TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 286, Number 2, December 1984

HERMITIAN FORMS IN FUNCTION THEORY

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ABSTRACT. Let f and g be analytic in the unit disk $|z| < 1$. We give a new derivation of the positive semidefinite Hermitian form equivalent to $|g(z)| \leq |f(z)|$, for $|z| < 1$, and use it to derive Hermitian forms for various classes of univalent functions. Sharp coefficient bounds for these classes are obtained from the Hermitian forms. We find the specific functions required to make the Hermitian forms equal to zero.

1. Introduction. In this paper[1] we derive positive semidefinite Hermitian forms that characterize various classes of functions. The derivations use a general result about majorization. We use the definition $f(z) = \sum_{k=0}^{\infty} a_k z^k$ majorizes $g(z) = \sum_{k=0}^{\infty} b_k z^k$ if $|\sum_{k=0}^{\infty} b_k z^k| \leq |\sum_{k=0}^{\infty} a_k z^k|$ for $|z| < 1$, and show that f majorizes g if and only if

$$\sum_{j=0}^{\infty} \left\{ \left| \sum_{k=0}^{\infty} a_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} b_k z_{k+j} \right|^2 \right\} \geq 0$$

whenever $\limsup_{k\to\infty} |z|^{1/k} < 1$. The majorization theorem is a little known result of Schur [4]. Hermitian forms are found for functions of positive real part, functions starlike of order α, close-to-convex functions, spirallike functions and typically real functions. Sharp coefficient bounds are also derived for $|a_0|$, $|a_1|$, $|a_2|$, where $|\sum_{k=0}^{\infty} a_k z^k| \leq 1$ and a new bound on $|a_n|$ is found.

The positive semidefinite Hermitian form that characterizes positive real part functions is identically zero for some functions; these functions are of the form $p(z) = \sum_{j=1}^{n} t_j(1 + x_j z)/(1 - x_j z)$, $t_j \geq 0$, $|x_j| = 1$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} t_j = 1$. For the Hermitian forms characterizing bounded functions and the class of functions majorized by a fixed function f we also find the functions which make the Hermitian forms identically zero. Finally we consider the following extremal problem: let

$$P = \left\{ f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \mid \operatorname{Re} f(z) \geq 0 \right\}$$

and F be an analytic function of n variables, and find $\max_{f\in P} |F(c_1, c_2, \ldots, c_n)|$. We show it is solved by a positive real part function of the form

$$f(z) = \sum_{j=1}^{n} t_j(1 + x_j z)/(1 - x_j z), \qquad |x_j| = 1,\ t_j \geq 0,\ \sum_{j=1}^{n} t_j = 1.$$

Received by the editors July 7, 1983.
1980 *Mathematics Subject Classification*. Primary 30C45, 10C05.

[1]This material constitutes a portion of the author's Ph.D. thesis from the University of Kentucky under the direction of Professor T. J. Suffridge. The author gratefully acknowledges Dr. Suffridge's assistance with this work.

2. Derivation of the Hermitian forms. We use the convention that $H(Z)$ is a Hermitian form defined on the vector space, V, of sequences, $Z = \{z_k\}_{k=0}^{\infty}$,

$$\limsup_{k\to\infty} |z_k|^{1/k} < 1$$

and $V_k \subset V$ represents those sequences such that $z_j = 0$ if $j > k$. Also $H(z_1, \ldots, z_k) = H(z_1, \ldots, z_k, 0, 0, \ldots)$. Further since $H(0) = 0$ we use the convention that $H(Z) > 0$ means $H(Z) > 0$ when $Z \neq 0$.

Our proof of the majorization result uses Cohn's Rule. For a proof see [**5**, p. 432].

THEOREM 1. *Cohn's Rule: The zeros of the polynomial* $g(z) = a_0 + a_1 z + \cdots + a_m z^m$, $a_m \neq 0$, *are in* $D = \{z \mid |z| < 1\}$ *if and only if the Hermitian form*

$$H(Z) = H(z_0, z_1, \ldots, z_{m-1}) = \sum_{j=0}^{m-1} \left\{ \left| \sum_{k=0}^{m-j-1} \bar{a}_{m-k} z_{j+k} \right|^2 - \left| \sum_{k=0}^{m-j-1} a_k z_{j+k} \right|^2 \right\}$$

is positive definite for all finite sequences, $Z = \{z_k\}_{k=0}^{m-1}$.

We will also use the next lemma in Theorem 3. Let $H_f^*(Z) = \sum_{j=0}^{\infty} |\sum_{k=0}^{\infty} a_k z_{k+j}|^2$ where $f(z) = \sum_{k=0}^{\infty} a_k z^k$.

LEMMA 2. *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *is analytic in* D, $|\alpha| \leq 1$, $Z \in V$, *then there exists* $W \in V$ *such that*

$$H^*_{f(z-\alpha)/(1-\bar{\alpha}z)}(Z) = H_f^*(W)$$

where W *depends only on* α *and* Z.

PROOF. Expand $f \cdot (z-\alpha)/(1-\bar{\alpha}z)$ in terms of z, then in $H^*_{f(z-\alpha)/(1-\bar{\alpha}z)}$ rearrange terms and choose $w_j = z_j + (1-|\alpha|^2) \sum_{k=1}^{\infty} \bar{\alpha}^{k-1} z_{k+j}$.

THEOREM 3. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *and* $g(z) = \sum_{k=0}^{\infty} b_k z^k$ *be analytic in* $D = \{z \mid |z| < 1\}$. *Then* $|f(z)| \geq |g(z)|$ *on* D *if and only if*

$$H(Z) = H_f^*(Z) - H_g^*(Z) = \sum_{j=0}^{\infty} \left\{ \left| \sum_{k=0}^{\infty} a_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} b_k z_{k+j} \right|^2 \right\}$$

is positive semidefinite on the family of all sequences $Z \in V$.

PROOF. Choose $z_k = w^k$ for $w \in D$, then $H(Z) \geq 0$ implies $|f(z)| \geq |g(z)|$.

