that α_ε is close to α_0 assumes that $\gamma = 0$. However, I claim that it is merely a technical modification to include the case $\gamma \neq 0$.

LEMMA 7.3. $\operatorname{sgn}(Q_0 \cdot K_0) = \operatorname{sgn}(P(T) \cdot N(T))$ *for small enough* ε, *where* $T = T(\varepsilon)$ *and* $U(\bar{c}(\varepsilon), T) = \omega(\varepsilon)$.

PROOF. $W^{cu}(\varepsilon) \cap \{u = a\}$ is a curve and can be parametrised by c. Moreover, there is a smooth function $\tau(c)$ so that it is given by $(U(c, \tau(c)), c)$ near $(\omega(\varepsilon), \bar{c}(\varepsilon))$. A tangent vector can be found by differentiating, with respect to c,

$$(\partial U/\partial c + (\partial U/\partial \xi)\tau', 1), \tag{7.13}$$

evaluated at $\bar{c}(\varepsilon)$. It follows that Q_ε is a scalar multiple of (7.13):

$$Q_\varepsilon = m(\partial U/\partial c + (\partial U/\partial \xi)\tau', 1).$$

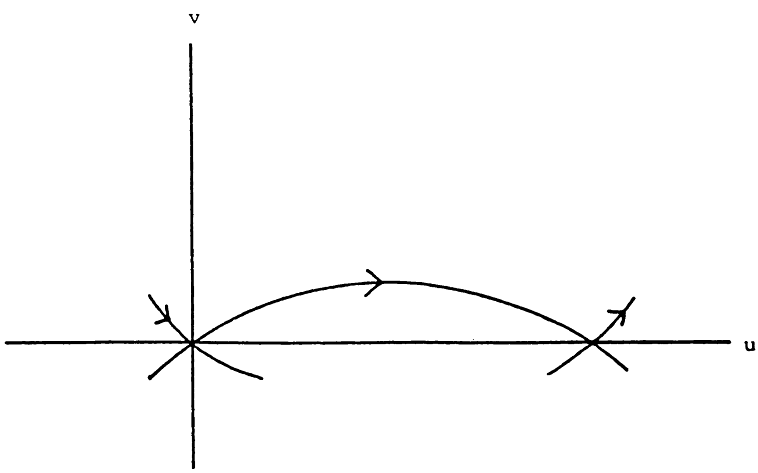
To check the sign of m, we must check the sign of $\partial w/\partial c = \tau'(\partial w/\partial \xi)$. Now $\tau(c)$ satisfies $u(c, \tau(c)) = a$, and so $\tau' = -(u_c/u_\xi)$. So we want the sign of

$$w_c - u_c(w_\xi/u_\xi). \tag{7.14}$$

Langer proves that if u_c is evaluated at a point near the right-hand slow manifold there is a k, α so that

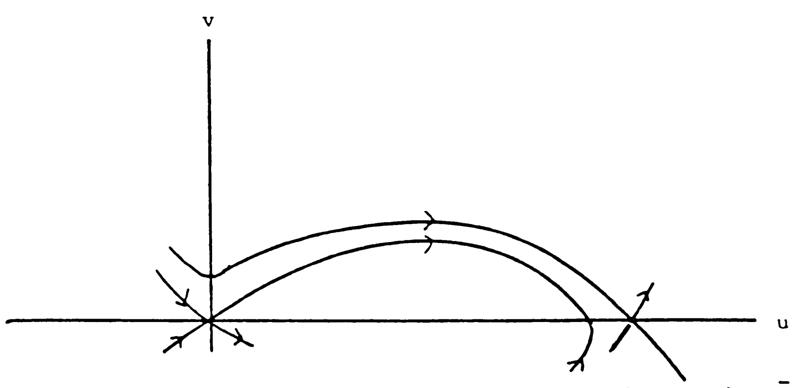
$$|u_c(\xi)| \geq ke^{\alpha\xi}. \tag{7.15}$$