The converse is divided into two parts; the first assumes that f is nonzero. Then since $|g(z)| \leq |f(z)|$ in D, we have $|g(z)/f(z)| < 1$ in D or $|g/f| = 1$. If $g = e^{i\theta} f$ then $H(Z) \equiv 0$. Otherwise, choose a real sequence, $\{r_m\}_{m=1}^{\infty}$, so that $0 \leq r_m < 1$ and $\lim_{m\to\infty} r_m = 1$.

If $f(z) \neq 0$ in D then $|f(r_m z)| > |g(r_m z)|$ for all $z \in \overline{D}$ and any m. Define the two sequences of polynomials of degree n with coefficients depending on r_m as follows: $P(n, m, z) = \sum_{k=0}^{n} a_k r_m^k z^k$ and $Q(n, m, z) = \sum_{k=0}^{n} b_k r_m^k z^k$. P and Q converge uniformly on $\overline{D}$ to $f(r_m z)$ and $g(r_m z)$ respectively. From these polynomials we derive one whose zeros are all in D and apply Cohn's Rule.

For r_m fixed and $N(m)$ large, $|Q(n,m,z)| < |P(n,m,z)|$, $n \geq N(m)$, $z \in \overline{D}$. Thus $P(n,m,z) \neq 0$ on $\overline{D}$ and $z^{2n+1}\overline{Q(n,m,1/\overline{z})}/P(n,m,z)$ is analytic on $\overline{D}$. By the maximum principle, $|z^{2n+1}\overline{Q(n,m,1/\overline{z})}/P(n,m,z)| < 1$, $z \in \overline{D}$, and by Rouché's theorem $R(n,m,z) = z^{2n+1}\overline{Q(n,m,1/\overline{z})} + P(n,m,z)$ has no zeros in $\overline{D}$. Thus the zeros of

$$z^{2n+1}\overline{R(n,m,1/\overline{z})} = \sum_{k=0}^{n} b_k r_m^k z^k + \sum_{k=n+1}^{2n+1} \overline{a}_{2n+1-k} r_m^{2n+1-k} z^k$$

are all in D. Apply Cohn's Rule to the polynomial $z^{2n+1}\overline{R(n,m,1/\overline{z})}$ to obtain the positive definite Hermitian form

$$\begin{aligned} H(n,m,z) = \sum_{j=0}^{2n} \Bigg\{ &\left| \sum_{k=0}^{n} r_m^k a_k z_{j+k} + \sum_{k=n+1}^{2n-j} r_m^{2n+1-k} \overline{b}_{2n+1-k} z_{j+k} \right|^2 \\ &- \left| \sum_{k=0}^{n} r_m^k b_k z_{j+k} + \sum_{k=n+1}^{2n-j} r_m^{2n+1-k} \overline{a}_{2n+1-k} z_{j+k} \right|^2 \Bigg\} \end{aligned}$$

where the sums from $n+1$ to $2n-j$ are understood to be zero if $j \geq n$. It follows that for fixed r_m,

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} H(n,m,Z) \\ &= \sum_{j=0}^{\infty} \left\{ \left| \sum_{k=0}^{\infty} a_k r_m^k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} b_k r_m^k z_{k+j} \right|^2 \right\}. \end{aligned}$$

Letting $r_m \to 1$ we have for all $Z \in V$, $H(Z) \geq 0$.

To remove the restriction that $f(z) \neq 0$ in D, we choose r_m so that $f(r_m z) \neq 0$ on $|z| = 1$. Then $|g(r_m z)| \leq |f(r_m z)|$, and $f(\alpha_1) = 0$ imply that $g(\alpha_1) = 0$. For each m, $f(r_m z)$ has finitely many zeros, $\{\alpha_j\}_{j=1}^{p}$ in D. Let

$$f(r_m z) = \left\{ \prod_{j=1}^{p} (\alpha_j - z)/(1 - \overline{\alpha}_j z) \right\} F(z)$$

and

$$g(r_m z) = \left\{ \prod_{j=1}^{p} (\alpha_j - z)/(1 - \overline{\alpha}_j z) \right\} G(z).$$

Since $|\alpha_j| < 1$, we have $|G(z)| \leq |F(z)|$, $z \in \overline{D}$. Also $F(z) \neq 0$ in $\overline{D}$. By Lemma 2, $0 \leq H_F^*(W) - H_G^*(W) = H_{f(r_m z)}^*(Z) - H_{g(r_m z)}^*(Z)$ where W depends only on Z and $\{\alpha_j\}_{j=1}^{p}$. To finish the proof let $r_m \to 1$.

An immediate corollary of this is that f is bounded, $|f(z)| = |\sum_{k=0}^{\infty} a_k z^k| \leq 1$, if and only if $H(Z) = \sum_{j=0}^{\infty}\{|z_j|^2 - |\sum_{k=0}^{\infty} b_k z_{k+j}|^2\} \geq 0$. A Hermitian form for $f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ of positive real part can be derived because $\operatorname{Re} f \geq 0$ is equivalent to $|f(z)-1|<|f(z)+1|$. Then by Schwarz's lemma $|\sum_{k=0}^{\infty} c_{k+1}z^k| < |2+\sum_{k=1}^{\infty} c_k z^k|$ and we have

$$0 \le \sum_{j=0}^{\infty}\left\{\left|2z_j + \sum_{k=1}^{\infty} c_k z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty} c_{k+1} z_{k+j}\right|^2\right\}.$$

This result is originally due to Carathéodory [**1**] (see also [**6**, p. 154]). Similar methods can be used to derive Hermitian forms for several classes of univalent functions as summarized in Theorem 4 below. Part (c) is Suffridge's result in [**5**, p. 436].

THEOREM 4. (a) *$g(z)=\sum_{k=0}^{\infty} b_k z^k$ is bounded, $|g(z)|\le 1$ in D, if and only if for all $Z\in V$,*

$$\sum_{j=0}^{\infty}\left\{|z_j|^2 - \left|\sum_{k=0}^{\infty} b_k z_{k+j}\right|^2\right\} \ge 0.$$

(b) *$f(z)=1+\sum_{k=1}^{\infty} c_k z^k$ has positive real part if and only if for all $Z\in V$*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j + \sum_{k=1}^{\infty} c_k z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty} c_{k+1} z_{k+j}\right|^2\right\} \ge 0.$$

(c) *$f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ is starlike of order α, that is* $\operatorname{Re} zf'/f \ge \alpha$, *$\alpha\le 1$, if and only if for all $Z\in V$*

$$\sum_{j=0}^{\infty}\left\{\left|2(1-\alpha)z_j + \sum_{k=1}^{\infty}(k+2-2\alpha)a_{k+1} z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty}(k+1)a_{k+2} z_{k+j}\right|^2\right\} \ge 0.$$

(d) *$f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ is convex, that is* $\operatorname{Re}(1+zf''(z)/f'(z)) \ge 0$ *if and only if for all $Z\in V$*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+1} z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2} z_{k_j}\right|^2\right\} \ge 0.$$

(e) *$f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ is close-to-convex, that is* $\operatorname{Re} e^{i\beta} f'/\Phi' \ge 0$, *$|\beta|<\pi/2$, for some convex $\Phi(z)=z+\sum_{k=2}^{\infty} b_k z^k$, if and only if for all $Z\in V$*

$$\sum_{j=0}^{\infty}\left\{\left|(1+e^{-2i\beta})z_j + \sum_{k=1}^{\infty}(k+1)(a_{k+1}+e^{-2i\beta}b_{k+1}) z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty}(k+2)(a_{k+2}-b_{k+2}) z_{k+j}\right|^2\right\} \ge 0.$$

(f) *$f(z)=z+\sum_{k=2}^{\infty} a_k z^k$ where each a_k is real, is typically real, that is* $\operatorname{Re}(1-z^2)f(z)/z \ge 0$ *if and only if for all $Z\in V$*

$$\sum_{j=0}^{\infty}\left\{\left|2z_j + \sum_{k=1}^{\infty}(a_{k+1}-a_{k-1}) z_{k+j}\right|^2 - \left|\sum_{k=0}^{\infty}(a_{k+2}-a_k) z_{k+j}\right|^2\right\} \ge 0.$$

(g) $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *is spirallike, that is* $\operatorname{Re} e^{i\alpha} z f'(z)/f(z) > 0$ *for some real* α, $|\alpha| < \pi/2$, *if and only if for all* $Z \in V$

$$\sum_{j=0}^{\infty} \left\{ \left| (1 + e^{-2i\alpha}) z_j + \sum_{k=1}^{\infty} (k + 1 + e^{-2i\alpha}) a_{k+1} z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} (k+1) a_{k+2} z_{k+j} \right|^2 \right\} \geq 0.$$

REMARK 1. The Hermitian forms can be used to derive coefficient bounds for each of the classes in (b)–(g). The bounds are known but have been proved by several different methods. Using Hermitian forms they may all be derived by induction on n where $z_n \neq 0$ and $z_k = 0$, $k \neq n$.

REMARK 2. Generally, any class of functions that can be expressed in terms of majorization of an analytic function G by an analytic function F where F and G are determined by f lends itself to this type of analysis. The nonnegative Hermitian form is found first and then used to derive sharp coefficient bounds by successively setting $z_n = 1$, $z_k = 0$, $k \neq n$, $n = 1, 2, 3, \ldots$.

For bounded functions, $|f(z)| = |\sum_{k=0}^{\infty} a_k z^k| \leq 1$, the sharp coefficient bounds are not determined by setting $z_n = 1$, $z_k = 0$, $k \neq n$, $n = 2, 3, \ldots$. When $n = 1$, $0 \leq H(1, 0, 0, \ldots) = 1 - |a_0|^2$ gives the sharp bound, $|a_0| \leq 1$. For $n = 2$, $|a_1| \leq 1 - |a_0|^2$. Equality occurs for $f(z) = (a_0 - z)/(1 - \bar{a}_0 z)$ where $a_1 = |a_0|^2 - 1$ and in this case $H(-\bar{a}_0 z_1, z_1) = 0$, $z_1 \neq 0$. Likewise, equality in the third coefficient bound occurs when $H(z_0, z_1, z_2) = 0$ for some choice of z_0, z_1, z_2 not all zero. We find that

$$|a_2(1 - |a_0|^2) + a_1^2 \bar{a}_0| \leq (1 - |a_0|^2)^2 - |a_1|^2$$

which implies that $|a_2| \leq (1 - |a_0|^2) - |a_1|^2/(1 + |a_0|)$. Theorem 13 expands on and gives a proof of these results.

Sharp inequalities for $|a_n|$, $n \geq 3$, are very complicated algebraically however an estimate for the nth coefficient is easily obtained from

$$0 \leq H(z_0, 0, \ldots, 0, z_n, 0, \ldots) = |z_0|^2 + |z_n|^2 - |a_0 z_0 + a_n z_n|^2 - \sum_{k=0}^{n-1} |a_k z_n|^2$$

where z_0 and z_n are not both zero. This is equivalent to $1 - |a_0|^2 \geq 0$ and

$$0 \leq \det \begin{bmatrix} 1 - |a_0|^2 & -a_0 \bar{a}_n \\ -\bar{a}_0 a_n & 1 - \sum_{k=0}^{n} |a_k|^2 \end{bmatrix}$$

which simplifies to $|a_n|^2 \leq (1 - |a_0|^2)(1 - \sum_{k=0}^{n-1} |a_k|^2)$. This is better than Landau's estimate, $|a_n| \leq 2(1 - |a_0|)$ [**2**, p. 34].

3. Solution of $H(Z) = 0$. Let $H(Z)$, $Z \in V$, denote the Hermitian form for functions of positive real part and let $H_1^*(Z) - H_f^*(Z)$, $Z \in V$, denote the form for a bounded function. We use the notation $D(b_1, \ldots, b_m)$ for the determinant associated with

$$H(z_1, \ldots, z_m) = \sum_{j=0}^{m} \left\{ \left| 2z_j + \sum_{k=1}^{m-j} b_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{m-j} b_{k+1} z_{k+j} \right|^2 \right\}.$$

Also $A_n = \{(z_1, \ldots z_n) \mid \sum_{j=1}^n |z_j|^2 = 1\}$ and for any form we will assume that unless otherwise stated, $Z \in A_n$.