I need to recover the sign of u_c. This information lies in the behavior of the front as c varies. Set $w = 0$, $\varepsilon = 0$; the phase portrait for $c = \bar{c}(0)$ is given in Figure 7. For $c > \bar{c}(0)$ but close to it, the phase portrait is that in Figure 8. So when $\varepsilon = 0$ and ξ is large, $u_c \ll 0$. By continuity it therefore follows for $c = \bar{c}(\varepsilon)$ and $\varepsilon \neq 0$. As $u(\bar{c}(\varepsilon), \xi)$ remains near the right-hand slow manifold, this does not change. Therefore $u_c(\xi) \leq -ke^{\alpha\xi}$.



phase portrait in w = 0 plane, c = $\bar{c}(0)$

FIGURE 7



phase portrait in w = 0 plane, c slightly larger than $\bar{c}$

FIGURE 8

The above described feature is, in fact, one of the main ingredients in proving α_ε is close to α_0.

The equation for w_c is

$$w_c' = -(\varepsilon/c)(u_c - \gamma w_c) + (\varepsilon/c^2)(u - \gamma w).$$

Near E_L, $|u - \gamma w| \ll 1$, so one sees that w_c' is essentially determined by u_c. Indeed, let T_0 be the time at which $U(\bar{c}(\varepsilon), \xi)$ enters B, and let T_1 be the exit time:

$$\left(e^{-(\varepsilon\gamma/c)\xi} w_c\right)' = -(\varepsilon/c)e^{-(\varepsilon\gamma/c)}\left(u_c - (u - \gamma w)/c\right),$$

$$e^{-(\varepsilon\gamma/c)} w_c(T_1) = e^{-(\varepsilon\gamma/c)T_0} w_c(T_0) - (\varepsilon/c)\int_{T_0}^{T_1} e^{-(\varepsilon\gamma/c)\xi}\left(u_c - \frac{u - \gamma w}{c}\right) d\xi.$$

Since $|u - \gamma w| \ll 1$, $u_c \ll 0$ and $T_1 - T_0$ is $O(1/\varepsilon)$ (the time spent on the slow manifold), $w_c(T_1) \to -\infty$ as $\varepsilon \to 0$ (recall $c < 0$).

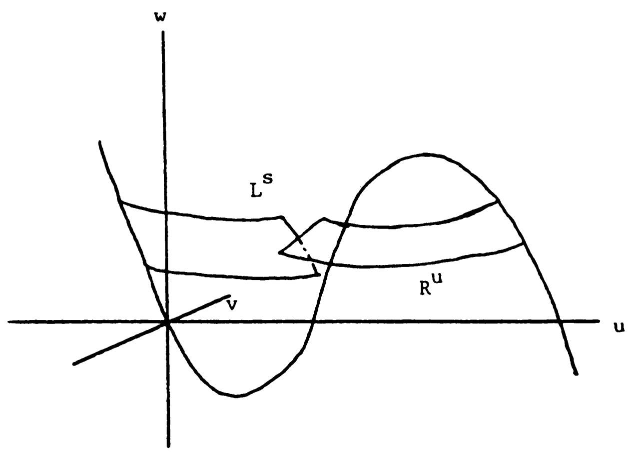
The time taken by $U(\bar{c}(\varepsilon), \xi)$ between leaving B and crossing $\{u = a\}$ is bounded independently of ε. Hence at $T = T(\varepsilon)$, if ε is small enough, $u_c < 0$ and $w_c < 0$.

Also at $u = a$, $u_\xi < 0$ and $w_\xi = -(\varepsilon/c)(u - \gamma w)$ with $\gamma \ll 1$, $w_\xi > 0$. From this, one sees that (7.14) is negative. This implies $m > 0$.

Let $K_\varepsilon = nN$. I must check that $n > 0$. For this we need that the second component of η is positive. As $\xi \to +\infty$, $\eta(0, \xi)$ is asymptotic to Y^-, whose second component is $(c - \beta^-)^{-1}$. β^- is close to $(c - (c^2 - 4f'(0))^{1/2})/2$, since $f'(0) < 0$ and $(c - \beta^-)^{-1}$ is positive. This component must stay positive as the solution moves up the left slow manifold. By continuity of $W^{cs}(\varepsilon)$ in ε, some scalar multiple of $\eta(0, \xi)$ will stay close to the normal of L^s. By the same argument that shows $K_\varepsilon \to K_0$, the second component of this vector can never be zero. Therefore the second component of η stays positive on $(T, +\infty)$. It follows that $n > 0$.