Schur [**4**, p. 229] and Carathéodory [**1**, p. 193] proved results similar to Theorem 5 below (see also [**6**, p. 154]).

THEOREM 5. *Let $(b_1, b_2, \ldots, b_n)$ be given. Then there exists $b_{n+1}, b_{n+2}, \ldots$ such that* $\mathrm{Re}(1 + \sum_{k=1}^\infty b_k z^k) \geq 0$, $|z| < 1$, *if and only if*

$$0 \leq H(z_1, z_2, \ldots, z_n) = \sum_{j=1}^n \left\{ \left| 2z_j + \sum_{k=1}^{n-j} b_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{n-j} b_{k+1} z_{k+j} \right|^2 \right\}$$

for all sequences $\{z_k\}_{k=1}^n$.

Furthermore, if there exists $Z \in A_n$ such that $H(Z) = 0$ then there is a unique function of positive real part, $p(z) = \sum_{k=1}^\infty b_k z^k$, which is necessarily of the form

$$p(z) = \sum_{j=1}^n t_j (t + x_j z)/(1 - x_j z),$$

$|x_j| = 1$, $t_j \geq 0$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^n t_j = 1$.

PROOF. If $\mathrm{Re}(1 + \sum_{k=1}^\infty b_k z^k) \geq 0$ then Theorem 4(b) shows that $H(z_1, \ldots, z_n) \geq 0$. The remainder of the proof is broken into several lemmas. We consider the case $H(Z) = 0$ next.

LEMMA 6. (a) *If $f(z) = \xi \prod_{i=1}^n (\alpha_i - z)/(1 - \overline{\alpha}_i z)$, $|\alpha_i| \leq 1$, $i = 1, 2, \ldots, n$, then there exists $W = (w_1, \ldots, w_{n+1}) \in A_{n+1}$ such that $H_1^*(W) - H_f^*(W) = 0$.*

(b) *If $p(z) = \sum_{i=1}^n t_i (1 + x_i z)/(1 - x_i z)$, $|x_i| = 1$, $t_i \geq 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^n t_i = 1$, then there exists $Z \in A_n$ such that $H(Z) = 0$.*

PROOF. If $f(z) = g(z) \cdot (\alpha - z)/(1 - \overline{\alpha} z) = (\sum_{k=0}^\infty b_k z^k)(\alpha - z)/(1 - \overline{\alpha} z)$ then

$$H_1^*(W) - H_f^*(W) = H_1^*(W) - H_g^*(W) + (1 - |\alpha|^2) \left| \sum_{j=0}^\infty \sum_{k=0}^j b_k \overline{\alpha}^{j-k} w_{j+1} \right|^2.$$

Using this reduction n times for $f(z) = \xi \prod_{i=1}^n (\alpha_i - z)/(1 - \overline{\alpha}_i z)$ we see that

$$\begin{aligned} H_1^*(W) - H_f^*(W) &= H_1^*(W) - H_\xi^*(W) + \sum_{i=1}^n (1 - |\alpha_i|^2) \left| \sum_{j=0}^n \sum_{k=0}^j b_k^{(i)} \overline{\alpha}^{j-k} w_{j+1} \right|^2 \\ &= \sum_{i=1}^n (1 - |\alpha_i|^2) \left| \sum_{j=0}^n B_{ij} w_{j+1} \right|^2 \end{aligned}$$

where $b_k^{(i)}$ and B_{ij} depend only on the α_j. Since $|\xi| = 1$, $H_1^*(W) - H_\xi^*(W) = 0$. The system, $\sum_{j=0}^n B_{ij} w_{j+1} = 0$, $i = 1, 2, \ldots, n$, of n equations in the $n + 1$ unknowns, $w_1, w_2, \ldots, w_{n+1}$, must have a nontrivial solution. Then if $w_j^* =$

$w_j/\sum_{j=1}^{n+1}|w_j|^2$, $(w_1^*,\dots,w_{n+1}^*)\in A_{n+1}$ such that

$$\sum_{j=1}^{n+1}\left\{|w_j^*|^2-\left|\sum_{j=0}^{n-j}a_kw_{k+j}^*\right|^2\right\}=0.$$

To prove part (b) we use the one-to-one correspondence between functions of positive real part and bounded functions given by

$$p(z)=\frac{1+zf(z)}{1-zf(z)} \tag{1}$$

where $\operatorname{Re}p(z)\ge 0$, $|f(z)|\le 1$, $z\in D$. Also

$$p(z)=\sum_{j=1}^{n+1}t_j\frac{1+x_jz}{1-x_jz},\qquad |x_j|=1,\ t_j\ge 0,\ \sum_{j=1}^{n+1}t_j=1 \tag{2}$$

(where if $t_j=0$, the jth term does not appear in the sum), corresponds to

$$f(z)=\xi\prod_{j=1}^{n}\frac{\alpha_j-z}{1-\bar\alpha_jz},\qquad |\xi|=1,\ |\alpha_j|\le 1,\ j=1,2,\dots,n \tag{3}$$

(where if $|\alpha_j|=1$ then the jth factor equals α_j and may be included in ξ). If $p(z)=1+\sum_{k=1}^{\infty}b_kz^k$ and $f(z)=\sum_{k=0}^{\infty}a_kz^k$ then

$$b_{k+1}=2a_k+\sum_{m=1}^{k}b_ma_{k-m}. \tag{4}$$

That and the choice of sequence

$$w_j=2z_j+\sum_{m=1}^{\infty}b_mz_{m+j} \tag{5}$$

show that $H_1^*(w_1,\dots,w_n)-H_f^*(w_1,\dots,w_n)=H(z_1,\dots,z_n)=(\sum_{j=1}^{n}|z_j|^2)\cdot H(z_1/\sum|z_j|^2,\dots,z_n/\sum|z_j|^2)$ where $(w_1,\dots,w_n)\in A_n$ and $(z_1/\sum|z_j|^2,\dots,z_n/\sum|z_j|^2)\in A_n$.