I now have proved that

$$Q_\varepsilon = m\left(\frac{\partial U}{\partial c} + \frac{\partial U}{\partial \xi}\tau', 1\right) \quad \text{and} \quad K_\varepsilon = nN$$



Transversal intersection of L^S and R^u

FIGURE 9

with both $m, n > 0$. Furthermore,

$$Q_\epsilon \cdot K_\epsilon = mn\left(\frac{\partial U}{\partial c}, 1\right) \cdot N + mn\left(\frac{\partial U}{\partial \xi}\tau', 0\right) \cdot N$$

$$= mn\left(\frac{\partial U}{\partial c} \cdot \eta + k\right) + mn\tau'\left(\frac{\partial U}{\partial \xi} \cdot \eta\right),$$

but $\partial U/\partial\xi \cdot \eta = 0$, since 0 is an eigenvalue and $\partial U/\partial \xi$ is an eigenfunction. Therefore,

$$Q_\epsilon \cdot K_\epsilon = mn(P(T) \cdot N(T)).$$

Since $m, n > 0$,

$$\operatorname{sgn}(Q_\epsilon \cdot K_\epsilon) = \operatorname{sgn}(P(T) \cdot N(T)).$$

Since $Q_\epsilon \to Q_0$ and $K_\epsilon \to K_0$, the lemma follows, as desired.

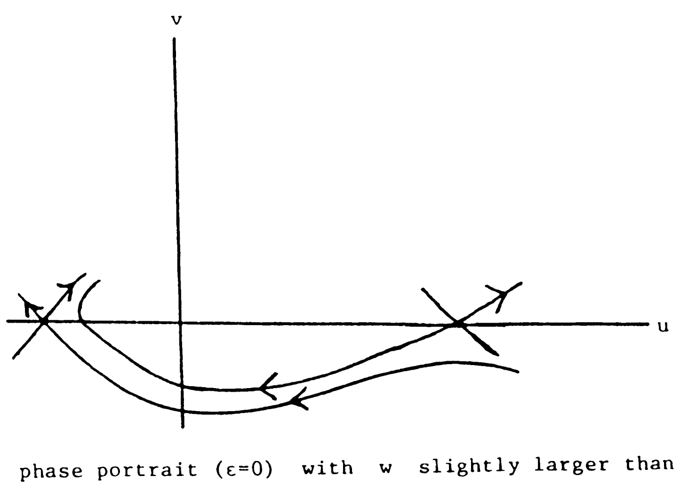
It now remains to prove Lemma 7.1. This will depend on an argument given by Langer, which, in this case, applies exactly, since $\epsilon = 0$, and the value of γ is therefore irrelevant.

PROOF OF LEMMA 7.1. To compute $Q_0 \cdot K_0$ we can project onto $\mathbf{R}^3$ as the fourth component of Q_0 is zero. Q_0 is tangent to $R^u \cap \{c = \bar{c}(0)\} \cap \{u = a\}$ and has a negative w component. R^u can be expressed as the graph of a function $v = h(u, w)$ in $\mathbf{R}^3$ near $\omega(0)$. Q_0 is therefore tangent to the curve $(a, h(a, w), w)$, and so a multiple of $(0, h_w, 1)$. It is a positive multiple of $(0, -h_w, -1)$.

Now K_0, projected onto $c = \bar{c}(0)$, is normal to $W^s(\epsilon)$; $c = \bar{c}(\epsilon)$ is given by $v = g(u, w)$. A normal with positive v component is therefore $(-g_u, 1, -g_w)$. It follows that $Q_0 \cdot K_0$ is a positive multiple of $g_w - h_w$. Langer proves the inequalities

(7.16) $$h_w > 0 \quad \text{and} \quad g_w < 0,$$

from which the lemma follows.



phase portrait (ε=0) with w slightly larger than w*

FIGURE 10

REMARK. Inequalities (7.16) have a very pretty and important geometrical interpretation. They quantify how the unstable manifold from the right slow manifold meets the stable manifold of the left one; see Figure 9. The direction is determined by the way the connection breaks as w changes. The phase portrait for w slightly larger than w^* is given in Figure 10. This is then easily seen to be the content of (7.16).

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