Thus if $p(z)$ is of the form (2), we find the corresponding f of form (1) and the sequence $W\in A_{n+1}$ such that $H_1^*(W)-H_f^*(W)=0$. Define the sequence $\{z_j\}_{j=1}^{n+1}$ by (5). Then $0=H_1^*(W)-H_f^*(W)=H(z_1,\dots,z_{n+1})$ where H is the form for the given function of positive real part.

Next we assume that the coefficients $b_1,b_2,\dots,b_n$ are given such that

$$H(z_1,\dots,z_n)>0,\qquad (z_1,\dots,z_n)\in A_n.$$

If $p(z)=1+b_1z+\cdots+b_nz^n+\cdots$ exists such that $\operatorname{Re}p(z)\ge 0$, then by Theorem 4(b), the coefficient b_{n+1} must be chosen to satisfy

$$\begin{aligned}0&\le H(z_1,\dots,z_{n+1})\\&=\sum_{j=1}^{n+1}\left\{\left|2z_j+\sum_{k=1}^{n+1-j}b_kz_{k+j}\right|^2-\left|\sum_{k=0}^{n+1-j}b_{k+1}z_{k+j}\right|^2\right\}.\end{aligned}$$

This is equivalent to the nonnegativity of the determinants of the principal submatrices of the matrix $B = (\beta_{ij})_{i,j=1}^{n+1}$ where $\beta_{ii} = 4 - |b_i|^2$, $\beta_{ij} = (2\bar{b}_{ij} - b_j\bar{b}_i)$, $i > j$, and $\beta_{ji} = \overline{\beta}_{ij}$.

$$\text{(6)} \qquad B = \begin{pmatrix} 4 - |b_1|^2 & \cdots & 2b_n - \bar{b}_1 b_{n+1} \\ & 4 - |b_2|^2 & 2b_{n-1} - \bar{b}_2 b_{n+1} \\ \vdots & \vdots & \vdots \\ 2\bar{b}_n - b_1\bar{b}_{n+1} & 2\bar{b}_{n-1} - b_2\bar{b}_{n+1} \quad \cdots & 4 - |b_{n+1}|^2 \end{pmatrix}.$$

The determinant of B can be written as a function of b_{n+1} and has the form

$$0 \le \det B = c_1|b_{n+1}|^2 + 2\operatorname{Re} c_2 b_{n+1} + c_3$$

where c_1, c_2, and c_3 depend only on $b_1, b_2, \dots, b_n$. Since $\det B < 0$ for large $|b_{n+1}|$, we have $c_1 < 0$ and we can write $|b_{n+1} - a| \le r$ where a and r depend only on c_1, c_2, c_3, that is, on $b_1, b_2, \dots, b_n$.

If $H(z_1, \dots, z_{n+1}) = 0$ for some $Z \in A_{n+1}$ then $\det B = 0$ and $|b_{n+1} - a| = r$. For these particular coefficients $b_1, b_2, \dots, b_n$ it is necessary to choose b_{n+1} on the boundary of the disk.

If $H(z_1, \dots, z_n) = 0$ for some $Z \in A_n$ then the choice of b_{n+1} is unique. To see this, we may assume that $z_n \ne 0$ and define $z^* = -\frac{1}{2}(b_1 z_1 + \cdots + b_n z_n)$. Then

$$0 \le H(z^*, z_1, \dots, z_n) = -|b_1 z^* + b_2 z_1 + \cdots + b_{n+1} z_n|^2.$$

Hence $b_1 z^* + \cdots + b_{n+1} z_n = 0$, $b_{n+1} = -(b_1 z^* + b_2 z_1 + \cdots + b_n z_{n-1})/z_n$ and $H(z^*, z_1, \dots, z_n) = 0$. This completes the proof of the next two lemmas.

LEMMA 7. *Let $b_1, b_2, \dots, b_n$ be given such that*

$$\begin{aligned} 0 &\le H(z_1, \dots, z_n) \\ &= \sum_{j=1}^{n} \left\{ \left| 2z_j + \sum_{k=1}^{n-j} b_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{n-j} b_{k+1} z_{k+j} \right|^2 \right\}. \end{aligned}$$

If there exists a function of positive real part, $p(z) = 1 + \sum_{k=0}^{\infty} b_k z^k$, then $|b_{n+1} - a| \le r$ where the values of a and r depend on $b_1, b_2, \dots, b_n$. If $H(Z) = 0$ for some $Z \in A_n$ then the choice for b_{n+1} is unique.

LEMMA 8. *If $H(z_1, \dots, z_n) = 0$ for some $(z_1, \dots, z_n) \in A_n$ and $H(z_1, \dots, z_{n+1}) \ge 0$ for all $(z_1, \dots, z_{n+1}) \in A_{n+1}$, then there exists $(z_1, z_2, \dots, z_{n+1}) \in A_{n+1}$, $z_{n+1} \ne 0$, such that $H(z_1, \dots, z_{n+1}) = 0$.*

LEMMA 9. *Let b_n be given such that $0 \le H(z_1, \dots, z_n) = \sum_{j=1}^{n} |2z_j|^2 - |b_n z_n|^2$, that is $b_1 = b_2 = \cdots = b_{n-1} = 0$. Then $p(z) = (1 + xz^n)/(1 - xz^n)$ where $x = \exp i(\arg b_n)$.*

PROOF. Since

$$D(b_1, b_2, \dots, b_n) = \begin{vmatrix} 4 & 0 & \cdots & 0 \\ 0 & 4 & \cdots & 0 \\ \vdots & & & \\ 0 & & \cdots & 4 - |b_n|^2 \end{vmatrix} = 0$$

we have $|b_n| = 2$. Choose θ so that $b_n = 2e^{in\theta}$ and let

$$p(z) = (1 + e^{in\theta} z^n)/(1 - e^{in\theta} z^n)$$
$$= \sum_{j=1}^{n} \left(\frac{1}{n}\right) (1 + ze^{i(\theta+2\pi j/n)})/(1 - ze^{i(\theta+2\pi j/n)}).$$

Then $\operatorname{Re} p(z) > 0$ and by Lemmas 7 and 8 it is unique.

We now assume that at least one $b_j \neq 0,\ 1 \leq j \leq n-1$. For this case the proof is by induction on n. For $n = 1, b_1$ is given and

$$0 = H(z_1, 0, \ldots, 0) = |2z_1|^2 - |b_1 z_1|^2$$

implies that $b_1 = 2e^{i\gamma}$. Then in order for $p(z)$ to have positive real part, it is necessary that

$$0 \leq H(z_1, z_2) = |2z_1 - 2e^{i\gamma} z_2|^2 - |2e^{i\gamma} z_1 + b_2 z_2|^2.$$

When $z_1 = -e^{i\gamma} z_2$, we find that $-|z_2|^2 |b_2 - 2e^{i2\gamma}|^2 \geq 0$ so that $b_2 = 2e^{i2\gamma}$. Similarly, $b_k = 2e^{ik\gamma}$, and $p(z) = (1 + e^{i\gamma} z)/(1 - e^{i\gamma} z)$.

Assume that if $H(Z) = 0,\ Z \in A_{n-1}$ then

$$p(z) = \sum_{j=1}^{n-1} t_j (1 + x_j z)/(1 - x_j z), \qquad t_j \geq 0,\ |x_j| = 1.$$

We prove the statement for n: if $b_1, b_2, \ldots, b_n$ are given and not all zero and $Z \in A_n$ satisfies $H(Z) = 0$ then p has the required decomposition. We will also assume that $H(z_1, \ldots, z_k) > 0$, for all $(z_1, \ldots, z_k) \in A_k,\ k = 1, 2, \ldots, n-1$. Otherwise the induction hypothesis applies. Lemma 10 explains how the induction hypothesis will be used and proves Theorem 5 in the case $H(z_1, \ldots, z_n) > 0$ for all $(z_1, \ldots, z_n) \in A_n$.

LEMMA 10. *If $H(z_1, \ldots, z_{n-1})$ has a positive minimum on the compact set A_{n-1} then there is a unique function*

$$Q(z) = \sum_{j=1}^{n-1} t_j \frac{1 + x_j z}{1 - x_j z} = 1 + c_1 z + c_2 z^2 + \cdots$$

of positive real part such that

$$tQ(z) + 1 - t = 1 + b_1 z + \cdots + b_{n-1} z^{n-1} + t c_n z^n + \cdots$$

where $0 < t < 1$.

PROOF. Consider the polynomial of degree $2m$, $t^{2m} D(b_1/t, \ldots, b_m/t)$ for each $m \leq n-1$, (see (6) for the matrix). If at least one $b_i \neq 0$ then $|b_i/t| > 2$ for $t > 0$ small enough. Thus $D(b_1/t, \ldots, b_m/t) < 0$ because

$$f(z) = 1 + (b_1/t)z + \cdots + (b_i/t)z^i + \cdots$$

cannot have positive real part. If $t = 1$ then since

$$\min_{A_{n-1}} H(z_1, \ldots, z_{n-1}) > 0$$

we have $D(b_1/t, \ldots, b_m/t) > 0$ and the polynomial $t^{2m} D(b_1/t, \ldots, b_m/t)$ has a root between $t = 0$ and $t = 1$. If $b_1 = b_2 = \cdots = b_m = 0$ then $t^{2m} D(b_1/t, \ldots, b_m/t) =$

$t^{2m} \cdot 4^m > 0$ for $t > 0$. Let t^* be the largest root such that for $t^* < t \leq 1$, $t^{2m}D(b_1/t, \ldots, b_m/t) > 0$ for each $m \leq n-1$. This implies that there exists $Z \in A_{n-1}$ such that

$$0 = H_{t^*}(Z) = \sum_{j=0}^{n-1} \left\{ \left| 2z_j + \sum_{k=1}^{n-1-j} (b_k/t^*) z_{k+j} \right|^2 - \left| \sum_{k=0}^{n-1-j} (b_{k+1}/t^*) z_{k+j} \right|^2 \right\}$$

and that $H_{t^*}(Z) \geq 0$ for all $Z \in A_n$. Thus by the induction hypothesis there exists

$$Q_{t^*}(z) = 1 + \frac{b_1}{t^*} z + \cdots + \frac{b_{n-1}}{t^*} z^{n-1} + \cdots = \sum_{j=1}^{n-1} t_j \frac{1 + x_j z}{1 - x_j z},$$

$t_j \geq 0$, $|x_j| = 1$, $\sum_{j=1}^n t_j = 1$.

Next we define for each $\phi \in [0, 2\pi]$ a function of the form (2) depending on ϕ. From this new function we obtain $b_n(\phi)$, that is the nth coefficient as a function of ϕ. The rest of the proof will show that every possible value of b_n is attained by $b_n(\phi)$.

LEMMA 11. *Given $\phi \in [0, 2\pi]$, there exists s, $0 < s < 1$, $b_n(\phi)$ and*

$$\begin{aligned} Q(z) &= 1 + \frac{b_1 - 2se^{i\phi}}{1 - s} z + \cdots + \frac{b_n(\phi) - 2se^{in\phi}}{1 - s} z^n + \cdots \\ &= \sum_{j=1}^{n-1} t_j \frac{1 + x_j z}{1 - x_j z}, \end{aligned}$$

$t_j \geq 0$, $|x_j| = 1$, $\sum_{j=1}^n t_j = 1$, *such that* $\operatorname{Re} Q(z) \geq 0$ *and*

$$\begin{aligned} (1-s)Q(z) + s \cdot \frac{1 + e^{i\phi} z}{1 - e^{i\phi} z} &= \sum_{j=1}^{n-1} (1-s) t_j \frac{1 + x_j z}{1 - x_j z} + s \cdot \frac{1 + e^{i\phi} z}{1 - e^{i\phi} z} \\ &= 1 + b_1 z + \cdots + b_{n-1} z^{n-1} + b_n(\phi) z^n + \cdots. \end{aligned}$$

Furthermore, $b_n(\phi)$ is a continuous function of ϕ.

PROOF. For each $\phi \in [0, 2\pi]$ and $s \in [0, 1]$ define $\beta_k = b_k - 2se^{ik\phi}$, $k = 1, 2, \ldots, n-1$. Then $(1-s)^{2m} D(\beta_1/(1-s), \ldots, \beta_m/(1-s))$ is a polynomial of degree $(2m)$ in s and a polynomial in $e^{i\phi}$. (The matrix is given in (6).) When $s = 0$, $\beta_k = b_k$, $k \leq m$ and $D(b_1, \ldots, b_m) > 0$ for each $m \leq n-1$. As $s \to 1$, $|\beta_m/(1-s)| > 2$ and $D(\beta_1/(1-s), \ldots, \beta_m/(1-s)) < 0$ for each m. Choose s^* so that for $0 \leq s < s^*$, $(1-s)^{2m} D(\beta\ /(1-s), \ldots, \beta_{\ n}/(1-s)) > 0$ for each $m \leq n-1$. When $s = s^*$, at least one of the polynomials $(1-s)^{2m} D(\beta_1/(1-s), \ldots, \beta_m/(1-s)) = 0$. Thus

$$H_{s^*}(Z) = \sum_{j=1}^{n-1} \left\{ \left| 2z_j + \sum_{k=1}^{n-1-j} \beta_k/(1-s^*) z_{k+j} \right|^2 - \left| \sum_{k=0}^{n-1-j} \beta_{k+1}/(1-s^*) z_{k+j} \right|^2 \right\} = 0$$

for some $Z \in A_{n-1}$ and $H_{s^*}(z) \geq 0$ for all $Z \in A_{n-1}$. Thus by the induction hypothesis

$$Q_{s^*}(z) = 1 + \frac{\beta_1}{1-s^*}z + \cdots + \frac{\beta_{n-1}}{1-s^*}z^{n-1} + \cdots = \sum_{j=1}^{n-1} t_j \frac{1+x_j z}{1-x_j z}$$

where $t_j \geq 0$, $|x_j| = 1$, $\sum_{j=1}^{n-1} t_j = 1$. Since $\beta_k = b_k - 2se^{ik\phi}$ we have $(1-s^*)Q_{s^*}(z) + s^*(1+e^{i\phi}z)/(1-e^{i\phi}z) = 1 + b_1 z + \cdots + b_n(\phi)z^n + \cdots$.

Also, since $(1-s)^{2m}D(\beta_1/(1-s), \ldots, \beta_m/(1-s))$ is a polynomial in s and in $e^{i\phi}$ we know that the choice of s^* depends continuously on the coefficients of the polynomial and hence on ϕ. Thus the coefficients of Q_{s^*}, $\beta_k/(1-s^*) = (b_k - 2s^*e^{ik\phi})/(1-s^*)$, $k \geq 1$, depend continuously on ϕ and so does $b_n(\phi)$.

By Lemma 7, $b_n(\phi) = a + re^{i\gamma(\phi)}$ where a and r depend only on $b_1, b_2, \ldots, b_{n-1}$. Let $\Gamma = \{e^{i\gamma(\phi)}\colon\ 0 \leq \phi \leq 2\pi\}$. The path Γ is closed because $b_n(\phi)$ continuous implies that $\gamma(\phi)$ is continuous. It is a nonempty subset of the unit circle by Lemma 11. If it is not an open subset of the unit circle then there exists $\gamma_0 \in \Gamma$ such that for all $\varepsilon > 0$, $\{e^{i\gamma}\colon\ |\gamma - \gamma_0| < \varepsilon\} \in \Gamma$. Essentially, the path reverses direction at γ_0 and retraces itself along the unit circle. Thus there exist ϕ_1 and ϕ_2 such that $\phi_1 < \phi_0 < \phi_2$ and $e^{i\gamma(\phi_1)} = e^{i\gamma(\phi_2)}$, that is $b_n(\phi_1) = b_n(\phi_2)$. From Lemma 7 we see that the two functions $1 + b_1 z + \cdots + b_n(\phi_i)z^n + \cdots$, $i = 1, 2$, are identical and by Lemma 11 we have

$$\begin{aligned}\sum_{j=1}^{n-1}(1-s)t_j\frac{1+x_j z}{1-x_j z} + s\cdot\frac{1+e^{i\phi_1}z}{1-e^{i\phi_1}z} &= 1 + b_1 z + \cdots + b_n(\phi_1)z^n + \cdots \\ &= 1 + b_1 z + \cdots + b_n(\phi_2)z^n + \cdots \\ &= \sum_{j=1}^{n-1}(1-r)\frac{1+y_j z}{1-y_j z} + r\cdot\frac{1+e^{i\phi_2}z}{1-e^{i\phi_2}z}.\end{aligned}$$

That is, the same function, $1 + b_1 z + \cdots + b_n(\phi_1)z^n + \cdots$, has two different decompositions (2). A function of the form (2) cannot have two such representations because of the uniqueness of the probability measure in the integral representation of functions of positive real part [**3**, p. 4]. This contradiction shows that Γ is an open subset of the unit circle.

Hence, Γ is closed, open, and nonempty and must be the entire unit circle. Therefore $b_n(\phi)$ is the entire circle, $|b_n - a| = r$, found in Lemma 7. Thus for each given b_n, we can find $b_n = b_n(\phi)$ and

$$1 + b_1 z + \cdots + b_n z^n + \cdots = \sum_{j=1}^{n} t_j \frac{1+x_j z}{1-x_j z},$$

$|x_j| = 1$, $t_j \geq 0$, $\sum_{j=1}^{n} t_j = 1$, as required. This finishes the proof of the next lemma and of Theorem 5.

LEMMA 12. *The path $b_n(\phi)$, $\phi \in [0, 2\pi]$ given in Lemma* 11 *maps to the entire circle, $|b_n - a| = r$ found in Lemma* 6.

REMARK 1. Because of the correspondence (1) a similar result holds for bounded functions. Schur proved a comparable result, [**4**, pp. 221–227] (see also [**6**, p. 159]).

THEOREM 13. *Let $(a_0, a_1, \dots, a_n)$ be given. Then there exists $a_{n+1}, a_{n+2}, \dots$ such that $|f(z)| \le 1$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ if and only if*

$$0 \le H_1^*(z_0, z_1, \dots, z_n) - H_f^*(z_0, z_1, \dots, z_n) = \sum_{j=0}^{n} \left\{ |z_j|^2 - \left| \sum_{k=0}^{n-j} z_k z_{k+j} \right|^2 \right\}.$$

Furthermore if $H_1^*(z_0, \dots, z_n) = H_f^*(z_0, \dots, z_n)$ for $(z_0, \dots, z_n) \in A_{n+1}$ then $f(z) = \xi \prod_{j=1}^{n} (\alpha_j - z)/(1 - \overline{\alpha}_j z)$, $|\alpha_j| \le 1$, $|\xi| = 1$.

REMARK 2. Similarly, there is a one-to-one correspondence between the Hermitian form for a bounded function and the Hermitian form for a function, $g(z) = \sum_{k=0}^{\infty} b_k z^k$, majorized by a fixed function, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, given by

$$H_f^*(Z) - H_g^*(Z) = H_1^*(W) - H_{g/f}^*(W)$$

where $w_j = \sum_{k=j}^{n} a_{k-j} z_k$ and $(g/f)(z) = \sum_{k=0}^{\infty} c_k z^k$ is defined by

$$b_k = \sum_{m=0}^{k} c_m a_{k-m}.$$

This correspondence and Remark 1 imply the following (see also [**4**, p. 221]).

THEOREM 14. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be fixed and $(b_0, b_1, \dots, b_n)$ be given. Then there exists $g(z) = \sum_{k=0}^{\infty} b_k z^k$ such that $|g(z)| \le |f(z)|$ if and only if*

$$0 \le H_f^*(z_0, \dots, z_n) - H_g^*(z_0, \dots, z_n) = \sum_{j=0}^{n} \left\{ \left| \sum_{k=0}^{n-j} a_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{n-j} b_k z_{k+j} \right|^2 \right\}$$

for all $Z \in V_n$. Furthermore if $H_f^*(z_0, \dots, z_n) = H_g^*(z_0, \dots, z_n)$ for some sequence $\{z_k\}_{k=0}^{n} \in A_{n+1}$ then

$$g(z) = \xi \cdot f(z) \cdot \prod_{i=1}^{n} \frac{\alpha_i - z}{1 - \overline{\alpha}_i z},$$

$|\xi| = 1$, $|\alpha_i| \le 1$, $i = 1, 2, \dots, n$.

REMARK 3. Similar results can be derived for the other classes of functions in Theorem 4 by using the correspondence between them and functions of positive real part. Also, Lemma 7 which shows that all possible values for the nth coefficient (given the preceding coefficients) lie in a disk can be proved for other classes of functions. The proof does not depend on any special characteristic of functions of positive real part. In the next theorem we use this lemma to show that the solution of certain kinds of extremal problems on the class of functions with positive real part is in the subclass of functions for which $H(Z) = 0$, $Z \in A_n$. By first proving an analogue of Lemma 7 for some other classes of functions we can also prove a theorem similar to Theorem 15 below, for these classes.

THEOREM 15. *Let F be a nonconstant, analytic function of $z_1, \dots, z_n$ and let*

$$F(b_1, \dots, b_n) = \max_{\operatorname{Re} f \geq 0} |F(c_1, c_2, \dots, c_n)|$$

where $f(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$, $f_0(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ *and* $\operatorname{Re} f \geq 0$, $\operatorname{Re} f_0 \geq 0$. *Then*

$$f_0(z) = \sum_{j=1}^{n} t_j \frac{1 + x_j z}{1 - x_j z},$$

$|x_j| = 1$, $t_j \geq 0$, $j = 1, 2, \dots, n$, $\sum_{j=1}^{n} t_j = 1$. *Equivalently, there exists $W \in V_n$ such that*

$$0 = H(W) = \sum_{j=1}^{n} \left\{ \left| 2w_j + \sum_{k=1}^{n-j} b_k w_{k+j} \right|^2 - \left| \sum_{k=0}^{n-j} b_{k+1} w_{k+j} \right|^2 \right\}.$$

PROOF. The maximum is attained because the class of functions with positive real part is compact, [**3**, p. 2]. Theorem 5 shows the equivalence of $H(W) = 0$ and the special form of f_0. We show that $H(W) \neq 0$ for every $W \in A_n$ implies that F is constant.

If $0 < \min_{W \in A_n} H(W)$ where H is the Hermitian form for $f_0(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$ then Lemma 7 shows that if $b_1, \dots, b_{n-1}$ are fixed, we have $|b_n - a| < r$ where a and r depend only on $b_1, \dots, b_{n-1}$ and are found from $D(b_1, \dots, b_n)$. That is, b_n lies in the interior of the disk of possible values for b_n. If $f(z) = 1 + \sum_{k=1}^{n-1} b_k z^k + \sum_{k=n}^{\infty} c_k z^k$ and $\operatorname{Re} f \geq 0$ then for $b_1, \dots, b_{n-1}$ fixed we have

$$\begin{aligned} |F(b_1, b_2, \dots, b_n)| &= \max_f |F(b_1, b_2, \dots, b_{n-1}, c_n)| \\ &= \max_{|z-a| \leq r} |F(b_1, b_2, \dots, b_{n-1}, z)|. \end{aligned}$$

Since the analytic function of z, $F(b_1, \dots, b_{n-1}, z)$, attains its maximum modulus in the interior of the disk, $|z - a| \leq r$, it must be a constant function. Thus F is a function of $z_1, z_2, \dots, z_{n-1}$ only.

Since

$$\left\{ \sum_{k=1}^{n-1} |w_k|^2 = 1, \ w_m = 0 \right\} \subseteq \left\{ \sum_{k=1}^{n} |w_k|^2 = 1 \right\} \subseteq A_n$$

we have

$$0 < \min_{W \in A_n} H(w_1, \dots, w_n) < \min_{W \in A_{n-1}} H(w_1, \dots, w_{n-1}, 0).$$

An application of the argument with n replaced by $n-1$ shows that F is a function of $z_1, z_2, \dots, z_{n-2}$ only and $0 < \min_{W \in A_{n-2}} H(w_1, \dots, w_{n-2}, 0, 0)$. Continuing in the same manner we find that F is a constant function.

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